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Algebraic Symmetry and Self–Duality of an Open ASEP

Jeffrey Kuan[1]

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Abstract
We consider the asymmetric simple exclusion process (ASEP) with open boundary condition at the left boundary, where particles exit at rate γ and enter at rate $\alpha = \gamma\tau^2$, and where τ is the asymmetry parameter in the bulk. At the right boundary, particles neither enter nor exit. By mapping the generator to the Hamiltonian of an XXZ quantum spin chain with reflection matrices, and using previously known results, we show algebraic symmetry and self–duality for the model.

1 Introduction

The asymmetric simple exclusion process (ASEP) is an interacting particle system on a one dimensional lattice, introduced in [1] and [2]. Particles jump one step to the right at rate p and one step to the left at rate q, and the jump is blocked if the site is already occupied. As first observed in [3], the generator of ASEP can be mapped to the Hamiltonian of the XXZ quantum spin chain, which (with closed boundary conditions) possesses a quantum group symmetry [4]. Using this symmetry, [5] proves a Markov self–duality for the ASEP with closed boundary conditions. Various modifications and generalizations of this self–duality have since been found [6–12].

A natural extension is to consider open boundary conditions, where particles may enter or exit the lattice. Let α, γ denote the entry and exit rates at the left boundary, and let β, δ denote the entry and exit rates at the right boundary. With open boundary conditions, the quantum group symmetry is broken. However, it turns out that for $\alpha/\gamma = p/q$, $\beta = \delta = 0$, a specific algebra element still commutes with the Hamiltonian [13]. Here, we use this algebra element to show a self–duality for this open ASEP.

✉ Jeffrey Kuan
jkuan@math.tamu.edu

1 Texas A & M University System, College Station, TX, USA

2 Preliminaries

2.1 Spin Chain Notation

Recall that $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which are a basis for $\mathfrak{sl}_2$, the traceless 2×2 matrices. Also define

$$\begin{aligned} \sigma^+ &= \tfrac{\sigma^x + i\sigma^y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \sigma^- &= \tfrac{\sigma^x - i\sigma^y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ n &= \tfrac{1-\sigma^z}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The subscripts $_j$ indicates that a matrix acts at lattice site j. For example, σ_j^x acts on $(\mathbb{C}^2)^{\otimes L}$ as $1^{\otimes j-1}\sigma^x \cdots 1^{\otimes L-j}$. The –ket vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponds to a *particle* and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to a *hole*. The operators σ_j^- and σ_j^+ are creation and annihilation operators, respectively.

Recall the Yang–Baxter equation

$$R_{12}(\lambda_1 - \lambda_2)R_{13}(\lambda_1 - \lambda_3)R_{23}(\lambda_2 - \lambda_3) = R_{23}(\lambda)R_{12}(\lambda_1)R_{12}(\lambda_1 - \lambda_2)$$

and the reflection equation [14]

$$R_{12}(\lambda_1 - \lambda_2)\mathcal{K}_1(\lambda_1)R_{21}(\lambda_1 + \lambda_2)\mathcal{K}_2(\lambda_2) = \mathcal{K}_2(\lambda_2)R_{12}(\lambda_1 + \lambda_2)\mathcal{K}_1(\lambda_1)R_{21}(\lambda_1 - \lambda_2)$$

The R–matrix of the XXZ model is a one–parameter solution to the Yang–Baxter equation, with the parameter denoted by μ (in addition to λ). A solution to the reflection equation is given in [13], which has three parameters, denoted μ, m, ζ (in addition to λ). In the most general setting, these parameters may be arbitrary complex numbers. For the probabilistic applications considered in this paper, we restrict to $\mu \in i\mathbb{R}$ and $m, \lambda, \zeta \in \mathbb{R}$.

The transfer matrix constructed from $R, \mathcal{K}$ leads to a Hamiltonian [15], which is stated as (1.3) from [13]:

$$\begin{aligned} \mathcal{H} = &-\frac{1}{4}\sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \cosh i\mu\, \sigma_i^z\sigma_{i+1}^z\right) - \frac{1}{4}\sinh i\mu\left(\sigma_L^z - \sigma_1^z\right) \\ &-\frac{L+1}{4}\cosh i\mu \\ &+\frac{\sinh i\mu}{4\sinh i\mu\left(\frac{m}{2}+\zeta\right)\cosh i\mu\left(\frac{m}{2}-\zeta\right)}\left(-\sinh(im\mu)\sigma_1^z + \sigma_1^x\right) + c_1 + c_2\sigma_L^z, . \end{aligned}$$

where $c_1 = (q + q^{-1})^{-1}/2$, $c_2 = 0$. This Hamiltonian is integrable, in the sense that it commutes with a family of transfer matrices: see (2.30) and (4.28) of [13].

2.2 Quantum Groups

Let A be the Cartan matrix of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

with entries denoted a_{ij}. The Drinfeld–Jimbo quantum group $\mathcal{U}_\tau(\widehat{\mathfrak{sl}}_2)$ is the bi–algebra generated by $\{e_i, f_i, k_i\}_{i=1,2}$ with relations

$$k_i k_j = k_j k_i, \qquad k_i e_j = \tau^{\frac{1}{2}a_{ij}} e_j k_i, \qquad k_i f_j = \tau^{-\frac{1}{2}a_{ij}} f_j k_i$$
$$\left[e_i, f_j\right] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{\tau - \tau^{-1}}, \quad i, j = 1, 2,$$

and

$$\chi_i^3 \chi_j - (\tau^2 + 1 + \tau^{-2}) \chi_i^2 \chi_j \chi_i + (\tau^2 + 1 + \tau^{-2}) \chi_i \chi_j \chi_i^2 - \chi_j \chi_i^3 = 0, \qquad \chi_i = e_i, f_i, \quad i \neq j.$$

The co–product is defined by

$$\Delta(\chi_i) = k_i \otimes \chi_i + \chi_i \otimes k_i^{-1}, \quad \chi_i = e_i, f_i, \qquad \Delta(k_i^{\pm}) = k_i^{\pm} \otimes k_i^{\pm},$$

and satisfies the co–associativity

$$\Delta^{(L)} = (\Delta^{(L-1)} \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta^{(L-1)}) \circ \Delta.$$

The evaluation module $\rho_\lambda : \mathcal{U}_\tau(\widehat{\mathfrak{sl}}_2) \to \mathrm{End}(\mathbb{C}^2)$ is given by

$$\rho_\lambda(k_1) = \tau^{\sigma^z/2}, \quad \rho_\lambda(e_1) = \sigma^+, \quad \rho_\lambda(f_1) = \sigma^-$$
$$\rho_\lambda(k_2) = \tau^{-\sigma^z/2}, \quad \rho_\lambda(e_2) = e^{-2\lambda}\sigma^-, \quad \rho_\lambda(f_2) = e^{2\lambda}\sigma^+.$$

2.3 Duality

Two Markov processes $X(t)$ and $Y(t)$ on state spaces $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, are *dual* with respect to a function $D(\cdot, \cdot)$ on $\mathfrak{X} \times \mathfrak{Y}$ if

$$\mathbb{E}_x[D(X(t), y)] = \mathbb{E}_y[D(x, Y(t))]$$

for all initial conditions $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$ and all times $t \geq 0$. An equivalent definition for duality on discrete state spaces $\mathfrak{X}$ and $\mathfrak{Y}$ can be given in terms of intertwining of generators. Namely, view the generator[1] $\mathcal{L}_X$ of $X(t)$ as a matrix with rows and columns indexed by $\mathfrak{X}$. Similarly, view the generator $\mathcal{L}_Y$ of $Y(t)$ as a matrix with

[1] Here, we use the mathematical physics convention that a stochastic matrix has columns that add up to 1, rather than its rows. Accordingly, the generator has columns that add up 1, rather than its rows.

rows and columns indexed by $\mathfrak{Y}$. Letting D be the matrix with rows indexed by $\mathfrak{X}$, columns indexed by $\mathfrak{Y}$, and entries $D(x, y)$, the duality can be stated as

$$\mathcal{L}_X^* D = D\mathcal{L}_Y.$$

Here the $*$ denotes transposition.

3 Main Results

Suppose that an ASEP on L lattice sites evolves with right jump rates p and left jump rates q (without assuming $p + q = 1$). Particles enter from the left at rate α, exit at the right at rate β, exit at the left at rate γ, and enter from the right at rate δ. Define the asymmetry parameter

$$\tau = \sqrt{\frac{p}{q}}.$$

We will assume that $\tau \neq 0, 1$, so that we are not considering the totally asymmetric case or the symmetric case. More precisely, define the generator of ASEP to be the operator

$$\mathcal{L} = -\sqrt{pq}\sum_{j=1}^{L}\Big(\tau^{-1}(\sigma_j^-\sigma_{j+1}^+ - (1-n_j)n_{j+1}) + \tau(\sigma_{j+1}^-\sigma_j^+ - n_j(1-n_{j+1}))\Big)$$
$$-\alpha(\sigma_1^- - 1 + n_1) - \gamma(\sigma_1^+ - n_1) - \delta(\sigma_L^- - 1 + n_L) - \beta(\sigma_L^+ - n_L).$$

In the ASEP to XXZ change of basis [3], the generator of ASEP becomes the XXZ Hamiltonian (see also e.g. Equations (2.12)–(2.14) of [16]). More specifically, let V denote the operator

$$V = \tau^{-\sum_{j=1}^{L} j n_j}.$$

Then

$$V\mathcal{L}V^{-1} = -\frac{1}{2}\sqrt{pq}\sum_{j=1}^{L-1}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \frac{\tau+\tau^{-1}}{2}\sigma_j^z\sigma_{j+1}^z - \frac{\tau+\tau^{-1}}{2}\right]$$
$$-A_1^+\sigma_1^x - iA_1^-\sigma_1^y - B_1\sigma_1^z - A_L^+\sigma_L^x - iA_L^-\sigma_L^y - B_L\sigma_L^z + \frac{1}{2}(\alpha+\beta+\gamma+\delta),$$

where

$$A_1^{\pm} = \frac{1}{2}(\gamma\tau \pm \alpha\tau^{-1}), \qquad B_1 = \frac{1}{2}(\gamma - \alpha) + \frac{1}{4}(\tau - \tau^{-1}),$$
$$A_L^{\pm} = \frac{1}{2}(\beta\tau^L \pm \delta\tau^{-L}), \qquad B_L = \frac{1}{2}(\beta - \delta) - \frac{1}{4}(\tau - \tau^{-1}).$$

Note that

$$A_1^- = 0 \text{ if and only if } \frac{\alpha}{\gamma} = \frac{p}{q} \text{ or } \alpha = \gamma = 0,$$
$$A_L^- = 0 \text{ if and only if } \frac{\delta}{\beta} = \left(\frac{p}{q}\right)^L \text{ or } \beta = \delta = 0.$$

Recall that $\mathcal{H}$ depends on μ, m and ζ.

Proposition 3.1 *Fix* $m = -1$, *and let* ζ *denote the unique real solution of the equation*

$$\gamma\tau = \frac{\tau - \tau^{-1}}{\tau + (\tau^{-2\zeta} - \tau^{2\zeta}) - \tau^{-1}}.$$

Then there exists a constant C *such that*

$$2\mathcal{H} = V\mathcal{L}V^{-1} + C\mathbb{I}$$

for the values $m = -1$, $\beta = \delta = 0$, $\alpha/\gamma = p/q$, $\sqrt{pq} = 1$, $\tau = e^{-i\mu}$.

Proof First we establish that the equation

$$\gamma\tau = \frac{\tau - \tau^{-1}}{\tau + (\tau^{-2\zeta} - \tau^{2\zeta}) - \tau^{-1}}.$$

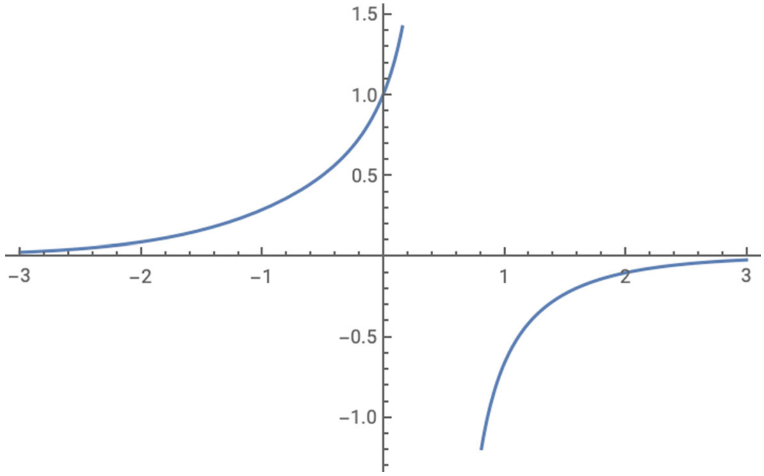
has a unique real solution in the parameter ζ. Let

$$f_\tau(\zeta) = \frac{\tau - \tau^{-1}}{\tau + (\tau^{-2\zeta} - \tau^{2\zeta}) - \tau^{-1}}$$

It suffices to snow that f_τ is a bijection from $\mathbb{R} - \{1/2\}$ to $\mathbb{R} - \{0\}$, because then $\zeta = f_\tau^{-1}(\gamma\tau)$ (recall that $\tau \neq 0$ by assumption). To do this, it is sufficient to show that

$$\begin{aligned} f_\tau'(\zeta) &> 0 \text{ for all } \zeta \neq 1/2, \\ \lim_{\zeta\to-\infty} f_\tau(\zeta) &= \lim_{\zeta\to\infty} f_\tau(\zeta) = 0, \\ \lim_{\zeta\to\frac{1}{2}^-} f_\tau(\zeta) &= \infty, \\ \lim_{\zeta\to\frac{1}{2}^+} f_\tau(\zeta) &= -\infty. \end{aligned}$$

More specifically, since $f_\tau(0) = 1$ and $f_\tau(\zeta) \neq 0$ for all ζ (by the assumption that $\tau \neq 1$), we have that f_τ maps $(-\infty, 1/2)$ bijectively to $(0, \infty)$ and maps $(1/2, \infty)$ bijectively to $(-\infty, 0)$. See the image below for the example of $f_2(\zeta)$.



It is straightforward to verify that $f_\tau'(\zeta) > 0$ for all $\zeta \neq 1/2$, since

$$f_\tau'(\zeta) = \frac{-(\tau - \tau^{-1})(\log(\tau^{-2})\tau^{-2\zeta} - \log(\tau^2)\tau^{2\zeta})}{(\tau + (\tau^{-2\zeta} - \tau^{2\zeta}) - \tau^{-1})^2},$$

which has non–negative numerator and denominator. The limits are likewise straightforward to verify.

Next, we need to match the coefficients of the operators in $2\mathcal{H}$ and $V\mathcal{L}V^{-1}$. It is immediate that the coefficients of $\sigma_j^x\sigma_{j+1}^x$ and $\sigma_j^y\sigma_{j+1}^y$ match. Setting $\tau = e^{-i\mu}$, we have

$$\cosh i\mu = \frac{e^{i\mu} + e^{-i\mu}}{2} = \frac{\tau + \tau^{-1}}{2}, \qquad \sinh i\mu = \frac{e^{i\mu} - e^{-i\mu}}{2} = -\frac{\tau - \tau^{-1}}{2},$$

showing that the coefficient of $\sigma_j^z\sigma_{j+1}^z$ matches. If furthermore, $\beta = \delta = 0$, then $A_L^\pm = 0$ and there is no σ_L^x, σ_L^y contribution, so the σ_L^x, σ_L^y coefficient matches. For $\alpha = (p/q)\gamma = \gamma\tau^2$, the term A_1^- equals zero, so the σ_1^y coefficient matches. Note that when $c_2 = 0$, the coefficient of σ_L^z in $2\mathcal{H}$ becomes

$$-\frac{1}{2}\sinh i\mu = \frac{1}{4}(\tau - \tau^{-1}),$$

which equals $-B_L$ with $\beta = \delta$. Thus the coefficient of σ_L^z matches.

To complete the proof, we next need to match the coefficients of σ_1^x and σ_1^z, meaning that there are two equalities to show. So comparing the σ_1^z terms, it remains to show

$$-B_1 = -\frac{1}{2}(\gamma - \alpha) - \frac{1}{4}(\tau - \tau^{-1}) = \frac{1}{2}\sinh i\mu - \frac{\sinh i\mu \ \sinh(im\mu)}{2\sinh i\mu\left(\frac{m}{2} + \zeta\right)\cosh i\mu\left(\frac{m}{2} - \zeta\right)}$$

and comparing the σ_1^x terms, it remains to show

$$-A_1^+ = -\frac{1}{2}(\gamma\tau + \alpha\tau^{-1}) = -\gamma\tau = \frac{\sinh i\mu}{2\sinh i\mu\left(\frac{m}{2} + \zeta\right)\cosh i\mu\left(\frac{m}{2} - \zeta\right)}.$$

In the former equality, the term $\frac{1}{2}\sinh i\mu$ cancels $-\frac{1}{4}(\tau - \tau^{-1})$, so we are left to show

$$\frac{1}{2}(\gamma - \alpha) = \frac{\sinh i\mu \ \sinh(im\mu)}{2\sinh i\mu\left(\frac{m}{2} + \zeta\right)\cosh i\mu\left(\frac{m}{2} - \zeta\right)}.$$

Now, using that $m = -1$, we have

$$\sinh(im\mu) = -\frac{\tau^m - \tau^{-m}}{2} = -\frac{1 - \tau^2}{2\tau} = -\frac{\gamma - \alpha}{2\gamma\tau},$$

which means that the two equalities are equivalent to each other. So it just remains to show that

$$\begin{aligned}\gamma\tau &= -\frac{\tau-\tau^{-1}}{(\tau^{-1/2}e^{i\mu\zeta}-\tau^{1/2}e^{-i\mu\zeta})(\tau^{-1/2}e^{-i\mu\zeta}+\tau^{1/2}e^{i\mu\zeta})}\\ &= -\frac{\tau-\tau^{-1}}{\tau^{-1}+(e^{2i\mu\zeta}-e^{-2i\mu\zeta})-\tau}\\ &= \frac{\tau-\tau^{-1}}{\tau+(\tau^{-2\zeta}-\tau^{2\zeta})-\tau^{-1}},\end{aligned}$$

which we have assumed to be true. □

We make a few remarks about the boundary condition $\alpha/\gamma = p/q$. In particular, this boundary condition has appeared in a few previous papers.

Remark 1 In [17], it is shown that on the half–line, stationary measures exist when $\alpha/p + \gamma/q = 1$. A phase transition occurs at $\alpha/p = 1/2$: for $\alpha/p < 1/2$, there exist stationary measures with i.i.d. Bernoulli random variables with parameter α/p, and when $\alpha/p > 1/2$ the stationary measures are spatially correlated. Under the additional condition that $\alpha/p + \gamma/q = 1$, the condition $\alpha/\gamma = p/q$ is equivalent to $\alpha/p = \gamma/q = 1/2$.

Remark 2 The condition $\alpha/\gamma = p/q$ had previously appeared in [18], which considered $\alpha/p = \gamma/q = 1/2$. Note that a variant of the reflection equation is satisfied in the stochastic vertex model of [18] – see Propositions 4.3 and 4.10 in that reference.

Remark 3 Note that duality can hold for other boundary conditions, such as those considered in [19], which proves duality (but not self–duality) for the ASEP without algebraic considerations.

Lemma 3.2 *For the values $\beta = \delta = 0$ and $\alpha/\gamma = p/q$, we have the detailed balance condition*

$$V^2\mathcal{L}V^{-2} = \mathcal{L}^*,$$

where the * denotes the transposition.

Proof Proof 1:

The Hamiltonian $\mathcal{H}$ is Hermitian[2], meaning that

$$\mathcal{H}^* = \mathcal{H}.$$

Therefore, by Proposition 3.1,

$$\begin{aligned}\mathcal{L}^* + C\mathbb{I} &= 2V\mathcal{H}^*\mathcal{V}^{-1}\\ &= 2V\mathcal{H}V^{-1}\\ &= V^2\mathcal{L}V^{-2} + C\mathbb{I},\end{aligned}$$

[2] Although this is not explicitly stated in [13], Hamiltonians in mathematical physics are always Hermitian.

implying the lemma.

Proof 2:

Note that [20] gives the stationary measures for open ASEP with generic $\alpha, \beta, \gamma, \delta$. For these generic parameters, the process is not reversible. However, for our choice of parameters ($\tau^2 = \alpha/\gamma$) the process does turn out to be reversible. The goal is to show that

$$\langle \eta | \mathcal{L} | \tilde{\eta} \rangle = \langle \eta | \mathcal{L}^* | \tilde{\eta} \rangle \frac{\langle \eta | V^{-2} | \eta \rangle}{\langle \tilde{\eta} | V^{-2} | \tilde{\eta} \rangle}$$

for any $\eta, \tilde{\eta}$. Since at most one particle may jump at a time, it suffices to consider two cases: the first is when $\eta, \tilde{\eta}$ differ at the left boundary, and the second is when $\tilde{\eta}$ is obtained from η by one particle jump.

For the first case, let $|\eta^+\rangle$ and $|\eta^-\rangle$ be two basis vectors which only different at the left boundary, where η^+ has a particle and η^- does not. Then

$$\langle \eta^- | \mathcal{L} | \eta^+ \rangle = \gamma = \frac{\alpha}{\alpha/\gamma} = \langle \eta^- | \mathcal{L}^* | \eta^+ \rangle \frac{\langle \eta^- | V^{-2} | \eta^- \rangle}{\langle \eta^+ | V^{-2} | \eta^+ \rangle},$$
$$\langle \eta^+ | \mathcal{L} | \eta^- \rangle = \alpha = \frac{\gamma}{(\alpha/\gamma)^{-1}} = \langle \eta^+ | \mathcal{L}^* | \eta^- \rangle \frac{\langle \eta^+ | V^{-2} | \eta^+ \rangle}{\langle \eta^- | V^{-2} | \eta^- \rangle},$$

which shows the detailed balance equation at the boundary.

In the second case, the detailed balance equation reduces to the detailed balance equation for ASEP with closed boundaries (because η and $\tilde{\eta}$ have the same number of particles), which is known to hold. □

Let $\mathcal{Q}^1(s) \in \mathcal{U}_\tau(\widehat{\mathfrak{sl}}_2)$ be the element from (4.1) of [13]

$$\mathcal{Q}^1(s) = s^{-1} k_1 e_1 + s k_1 f_1 + x_1 k_1^2 - x_1 \mathbb{I}.$$

For $m = -1$ the value of $x_1 = \frac{e^{i\mu\xi}}{2\kappa \sinh i\mu}$ is simply equal to 1, by (2.17) of [13]. By (4.30) of [13][3], there is the commutation

$$\left[\mathcal{H}, \rho_0^{\otimes L}(\Delta^{(L)}(\mathcal{Q}^1(\tau^{-1/2}))) \right] = 0.$$

For any value of λ, the evaluation representation ρ_λ maps $\mathcal{Q}^1(\tau^{-1/2})$ to

$$\begin{pmatrix} \tau^{-1} - 1 & 1 \\ 1 & \tau - 1 \end{pmatrix}. \tag{1}$$

By the relations in the quantum group $\mathcal{U}_\tau(\widehat{\mathfrak{sl}}_2)$,

$$\mathcal{Q}^1(s\tau^{-1}) k_1^2 = k_1^2 \mathcal{Q}^1(s).$$

[3] The paper [13] uses a different co–product than the one here: the left and right tensor products are reversed. This is due to the choice of the direction of asymmetry in the ASEP; here, we have an open boundary at the left and a closed boundary at the right, whereas the choice of reflection matrices in [13] would have a closed boundary at the left (diagonal reflection matrix) and an open boundary at the right (non–diagonal reflection matrix). The examples in Section 4 will demonstrate that this is the correct choice of co–product for our present case.

One can see directly that

$$\Delta(\mathcal{Q}^1(\tau^{-1/2})) = k_1^2 \otimes \mathcal{Q}^1(\tau^{-1/2}) + \mathcal{Q}^1(\tau^{-1/2}) \otimes 1, \tag{2}$$

so by co–associativity

$$\Delta^{(L)}(\mathcal{Q}^1(\tau^{-1/2})) = \sum_{x=1}^{L} \underbrace{k_1^2 \otimes \cdots \otimes k_1^2}_{x-1} \otimes \mathcal{Q}^1(\tau^{-1/2}) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L-x}$$

For $N \geq 1$, let S_N be the operator

$$S_N := \rho_0^{\otimes L}(\Delta^{(L)}(\mathcal{Q}^1(\tau^{-1/2}))^N)$$

and let D_N be the operator

$$D_N = V S_N V.$$

Theorem 3.3 *For the values $\beta = \delta = 0$ and $\alpha/\gamma = p/q$, and for any $N \geq 1$, we have the duality result*

$$\mathcal{L}^* D_N = D_N \mathcal{L}.$$

Proof Once Proposition 3.1 and Lemma 3.2 are proven, this is similar to the argument made in [21]. We briefly recall the proof again for completeness.

Combine the two identities

$$2\mathcal{H} = V\mathcal{L}V^{-1} + C\mathbb{I}$$

and

$$\mathcal{H}S_N = S_N\mathcal{H},$$

to get that

$$V\mathcal{L}V^{-1}S_N + CS_N = S_N V\mathcal{L}V^{-1} + CS_N.$$

Now, using that

$$V^2\mathcal{L}V^{-2} = \mathcal{L}^*,$$

we have

$$V^2\mathcal{L}V^{-2}VS_N = VS_N V\mathcal{L}V^{-1}$$

is equivalent to

$$\mathcal{L}^* V S_N V = V S_N V \mathcal{L}.$$

□

By applying additional symmetries, we obtain two more duality functions. Let Π be the particle hole involution, defined by

$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes L}.$$

Let $\widetilde{\mathcal{L}}$ be the generator for ASEP, where particles jump to the *left* at rate p and right at rate q, particles *exit* at the left boundary at rate α and enter at the left boundary at rate γ, with closed boundary conditions at the right boundary.

Corollary 3.4 *We have*

$$\mathcal{L}^* D_N \Pi = D_N \Pi \widetilde{\mathcal{L}}.$$

On the semi–infinite lattice $\mathbb{Z}_{>0}$*, we have*

$$\mathcal{L}^* D_N V^{-2} = D_N V^{-2} \widetilde{\mathcal{L}}.$$

Proof The holes of ASEP evolving under $\mathcal{L}$ have the same evolution as the particles of ASEP evolving under $\tilde{\mathcal{L}}$. In other words, $\Pi\mathcal{L}\Pi = \mathcal{L}^*$. This implies the first statement. It can be checked directly that on the semi–infinite lattice, $V^{-2}\widetilde{\mathcal{L}}V^2 = \mathcal{L}$, implying the second statement. □

We now proceed to an explicit expression for S_N (and hence of D_N). For non–negative integers a and b, let $Q^{a,b}$ be the element of $\mathcal{U}_\tau(\widehat{\mathfrak{sl}_2})$ defined by

$$Q^{a,b} = \sum_{l_1+\ldots+l_a \le b} (k_1^2)^b \mathcal{Q}^1(\tau^{l_1+\ldots+l_a-1/2}) \cdots \mathcal{Q}^1(\tau^{l_2+\ldots+l_a-1/2}) \cdots \mathcal{Q}^1(\tau^{l_a-1/2})$$

For any set of integers $m_1, \ldots, m_L$ and $1 \le i \le j \le L$, let $m_{[i,j]} = m_i + \ldots + m_j$.

Proposition 3.5 *The symmetry operator* S_N *has the form*

$$\sum_{m_1+\ldots+m_L=N} \sum_{j=1}^{L} \rho_0(Q_j^{m_j, m_{[j+1,L]}})$$

Proof When expanding $\Delta^{(L)}(\mathcal{Q}^1(\tau^{-1/2}))^N$, let m_j denote the number of times that $\mathcal{Q}^1$ acts on lattice site j for $1 \le j \le L$. We must have that $m_1 + \ldots + m_L = N$. At lattice site x, the operator k_1^2 acts $m_{[j+1,L]}$ times, corresponding to the $m_{[j+1,L]}$ times that $\mathcal{Q}^1$ acts to the right of j. So the action at lattice site j is of the form

$$k_1^2 \cdots k_1^2 \mathcal{Q}^1(\tau^{-1/2}) k_1^2 \cdots\cdots k_1^2 \mathcal{Q}^1(\tau^{-1/2}) k_1^2 \cdots k_1^2 \mathcal{Q}^1(\tau^{-1/2}) k_1^2 \cdots k_1^2.$$

Let l_0 denote the length of the first block of k_1^2, and let l_1 denote the length of the second block, and so forth, up to l_{m_j}. We must have $l_0 + \ldots + l_{m_j} = m_{[j+1,L]}$. By repeated applications of (2), the result follows. □

4 Simple Cases

Suppose that $L = 1$ and N is arbitrary. Let $|1\rangle$ denote the vector (0 1) and $|0\rangle$ denote the vector (1 0). Taking M to be the matrix in (1), the identity $\langle 0|\mathcal{L}^* D|1\rangle = \langle 0|D\mathcal{L}|1\rangle$ becomes

$$-\gamma(M^N)_{12}\tau^{-1} + \gamma(M^N)_{11} = -\alpha(M^N)_{21}\tau^{-1} + \alpha(M^N)_{22}\tau^{-2}.$$

For $\alpha = \gamma\tau^2$, one can check that both sides equal $-\gamma$ for N odd and γ for N even.

Suppose that $N = 1$ and L is arbitrary. Let $|x\rangle$ denote the particle configuration with a single particle at site x, and $|\emptyset\rangle$ denote the particle configuration with no particles. When $\mathcal{Q}^1$ is applied to lattice site y, the operator k_1^2 acts on the $y-1$ sites to the left, as the constant τ on particles and τ^{-1} on holes. The operator $V = \tau^{-\sum_j jn_j}$ acts on all sites, but only has a nonzero contribution at particles. Thus $\langle\emptyset|\mathcal{L}^* D|x\rangle = \langle\emptyset|D\mathcal{L}|x\rangle$ amounts to the identity

$$p(\tau^{-1})^x(\tau^{-1})^{L-(x+1)} + q(\tau^{-1})^{x-2}(\tau^{-1})^{L-(x-1)} - (p+q)(\tau^{-1})^{x-1}(\tau^{-1})^{L-x} = 0.$$

And indeed, for $\tau = \sqrt{p/q}$, the left–hand–side is

$$\tau^{-(L-1)}(p+q-p-q),$$

which equals 0.

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