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Twisted Quadrics and Algebraic Submanifolds in $\mathbb{R}^n$

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Abstract

We propose a general procedure to construct noncommutative deformations of an algebraic submanifold M of $\mathbb{R}^n$, specializing the procedure [G. Fiore, T. Weber, *Twisted submanifolds of* $\mathbb{R}^n$, arXiv:2003.03854] valid for smooth submanifolds. We use the framework of twisted differential geometry of Aschieri et al. (Class. Quantum Grav. 23, 1883–1911, 2006), whereby the commutative pointwise product is replaced by the $\star$-product determined by a Drinfel'd twist. We actually simultaneously construct noncommutative deformations of *all* the algebraic submanifolds M_c that are level sets of the $f^a(x)$, where $f^a(x) = 0$ are the polynomial equations solved by the points of M, employing twists based on the Lie algebra Ξ_t of vector fields that are tangent to *all* the M_c. The twisted Cartan calculus is automatically equivariant under twisted Ξ_t. If we endow $\mathbb{R}^n$ with a metric, then twisting and projecting to normal or tangent components commute, projecting the Levi-Civita connection to the twisted M is consistent, and in particular a twisted Gauss theorem holds, provided the twist is based on Killing vector fields. Twisted algebraic quadrics can be characterized in terms of generators and $\star$-polynomial relations. We explicitly work out deformations based on abelian or Jordanian twists of all quadrics in $\mathbb{R}^3$ except ellipsoids, in particular twisted cylinders embedded in twisted Euclidean $\mathbb{R}^3$ and twisted hyperboloids embedded in twisted Minkowski $\mathbb{R}^3$ [the latter are twisted (anti-)de Sitter spaces dS_2, AdS_2].

Keywords Drinfel'd twist · Deformation quantization · Noncommutative riemannian submanifold

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1 Introduction

The concept of a submanifold N of a manifold M plays a fundamental role in mathematics and physics. A metric, connection, ..., on M uniquely induces a metric, connection, ..., on N. Algebraic submanifolds of affine spaces such as $\mathbb{R}^n$ or $\mathbb{C}^n$ are paramount for their simplicity and their special properties. In the last few decades the program of generalizing differential geometry into so-called Noncommutative Geometry (NCG) has made a remarkable progress [14, 35, 41–43]; NCG might provide a suitable framework for a theory of quantum spacetime allowing the quantization of gravity (see e.g. [1, 20]) or for unifying fundamental interactions (see e.g. [12, 15]). Surprisingly, the question whether, and to what extent, a notion of a submanifold is possible in NCG has received little systematic attention (rather isolated exceptions are e.g. Ref. [17, 45, 48, 54]). On several noncommutative (NC) spaces one can make sense of special classes of NC submanifolds, but some aspects of the latter may depart from their commutative counterparts. For instance, from the $SO_q(n)$-equivariant noncommutative algebra "of functions on the quantum Euclidean space $\mathbb{R}^n_q$", which is generated by n non-commuting coordinates x^i, one can obtain the one $\mathcal{A}$ on the quantum Euclidean sphere S^{n-1}_q by imposing that the [central and $SO_q(n)$-invariant] "square distance from the origin" $r^2 = x^i x_i$ be 1. But the $SO_q(n)$-equivariant differential calculus on $\mathcal{A}$ (i.e. the corresponding $\mathcal{A}$-bimodule Ω of 1-forms) remains of dimension n instead of $n-1$; the 1-form dr^2 cannot be set to zero, and actually the graded commutator $\left[\frac{1}{q^2-1} r^{-2} dr^2,\ \cdot\right]$ acts as the exterior derivative [11, 26, 28, 53].

In [32] the above question is systematically addressed within the framework of deformation quantization [6], in the particular approach based on Drinfel'd twisting [21] of Hopf algebras; a general procedure to construct noncommutative generalizations of smooth submanifolds $M \subset \mathbb{R}^n$, of the Cartan calculus, and of (pseudo)Riemannian geometry on M is proposed. In the present work we proceed studying more in detail algebraic submanifolds $M \subset \mathbb{R}^n$, in particular quadrics, using tools of algebraic geometry. Considering $\mathbb{C}^n$ instead of $\mathbb{R}^n$ seems viable, too.

Assume that the algebraic submanifold $M \subset \mathbb{R}^n$ consists of solutions x of the equations

$$f^a(x) = 0, \qquad a = 1, 2, ..., k < n, \tag{1}$$

where $f \equiv (f^1, ..., f^k) : \mathbb{R}^n \mapsto \mathbb{R}^k$ are polynomial functions fulfilling the irreducibility conditions listed in Theorem 1; in particular, the Jacobian matrix $J = \partial f/\partial x$ is of rank k on some non-empty open subset $\mathcal{D}_f \subset \mathbb{R}^n$, and M more precisely consists of the points of $\mathcal{D}_f$ fulfilling (1). One easily shows that $\mathcal{E}_f := \mathbb{R}^n \setminus \mathcal{D}_f$ is empty or of zero measure[1]. By replacing in (1) $f^a(x) \mapsto f^a_c(x) := f^a(x) - c^a$, with $c \equiv (c^1, ..., c^k) \in f\left(\mathcal{D}_f\right)$, we obtain a k-parameter family of embedded manifolds M_c ($M_0 = M$) of dimension $n-k$ that are level sets of f. Embedded algebraic submanifolds $N \subset M$ can be obtained by adding more polynomial equations of the same

[1] Let J_α be the $k\times k$ submatrices of J, j_α their determinants, $\mathcal{E}_\alpha := \{x \in \mathbb{R}^n \mid j_\alpha(x) = 0\}$, $\alpha = 1, 2, ..., \binom{n}{k}$. $\mathcal{E}_f = \bigcap_\alpha \mathcal{E}_\alpha$. At least one polynomial function $j_\alpha(x)$ is not identically zero; hence $\mathcal{E}_\alpha$ has codimension 1 and zero measure, and so has $\mathcal{E}_f$.

type to (1). Let $\mathcal{X}$ be the $*$-algebra (over $\mathbb{C}$) of polynomial functions $P:\mathbb{R}^n\to\mathbb{C}$, restricted to $\mathcal{D}_f$. The $*$-algebra $\mathcal{X}^M$ of complex-valued polynomial functions on M can be expressed as the quotient of $\mathcal{X}$ over the ideal $\mathcal{C}\subset\mathcal{X}$ of polynomial functions vanishing on M:

$$\mathcal{X}^M:=\mathcal{X}/\mathcal{C}\equiv\left\{[\alpha]:=\alpha+\mathcal{C}\ \mid\ \alpha\in\mathcal{X}\right\};\tag{2}$$

In Appendix A, after recalling some basic notions and notation in algebraic geometry, we prove

Theorem 1 *Assume that J is of rank k on a non-empty open subset $\mathcal{D}_f\subset\mathbb{R}^n$, so that the system (1) defines an algebraic submanifold $M\subset\mathcal{D}_f$ of dimension $n-k$. In addition, assume that M is irreducible in $\mathbb{C}^n$; this is the case e.g. if there exists a k-dimensional affine subspace $\pi\subset\mathbb{R}^n$ meeting M in $s:=\prod_{a=1}^k\deg f^a$ points. Then $\mathcal{C}$ is the complexification of the ideal generated by the f^a in $\mathbb{R}[x^1,...,x^n]$, i.e. for all $h\in\mathcal{C}$ there exist $h^a\in\mathcal{X}$ such that*

$$h(x)=\sum_{a=1}^k h^a(x)f^a(x)=\sum_{a=1}^k f^a(x)h^a(x).\tag{3}$$

(In the smooth context, i.e. with $f^a,h,h^a\in C^\infty(\mathcal{D}_f)$, (3) holds if J is of rank k on $\mathcal{D}_f$ [32].) $\mathcal{X}^N$ is the quotient of $\mathcal{X}^M$ over the ideal generated by further equations of type (1), or equivalently of $\mathcal{X}$ over the ideal generated by all such equations. Identifying vector fields with derivations (first order differential operators), we denote as $\Xi:=\{X=X^i\partial_i\ \mid\ X^i\in\mathcal{X}\}$ the Lie algebra of polynomial vector fields X on $\mathcal{D}_f$ (here and below we abbreviate $\partial_i\equiv\partial/\partial x^i$) and

$$\begin{aligned}\Xi_{\mathcal{C}}&=\{X\in\Xi\mid X(f^a)\in\mathcal{C}\text{ for all }a\in\{1,\dots,k\}\},\\ \Xi_{\mathcal{CC}}&=\{X\in\Xi\mid X(h)\in\mathcal{C}\text{ for all }h\in\mathcal{X}\}\subset\Xi_{\mathcal{C}}.\end{aligned}\tag{4}$$

The former is a Lie $*$-subalgebra of Ξ, while the latter is a Lie $*$-ideal; both are $\mathcal{X}$-$*$-subbimodules. By Theorem 1 the latter decomposes as $\Xi_{\mathcal{CC}}=\bigoplus_{a=1}^k f^a\,\Xi$. We identify the Lie algebra Ξ_M of vector fields tangent to M with that of derivations of $\mathcal{X}^M$, namely with

$$\Xi_M:=\Xi_{\mathcal{C}}/\Xi_{\mathcal{CC}}\equiv\left\{[X]:=X+\Xi_{\mathcal{CC}}\ \mid\ X\in\Xi_{\mathcal{C}}\right\}.\tag{5}$$

A general framework for deforming $\mathcal{X}$ into a family - depending on a formal parameter ν - of noncommutative algebras $\mathcal{X}_\star$ over $\mathbb{C}[[\nu]]$ (the ring of formal power series in ν with coefficients in $\mathbb{C}$) is Deformation Quantization [6, 40]: as a module over $\mathbb{C}[[\nu]]$ $\mathcal{X}_\star$ coincides with $\mathcal{X}[[\nu]]$, but the commutative pointwise product $\alpha\beta$ of $\alpha,\beta\in\mathcal{X}$ ($\mathbb{C}[[\nu]]$-bilinearly extended to $\mathcal{X}[[\nu]]$) is deformed into a possibly noncommutative (but still associative) product,

$$\alpha\star\beta=\alpha\beta+\sum\nolimits_{l=1}^\infty\nu^l B_l(\alpha,\beta),\tag{6}$$

where B_l are suitable bidifferential operators of degree l at most. We wish to deform $\mathcal{X}^M$ into a noncommutative algebra $\mathcal{X}^M_\star$ in the form of a quotient

$$\mathcal{X}^M_\star:=\mathcal{X}_\star/\mathcal{C}_\star\equiv\left\{[\alpha]:=\alpha+\mathcal{C}_\star\ \mid\ \alpha\in\mathcal{X}_\star\right\},\tag{7}$$

with $\mathcal{C}_\star$ a two-sided ideal of $\mathcal{X}_\star$, and fulfilling itself $\mathcal{X}^M_\star = \mathcal{X}^M[[\nu]]$ as an equality of $\mathbb{C}[[\nu]]$-modules. To this end we require that $\mathcal{C}_\star = \mathcal{C}[[\nu]]$, i.e. that $c\star\alpha,\ \alpha\star c \in \mathcal{C}[[\nu]]$ for all $\alpha \in \mathcal{X}, c \in \mathcal{C}$, so that $(\alpha+c)\star(\alpha'+c')-\alpha\star\alpha' \in \mathcal{C}[[\nu]]$ for all $\alpha, \alpha' \in \mathcal{X}[[\nu]]$ and $c, c' \in \mathcal{C}[[\nu]]$. As a result, taking the quotient would commute with deforming the product: $(\mathcal{X}/\mathcal{C})_\star = \mathcal{X}_\star/\mathcal{C}_\star$. As argued in [32], these conditions are fulfilled if[2], for all $\alpha \in \mathcal{X}$, $a = 1, .., k$,

$$\alpha \star f^a = \alpha f^a = f^a \star \alpha \qquad \Leftrightarrow \qquad B_l(\alpha, f^a) = 0 = B_l(f^a, \alpha) \qquad \forall l \in \mathbb{N} \quad (8)$$

(this implies that the f^a are central in $\mathcal{X}_\star$, again). The quotient (7) also appears in the context of deformation quantization of Marsden-Weinstein reduction [10, 37]. A more algebraic approach to deformation quantization of reduced spaces is given in the recent article [18].

In [21] Drinfel'd introduced a general deformation quantization procedure of universal enveloping algebras $U\mathfrak{g}$ (seen as Hopf algebras) of Lie groups G and of their module algebras, based on *twisting*; a *twist* is a suitable element (a 2-cocycle, see Section 2.1)

$$\mathcal{F} = \mathbf{1}\otimes\mathbf{1} + \sum_{l=1}^{\infty} \nu^l \sum_{I_l} \mathcal{F}_1^{I_l} \otimes \mathcal{F}_2^{I_l} \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\nu]] \tag{9}$$

(here $\otimes = \otimes_{\mathbb{C}[[\nu]]}$, and tensor products are meant completed in the ν-adic topology); $\mathcal{F}$ acts on the tensor product of any two $U\mathfrak{g}$-modules or module algebras, in particular algebras of functions on any smooth manifolds G acts on, including some symplectic manifolds[3] [3]. Given a generic smooth manifold M, the authors of [1] pick up $\mathfrak{g} \equiv \Xi_M$, the Lie algebra of smooth vector fields on M (and of the infinite-dimensional Lie group of diffeomorphisms of M), and the $U\Xi_M$-module algebra $\mathcal{X}^M = C^\infty(M)$; $\mathcal{F}_1^{I_l}, \mathcal{F}_2^{I_l}$ seen as differential operators acting on $\mathcal{X}^M$ have order l at most and no zero-order term. The corresponding deformed product reads

$$\alpha \star \beta := \alpha\beta + \sum\nolimits_{l=1}^{\infty} \nu^l \sum\nolimits_{I_l} \overline{\mathcal{F}}_1^{I_l}(\alpha)\ \overline{\mathcal{F}}_2^{I_l}(\beta)\,, \tag{10}$$

where $\overline{\mathcal{F}} \equiv \mathcal{F}^{-1} = \mathbf{1}\otimes\mathbf{1} + \sum_{l=1}^{\infty} \nu^l \sum_{I_l} \overline{\mathcal{F}}_1^{I_l} \otimes \overline{\mathcal{F}}_2^{I_l}$ is the inverse of the twist. In the sequel we will use Sweedler notation with suppressed summation symbols and abbreviate $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, $\overline{\mathcal{F}} = \overline{\mathcal{F}}_1 \otimes \overline{\mathcal{F}}_2$; in the presence of several copies of $\mathcal{F}$ we distinguish the summations by writing $\mathcal{F}_1\otimes\mathcal{F}_2$, $\mathcal{F}_1'\otimes\mathcal{F}_2'$, etc. Actually Ref. [1] twists not only $U\Xi_M$, $\mathcal{X}^M$ into new Hopf algebra $U\Xi_M^{\mathcal{F}}$ and $U\Xi_M^{\mathcal{F}}$-equivariant module algebra $\mathcal{X}^M_\star$, but also the $U\Xi_M$-equivariant $\mathcal{X}^M$-bimodule of differential forms on M,

[2]In fact, for all $c \equiv \sum_{a=1}^k f^a c^a \in \mathcal{C}$ ($c^a \in \mathcal{X}$) (8) implies $c = \sum_{a=1}^k f^a \star c^a$ and, for all $\alpha \in \mathcal{X}$, by the associativity of $\star$, $c \star \alpha = (\sum_{a=1}^k f^a \star c^a) \star \alpha = \sum_{a=1}^k f^a \star (c^a \star \alpha) = \sum_{a=1}^k f^a (c^a \star \alpha) \in \mathcal{C}[[\nu]]$; and similarly for $\alpha \star c$.
It is not sufficient to require that $\alpha \star f^a - \alpha f^a$, $f^a \star \alpha - f^a \alpha$ belong to $\mathcal{C}[[\nu]]$ to obtain the same results.
[3]However this quantization procedure does not apply to every Poisson manifold: there are several symplectic manifolds, e.g. the symplectic 2-sphere and the symplectic Riemann surfaces of genus $g > 1$, which do not admit a $\star$-product induced by a Drinfel'd twist (c.f. [9, 16]). Nevertheless, if one is not taking into account the Poisson structure, every G-manifold can be quantized via the above approach.

their tensor powers, the Lie derivative, and the geometry on M (metric, connection, curvature, torsion,...) - if present -, into deformed counterparts.

Here and in [32], as in [45], we take the algebraic characterization (2), (5) as the starting point for defining submanifolds in NCG, but use a twist-deformed differential calculus on it. Our twist is based on the Lie subalgebra (and $\mathcal{X}$-bimodule) $\mathfrak{g} \equiv \Xi_t \subset \Xi$ defined by

$$\Xi_t := \{X \in \Xi \;\mid\; X(f^1) = 0, \;..., \; X(f^k) = 0\} \subset \Xi_{\mathcal{C}}, \tag{11}$$

which consists of vector fields tangent to *all* submanifolds M_c (because they fulfill $X(f_c^a) = 0$ for all $c \in \mathbb{R}^k$) at all points. As in [32], we note that, applying this deformation procedure to the previously defined $\mathcal{X}$ with a twist $\mathcal{F} \in U\Xi_t \otimes U\Xi_t[[\nu]]$, we satisfy (8) and therefore obtain a deformation $\mathcal{X}_\star$ of $\mathcal{X}$ such that for all $c \in f(\mathcal{D}_f)$ $\mathcal{X}_\star^{M_c} = \mathcal{X}^{M_c}[[\nu]] = \mathcal{X}_\star/\mathcal{C}_\star^c$; moreover, $\Xi_{M_c\star} = \Xi_{M_c}[[\nu]] = \Xi_{\mathcal{C}^c\star}/\Xi_{\mathcal{C}\mathcal{C}^c\star}$, see Section 2.3. In other words, we obtain a noncommutative deformation, in the sense of deformation quantization and in the form of quotients as in (2), (5), of the k-parameter family of embedded algebraic manifolds $M_c \subset \mathbb{R}^n$. For every $X \in \Xi_{\mathcal{C}}$ there is an element in the equivalence class $[X]$ that belongs to Ξ_t, namely its tangent projection X_t; hence we can work with the latter. $\mathcal{X}_\star$, $\Xi_\star$, ... are $U\Xi^{\mathcal{F}}$-equivariant, while $\mathcal{X}_\star^{M_c}$, $\Xi_{M\star}$, $\Xi_{t\star}$, ... are $U\Xi_t^{\mathcal{F}}$-equivariant. If $\mathcal{F}$ is unitary or real, then $U\Xi^{\mathcal{F}}$ and $\mathcal{X}_\star$, $\Xi_\star$, ... admit $*$-structures (involutions) making them a Hopf $*$-algebra and $U\Xi^{\mathcal{F}}$-equivariant (Lie) $*$-algebras respectively; thereby $U\Xi_t^{\mathcal{F}}$ is a Hopf $*$-subalgebra and $\mathcal{X}_\star^{M_c}$, $\Xi_{t\star}$, ... are $U\Xi_t^{\mathcal{F}}$-equivariant (Lie) $*$-subalgebras.

In passing, we recall that sometimes, if a Poisson manifold M is symmetric under a solvable Lie group G like $\mathbb{R}^d$, the Heisenberg or the "$ax + b$" group, one can construct even a *strict* (i.e. non-formal) deformation quantization [52] of $C^\infty(M)$ such that the $\star$-product remains invariant under G itself (or a cocommutative Hopf algebra), see e.g. [7, 52].

The plan of the paper will be as follows.

Section 2 reviews: Hopf algebras, their module algebras and twisting[4, 13, 21, 22, 34, 39, 43, 47] (Section 2.1); their application [1, 2] to the differential geometry on a generic manifold (Section 2.2); twisting of smooth submanifolds of $\mathbb{R}^n$ as developed in [32] (Section 2.3).

In Section 3 we apply this procedure to algebraic submanifolds $M \subset \mathbb{R}^n$. For simplicity we stick to M of codimension 1, and we assume that there is a Lie subalgebra $\mathfrak{g}$ (of dimension at least 2) of both Ξ_t and the Lie algebra $\mathsf{aff}(n)$ of the affine group $\mathsf{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(n)$ of $\mathbb{R}^n$; the level sets of $f(x)$ of degree 1 (hyperplanes) or 2 (quadrics) are of this type. Choosing a twist $\mathcal{F} \in U\mathfrak{g} \otimes U\mathfrak{g}[[\nu]]$ we find that the algebra $\mathcal{X}$ of polynomial functions (with complex coefficients) in the set of Cartesian coordinates $x^1, ..., x^n$ is deformed so that every $\star$-polynomial of degree k in x equals an ordinary polynomial of the same degree in x, and vice versa. This implies in particular that the polynomial relations $x^i x^j - x^j x^i = 0$ (whence the commutativity of $\mathcal{X}$), as well as the ones (1) defining the ideal $\mathcal{C}$, can be expressed as $\star$-polynomial relations *of the same degree*, so that $\mathcal{X}_\star$, $\mathcal{X}_\star^M = \mathcal{X}_\star/\mathcal{C}_\star$ can be defined globally in terms of generators and polynomial relations, and moreover the subspaces $\widetilde{\mathcal{X}}^q$, $\widetilde{\mathcal{X}^M}^q$ of $\mathcal{X}$, $\mathcal{X}^M = \mathcal{X}/\mathcal{C}$ consisting of polynomials of any degree q in x^i coincide as $\mathbb{C}[[\nu]]$-modules with their deformed counterparts $\widetilde{\mathcal{X}}_\star^q$, $\widetilde{\mathcal{X}^M}_\star^q$; in particular their dimensions (hence the Hilbert-Poincaré series of both $\mathcal{X}$ and $\mathcal{X}^M$) remain the same under deformation - an important (and often overlooked) property that guarantees the smoothness of the deformation. The same occurs with the $\mathcal{X}_\star$-bimodules and algebras $\Omega^\bullet_\star$ of differential forms, that of differential operators, etc. We convey all these informations into what we name the *differential calculus algebras* $\mathcal{Q}^\bullet$, $\mathcal{Q}^\bullet_{M_c}$ on $\mathbb{R}^n$, M respectively (generated by the Cartesian coordinates, their differentials, and a basis of vector fields, subject to appropriate relations; they are graded by the form degree and filtered by both the degrees in the x^i and in the vector fields), and their deformations $\mathcal{Q}^\bullet_\star$, $\mathcal{Q}^\bullet_{M_c\star}$ (see Sections 3.1 and 3.2).

In Section 4 we discuss in detail deformations, induced by unitary twists of abelian [51] or Jordanian [49] type, of all families of quadric surfaces embedded in $\mathbb{R}^3$, except ellipsoids. The deformation of each element of every class is interesting by itself, as a novel example of a NC manifold. Endowing $\mathbb{R}^3$ with the Euclidean (resp. Minkowski) metric gives the circular cylinders (resp. hyperboloids and cone) a Lie algebra $\mathfrak{k} \subset \Xi_t$ of isometries of dimension at least 2; choosing a twist $\mathcal{F} \in U\mathfrak{k} \otimes U\mathfrak{k}[[\nu]]$ we thus find twisted (pseudo)Riemannian M_c (with the metric given by the twisted first fundamental form) that are symmetric under the Hopf algebra $U\mathfrak{k}^{\mathcal{F}}$ (the "quantum group of isometries"); the twisted Levi-Civita connection on $\mathbb{R}^3$ (the exterior derivative) projects to the twisted Levi-Civita connection on M_c, while the twisted curvature can be expressed in terms of the twisted second fundamental form through a twisted Gauss theorem. Actually, the metric, Levi-Civita connection, intrinsic and extrinsic curvatures of any circular cylinder or hyperboloid, as elements in the appropriate tensor spaces, remain undeformed; the twist enters only their action on twisted tensor products of vector fields. The twisted hyperboloids can be seen as twisted (anti-)de Sitter spaces dS_2, AdS_2.

In Appendices A, B we recall basic notions in algebraic geometry and prove most theorems.

We recall that (anti-)de Sitter spaces, which can be represented as solutions of $2f_c(x) \equiv (x^1)^2+...+(x^{n-1})^2-(x^n)^2-2c=0$ in Minkowski $\mathbb{R}^n$, are maximally symmetric cosmological solutions to the Einstein equations of general relativity with a nonzero cosmological constant Λ in spacetime dimension $n-1$, and play a prominent role in present cosmology and theoretical physics (see e.g. [19, 44]). Interpreting x in Minkowski $\mathbb{R}^n$ as relativistic n-momentum, rather than position in spacetime, then the same equation represents the dispersion relation of a relativistic particle of square mass $2c$. In either case it would be interesting to study the physical consequences of twist deformations. On the mathematical side, directions for further investigations include: submanifolds of $\mathbb{C}^n$ (rather than $\mathbb{R}^n$), just dropping $*$-structures and the related constraints on the twist; twist deformations of the (zero-measure) algebraic set $\mathcal{E}_f$.

Finally, we mention that in [30, 31, 50] an alternative approach to introduce NC (more precisely, fuzzy) submanifolds $S \subset \mathbb{R}^n$ has been proposed and applied to spheres, projecting the algebra of observables of a quantum particle in $\mathbb{R}^n$, subject to a confining potential with a very sharp minimum on S, to the Hilbert subspace with energy below a certain cutoff.

Everywhere we consider vector spaces V over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$; we denote by $V[[\nu]]$ the $\mathbb{K}[[\nu]]$-module of formal power series in ν with coefficients in $\mathbb{K}$. We shall denote by the same symbol a $\mathbb{K}$-linear map $\phi\colon V \to W$ and its $\mathbb{K}[[\nu]]$-linear extension $\phi\colon V[[\nu]] \to W[[\nu]]$.

2 Preliminaries

2.1 Hopf Algebras and their Representations

Hopf algebras. We recall that a *Hopf algebra* $(H, \mu, \eta, \Delta, \epsilon, S)$ over $\mathbb{K}$ is an associative unital algebra (H, μ, η) over $\mathbb{K}$ [$\mu\colon H\otimes H \to H$ is the product: $\mu(a\otimes b) \equiv a \cdot b$ for $a, b \in H$, $\eta\colon \mathbb{K} \to H$ with $\eta(1) =: \mathbf{1}$ is the unit] endowed with a coproduct, counit, antipode Δ, ϵ, S. While Δ, ϵ are algebra maps, S is an anti-algebra map; they have to fulfill a number of properties (see e.g. [13, 22, 43]), namely $(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta =: \Delta^{(2)}$ (coassociativity), $(\epsilon\otimes\mathrm{id})\circ\Delta = \mathrm{id} = (\mathrm{id}\otimes\epsilon)\circ\Delta$ (counitality), $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta$ (antipode property). We shall use Sweedler's notation with suppressed summation symbols for the coproduct Δ and its $(n-1)$-fold iteration

$$\Delta^{(n)} : H \to (H)^{\otimes n}, \qquad \Delta^{(n)}(a) = a_{(1)} \otimes a_{(2)} \otimes ... \otimes a_{(n)}. \tag{12}$$

A $*$-involution on a $\mathbb{K}$-algebra $\mathcal{A}$ is an involutive, anti-algebra map $*\colon \mathcal{A} \to \mathcal{A}$ such that $(\lambda a + \rho b)^* = \overline{\lambda}a^* + \overline{\rho}b^*$ for all $a, b \in \mathcal{A}$ and $\lambda, \rho \in \mathbb{K}$ (here $\overline{\lambda}$ denotes the complex conjugation of λ). A *Hopf $*$-algebra* $(H, \mu, \eta, \Delta, \epsilon, S, *)$ over $\mathbb{K}$ is a Hopf algebra endowed with a $*$-involution such that, for all $a, b \in H$,

$$\mathbf{1}^* = \mathbf{1}, \quad \Delta(a)^{*\otimes *} = \Delta(a^*), \quad \epsilon(a^*) = \overline{\epsilon(a)} \quad \text{and} \quad S[S(a^*)^*] = a. \tag{13}$$

The universal enveloping algebra (UEA) $U\mathfrak{g}$ of a $\mathbb{K}$-Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is a Hopf algebra; Δ, ϵ, S are determined by their actions on $\mathbf{1}$ and on *primitive* elements, i.e. $g \in \mathfrak{g}$:

$$\Delta(g) = g\otimes 1 + 1\otimes g, \quad \epsilon(g) = 0 \quad \text{and} \quad S(g) = -g. \tag{14}$$

It is cocommutative, i.e. $\tau \circ \Delta = \Delta$, where τ is the flip, $\tau(a \otimes b) = b \otimes a$. If there is a $*$-involution $*\colon \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ such that $[g, h]^* = [h^*, g^*]$ for all $g, h \in \mathfrak{g}$, the UEA $U\mathfrak{g}$ becomes a Hopf $*$-algebra with respect to the extension $*\colon U\mathfrak{g} \to U\mathfrak{g}$.

Replacing everywhere in the above definition $\mathbb{K}$ by the commutative ring $\mathbb{K}[[\nu]]$ one obtains the definition of a Hopf ($*$-)algebra over $\mathbb{K}[[\nu]]$. For any Hopf ($*$-)algebra over $\mathbb{K}$ the $\mathbb{K}[[\nu]]$-linear extension (with completed tensor product in the ν-adic topology) is trivially a Hopf ($*$-)algebra over $\mathbb{K}[[\nu]]$. Other ones can be obtained by twisting (see below).

Hopf algebra modules and module algebras. Given an associative unital algebra $\mathcal{A}$ over $\mathbb{K}$, a $\mathbb{K}$-vector space $\mathcal{M}$ is said to be a *left $\mathcal{A}$-module* if it is endowed with a $\mathbb{K}$-linear map $\triangleright\colon \mathcal{A}\otimes\mathcal{M} \to \mathcal{M}$ such that $a \triangleright (b \triangleright s) = (a \cdot b) \triangleright s$ and $\mathbf{1} \triangleright s = s$ for all $a, b \in \mathcal{A}$ and $s \in \mathcal{M}$. Similarly right $\mathcal{A}$-modules are defined. An $\mathcal{A}$-bimodule is a left and a right $\mathcal{A}$-module with commuting module actions. A $\mathbb{K}$-linear map $\phi\colon \mathcal{M} \to \mathcal{M}'$ between left $\mathcal{A}$-modules is said to be *$\mathcal{A}$-equivariant* if ϕ intertwines the $\mathcal{A}$-module actions, i.e. if $\phi(a \triangleright s) = a \triangleright \phi(s)$ for all $a \in \mathcal{A}$ and $s \in \mathcal{M}$. For a

Hopf $*$-algebra H, a left H-module $\mathcal{M}$ is said to be a *left H-$*$-module* if there is a $*$-involution $*\colon \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$ such that

$$(a \triangleright s)^* = S(a)^* \triangleright s^* \text{ for all } a \in H \text{ and } s \in \mathcal{M}. \tag{15}$$

Similarly, right $\mathcal{A}$-$*$-modules and $\mathcal{A}$-$*$-bimodules are defined. An element $s \in \mathcal{M}$ of a left H-module is said to be *H-invariant* if $a \triangleright s = \epsilon(a)s$ for all $a \in H$. An associative unital ($*$-)algebra $\mathcal{A}$ is said to be a *left H-($*$-)module algebra* if $\mathcal{A}$ is a left H-($*$-)module such that

$$\xi \triangleright (a \cdot b) = (\xi_{(1)} \triangleright a) \cdot (\xi_{(2)} \triangleright b) \quad \text{and} \quad \xi \triangleright 1 = \epsilon(\xi)1 \tag{16}$$

for all $\xi \in H$ and $a, b \in \mathcal{A}$. More generally, an $\mathcal{A}$-($*$-)bimodule $\mathcal{M}$ for a left H-module ($*$-)algebra $\mathcal{A}$ is said to be an *H-equivariant $\mathcal{A}$-($*$-)bimodule* if $\mathcal{M}$ is a left H-module such that

$$\xi \triangleright (a \cdot s \cdot b) = (\xi_{(1)} \triangleright a) \cdot (\xi_{(2)} \triangleright s) \cdot (\xi_{(3)} \triangleright b) \quad [\text{and } (a \cdot s \cdot b)^* = b^* \cdot s^* \cdot a^*] \tag{17}$$

hold for all $\xi \in H$, $a, b \in \mathcal{A}$ and $s \in \mathcal{M}$, where we denoted the $\mathcal{A}$-($*$-)module actions by $\cdot$.

Similarly one defines module ($*$-)algebras and (equivariant) (bi-)($*$-)modules over $\mathbb{K}[[\nu]]$, and trivially obtains istances of them from their $\mathbb{K}$-counterparts by $\mathbb{K}[[\nu]]$-linear extension.

Drinfel'd twist deformation. Fix a Hopf algebra H over $\mathbb{K}$. A *(Drinfel'd) twist* on H is an element $\mathcal{F} = \mathbf{1} \otimes \mathbf{1} + \mathcal{O}(\nu) \in (H \otimes H)[[\nu]]$ of the form (9) satisfying the 2-cocycle property

$$(\mathcal{F} \otimes \mathbf{1})(\Delta \otimes \mathrm{id})(\mathcal{F}) = (\mathbf{1} \otimes \mathcal{F})(\mathrm{id} \otimes \Delta)(\mathcal{F}) \tag{18}$$

and the normalization property $(\epsilon \otimes \mathrm{id})(\mathcal{F}) = \mathbf{1} = (\mathrm{id} \otimes \epsilon)(\mathcal{F})$. Every twist is invertible as a formal power series. We denote the inverse twist by $\overline{\mathcal{F}}$ and suppress summation symbols, employing the *leg notation*: $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, $\overline{\mathcal{F}} = \overline{\mathcal{F}}_1 \otimes \overline{\mathcal{F}}_2$, and $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3$ for the expression at both sides of (18). In the presence of several copies of $\mathcal{F}$ we write $\mathcal{F} = \mathcal{F}'_1 \otimes \mathcal{F}'_2$ for the second copy etc. to distinguish the summations. To every twist we assign an element $\beta := \mathcal{F}_1 \cdot S(\mathcal{F}_2) \in H[[\nu]]$. It is invertible with inverse given by $\beta^{-1} = S(\overline{\mathcal{F}}_1) \cdot \overline{\mathcal{F}}_2 \in H[[\nu]]$.

Let $\mathcal{F}$ be a Drinfel'd twist on H. Then $H^{\mathcal{F}} = (H[[\nu]], \mu, \eta, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}})$ is a Hopf algebra over $\mathbb{K}[[\nu]]$, where the twisted coproduct and antipode are defined by

$$\Delta_{\mathcal{F}}(\xi) = \mathcal{F}\Delta(\xi)\overline{\mathcal{F}} \quad \text{and} \quad S_{\mathcal{F}}(\xi) = \beta S(\xi)\beta^{-1} \tag{19}$$

for all $\xi \in H$[4]. Again, we shall use Sweedler's notation with suppressed summation symbols for the coproduct $\Delta_{\mathcal{F}}$ and its $(n-1)$-fold iteration

$$\Delta_{\mathcal{F}}^{(n)} : H \to (H)^{\otimes n}, \qquad \Delta_{\mathcal{F}}^{(n)}(a) = a_{\widehat{(1)}} \otimes a_{\widehat{(2)}} \otimes \ldots \otimes a_{\widehat{(n)}}. \tag{20}$$

If $\mathcal{A}$ is a left H-module algebra then $\mathcal{A}_\star = (\mathcal{A}[[\nu]], \star, 1)$ is a left $H^{\mathcal{F}}$-module algebra with respect to the product (10), [now abbreviated as $a \star b = (\overline{\mathcal{F}}_1 \triangleright a) \cdot (\overline{\mathcal{F}}_2 \triangleright b)$]

[4]Here one could replace β^{-1} by $S(\beta)$, as $S(\beta)\beta \in \mathrm{Centre}(H)[[\lambda]]$.

for $a, b \in \mathcal{A}[[\nu]]$; this implies the twisted Leibniz rule

$$g \triangleright (a \star b) = \big(g_{\widehat{(1)}} \triangleright a\big) \star \big(g_{\widehat{(2)}} \triangleright b\big), \text{ for all } g \in H^{\mathcal{F}}. \tag{21}$$

More generally, if $\mathcal{A}$ is a left H-module algebra and $\mathcal{M}$ an H-equivariant $\mathcal{A}$-bimodule, then $M_\star = \mathcal{M}[[\nu]]$ becomes (cf. [4] Theorem 3.5) an $H^{\mathcal{F}}$-equivariant $\mathcal{A}_\star$-bimodule, with respect to the undeformed Hopf algebra action and the twisted module actions

$$a\star s = (\overline{\mathcal{F}}_1 \triangleright a)\cdot(\overline{\mathcal{F}}_2 \triangleright s) \quad \text{and} \quad s\star a = (\overline{\mathcal{F}}_1 \triangleright s)\cdot(\overline{\mathcal{F}}_2 \triangleright a) \text{ for all } a \in \mathcal{A} \text{ and } s \in \mathcal{M} \tag{22}$$

on $\mathcal{M}_\star$. If H is cocommutative then in general $H^{\mathcal{F}}$ is not, but it is *quasi-cocommutative*, i.e.

$$\xi_{\widehat{(2)}} \otimes \xi_{\widehat{(1)}} = \mathcal{R} \cdot \Delta_{\mathcal{F}}(\xi) \cdot \overline{\mathcal{R}} \quad \text{for all } \xi \in H^{\mathcal{F}}, \tag{23}$$

where $\mathcal{R} := \mathcal{F}_{21}\overline{\mathcal{F}} \in (H \otimes H)[[\nu]]$ is the *triangular structure* or *universal $\mathcal{R}$-matrix*. $\mathcal{R}$ has inverse $\overline{\mathcal{R}} = \mathcal{F}\overline{\mathcal{F}}_{21} = \mathcal{R}_{21} \in (H \otimes H)[[\nu]]$ and further satisfies the so-called *hexagon relations*

$$(\Delta_{\mathcal{F}} \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta_{\mathcal{F}})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{24}$$

As the representation theory of a Hopf algebra H is monoidal, the $\mathbb{K}$-tensor product $\mathcal{M} \otimes \mathcal{M}'$ of two left H-modules is also a left H-module, via the H action $\xi \triangleright (s \otimes s') = \xi_{(1)} \triangleright s \otimes \xi_{(2)} \triangleright s'$. The $\star$-tensor product

$$s \otimes_\star s' := \overline{\mathcal{F}}_1 \triangleright s \otimes \overline{\mathcal{F}}_2 \triangleright s', \quad s \in \mathcal{M},\ s' \in \mathcal{M}' \tag{25}$$

is the corresponding monoidal structure on the representation theory of $H^{\mathcal{F}}$, since

$$\xi \triangleright (s \otimes_\star s') = \xi_{\widehat{(1)}} \triangleright s \otimes_\star \xi_{\widehat{(2)}} \triangleright s' \tag{26}$$

for all $\xi \in H^{\mathcal{F}}$, i.e. $(\mathcal{M} \otimes \mathcal{M}')_\star = \mathcal{M}_\star \otimes_\star \mathcal{M}'_\star$. Consider [4] for more information.

The algebra $(H[[\nu]], \star)$ itself is a $H^{\mathcal{F}}$-module algebra, and one can build a triangular Hopf algebra $H_\star = (H[[\nu]], \star, \eta, \Delta_\star, \epsilon, S_\star, \mathcal{R}_\star)$ isomorphic to $H^{\mathcal{F}} = (H[[\nu]], \mu, \eta, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}, \mathcal{R})$, with isomorphism $D : H_\star \to H^{\mathcal{F}}$ given by $D(\xi) := (\overline{\mathcal{F}}_1 \triangleright \xi)\overline{\mathcal{F}}_2 = \mathcal{F}_1\, \xi\, S(\mathcal{F}_2)\ \beta^{-1}$ and inverse by $D^{-1}(\phi) = \overline{\mathcal{F}}_1\, \phi\, \beta\, S\big(\overline{\mathcal{F}}_2\big)$ [1, 36] (cf. also [24, 29]). In other words, $D(\xi \star \xi') = D(\xi)D(\xi')$, and $\Delta_\star, S_\star, \mathcal{R}_\star$ are related to $\Delta_{\mathcal{F}}, S_{\mathcal{F}}, \mathcal{R}$ by the relations

$$\Delta_\star = (D^{-1} \otimes D^{-1}) \circ \Delta_{\mathcal{F}} \circ D, \qquad S_\star = D^{-1} \circ S_{\mathcal{F}} \circ D, \qquad \mathcal{R}_\star = (D^{-1} \otimes D^{-1})(\mathcal{R}). \tag{27}$$

One can think of D also as a change of generators within $H[[\nu]]$.

If H is a Hopf $*$-algebra, and the twist is either *real* [namely, if $\mathcal{F}^{*\otimes *} = (S \otimes S)(\mathcal{F}_{21})$] or *unitary* (namely, if $\mathcal{F}^{*\otimes *} = \overline{\mathcal{F}}$), then one can make both $H^{\mathcal{F}}$ and $H_\star$ into Hopf $*$-algebras in such a way that twisting transforms the H $*$-modules and module $*$-algebras into $H^{\mathcal{F}}$ and $H_\star$ $*$-modules and module $*$-algebras, respectively. In fact, if $\mathcal{F}$ is real then also $\beta^* = \beta$, while $\mathcal{R}^{*\otimes *} = (\beta \otimes \beta)^{-1}\overline{\mathcal{R}}(\beta \otimes \beta) = (\beta \otimes \beta)\overline{\mathcal{R}}(\beta \otimes \beta)^{-1}$, and $H^{\mathcal{F}}$ endowed with the $*$-involution

$$\xi^{*\mathcal{F}} := \beta \xi^* \beta^{-1}, \quad \text{for } \xi \in H^{\mathcal{F}}, \tag{28}$$

is a triangular Hopf $*$-algebra (in fact, $\mathcal{R}^{*_{\mathcal{F}}\otimes *_{\mathcal{F}}} = \overline{\mathcal{R}}$); moreover, $\mathcal{A}_\star$, $\mathcal{M}_\star$ are a left $H^{\mathcal{F}}$-module $*$-algebra and a $H^{\mathcal{F}}$-equivariant $\mathcal{A}_\star$-$*$-bimodule when endowed with the undeformed $*$-involutions (cf. [43] Proposition 2.3.7). In particular $(H[[\nu]], \star, *)$ is a left $H^{\mathcal{F}}$-module $*$-algebra. Actually D is an isomorphism of the triangular Hopf $*$-algebra $(H_\star, *)$ onto the one $(H^{\mathcal{F}}, *_{\mathcal{F}})$, see [1, 43] for more information. If $\mathcal{F}$ is a unitary twist, then also $\mathcal{R}$ is, $\beta^* = S(\beta^{-1})$, and $H^{\mathcal{F}}$ endowed with the undeformed $*$-involution is a Hopf $*$-algebra; moreover, $\mathcal{A}_\star$, $\mathcal{M}_\star$ are respectively a left $H^{\mathcal{F}}$-module $*$-algebra and an $H^{\mathcal{F}}$-equivariant $\mathcal{A}_\star$-$*$-bimodule when endowed with the twisted $*$-involutions

$$a^{*_\star} = S(\beta) \triangleright a^*, \qquad s^{*_\star} = S(\beta) \triangleright s^*, \tag{29}$$

where $a \in \mathcal{A}[[\nu]]$ and $s \in \mathcal{M}[[\nu]]$ (cf. [29]). In particular $(H[[\nu]], \star, *_\star)$ is a left $H^{\mathcal{F}}$-module $*$-algebra. Actually, one finds that $(H_\star, *_\star)$ is a triangular Hopf $*$-algebra, in particular $\Delta_\star \circ *_\star = (*_\star \otimes *_\star) \circ \Delta_\star$, $S_\star \circ *_\star \circ S_\star \circ *_\star = \mathrm{id}$, and $D : (H_\star, *_\star) \to (H^{\mathcal{F}}, *)$ is an isomorphism of triangular Hopf $*$-algebras, see Proposition 18 in [32].

For their simplicity, here we shall only use *abelian* or the following *Jordanian* Drinfel'd twists on UEAs:

i.) For a finite number $n \in \mathbb{N}$ of pairwise commuting elements $e_1, \ldots, e_n, f_1, \ldots, f_n \in \mathfrak{g}$ we set $P := \sum_{i=1}^n e_i \otimes f_i \in \mathfrak{g} \otimes \mathfrak{g}$, $P' = \frac{1}{2} \sum_{i=1}^n (e_i \otimes f_i - f_i \otimes e_i)$. Then

$$\mathcal{F} = \exp(i\nu P) \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\nu]] \tag{30}$$

is a Drinfel'd twist on $U\mathfrak{g}$ ([51]); it is said of *abelian* (or *Reshetikhin*) type. It is unitary if $P^{*\otimes *} = P$; this is e.g. the case if the e_i, f_i are anti-Hermitian or Hermitian. The twist $\mathcal{F}' = \exp(i\nu P')$ is both unitary and real, leads to the same $\mathcal{R}$ and makes $\beta = \mathbf{1}$, whence $S_{\mathcal{F}} = S$, and the $*$-structure remains undeformed also for H-$*$-modules and module algebras, see (29).

ii.) Let $H, E \in \mathfrak{g}$ be elements of a Lie algebra such that $[H, E] = 2E$. Then

$$\mathcal{F} = \exp\left[\frac{1}{2} H \otimes \log(\mathbf{1} + i\nu E)\right] \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\nu]] \tag{31}$$

defines a *Jordanian* Drinfel'd twist [49]. If H and E are anti-Hermitian, $\mathcal{F}$ is unitary.

2.2 Twisted Differential Geometry

Here we recall some results obtained in [1, 2]. We apply the notions overviewed in the previous section choosing as Hopf $*$-algebra $H = U\Xi$, where $\Xi := \Gamma^\infty(TM)$ denotes the Lie $*$-algebra of smooth vector fields on a smooth manifold M, as a left H-module $*$-algebra the $*$-algebra $\mathcal{X} = \mathcal{C}^\infty(M)$ of smooth $\mathbb{K}$-valued functions on M, as H-equivariant symmetric $\mathcal{X}$-$*$-bimodules Ξ itself, the space $\Omega = \Gamma^\infty(T^*M)$ of differential 1-forms on M, as well as their tensor (or wedge) powers. The Hopf $*$-algebra action on $\mathcal{X}$, Ξ and Ω is given by the extension of the Lie derivative: for $X, Y \in \Xi$, $f \in \mathcal{X}$ and $\omega \in \Omega$ we have

$$\mathcal{L}_X f =: X(f), \qquad \mathcal{L}_X Y = [X, Y], \qquad \mathcal{L}_X \omega = (\mathrm{i}_X \mathrm{d} + \mathrm{d}\mathrm{i}_X)\omega \tag{32}$$

and we set $\mathcal{L}_{XY} = \mathcal{L}_X\mathcal{L}_Y$, $\mathcal{L}_{\mathbf{1}} = \mathrm{id}$. Henceforth we denote such an extension by $\triangleright$.

2.2.1 Twisted Tensor Fields

The *tensor algebra* $\mathcal{T} := \bigoplus_{p,r\in\mathbb{N}_0} \mathcal{T}^{p,r}$ on M is defined as the direct sum of the $\mathbb{K}$-modules

$$\mathcal{T}^{p,r} := \underbrace{\Omega\otimes\ldots\otimes\Omega}_{p\text{-times}}\otimes\underbrace{\Xi\otimes\ldots\otimes\Xi}_{r\text{-times}} \tag{33}$$

for $p, r \geq 0$, $p + r > 0$, where we set $\mathcal{T}^{0,0} := \mathcal{X}$. Here and below $\otimes$ stands for $\otimes_{\mathcal{X}}$ (rather than $\otimes_{\mathbb{K}}$), namely $T \otimes fT' = Tf \otimes T'$ for all $f \in \mathcal{X}$. Every $\mathcal{T}^{p,r}$ is an H-equivariant $\mathcal{X}$-$*$-bimodule with respect to the module actions

$$\xi\triangleright(\omega_1\otimes\ldots\otimes\omega_p\otimes X_1\otimes\ldots\otimes X_r) = [\xi_{(1)}\triangleright\omega_1]\otimes\ldots\otimes[\xi_{(p)}\triangleright\omega_p]\otimes[\xi_{(p+1)}\triangleright X_1]\otimes\ldots\otimes[\xi_{(p+r)}\triangleright X_r],$$

$$h\cdot(\omega_1\otimes\ldots\otimes\omega_p\otimes X_1\otimes\ldots\otimes X_r)\cdot k = (h\cdot\omega_1)\otimes\ldots\otimes\omega_p\otimes X_1\otimes\ldots\otimes(X_r\cdot k)$$

for all $\xi \in H$ and $h, k \in \mathcal{X}$. This induces the structure of an H-equivariant $\mathcal{X}$-$*$-bimodule on $\mathcal{T}$. In particular, for all $T, T' \in \mathcal{T}$, $\xi \in H$ and $h, k \in \mathcal{X}$ the relations

$$\begin{aligned} \xi \triangleright (T\otimes T') &= \xi_{(1)} \triangleright T\otimes\xi_{(2)} \triangleright T', \\ h\cdot(T\otimes T')\cdot k &= (h\cdot T)\otimes(T'\cdot k), \\ (T\cdot h)\otimes T' &= T\otimes(h\cdot T') \end{aligned} \tag{34}$$

hold. Let $T \in \mathcal{T}^{p,r}$. On a local chart (U, x) of M there are unique functions $T^{\lambda_1,\ldots,\lambda_r}_{\mu_1,\ldots,\mu_p} \in \mathcal{C}^\infty(U)$ such that $T = T^{\lambda_1,\ldots,\lambda_r}_{\mu_1,\ldots,\mu_p}\mathrm{d}x^{\mu_1}\otimes\ldots\otimes\mathrm{d}x^{\mu_p}\otimes\partial_{\lambda_1}\otimes\ldots\otimes\partial_{\lambda_r}$, where $\{\partial_i\}$ is the dual frame of vector fields on U corresponding to $\{x^i\}$, i.e. $\langle\partial_i, \mathrm{d}x^j\rangle = \delta_i^j$ and we sum over repeated indices.

Consider a (in particular, unitary or real) Drinfel'd twist $\mathcal{F}$ on H. Applying the results of Section 2.1 to H, $\mathcal{X}$, Ξ, Ω and $\mathcal{T}$ we obtain the following: $H^{\mathcal{F}} = U\Xi^{\mathcal{F}}$ is a Hopf ($*$-)algebra, $\mathcal{X}_\star$ is a left $H^{\mathcal{F}}$-module ($*$-)algebra, while $\Xi_\star$, $\Omega_\star$, $\mathcal{T}_\star$ are $H^{\mathcal{F}}$-equivariant $\mathcal{X}_\star$-($*$-)bimodules. The $H^{\mathcal{F}}$-actions are given by the $\star$-*Lie derivative* $\mathcal{L}^\star_\xi T := \mathcal{L}_{\overline{\mathcal{F}}_1\triangleright\xi}(\overline{\mathcal{F}}_2 \triangleright T)$ for all $\xi \in H^{\mathcal{F}}$ and $T \in \mathcal{T}_\star$. On $\star$-vector fields $X, Y \in \Xi_\star$, the $\star$-Lie derivative

$$\mathcal{L}^\star_X Y = [\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y] = X \star Y - (\overline{\mathcal{R}}_1 \triangleright Y) \star (\overline{\mathcal{R}}_2 \triangleright X) =: [X, Y]_\star \tag{35}$$

structures $\Xi_\star$ as a $\star$-*Lie algebra*. This means that $[\cdot,\cdot]_\star$ is twisted skew-symmetric, i.e. $[Y, X]_\star = -[\overline{\mathcal{R}}_1 \triangleright X, \overline{\mathcal{R}}_2 \triangleright Y]_\star$ and satisfies the twisted Jacobi identity $[X, [Y, Z]_\star]_\star = [[X, Y]_\star, Z]_\star + [\overline{\mathcal{R}}_1 \triangleright Y, [\overline{\mathcal{R}}_2 \triangleright X, Z]_\star]_\star$ for all $X, Y, Z \in \Xi_\star$. Furthermore, $[\cdot,\cdot]_\star$ is $H^{\mathcal{F}}$-equivariant, i.e. $\xi\triangleright[X, Y]_\star = [\xi_{\widehat{(1)}}\triangleright X, \xi_{\widehat{(2)}}\triangleright Y]_\star$ and $\star$-vector fields act on $\mathcal{X}_\star$ as *twisted derivations*, i.e.

$$\mathcal{L}^\star_X(f \star f') = \mathcal{L}^\star_X(f) \star f' + (\overline{\mathcal{R}}_1 \triangleright f) \star \mathcal{L}^\star_{\overline{\mathcal{R}}_2\triangleright X} f' \tag{36}$$

for all $X \in \Xi_\star$ and $f, f' \in \mathcal{X}_\star$. By setting $\mathcal{A} = \mathcal{T}$ we can apply the results of Section 2.1, in particular define a deformed tensor algebra $\mathcal{T}_\star$ with associative $\star$-tensor product defined by (25). This can be decomposed as $\mathcal{T}_\star = \bigoplus_{p,r\in\mathbb{N}_0} \mathcal{T}^{p,r}_\star$, where $\mathcal{T}_\star^{0,0} := \mathcal{X}_\star$ and for $p + r > 0$

$$\mathcal{T}_\star^{p,r} := \underbrace{\Omega_\star \otimes_\star \ldots \otimes_\star \Omega_\star}_{p\text{-times}} \otimes_\star \underbrace{\Xi_\star \otimes_\star \ldots \otimes_\star \Xi_\star}_{r\text{-times}} . \tag{37}$$

In particular, for all $T, T' \in \mathcal{T}_\star$, $h, k \in \mathcal{X}_\star$ and $\xi \in H^{\mathcal{F}}$

$$\begin{aligned} \xi \triangleright (T \otimes_\star T') &= \xi_{\widehat{(1)}} \triangleright T \otimes_\star \xi_{\widehat{(2)}} \triangleright T', \\ h \star (T \otimes_\star T') \star k &= (h \star T) \otimes_\star (T' \star k), \\ (T \star h) \otimes_\star T' &= T \otimes_\star (h \star T'). \end{aligned} \tag{38}$$

The third formula shows that $\otimes_\star$ is actually $\otimes_{\mathcal{X}_\star}$, the tensor product over $\mathcal{X}_\star$. Let $T \in \mathcal{T}_\star^{p,r}$. On any local chart (U, x) of M there unique functions $T_{\star\mu_1,\ldots,\mu_p}^{\lambda_1,\ldots,\lambda_r} \in \mathcal{C}^\infty(U)[[\nu]]$ such that

$$T = T_{\star\mu_1,\ldots,\mu_p}^{\lambda_1,\ldots,\lambda_r} \star \mathrm{d}x^{\mu_1} \otimes_\star \ldots \otimes_\star \mathrm{d}x^{\mu_p} \otimes_\star \partial_{\lambda_1} \otimes_\star \ldots \otimes_\star \partial_{\lambda_r} . \tag{39}$$

Higher order differential forms are defined by the twisted skew-symmetrization of $\otimes_\star$

$$\omega \wedge_\star \omega' := (\overline{\mathcal{F}}_1 \triangleright \omega) \wedge (\overline{\mathcal{F}}_2 \triangleright \omega') = \omega \otimes_\star \omega' - \overline{\mathcal{R}}_1 \triangleright \omega' \otimes_\star \overline{\mathcal{R}}_2 \triangleright \omega \tag{40}$$

($\star$-*wedge product*, an associative unital product), and we define $\Omega_\star^\bullet := (\Lambda^\bullet \Omega_\star, \wedge_\star)$ to be the twisted exterior algebra of Ω (see [54] for more information).

The dual pairing $\langle\ ,\ \rangle$ between vector fields and 1-forms can be equivalently considered as $\mathcal{X}$-bilinear maps $\Xi \otimes \Omega \to \mathcal{X}$ or $\Omega \otimes \Xi \to \mathcal{X}$; for all arguments $X \in \Xi$, $\omega \in \Omega$ these maps have the same images, which we respectively denote by the lhs and right-hand side (rhs) of the identity $\langle X, \omega\rangle = \langle \omega, X\rangle$. They have distinct twist deformations ($\star$-*pairings*) defined by

$$(T, T') \mapsto \langle T, T'\rangle_\star := \left\langle \overline{\mathcal{F}}_1 \triangleright T, \overline{\mathcal{F}}_2 \triangleright T' \right\rangle , \tag{41}$$

with $(T, T') = (X, \omega)$ and $(T, T') = (\omega, X)$ respectively. They satisfy

$$\begin{aligned} \langle T, T'\rangle_\star &= \langle \overline{\mathcal{R}}_1 \triangleright T', \overline{\mathcal{R}}_2 \triangleright T\rangle_\star, \\ \xi \triangleright \langle T, T'\rangle_\star &= \langle \xi_{\widehat{(1)}} \triangleright X, \xi_{\widehat{(2)}} \triangleright \omega\rangle_\star, \\ \langle h_1 \star T \star h_2, T' \star h_3\rangle_\star &= h_1 \star \langle T, h_2 \star T'\rangle_\star \star h_3 \end{aligned} \tag{42}$$

for all $\xi \in H^{\mathcal{F}}$, $X \in \Xi_\star$, $\omega \in \Omega_\star$, $(T, T') = (X, \omega)$ or $(T, T') = (\omega, X)$, and $h, h_1, h_2, h_3 \in \mathcal{X}_\star$. Moreover, $\langle X, \mathrm{d}h\rangle_\star = \mathcal{L}_X^\star h$. As one can extend the ordinary pairing to higher tensor powers setting

$$\langle T_p \otimes \ldots \otimes T_1, T_1' \otimes \ldots \otimes T_p' \otimes \tau\rangle := \langle T_p \langle \ldots \langle T_1, T_1'\rangle, \ldots\rangle, T_p'\rangle\, \tau , \tag{43}$$

for all $\tau \in \mathcal{T}^{p,r}$ (the image will belong again to $\mathcal{T}^{p,r}$) provided $(T_i, T_i') \in \Xi \otimes \Omega$ or $(T_i, T_i') \in \Omega \otimes \Xi$ for all i, so can one extend $\langle\ ,\ \rangle_\star$ to the corresponding twisted tensor powers using the same formula (41). Due to the 'onion structure' of (43) (i.e. the order of the T_i and of the T_i' are opposite of each other), properties (42) are preserved, namely the $\star$-paring is $H^{\mathcal{F}}$-equivariant, as well as left, right and middle $\mathcal{X}_\star$-linear (if we chose a different order in (43) the deformed definition would need copies of $\mathcal{R}$ acting on the T_i, T_i').

2.2.2 Twisted Covariant Derivatives and Metrics

A *twisted covariant derivative* (or *connection*) is a $\mathbb{K}[[\nu]]$-linear map $\nabla^{\mathcal{F}}\colon \Xi_\star\otimes_{\mathbb{K}[[\nu]]}\mathcal{T}_\star\to\mathcal{T}_\star$ fulfilling, for all $X,Y\in\Xi_\star$, $h\in\mathcal{X}_\star$, $T,T'\in\mathcal{T}_\star$ and $\omega\in\Omega_\star$,

$$\nabla^{\mathcal{F}}_X h=\mathcal{L}^\star_X h, \tag{44}$$

$$\nabla^{\mathcal{F}}_{h\star X}T=h\star(\nabla^{\mathcal{F}}_X T), \tag{45}$$

$$\nabla^{\mathcal{F}}_X(T\otimes_\star T')=[\overline{\mathcal{R}}_1\triangleright\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2\triangleright X}(\overline{\mathcal{R}}''_2\triangleright T)]\otimes_\star[(\overline{\mathcal{R}}_2\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_1)\triangleright T']+(\overline{\mathcal{R}}_1\triangleright T)\otimes_\star(\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\triangleright X}T), \tag{46}$$

$$\nabla^{\mathcal{F}}_X\langle Y,\omega\rangle_\star=\langle\overline{\mathcal{R}}_1\triangleright[\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2\triangleright X}(\overline{\mathcal{R}}''_2\triangleright Y)],(\overline{\mathcal{R}}_2\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_2)\triangleright\omega\rangle_\star+\langle\overline{\mathcal{R}}_1\triangleright Y,\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\triangleright X}\omega\rangle_\star. \tag{47}$$

Its *curvature* $\mathsf{R}^{\mathcal{F}}_\star$ and *torsion* $\mathsf{T}^{\mathcal{F}}_\star$ maps respectively act on all $X,Y,Z\in\Xi_\star$ through

$$\begin{aligned}\mathsf{T}^{\mathcal{F}}_\star(X,Y)&:=\nabla^{\mathcal{F}}_X Y-\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_1\triangleright Y}(\overline{\mathcal{R}}_2\triangleright X)-[X,Y]_\star,\\ \mathsf{R}^{\mathcal{F}}_\star(X,Y,Z)&:=\nabla^{\mathcal{F}}_X\nabla^{\mathcal{F}}_Y Z-\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_1\triangleright Y}\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\triangleright X}Z-\nabla^{\mathcal{F}}_{[X,Y]_\star}Z\end{aligned} \tag{48}$$

and are left $\mathcal{X}_\star$-linear maps $\mathsf{T}^{\mathcal{F}}_\star\colon\Xi_\star\otimes_\star\Xi_\star\to\Xi_\star$ and $\mathsf{R}^{\mathcal{F}}_\star\colon\Xi_\star\otimes_\star\Xi_\star\otimes_\star\Xi_\star\to\Xi_\star$ fulfilling

$$\mathsf{T}^{\mathcal{F}}_\star(Y,X)=-\mathsf{T}^{\mathcal{F}}_\star(\overline{\mathcal{R}}_1\triangleright X,\overline{\mathcal{R}}_2\triangleright Y),\qquad \mathsf{R}^{\mathcal{F}}_\star(Y,X,Z)=-\mathsf{R}^{\mathcal{F}}_\star(\overline{\mathcal{R}}_1\triangleright X,\overline{\mathcal{R}}_2\triangleright Y,Z). \tag{49}$$

They are in one-to-one correspondence with elements $\mathsf{T}^{\mathcal{F}}\in\Omega^2_\star\otimes_\star\Xi_\star$, $\mathsf{R}^{\mathcal{F}}\in\Omega_\star\otimes_\star\Omega^2_\star\otimes_\star\Xi_\star$ such that

$$\mathsf{T}^{\mathcal{F}}_\star(X,Y)=\langle X\otimes_\star Y,\mathsf{T}^{\mathcal{F}}\rangle_\star,\qquad \mathsf{R}^{\mathcal{F}}_\star(X,Y,Z)=\langle X\otimes_\star Y\otimes_\star Z,\mathsf{R}^{\mathcal{F}}\rangle_\star. \tag{50}$$

Setting $\mathcal{F}=1\otimes1$ it follows that $\mathcal{R}=1\otimes1$ and the definitions of twisted connection, torsion, curvature give the algebraic notion of connection, torsion, curvature of differential geometry. Consider a (classical) connection $\nabla\colon\Xi\otimes\mathcal{T}\to\mathcal{T}$ on M and its *equivariance Lie algebra* $\mathfrak{e}\subseteq\Xi$ (cf. [32]). The latter is a Lie subalgebra of the Lie algebra of vector fields defined by

$$\mathfrak{e}=\{\xi\in\Xi\mid\xi\triangleright(\nabla_X T)=\nabla_{\xi\triangleright X}T+\nabla_X(\xi\triangleright T)\text{ for all }X\in\Xi,\ T\in\mathcal{T}\}. \tag{51}$$

It follows that ∇ is $U\mathfrak{e}$-equivariant, i.e. $\xi\triangleright(\nabla_X T)=\nabla_{\xi_{(1)}\triangleright X}[\xi_{(2)}\triangleright T]$ for all $\xi\in U\mathfrak{e}$, $X\in\Xi$ and $T\in\mathcal{T}$. If $\mathcal{F}\in(U\mathfrak{e}\otimes U\mathfrak{e})[[\nu]]$ is a Drinfel'd twist, then

$$\nabla^{\mathcal{F}}_X T:=\nabla_{\overline{\mathcal{F}}_1\triangleright X}(\overline{\mathcal{F}}_2\triangleright T) \tag{52}$$

defines an $U\mathfrak{e}^{\mathcal{F}}$-equivariant twisted connection $\nabla^{\mathcal{F}}\colon\Xi_\star\otimes_{\mathbb{K}[[\nu]]}\mathcal{T}_\star\to\mathcal{T}_\star$; then (46–47) reduce to

$$\begin{aligned}\nabla^{\mathcal{F}}_X(T\otimes_\star T')&=(\nabla^{\mathcal{F}}_X T)\otimes_\star T'+(\overline{\mathcal{R}}_1\triangleright T)\otimes_\star(\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\triangleright X}T'),\\ \nabla^{\mathcal{F}}_X\langle Y,\omega\rangle_\star&=\langle\nabla^{\mathcal{F}}_X Y,\omega\rangle_\star+\langle\overline{\mathcal{R}}_1\triangleright Y,\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\triangleright X}\omega\rangle_\star\end{aligned} \tag{53}$$

for all $X,Y\in\Xi_\star$, $T,T'\in\mathcal{T}_\star$ and $\omega\in\Omega_\star$ (cf. [32] Proposition 2).

A *metric* on M is a non-degenerate element $\mathbf{g}=\mathbf{g}^\alpha\otimes\mathbf{g}_\alpha\in(\Omega\otimes\Omega)[[\nu]]$ such that $\mathbf{g}=\mathbf{g}_\alpha\otimes\mathbf{g}^\alpha$. We can view $\mathbf{g}$ as an element $\mathbf{g}=\mathbf{g}^A\otimes_\star\mathbf{g}_A\in\Omega_\star\otimes_\star\Omega_\star$ with $\mathbf{g}^A\otimes\mathbf{g}_A=\mathcal{F}_1\triangleright\mathbf{g}^\alpha\otimes\mathcal{F}_2\triangleright\mathbf{g}_\alpha$. A twisted connection $\nabla^{\mathcal{F}}$ such that $\mathsf{T}^{\mathcal{F}}=0$ and $\nabla^{\mathcal{F}}\mathbf{g}=0$

is said to be a *Levi-Civita* (LC) connection for $\mathbf{g}$. The associated Ricci tensor map and Ricci scalar of $\nabla^{\mathcal{F}}$ are respectively defined by

$$\mathsf{Ric}^{\mathcal{F}}_\star:\ \Xi_\star\otimes_\star\Xi_\star\to\Xi_\star,\quad \mathsf{Ric}^{\mathcal{F}}_\star(X,Y):=\langle\theta^i,\mathsf{R}^{\mathcal{F}}_\star(e_i,X,Y)\rangle_\star,\qquad \mathfrak{R}^{\mathcal{F}}:=\mathsf{Ric}^{\mathcal{F}}\left(\mathbf{g}^{-1A},\mathbf{g}^{-1}{}_A\right) \tag{54}$$

(sum over α, A, i), where $\{e_i\}$, $\{\theta^i\}$ are $\star$-dual bases of $\Xi_\star,\Omega_\star$, in the sense $\langle e_i,\theta^j\rangle_\star=\delta_i^j$. One easily finds $\mathsf{Ric}^{\mathcal{F}}(X,Y)=\langle\theta^i\otimes_\star e_i\otimes_\star X\otimes_\star Y,\mathsf{R}^{\mathcal{F}}\rangle_\star$.

For a (pseudo-)Riemannian manifold $(M,\mathbf{g})$ we define the Lie subalgebra

$$\mathfrak{k}:=\{\xi\in\Xi\mid \xi\triangleright\mathbf{g}(X,Y)=\mathbf{g}(\xi\triangleright X,Y)+\mathbf{g}(X,\xi\triangleright Y)\text{ for all }X,Y\in\Xi\}\subseteq\Xi \tag{55}$$

of *Killing vector fields*. If $\nabla:\ \Xi\otimes\mathcal{T}\to\mathcal{T}$ is the *Levi-Civita* (LC) covariant derivative on $(M,\mathbf{g})$ [i.e. $\mathsf{T}=0$ and $\mathcal{L}_X\mathbf{g}(Y,Z)=\mathbf{g}(\nabla_XY,Z)+\mathbf{g}(Y,\nabla_XZ)$ for all $X,Y,Z\in\Xi$] and $\mathfrak{e}$ the corresponding equivariance Lie algebra, we obtain $\mathfrak{k}\subseteq\mathfrak{e}$ by the Koszul formula.

The following results are taken from [2, 32]. If $\mathcal{F}\in(U\mathfrak{k}\otimes U\mathfrak{k})[[\nu]]$ is a twist "based on Killing vector fields", then (52) defines a twisted LC connection $\nabla^{\mathcal{F}}:\ \Xi_\star\otimes_{\mathbb{K}[[\nu]]}\mathcal{T}_\star\to\mathcal{T}_\star$, and moreover

$$\mathbf{g}_\star(X,Y):=\left\langle X,\left\langle Y,\mathbf{g}^A\right\rangle_\star\mathbf{g}_A\right\rangle_\star=\mathbf{g}\left(\overline{\mathcal{F}}_1\triangleright X,\overline{\mathcal{F}}_2\triangleright Y\right)=\langle\, X\otimes_\star Y\,,\,\mathbf{g}\,\rangle \tag{56}$$

for all $X,Y\in\Xi_\star$. $\nabla^{\mathcal{F}}$ is the unique LC connection with respect to $\mathbf{g}_\star$; equivalently

$$\mathcal{L}^\star_X[\mathbf{g}_\star(Y,Z)]=\mathbf{g}_\star(\nabla^{\mathcal{F}}_XY,Z)+\mathbf{g}_\star(\overline{\mathcal{R}}_1\triangleright Y,\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\triangleright X}Z) \tag{57}$$

for all $X,Y,Z\in\Xi_\star$. This *twisted metric map* $\mathbf{g}_\star:\ \Xi_\star\otimes_\star\Xi_\star\to\mathcal{X}_\star$ as well as the twisted curvature and Ricci tensor maps, are left $\mathcal{X}_\star$-linear in the first argument and right $\mathcal{X}_\star$-linear in the last argument. Also the twisted Ricci tensor map is in one-to-one correspondence with an element $\mathsf{Ric}^{\mathcal{F}}\in\Omega_\star\otimes_\star\Omega_\star$ such that $\mathsf{Ric}^{\mathcal{F}}_\star(X,Y)=\langle X\otimes_\star Y,\mathsf{Ric}^{\mathcal{F}}\rangle_\star$, by the non-degeneracy of the $\star$-pairing. The twisted curvature, Ricci tensor and Ricci scalar are $U\mathfrak{k}^{\mathcal{F}}$-invariant and coincide with their undeformed counterparts as elements

$$\mathsf{R}^{\mathcal{F}}=\mathsf{R}\in(\Omega\otimes\Omega^2\otimes\Xi)[[\nu]],\qquad \mathsf{Ric}^{\mathcal{F}}=\mathsf{Ric}\in(\Omega\otimes\Omega)[[\nu]],\qquad \mathfrak{R}^{\mathcal{F}}=\mathfrak{R}\in\mathcal{X}. \tag{58}$$

2.3 Twisted Smooth Submanifolds of $\mathbb{R}^n$ of Codimension 1

Here we collect the main results of [32] regarding a smooth submanifold $M\subset\mathcal{D}_f\subseteq\mathbb{R}^n$ whose points x solve the single equation $f(x)=0$. More generally, the solutions $x\in\mathcal{D}_f$ of

$$f_c(x):=f(x)-c=0,\qquad\qquad c\in f(\mathcal{D}_f)\subseteq\mathbb{R}, \tag{59}$$

define a smooth manifold M_c; varying c we obtain a whole 1-parameter family of embedded submanifolds $M_c\subseteq\mathbb{R}^n$ of dimension $n-1$. In [32] $\mathcal{X}$ stands for the $*$-algebra of smooth functions on $\mathcal{D}_f$, and also $\mathcal{X}^M=\mathcal{C}^\infty(M)$, Ξ_M, Ξ_t, ... are understood in the smooth context.

Twist deformation of tangent and normal vector fields. According to Section 2.1 $\Xi_\star$ is a $\mathcal{X}_\star$-bimodule with $\mathcal{X}_\star$-subbimodules $\Xi_{\mathcal{C}\star}$, $\Xi_{\mathcal{C}\mathcal{C}\star}$, $\Xi_{t\star}$. We further define the

$\mathcal{X}_\star$-bimodule

$$\Omega_{\perp\star} := \{\omega \in \Omega_\star \mid \langle \Xi_{t\star}, \omega\rangle_\star = 0\}. \tag{60}$$

By Proposition 9 in [32], the $\mathcal{X}_\star$-bimodules $\Xi_{\mathcal{C}\star}$, $\Xi_{t\star}$ and $\Xi_{M\star} =: \Xi_{\mathcal{C}\star}/\Xi_{\mathcal{CC}\star}$ are $\star$-Lie subalgebras of $\Xi_\star$ while $\Xi_{\mathcal{CC}\star}$ is a $\star$-Lie ideal. Furthermore, we obtain the decomposition $\Omega_{\perp\star} = \mathcal{X}_\star \star \mathrm{d}f = \mathrm{d}f \star \mathcal{X}_\star$, and the twisted exterior algebras $\Xi^\bullet_\star, \Xi^\bullet_{t\star}, \Xi^\bullet_{\mathcal{C}\star}, \Xi^\bullet_{\mathcal{CC}\star}, \Xi^\bullet_{M\star}, \Omega^\bullet_{\perp\star}$ are $U\Xi_t^{\mathcal{F}}$-equivariant $\mathcal{X}_\star$-bimodules. $\Xi_{t\star}, \Xi_{\mathcal{C}\star}, \Xi_{\mathcal{CC}\star}, \Xi_{M\star}, \Omega_{\perp\star}$ resp. coincide as $\mathbb{C}[[\nu]]$-modules with $\Xi_t[[\nu]], \Xi_{\mathcal{C}}[[\nu]], \Xi_{\mathcal{CC}}[[\nu]], \Xi_M[[\nu]], \Omega_\perp[[\nu]]$.

Let $\mathbf{g} = \mathbf{g}^\alpha \otimes \mathbf{g}_\alpha \in \Omega\otimes\Omega$ be a (non-degenerate) metric on $\mathcal{D}_f$ with inverse $\mathbf{g}^{-1} = \mathbf{g}^{-1\alpha} \otimes \mathbf{g}_\alpha^{-1}$.

$$\Xi_\perp := \{X \in \Xi \mid \mathbf{g}(X, \Xi_t) = 0\}, \qquad \Omega_t := \{\omega \in \Omega \mid \mathbf{g}^{-1}(\omega, \Omega_\perp) = 0\} \tag{61}$$

are the $\mathcal{X}$-bimodules of normal vector fields and tangent differential forms. The open subset where the restriction $\mathbf{g}_\perp^{-1} := \mathbf{g}^{-1}|_{\Omega_\perp\otimes\Omega_\perp} : \Omega_\perp\otimes\Omega_\perp \to \mathcal{X}$ is non-degenerate is denoted by $\mathcal{D}'_f \subset \mathcal{D}_f$. If $\mathbf{g}$ is Riemannian $\mathcal{D}'_f = \mathcal{D}_f$. From now on we denote the restrictions of $\Xi, \Xi_t, \Xi_\perp, \Omega, \Omega_t, \Omega_\perp$ to $\mathcal{D}'_f$ by the same symbols and by $\mathfrak{k} \subseteq \Xi_t$ the Lie subalgebra of Killing vector fields with respect to $\mathbf{g}$ which are also tangent to $M_c \subseteq \mathcal{D}'_f$. The deformed analogues of (61)

$$\Xi_{\perp\star} := \{X \in \Xi_\star \mid \mathbf{g}_\star(X, \Xi_{t,\star}) = 0\}, \qquad \Omega_{t\star} := \{\omega \in \Omega_\star \mid \mathbf{g}_\star^{-1}(\omega, \Omega_{\perp\star}) = 0\} \tag{62}$$

can be defined for any twist $\mathcal{F} \in (U\Xi_t \otimes U\Xi_t)[[\nu]]$. Henceforth in this section $\mathcal{F} \in (U\mathfrak{k}\otimes U\mathfrak{k})[[\nu]]$.

If $\mathcal{F} \in (U\mathfrak{k}\otimes U\mathfrak{k})[[\nu]]$ then by Proposition 10 in [32], there are direct sum decompositions

$$\Xi_\star = \Xi_{t\star} \oplus \Xi_{\perp\star}, \qquad \Omega_\star = \Omega_{t\star} \oplus \Omega_{\perp\star} \tag{63}$$

into orthogonal $\mathcal{X}_\star$-bimodules, with respect to $\mathbf{g}_\star$ and $\mathbf{g}_\star^{-1}$ respectively. $\Xi_{t\star}$ is a $\star$-Lie subalgebra of $\Xi_\star$, $\Omega_{t\star}$ and $\Xi_{\perp\star}$ are orthogonal with respect to the $\star$-pairing and actually $\Omega_{t\star} = \{\omega \in \Omega_\star \mid \langle \Xi_{\perp\star}, \omega\rangle_\star = 0\}$. Furthermore, the restrictions

$$\begin{aligned} \mathbf{g}_{\perp\star} &:= \mathbf{g}|_{\Xi_{\perp\star}\otimes_\star\Xi_{\perp\star}} : \Xi_{\perp\star}\otimes_\star\Xi_{\perp\star} \to \mathcal{X}_\star, \qquad & \mathbf{g}_{t\star} &:= \mathbf{g}|_{\Xi_{t\star}\otimes_\star\Xi_{t\star}} : \Xi_{t\star}\otimes_\star\Xi_{t\star} \to \mathcal{X}_\star, \\ \mathbf{g}_{\perp\star}^{-1} &:= \mathbf{g}^{-1}|_{\Omega_{\perp\star}\otimes_\star\Omega_{\perp\star}} : \Omega_{\perp\star}\otimes_\star\Omega_{\perp\star} \to \mathcal{X}_\star, \qquad & \mathbf{g}_{t\star}^{-1} &:= \mathbf{g}^{-1}|_{\Omega_{t\star}\otimes_\star\Omega_{t\star}} : \Omega_{t\star}\otimes_\star\Omega_{t\star} \to \mathcal{X}_\star, \end{aligned} \tag{64}$$

are non-degenerate. $\Xi_{t\star}, \Omega_{\perp\star}, \Xi_{\perp\star}, \Omega_{t\star}$ resp. coincide with $\Xi_t[[\nu]], \Omega_\perp[[\nu]], \Xi_\perp[[\nu]], \Omega_t[[\nu]]$ as $\mathbb{C}[[\nu]]$-modules; and similarly for their $\star$-tensor (and -wedge) powers. The orthogonal projections $\mathrm{pr}_{t\star} : \Xi_\star \to \Xi_{t\star}$, $\mathrm{pr}_{\perp\star} : \Xi_\star \to \Xi_{\perp\star}$, $\mathrm{pr}_{t\star} : \Omega_\star \to \Omega_{t\star}$ and $\mathrm{pr}_{\perp\star} : \Omega_\star \to \Omega_{\perp\star}$ and their (unique) extensions to multivector fields and higher rank forms are the $\mathbb{C}[[\nu]]$-linear extensions of their classical counterparts. They, as well as $\Xi^\bullet_\star, \Xi^\bullet_{t\star}, \Xi^\bullet_{\perp\star}, \Omega^\bullet_\star, \Omega^\bullet_{t\star}, \Omega^\bullet_{\perp\star}$, are $U\mathfrak{k}^{\mathcal{F}}$-equivariant.

The induced metric (*first fundamental form*) for the family of submanifolds $M_c \subseteq \mathcal{D}'_f$, where $c \in f(\mathcal{D}'_f)$, stays undeformed: $\mathbf{g}_t^{\mathcal{F}} := (\mathrm{pr}_{t\star}\otimes\mathrm{pr}_{t\star})(\mathbf{g}) = (\mathrm{pr}_t\otimes\mathrm{pr}_t)(\mathbf{g}) =: \mathbf{g}_t$.

Defining $\Omega_{\mathcal{C}\star} := \{\omega \in \Omega_\star \mid \langle \Xi_{\perp\star}, \omega\rangle_\star \subseteq \mathcal{C}[[\nu]]\}$ and $\Omega_{\mathcal{CC}\star} := \Omega_\star \star f = f \star \Omega_\star$, we further obtain

$$\Omega_{M\star} = \Omega_{\mathcal{C}\star}/\Omega_{\mathcal{CC}\star} = \{[\omega] = \omega + \Omega_{\mathcal{CC}\star} \mid \omega \in \Omega_{\mathcal{C}\star}\}. \tag{65}$$

The following proposition assures that every element of $\Xi_{M\star}$ can be represented by an element in $\Xi_{t\star}$ and every element of $\Omega_{M\star}$ can be represented by an element in $\Omega_{t\star}$.

Proposition 11 in [32]. *For* $X \in \Xi_{\mathcal{C}\star}$, $\omega \in \Omega_{\mathcal{C}\star}$ *the tangent projections* $X_{t\star} := \mathrm{pr}_{t\star}(X) \in \Xi_{t\star}$, $\omega_{t\star} := \mathrm{pr}_{t\star}(\omega) \in \Omega_{t\star}$ *respectively belong to* $[X] \in \Xi_{M\star}$ *and* $[\omega] \in \Omega_{M\star}$.

Let ∇ be the LC connection corresponding to $(\mathcal{D}_f, \mathbf{g})$ and $\nabla^{\mathcal{F}}$ be the twisted LC connection corresponding to $\mathbf{g}_\star$. The induced twisted *second fundamental form* and *LC connection* on the family of submanifolds M_c are $II_\star^{\mathcal{F}} := \mathrm{pr}_{\perp\star} \circ \nabla^{\mathcal{F}}|_{\Xi_{t\star}\otimes_\star\Xi_{t\star}} : \Xi_{t\star}\otimes_\star\Xi_{t\star} \to \Xi_{\perp\star}$ and $\nabla_t^{\mathcal{F}} := \mathrm{pr}_{t\star}\circ\nabla^{\mathcal{F}}|_{\Xi_{t\star}\otimes_{\mathbb{K}[[\nu]]}\Xi_{t\star}} : \Xi_{t\star}\otimes_{\mathbb{K}[[\nu]]}\Xi_{t\star} \to \Xi_{t\star}$ respectively; the latter yields the curvature $\mathsf{R}_{t\star}^{\mathcal{F}}$ via (48). We now summarize results of Propositions 3, 12 and 13 in [32]. As $\mathbf{g}_t^{\mathcal{F}} = \mathbf{g}_t$, also the twisted second fundamental form, curvature, Ricci tensor and Ricci scalar on M are $U\mathfrak{k}^{\mathcal{F}}$-invariant and coincide with the undeformed ones as elements

$$
\begin{aligned}
&II^{\mathcal{F}} = II \in (\Omega_t \otimes \Omega_t \otimes \Xi_\perp)[[\nu]], \quad \mathsf{R}_t^{\mathcal{F}} = \mathsf{R}_t \in (\Omega_t \otimes \Omega_t^2 \otimes \Xi_t)[[\nu]],\\
&\mathsf{Ric}_t^{\mathcal{F}} = \mathsf{Ric}_t \in (\Omega\otimes\Omega)[[\nu]], \qquad \mathfrak{R}_t^{\mathcal{F}} = \mathfrak{R}_t \in \mathcal{X}.
\end{aligned}
\tag{66}
$$

Hence $\mathbf{g}_{t\star} = \left\langle \cdot \otimes_\star \cdot, \mathbf{g}_t^{\mathcal{F}} \right\rangle_\star : \Xi_{t\star} \otimes_\star \Xi_{t\star} \to \mathcal{X}_\star$, $II_\star^{\mathcal{F}} = \left\langle \cdot \otimes_\star \cdot, II^{\mathcal{F}} \right\rangle_\star : \Xi_{t\star} \otimes_\star \Xi_{t\star} \to \Xi_{\perp\star}$, $\mathsf{R}_{t\star}^{\mathcal{F}} = \left\langle \cdot \otimes_\star \cdot \otimes_\star \cdot, \mathsf{R}_t^{\mathcal{F}} \right\rangle_\star : \Xi_{t\star} \otimes_\star \Xi_{t\star} \otimes_\star \Xi_{t\star} \to \Xi_{t\star}$, $\mathsf{Ric}_{t\star}^{\mathcal{F}} = \left\langle \cdot \otimes_\star \cdot, \mathsf{Ric}_t^{\mathcal{F}} \right\rangle_\star : \Xi_{t\star}\otimes_\star\Xi_{t\star} \to \mathcal{X}_\star$, are $U\mathfrak{k}^{\mathcal{F}}$-equivariant maps, and for all $X, Y, Z \in \Xi_{t\star}$ they actually reduce to

$$
\begin{aligned}
\mathbf{g}_{t\star}(X,Y) &= \mathbf{g}_t\big(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y\big), \qquad \mathsf{R}_{t\star}^{\mathcal{F}}(X,Y,Z) = \mathsf{R}_t\big(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y, \overline{\mathcal{F}}_3 \triangleright Z\big),\\
II_\star^{\mathcal{F}}(X,Y) &= II\big(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y\big), \qquad \mathsf{Ric}_{t\star}^{\mathcal{F}}(X,Y) = \mathsf{Ric}_t\big(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y\big),
\end{aligned}
\tag{67}
$$

where $\overline{\mathcal{F}}_1 \otimes \overline{\mathcal{F}}_2 \otimes \overline{\mathcal{F}}_3$ is the inverse of (18); these maps are left (resp. right) $\mathcal{X}_\star$-linear in the first (resp. last) argument, 'middle' $\mathcal{X}_\star$-linear otherwise, in the sense $\mathbf{g}_{t\star}(X\star h, Y) = \mathbf{g}_{t\star}(X, h\star Y)$, etc. Furthermore, the following *twisted Gauss equation* holds for all $X, Y, Z, W \in \Xi_{t\star}$

$$
\begin{aligned}
\mathbf{g}_\star\left(\mathsf{R}_\star^{\mathcal{F}}(X,Y,Z), W\right) = {}& \mathbf{g}_\star\left(\mathsf{R}_{t\star}^{\mathcal{F}}(X,Y,Z), W\right) + \mathbf{g}_\star\big(II_\star^{\mathcal{F}}(X, \overline{\mathcal{R}}_1 \triangleright Z), II_\star^{\mathcal{F}}(\overline{\mathcal{R}}_2 \triangleright Y, W)\big)\\
&- \mathbf{g}_\star\big(II_\star^{\mathcal{F}}(\overline{\mathcal{R}}_{1\widehat{(1)}} \triangleright Y, \overline{\mathcal{R}}_{1\widehat{(2)}} \triangleright Z), II_\star^{\mathcal{F}}(\overline{\mathcal{R}}_2 \triangleright X, W)\big).
\end{aligned}
\tag{68}
$$

The twisted first and second fundamental forms, Levi-Civita connection, curvature tensor, Ricci tensor, Ricci scalar *on* M are finally obtained from the above by applying the further projection $\mathcal{X}_\star \to \mathcal{X}_\star^M$, which amounts to choosing the $c = 0$ manifold M out of the M_c family. Of course, one can do the same on any other M_c.

Decompositions (63) in terms of bases or complete sets. In terms of Cartesian coordinates $(x^1, \ldots, x^n)$ of $\mathbb{R}^n$ the components of the metric and of the inverse metric on $\mathbb{R}^n$ are denoted by $g_{ij} = \mathbf{g}(\partial_i, \partial_j)$ and $g^{ij} = \mathbf{g}^{-1}(\mathrm{d}x^i, \mathrm{d}x^j)$ (as before $\partial_i \equiv \partial/\partial x^i$). Using them we lower and raise indices: $\mathrm{d}x_i := g_{ij}\mathrm{d}x^j$, $Y_i := g_{ij}Y^j$, $\partial^i := g^{ij}\partial_j$, etc. In particular

$$
\begin{aligned}
\mathbf{g} &= \mathrm{d}x^i \otimes \mathrm{d}x_i, & \mathbf{g}(X,Y) &= X^i Y_i,\\
\mathbf{g}^{-1} &= \partial^i \otimes \partial_i, & \mathbf{g}^{-1}(\omega,\eta) &= \omega^i \eta_i.
\end{aligned}
\tag{69}
$$

Let $\mathsf{E} := f^i f_i$ ($f_i \equiv \partial_i f$), $\mathcal{D}'_f \subset \mathcal{D}_f \subset \mathbb{R}^n$ be the subset where $\mathsf{E} \neq 0$, and $K := \mathsf{E}^{-1}$ on $\mathcal{D}'_f$. If $\mathbf{g}$ is *Riemannian* then $\mathcal{D}'_f = \mathcal{D}_f$, because $\mathsf{E} > 0$ on all of $\mathcal{D}_f$ (as $\mathbf{g}^{-1}$ is positive-definite). Let

$$V_\perp := \mathbf{g}^{-1}(df, dx^i)\,\partial_i = f^i\partial_i, \qquad U_\perp := \sqrt{|K|}V_\perp, \qquad \theta := \sqrt{|K|}df; \quad (70)$$

$V_\perp$, $N_\perp := KV_\perp = K\star V_\perp$ or $U_\perp$ spans $\Xi_\perp$ (and $\Xi_{\perp\star}$), while df or θ spans $\Omega_\perp$ (and $\Omega_{\perp\star}$). All are $U\mathfrak{k}$-invariant. $N_\perp, df$ are $\star$-dual, $\langle N_\perp, df\rangle_\star = 1$, but $\mathbf{g}_\star^{-1}(df, df) = \mathsf{E}$, $\mathbf{g}_\star(N_\perp, N_\perp) = K$, while

$$\langle U_\perp, \theta\rangle_\star = 1, \qquad \mathbf{g}_\star(U_\perp, U_\perp) = \epsilon, \qquad \mathbf{g}_\star^{-1}(\theta, \theta) = \epsilon, \qquad \epsilon := \mathrm{sign}(\mathsf{E}) \quad (71)$$

(see Proposition 8 in [32]); these relations hold also without $\star$. The projection $\mathrm{pr}_{\perp\star}$ ($\mathbb{C}[[\nu]]$-linear extension of $\mathrm{pr}_\perp$) on $X \in \Xi_\star$, $\omega \in \Omega_\star$ can be equivalently expressed as

$$\begin{aligned}\mathrm{pr}_{\perp\star}(\omega) &= \omega_\perp = \epsilon\,\theta\star\mathbf{g}_\star^{-1}(\theta,\omega) = df\star K\star\mathbf{g}_\star^{-1}(df,\omega) = \mathbf{g}_\star^{-1}(\omega, df)\star K\star df,\\ \mathrm{pr}_{\perp\star}(X) &= X_\perp = \epsilon\,\mathbf{g}_\star(X, U_\perp)\star U_\perp = \mathbf{g}_\star(X, V_\perp)\star K\star V_\perp = V_\perp\star K\star\mathbf{g}_\star(V_\perp, X)\end{aligned} \quad (72)$$

(see Proposition 14 in [32]). By the $\star$-bilinearity of $\mathbf{g}_\star$ these equations imply in particular

$$\begin{aligned}\omega_\perp &= df\star K\star\mathbf{g}_\star^{-1}(df, dx^i)\star\check{\omega}_i = \hat{\omega}_i\star\mathbf{g}_\star^{-1}(dx^i, df)\star K\star df,\\ X_\perp &= \hat{X}^i\star\mathbf{g}_\star(\partial_i, V_\perp)\star K\star V_\perp = V_\perp\star K\star\mathbf{g}_\star(V_\perp,\partial_i)\star\check{X}^i,\end{aligned} \quad (73)$$

in terms of the left and right decompositions $\omega = \hat{\omega}_i\star dx^i = dx^i\star\check{\omega}_i \in \Omega_\star$, $X = \hat{X}^i\star\partial_i = \partial_i\star\check{X}^i \in \Xi_\star$ in the bases $\{dx^i\}_{i=1}^n$, $\{\partial_i\}_{i=1}^n$. One can decompose $df, N_\perp, \theta, U_\perp$ themselves in the same way, if one wishes. If the metric is Euclidean ($g_{ij} = \delta_{ij}$) or Minkowski [$g_{ij} = g^{ij} = \eta_{ij} = \mathrm{diag}(1, ..., 1, -1)$] one makes (73) more explicit replacing

$$\begin{aligned}\mathbf{g}_\star^{-1}(dx^i, df) &= \mathbf{g}_\star^{-1}(df, dx^i) = \mathbf{g}^{-1}(dx^i, df) = f^i, \quad ,\\ \mathbf{g}_\star(\partial_i, N_\perp) &= \mathbf{g}_\star(N_\perp^a, \partial_i) = \mathbf{g}(\partial_i, N_\perp) = Kf_i = K\star f_i.\end{aligned} \quad (74)$$

Finally, we can express the tangent projection acting on $X \in \Xi_\star$, $\omega \in \Omega_\star$ simply as $\mathrm{pr}_{t\star}(X) = X_t := X - X_\perp$, $\mathrm{pr}_{t\star}(\omega) = \omega_t := \omega - \omega_\perp$. All the above formulae hold also if we drop all $\star$.

Having determined bases of $\Xi_{\perp\star}, \Omega_{\perp\star}$ we now consider $\Xi_{t\star}, \Omega_{t\star}$. The globally defined sets $\Theta_t := \{\vartheta^j\}_{j=1}^n$, $S_W := \{W_j\}_{j=1}^n$, where $\vartheta^j := \mathrm{pr}_t(dx^j)$, $W_j := \mathrm{pr}_t(\partial_j) =: K\,V_j$, are respectively complete in Ω_t, Ξ_t; they are not bases, because of the linear dependence relations $\vartheta^j f_j = 0$, $f^j W_j = 0$. An alternative complete set (of globally defined vector fields) in Ξ_t is

$$S_L := \{L_{ij}\}_{i,j=1}^n, \qquad \text{where} \qquad L_{ij} := f_i\partial_j - f_j\partial_i. \quad (75)$$

In fact, L_{ij} manifestly annihilate f, and S_L is complete because the combinations $Kf^iL_{ij} = W_j$ make up S_W. Clearly $L_{ij} = -L_{ji}$, so at most $n(n-1)/2$ L_{ij} (e.g. those with $i < j$) are linearly independent over $\mathbb{R}$ (or $\mathbb{C}$), while S_L is of rank $n-1$ over $\mathcal{X}$ because of the dependence relations

$$f_{[i}L_{jk]} = 0 \quad (76)$$

(square brackets enclosing indices mean a complete antisymmetrization of the latter). Contrary to the W_j, the L_{ij} are anti-Hermitian under the $*$-structure $L_{ij}^* = -L_{ij}$

and *do not involve* $\mathbf{g}$, so they can be used even if we introduce no metric. Setting $f_{ih} = \partial_i \partial_h f$, their Lie brackets are

$$[L_{ij}, L_{hk}] = f_{jh} L_{ik} - f_{ih} L_{jk} - f_{jk} L_{ih} + f_{ik} L_{jh}. \tag{77}$$

By the mentioned propositions, every complete set of Ω_t, e.g. Θ_t, is also a complete set of $\Omega_{t\star}$; similarly, every complete set of Ξ_t, e.g. S_W or S_L, is also a complete set of $\Xi_{t\star}$.

3 Twisted Algebraic Submanifolds of $\mathbb{R}^n$: the Quadrics

We can apply the whole machinery developed in the previous chapter to twist deform algebraic manifolds of codimension 1 embedded in $\mathbb{R}^n$ provided we adopt $\mathcal{X} = \mathrm{Pol}^\bullet(\mathbb{R}^n)$, etc. everywhere. We can assume without loss of generality that the f be an irreducible polynomial function[5]. It is interesting to ask for which algebraic submanifolds $M_c \subset \mathbb{R}^n$ the infinite-dimensional Lie algebra Ξ_t admits a nontrivial finite-dimensional subalgebra $\mathfrak{g}$ over $\mathbb{R}$ (or $\mathbb{C}$), so that we can build concrete examples of twisted M_c by choosing a twist $\mathcal{F} \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\nu]]$ of a known type. If M_c are manifestly symmetric under a Lie group[6] $\mathfrak{K}$, then such a $\mathfrak{g}$ exists and contains the Lie algebra $\mathfrak{k}$ of $\mathfrak{K}$ (if M is maximally symmetric then $\mathfrak{k}$ is even complete - over $\mathcal{X}$ - in Ξ_t). In general, given any set S of vector fields that is complete in Ξ_t the question is whether there are combinations of them (with coefficients in $\mathcal{X}$) that close a finite-dimensional Lie algebra $\mathfrak{g}$.

Here we answer this question in the simple situation where the L_{ij} themselves close a finite-dimensional Lie algebra $\mathfrak{g}$. This means that in (77) f_{ij} =const, hence $f(x)$ is a quadratic polynomial, and M is either a quadric or the union of two hyperplanes (reducible case); moreover $\mathfrak{g}$ is a Lie subalgebra of the affine Lie algebra $\mathsf{aff}(n)$ of $\mathbb{R}^n$. In the next subsection we find some results valid for all $n \geq 3$ drawing some general consequences from the only assumptions $\mathcal{X} = \mathrm{Pol}^\bullet(\mathbb{R}^n)$ and $\mathfrak{g} \subset \mathsf{aff}(n)$; in particular, in Sections 3.1, 3.2 we show that the *global* description of differential geometry on $\mathbb{R}^n$, M_c in terms of generators and relations extends to their twist deformations, in such a way to preserve the spaces consisting of polynomials of any fixed degrees in the coordinates x^i, differential dx^i and vector fields chosen as generators. In Section 4 we shall analyze in detail the twisted quadrics embedded in $\mathbb{R}^3$.

[5]If $f(x) = g(x)h(x)$, we find

$$L_{ij} = h(x)[g_i \partial_j - g_j \partial_i] + g(x)[h_i \partial_j - h_j \partial_i];$$

on M_g the second term vanishes and the first is tangent to M_g, as it must be; and similarly on M_h. Having assumed the Jacobian everywhere of maximal rank M_g, M_h have empty intersection and can be analyzed separately. Otherwise L_{ij} vanishes on $M_g \cap M_h \neq \emptyset$ (the singular part of M), so that on the latter a twist built using the L_{ij} will reduce to the identity, and the $\star$-product to the pointwise product (see the conclusions).

[6]For instance, the sphere S^{n-1} is $SO(n)$ invariant; a cylinder in $\mathbb{R}^3$ is invariant under $SO(2) \times \mathbb{R}$; the hyperellipsoid of equation $(x^1)^2 + (x^2)^2 + 2[(x^3)^2 + (x^4)^2] = 1$ is invariant under $SO(2) \times SO(2)$; etc.

If f is of degree two then there are real constants $a_{\mu\nu} = a_{\nu\mu}$ ($\mu, \nu = 0, 1, ..., n$) such that

$$f(x) \equiv \frac{1}{2} a_{ij} x^i x^j + a_{0i} x^i + \frac{1}{2} a_{00} = 0; \tag{78}$$

hence $f_i = a_{ij} x^j + a_{i0}$, all $f_{ij} = a_{ij}$ are constant, and (77) has already the desired form

$$[L_{ij}, L_{hk}] = a_{jh} L_{ik} - a_{ih} L_{jk} - a_{jk} L_{ih} + a_{ik} L_{jh}, \tag{79}$$

i.e. the L_{ij} span a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. This is a Lie subalgebra of the affine Lie algebra of $\mathbb{R}^n$, because all $L_{ih}\triangleright$ act as linear transformations of the coordinates x^k:

$$L_{ij} \triangleright x^h = (a_{ix} x^k + a_{0i}) \delta^h_j - (a_{jx} x^k + a_{j0}) \delta^h_i \tag{80}$$

Let $r := \text{rank}(a_{\mu\nu})$. To identify $\mathfrak{g}$ for irreducible f's ($r > 2$)[7] we note that by a suitable Euclidean transformation (this will be also an affine one) one can always make the x^i canonical coordinates for the quadric, so that $a_{ij} = a_i \delta_{ij}$ (no sum over i), $b_i := a_{0i} = 0$ if $a_i \neq 0$, and coordinates are ordered so that

$$a_1 > 0, \quad ..., \quad a_l > 0, \quad a_{l+1} < 0, \quad ..., \quad a_m < 0, \quad \begin{cases} a_{m+1} = 0, \\ b_{m+1} < 0, \end{cases} \quad ..., \quad \begin{cases} a_n = 0, \\ b_n < 0, \end{cases} \tag{81}$$

with $l \leq m \leq n$; moreover, if $m < n$ one can make $a_{00} = 0$ by translation of a x^j with $j > m$. The associated new L_{ij} (which are related to the old by a linear transformation) fulfill

$$[L_{ij}, L_{hk}] = a_j [\delta_{jh} L_{ik} - \delta_{jk} L_{ih}] - a_i [\delta_{ih} L_{jk} - \delta_{ik} L_{jh}]. \tag{82}$$

It is easy to check that $r = n+1$ if $m = n$, $r = m+2$ if $m < n$. One can always make $a_1 = 1$ by replacing $f \mapsto f/a_1$; one can make also the other nonzero a_i's in (82) be ± 1 by the rescalings $x^i \mapsto y^i := |a_i|^{1/2} x^i$ of the corresponding coordinates (another affine transformation). So the associated new L_{ij} fulfill (82) with the $a_i \in \{-1, 0, 1\}$. Then:

- If $k > j > m$ (what is possible only if $m < n-1$), then $[L_{jk}, L_{hi}] = 0$. Hence the center $\mathcal{Z}(\mathfrak{g})$ of $\mathfrak{g}$ is trivial if $m = n, n-1$; otherwise it contains all such $L_{jk} = a_{0j} \partial_k - a_{0k} \partial_j$, and $\mathcal{Z}(\mathfrak{g}) \simeq \mathbb{R}^{n-m-1}$; a basis of $\mathcal{Z}(\mathfrak{g})$ is $\mathcal{B} = \{L_{(m+1)(m+2)}, L_{(m+2)(m+3)}, ..., L_{(n-1)n}\}$.
- The L_{ij} with $j > m$ span an ideal $\mathcal{I}(\mathfrak{g}) \supset \mathcal{Z}(\mathfrak{g})$ of $\mathfrak{g}$, because (82) becomes $[L_{ij}, L_{hk}] = a_i [\delta_{ih} L_{kj} - \delta_{ik} L_{hj}]$; adding the $m(n-m)$ elements L_{ij} with $i \leq m < j$ to $\mathcal{B}$ one obtains a basis of $\mathcal{I}(\mathfrak{g})$, hence $\dim[\mathcal{I}(\mathfrak{g})] = m(n-m) + (n-m-1)\theta(n-m-1)$. $\mathcal{I}(\mathfrak{g})$ is a nilpotent Lie subalgebra, the radical $\mathcal{R}(\mathfrak{g})$ (the largest solvable ideal) of $\mathfrak{g}$.
- Finally, the L_{ij} with $i < j \leq m$ make up a basis of a $m(m-1)/2$-dimensional simple Lie-subalgebra $\mathfrak{g}_s \simeq so(l, m-l)$, in view of the signs of a_i, a_j.

[7] If all $a_{ij} = 0$ vanish, but $a_{0i} \neq 0$ for some i then $r = 1$, M is a (hyper)plane, and rhs(79) vanishes; one can express all L_{ij} (or V_j) as combinations with constant coefficients of $(n-1)$ independent ones: i.e. $\mathfrak{g} \sim \mathbb{R}^{n-1}$ is the abelian group of translations in the (hyper)plane. $r = 2$ corresponds to a reducible f, i.e. two (hyper)planes.

Summing up, the Levi decomposition of $\mathfrak{g}$ becomes $\mathfrak{g} \simeq so(l, m-l) \ltimes \mathcal{R}$.

The cones, which in the y coordinates are represented by the homogeneous equations

$$f(y) := (y^1)^2 + ... + (y^l)^2 - (y^{l+1})^2 - ... - (y^n)^2 = 0,$$

strictly speaking are not encompassed in the above analysis because the Jacobian matrix $(f_i)(y)$ vanishes at the apex $y = 0$ (the only singular point). They are algebraic varieties that are limits of the hyperboloids $f_c(y) = 0$ as $c \to 0$. If we omit the apex, a cone becomes a disconnected union of two nappes (which are open in $\mathbb{R}^n$), and $\mathfrak{g}$ is spanned not only by the L_{ij}, but also by the central anti-Hermitian element $\eta = x^i \partial_i + n/2$ generating dilatations; note that all of them vanish on the apex. Hence $\mathfrak{g} \simeq so(l, n-l) \times \mathbb{R}$ in this case.

If we endow $\mathbb{R}^n$ with the Euclidean metric, the metric matrix $g_{ij} = \delta_{ij}$ is not changed by the above Euclidean changes of coordinates, because the Euclidean group is the isometry group $\mathfrak{H}$ of $\mathbb{R}^n$, whereas its nonzero (diagonal) elements are rescaled if we rescale $x^i \mapsto |a_i|^{1/2} x^i$. Similarly, if we endow $\mathbb{R}^n$ with the Minkowski metric, Euclidean changes of coordinates involving only the space ones, or a translation of the time coordinate, do not alter the metric matrix $g_{ij} = \eta_{ij}$.

3.1 Twisted Differential Calculus on $\mathbb{R}^n$ by Generators, Relations

Let us abbreviate $\xi^i := dx^i$. We name *differential calculus algebra on* $\mathbb{R}^n$ the unital associative $*$-algebra $\mathcal{Q}^\bullet$ over $\mathbb{C}$ generated by Hermitian elements $\{\mathbf{1}, x^i, \xi^i, \mathrm{i}\partial_i\}_{i=1}^n$ fulfilling

$$\begin{aligned} &\mathbf{1}\eta^i - \eta^i = \eta^i \mathbf{1} - \eta^i = 0, \qquad \text{for } \eta^i = x^i, \xi^i, \partial_i \\ &x^i x^j - x^j x^i = 0, \\ &\xi^i x^j - x^j \xi^i = 0, \end{aligned} \tag{83}$$

$$\begin{aligned} &\partial_i \partial_j - \partial_j \partial_i = 0, \\ &\partial_j \xi^i - \xi^i \partial_j = 0, \\ &\xi^i \xi^j + \xi^j \xi^i = 0, \\ &\partial_i x^j - \delta_i^j \mathbf{1} - x^j \partial_i = 0. \end{aligned} \tag{84}$$

The $x^0 \equiv \mathbf{1}, x^i, \xi^i, \partial_i$ play respectively the role of the unit, of Cartesian coordinate functions on $\mathbb{R}^n$, of differentials dx^i of x^i, of partial derivatives $\partial/\partial x^i$ with respect to x^i. This is the adaptation of the definition of $\mathcal{Q}^\bullet$ in the smooth context (Sections 3.1.3, 3.2.3 in [32]) to the polynomial one: the relations in the first two lines define the algebra structure of $\mathcal{X}$, the other ones determine the relations (113-114) of [32] for the current choice of $\mathcal{X}$ and of the pair $\{\xi^i\}, \{\partial_i\}$ of dual frames. The x^μ ($\mu = 0, ..., n$) span the fundamental module $(\check{\mathcal{M}}, \tau)$ of $U\mathsf{aff}(n)$ (the invariant element $\mathbf{1}$ itself spans a 1-dim, non-faithful submodule), the ξ^i span a related module $(\mathcal{M}, \tau)$, the ∂_i the contragredient one $(\mathcal{M}^\vee, \tau^\vee)$. More precisely they are related by

$$\begin{aligned}
&g \triangleright \mathbf{1} = \varepsilon(g)\mathbf{1},\\
&g \triangleright x^i = x^\mu \check{\tau}^{\mu i}(g) =: x^j \tau^{ji}(g) + \mathbf{1}\check{\tau}^{0i}(g),\\
&g \triangleright \xi^i = \xi^j \tau^{ji}(g),\\
&g \triangleright \partial_i = \tau^{\vee ji}(g)\partial_j = \tau^{ij}(Sg)\partial_j;
\end{aligned} \tag{85}$$

the first relation and $g \triangleright x^0 = x^\mu \check{\tau}^{\mu 0}(g)$ imply $\check{\tau}^{\mu 0}(g) = \varepsilon(g)\delta^{\mu 0}$. We encompass these $U\mathsf{aff}(n)$-modules into a single one $(\widetilde{\mathcal{M}}, \rho)$ spanned by $(a^0, a^1, ..., a^{3n}) \equiv (\mathbf{1}, x^1, ..., x^n, \xi^1, ..., \xi^n, \partial_1, ..., \partial_n)$. All are trivially also $U\mathfrak{g}$-modules; also $\mathfrak{g}$ is, under the adjoint action. Of course, this $U\mathsf{aff}(n)$ action is compatible with the relations (83-84); the ideal $\mathcal{I}$ generated by their left-hand sides in the free $*$-algebra $\mathcal{A}^f$ generated by $\{a^0, a^1, ..., a^{3n}\}$ is $U\mathsf{aff}(n)$-invariant. The $U\mathsf{aff}(n)$-action is also compatible with the invariance of the exterior derivative, because $g \triangleright \xi^i = d(g \triangleright x^i)$.

In the $\mathcal{Q}^\bullet$ framework $Xh = hX + X(h)$ is the inhomogeneous first order differential operator sum of a first order part (the vector field hX) and a zero order part (the multiplication operator by $X(h)$); it must not be confused with the product of X by h from the right, which is equal to hX and so far has been denoted in the same way. In the $\mathcal{Q}^\bullet$ framework we denote the latter by $X \triangleleft h$ (of course $(X \triangleleft h)(h') = X(h')h = hX(h')$, $X \triangleleft (hh') = hh'X$ remain valid).

When choosing a basis $\mathcal{B}$ of $\mathcal{Q}^\bullet$ made out of monomials in these generators, relations (83-84) allow to order them in any prescribed way; in particular we may choose

$$\mathcal{B} := \left\{\beta^{\vec{p},\vec{q},\vec{r}} := (\xi^1)^{p_1}...(\xi^n)^{p_n}(x^1)^{q_1}...(x^n)^{q_n}\partial_1^{r_1}...\partial_n^{r_n} \;\mid\; \vec{p} \in \{0,1\}^n,\ \vec{q},\vec{r} \in \mathbb{N}_0^n\right\}$$

(we define $\beta^{\vec{0},\vec{0},\vec{0}} := \mathbf{1}$). The $*$-algebra structure of $\mathcal{Q}^\bullet$ is compatible with the form grading $\natural$

$$\natural\left(\beta^{\vec{p},\vec{q},\vec{r}}\right) = p, \qquad p := \sum_{i=1}^n p_i, \quad q := \sum_{i=1}^n q_i, \quad r := \sum_{i=1}^n r_i \tag{86}$$

and the one $\sharp$ defined by $\sharp\left(\beta^{\vec{p},\vec{q},\vec{r}}\right) = q - r$ (p, q, r are the total degrees in ξ^i, x^i, ∂_i respectively). Fixing part or all of p, q, r we obtain the various relevant $U\mathsf{aff}(n)$ modules or module subalgebras or $\mathcal{X}$-bimodules: $\Lambda^\bullet, \Lambda^p, \Omega^\bullet, \Omega^p, ...$ For instance the exterior algebra $\Lambda^\bullet$ is generated by the ξ^i alone ($q = r = 0$) and its $\natural = p$ component is the $U\mathsf{aff}(n)$-submodule of exterior p-forms Λ^p; by $(84)_3$ $\dim(\Lambda^p) = \binom{n}{p}$; in particular this is zero for $p > n$, 1 for $p = n$, and $\Lambda^\bullet = \bigoplus_{p=0}^n \Lambda^p$. Let $\mathcal{X}^q$ be the component of $\mathcal{X}$ of degree q, and $\widetilde{\mathcal{X}}^q := \bigoplus_{h=0}^q \mathcal{X}^q$ (i.e. $\mathcal{X}^q, \widetilde{\mathcal{X}}^q$ consist resp. of homogeneous and inhomogenous polynomials in x^i of degree q); $\mathcal{X} = \bigoplus_{q=0}^\infty \mathcal{X}^q$ is trivially a filtered algebra $\mathcal{X} = \biguplus_{q=0}^\infty \widetilde{\mathcal{X}}^q$. Let $\mathcal{D}$ be the unital subalgebra generated by the ∂_i alone, $\mathcal{D}^r$ its component of degree r, and $\widetilde{\mathcal{D}}^r := \bigoplus_{h=0}^r \mathcal{D}^r$; then $\mathcal{D} = \bigoplus_{r=0}^\infty \mathcal{D}^r$ is trivially a filtered algebra $\mathcal{D} = \bigoplus_{r=0}^\infty \widetilde{\mathcal{D}}^r$. Finally, let

$$\mathcal{Q}^{pqr} := \Lambda^p \widetilde{\mathcal{X}}^q \widetilde{\mathcal{D}}^r. \tag{87}$$

By (85) the $U\mathsf{aff}(n)$ action maps $\Lambda^p, \widetilde{\mathcal{X}}^q, \mathcal{D}^r$ into themselves, and all $\mathcal{Q}^{pqr}$ are $U\mathsf{aff}(n)$-$*$-modules. By (83-84), $\widetilde{\mathcal{D}}^r\widetilde{\mathcal{X}}^{q'} = \widetilde{\mathcal{X}}^{q'}\widetilde{\mathcal{D}}^r$, whence

$$\mathcal{Q}^{pqr}\mathcal{Q}^{p'q'r'} \subseteq \mathcal{Q}^{(p+p')(q+q')(r+r')} \tag{88}$$

(this multiplication rule would not hold if we had defined $\mathcal{Q}^{pqr} := \Lambda^p\mathcal{X}^q\mathcal{D}^r$, because, $\mathcal{D}^r\mathcal{X}^{q'} \neq \mathcal{X}^{q'}\mathcal{D}^r$). A basis of $\mathcal{Q}^{pqr}$ is $\mathcal{B}^{pqr} := \{\beta^{\vec{p},\vec{q},\vec{r}} \mid p = \sum_{i=1}^n p_i, \ \sum_{i=1}^n q_i \leq q, \ \sum_{i=1}^n r_i \leq r\}$. $\mathcal{Q}^\bullet$ is graded by p and filtered by both q, r; it decomposes as

$$\mathcal{Q}^\bullet = \bigoplus_{p=0}^{n} \biguplus_{q=0}^{\infty} \biguplus_{r=0}^{\infty} \mathcal{Q}^{pqr}. \tag{89}$$

Choosing a twist $\mathcal{F}$ based on $U\mathsf{aff}(n)$ (in particular, on $U\mathfrak{g}$) and setting (10) for all $a, b \in \mathcal{Q}^\bullet$ one makes $\mathcal{Q}^\bullet$ into a $U\mathsf{aff}(n)^{\mathcal{F}}$-module (resp. $U\mathfrak{g}^{\mathcal{F}}$-module) algebra $\mathcal{Q}^\bullet_\star$ with grading $\natural$ (whereas the grading $\sharp$ is not preserved). In the appendix we prove

Proposition 2 *The vector fields $\partial'_i := S(\beta) \triangleright \partial_i = \tau^{ij}(\beta)\partial_j$ are the $\star$-dual ones to the $\xi^i = dx^i$; under the $U\mathsf{aff}(n)$ (and $U\mathfrak{g}$) action they transform according to $g \triangleright \partial'_i = \tau^{ij}[S_{\mathcal{F}}(g)]$. The polynomials relations (83-84) are deformed into the ones*

$$\begin{aligned}
&\boldsymbol{1} \star \eta^i - \eta^i = \eta^i \star \boldsymbol{1} - \eta^i = 0, \qquad \text{for } \eta^i = x^i, \xi^i,\\
&x^i \star x^j - x^\nu \star x^\mu R^{\mu\nu}_{ij} = 0,\\
&\xi^i \star x^j - x^\nu \star \xi^h R^{h\nu}_{ij} = 0,\\
&\xi^i \star \xi^j + \xi^k \star \xi^h R^{hk}_{ij} = 0,
\end{aligned} \tag{90}$$

$$\begin{aligned}
&\boldsymbol{1} \star \partial'_i - \partial'_i = \partial'_i \star \boldsymbol{1} - \partial'_i = 0,\\
&\partial'_i \star \partial'_j - R^{ij}_{hk}\partial'_k \star \partial'_h = 0,\\
&\partial'_i \star \xi^j - R^{hi}_{jk}\xi^h \star \partial'_k = 0,\\
&\partial'_i \star x^j - \delta^j_i \boldsymbol{1} - R^{\mu i}_{jk}x^\mu \star \partial'_k = 0.
\end{aligned} \tag{91}$$

where $R^{\mu\nu}_{ij} := (\tau^{\mu i} \otimes \tau^{\nu j})(\mathcal{R})$. Defining $\mathcal{Q}^{pqr}_\star := \Lambda^p_\star \widetilde{\mathcal{X}}^q_\star \widetilde{\mathcal{D}}^r_\star$, we find not only $\mathcal{Q}^\bullet_\star = \mathcal{Q}^\bullet[[\nu]]$, but that for all $p, q, r \in \mathbb{N}_0$ also

$$\mathcal{Q}^{pqr}_\star = \mathcal{Q}^{pqr}[[\nu]] \tag{92}$$

hold as equalities of $\mathbb{C}[[\nu]]$-modules. A basis $\mathcal{B}^{pqr}_\star$ of $\mathcal{Q}^{pqr}_\star$ is obtained replacing all products in the definition of $\mathcal{B}^{pqr}$ by $\star$-products. $\mathcal{Q}^\bullet_\star$ is graded by p, filtered by both q, r, and

$$\mathcal{Q}^\bullet_\star = \bigoplus_{p=0}^{n} \biguplus_{q=0}^{\infty} \biguplus_{r=0}^{\infty} \mathcal{Q}^{pqr}_\star, \qquad \mathcal{Q}^{pqr}_\star \star \mathcal{Q}^{p'q'r'}_\star \subseteq \mathcal{Q}^{(p+p')(q+q')(r+r')}_\star. \tag{93}$$

$\mathcal{Q}^\bullet_\star$ is a $U\mathfrak{g}^{\mathcal{F}}$-module $*$-algebra with the $\mathcal{Q}^{pqr}_\star$ as $*$-submodules, if $\mathcal{F}$ is either real or unitary; correspondingly the involution is the undeformed one $*$, respectively is given by (29)*, i.e.*

$$\mathbf{1}^{*_\star}=\mathbf{1},\quad x^{i\,*_\star}=x^\mu\check{\tau}^{\mu i}[S(\beta)],\quad \xi^{i\,*_\star}=\xi^k\tau^{ki}[S(\beta)],\quad \partial_i'^{\,*_\star}=-\tau^{ik}\left(\beta^{-1}\right)\hat{\partial}_k'. \tag{94}$$

In the $\mathcal{Q}^\bullet_\star$ framework $X\star h=(\overline{\mathcal{R}}_1\triangleright h)\star(\overline{\mathcal{R}}_2\triangleright X)+X_\star(h)$, while so far it stood just for the $\star$-product of the vector field X by the function h from the right, i.e. for the first term at the rhs; denoting the latter by $X\triangleleft_\star h:=(\overline{\mathcal{R}}_1\triangleright h)\star(\overline{\mathcal{R}}_2\triangleright X)$, we can abbreviate $X\star h=X\triangleleft_\star h+X_\star(h)$. Of course $(X\triangleleft_\star h)_\star(h')=[X_\star(\overline{\mathcal{R}}_1\triangleright h')]\star(\overline{\mathcal{R}}_2\triangleright h)$, $(X\triangleleft_\star h)\triangleleft_\star h'=X\triangleleft_\star(h\star h')$ remain valid.

These results are the strict analogues of their untwisted counterparts. Relation (92) is much stronger than the equality of infinite-dimensional $\mathbb{C}[[\nu]]$-modules $\mathcal{Q}^\bullet_\star=\mathcal{Q}^\bullet[[\nu]]$; it implies $\dim\left(\mathcal{Q}^{pqr}_\star\right)=\dim\left(\mathcal{Q}^{pqr}\right)$ over $\mathbb{C}[[\nu]]$, so that the Hilbert-Poincaré series of the p-graded and (q,r)-filtered algebras $\mathcal{Q}^\bullet_\star$, $\mathcal{Q}^\bullet[[\nu]]$ coincide. In particular, $p=r=0$ yields $\dim\left(\widetilde{\mathcal{X}}^q_\star\right)=\dim\left(\widetilde{\mathcal{X}}^q\right)$.

The $U\mathsf{aff}^{\mathcal{F}}$-equivariant relations (90-91) defining $\mathcal{Q}^\bullet_\star$ have the same form (see e.g. formulae (1.10-15) in [27]) as the quantum group equivariant ones defining the differential calculus algebras on the celebrated 'quantum spaces' introduced in [23]. The relations, among (90-91), that involve only the generators x^i, ∂_j' of the twisted Heisenberg algebra on $\mathbb{R}^n$ (the $p=0$ component of $\mathcal{Q}^\bullet_\star$) were already determined in [24, 25], while (92) extends results of [29].

3.2 Twisted Differential Calculus on *M* by Generators, Relations

Chosen a basis $\{e_1,...,e_B\}$ of $\mathfrak{g}$ (e.g. consisting of L_{ij}), on $\mathcal{D}_f\subseteq\mathbb{R}^n$ one can use $S'\equiv\{e_1,...,e_B,e_{B+1}=V_\perp\}$, instead of $S\equiv\{\partial_1,...,\partial_n\}$, as a complete set of vector fields in Ξ. They fulfill the following commutation relations with the coordinates

$$V_\perp x^h-x^hV_\perp-f^i=0,\qquad e_\alpha x^h-x^he_\alpha-x^\mu\check{\tau}^{\mu h}(e_\alpha)=0,\quad \alpha=1,...,B \tag{95}$$

and the remaining relations of the type (113) in [32], i.e.

$$\begin{aligned}&\textstyle\sum_{\alpha=1}^A t_l^\alpha\, e_\alpha=0,\qquad l=1,...,B+1-n,\\ &e_\alpha e_\beta-e_\beta e_\alpha-C^\gamma_{\alpha\beta}\,e_\gamma=0,\\ &e_\alpha\xi^i-\xi^ie_\alpha=0,\end{aligned} \tag{96}$$

with suitable $t_l^a,C^\gamma_{\alpha\beta}\in\mathcal{X}$. For instance, if $S\equiv\{L_{ij},V_\perp\}$ then the dependence relations in the first line amount to (76), while the commutation relations in the second line have constant $C^\gamma_{\alpha\beta}$ and amount to (79) for $\alpha,\beta\le B$. We collectively rename $\mathbf{1},x^1,...,x^n,\xi^1,...,\xi^n,e_1,...,e_B$ as $a^0,a^1,...,a^N$; we denote as $\mathcal{A}'^\bullet$ the free algebra generated by $a^0,...,a^N$, and as $\mathcal{A}'^{pqr}$ the subspace consisting of polynomials in the a^A of degree q in the x^i, of degree r in the e_α and homogeneous of degree p in the ξ^i. Clearly

$$\mathcal{A}'^{pqr}\mathcal{A}'^{p'q'r'}\subseteq\mathcal{A}'^{(p+p')(q+q')(r+r')}. \tag{97}$$

$\mathcal{A}'^{\bullet}$ is graded by p and filtered by both q, r; it decomposes as

$$\mathcal{A}'^{\bullet} = \bigoplus_{p=0}^{\infty} \biguplus_{q=0}^{\infty} \biguplus_{r=0}^{\infty} \mathcal{A}'^{pqr}. \tag{98}$$

For all $c \in \mathbb{R}$ denote as $\{f_c^J(a^0, ..., a^N)\}_{J\in\mathcal{J}}$ the set of polynomial functions at the lhs of (96), (83) , (95) involving only e_α with $\alpha \le B$, together with

$$\begin{aligned} f_c &\equiv f(x) - c = 0, \\ df(x) &\equiv \xi^h f_h = 0, \end{aligned} \tag{99}$$

which are (78) and its exterior derivative. Let $\mathcal{I}_{M_c}$ be the ideal generated by all the $f_c^J(a)$ in $\mathcal{A}'^{\bullet}$. We define the *differential calculus algebra on M_c* as the quotient

$$\mathcal{Q}_{M_c}^{\bullet} := \mathcal{A}'^{\bullet}/\mathcal{I}_{M_c}. \tag{100}$$

$\mathcal{I}_{M_c}^{pqr} := \mathcal{I}_{M_c} \cap \mathcal{A}'^{pqr}$ is a subspace of $\mathcal{A}'^{pqr}$. The quotient subspaces $\mathcal{Q}_{M_c}^{pqr} := \mathcal{A}'^{pqr}/\mathcal{I}_{M_c}^{pqr}$ fulfill

$$\mathcal{Q}_{M_c}^{pqr}\mathcal{Q}_{M_c}^{p'q'r'} \subseteq \mathcal{Q}_{M_c}^{(p+p')(q+q')(r+r')} \tag{101}$$

because of the equations $f_c^J(a) = 0$, in particular because $x^\mu \check{\tau}^{\mu h}(e_\alpha)$ in (95) are polynomial functions of first degree in x^i. $\mathcal{Q}_{M_c}^{\bullet}$ is graded by p and filtered by both q, r; it decomposes as

$$\mathcal{Q}_{M_c}^{\bullet} = \bigoplus_{p=0}^{n-1} \biguplus_{q=0}^{\infty} \biguplus_{r=0}^{\infty} \mathcal{Q}_{M_c}^{pqr}. \tag{102}$$

By (85), $(96)_2$ the a^i span a (reducible) $U\mathfrak{g}$-$*$-module. Hence $\mathcal{A}'^{\bullet}$, which is generated by them, is a $U\mathfrak{g}$-module $*$-algebra, and the $\mathcal{A}'^{pqr}$ are $U\mathfrak{g}$-$*$-submodules. It is immediate to check that also the $f_c^J(a)$ span a (reducible) $U\mathfrak{g}$-$*$-module,

$$[f_c^J(a)]^* = f_c^J(a), \qquad g \triangleright f_c^J(a) = \sum_{J'\in\mathcal{J}} f_c^{J'}(a)\tau_{J'}^J(g), \tag{103}$$

more precisely $g \triangleright f^c = \varepsilon(g) f^c$, while more generally $g \triangleright f_c^J(a)$ is a numerical combination of the $f_c^{J'}(a)$ appearing in the same equation where $f_c^J(a)$ appears, e.g. $g \triangleright (\xi^i\xi^j + \xi^j\xi^i) = (\xi^h\xi^k + \xi^k\xi^h)\tau^{hi}(g_{(1)})\tau^{kj}(g_{(2)})$. Therefore $\mathcal{I}_{M_c}$ is a $U\mathfrak{g}$-$*$-module, and $\mathcal{Q}_{M_c}^{\bullet}$ is a $U\mathfrak{g}$-module $*$-algebra as well; moreover $\mathcal{I}_{M_c}^{pqr} \subset \mathcal{I}_{M_c}$ and $\mathcal{Q}_{M_c}^{pqr} \subset \mathcal{Q}_{M_c}^{\bullet}$ are $U\mathfrak{g}$ $*$-submodules as well.

Equations (85) and (103) with a twist $\mathcal{F} \in U\mathfrak{g} \otimes U\mathfrak{g}[[\nu]]$ imply that:

1. $\mathcal{A}_\star'^{\bullet}$ is a $U\mathfrak{g}^{\mathcal{F}}$-module $*_\star$-algebra; each component $\mathcal{A}_\star'^{pqr}$ consisting of polynomials in the a^A of degree q in the x^i, of degree r in the e_α and homogeneous of degree p in the ξ^i is a $U\mathfrak{g}^{\mathcal{F}}$-$*_\star$-submodule; $\mathcal{A}_\star'^{pqr} = \mathcal{A}'^{pqr}[[\nu]]$, $\mathcal{A}_\star'^{\bullet} = \mathcal{A}'^{\bullet}[[\nu]]$ hold as equalities of $\mathbb{C}[[\nu]]$-modules.
2. For all $J \in \mathcal{J}, \alpha, \alpha' \in \mathcal{A}'^{\bullet}[[\nu]], \beta, \beta' \in \mathcal{I}_{M_c}[[\nu]]$, also

$$f_c^J(a) \star \alpha, \quad \alpha \star f_c^J(a), \quad \beta \star \alpha, \quad \alpha \star \beta, \quad (\alpha + \beta) \star (\alpha' + \beta') - \alpha \star \alpha'$$

belong to $\mathcal{I}_{M_c}[[\nu]]$; if the twist $\mathcal{F}$ is either real or unitary then also $[f_c^J(a)]^{*_\star}$, $\beta^{*_\star}$ do. Therefore $\mathcal{I}_{M_c\star} := \mathcal{I}_{M_c}[[\nu]]$ is a two-sided ($*_\star$-)ideal of $\mathcal{A}_\star'^{\bullet}$. For each component $\mathcal{I}^{pqr}_{M_c\star} := \mathcal{I}_{M_c\star} \cap \mathcal{A}'^{pqr}_{\star}$ we find $\mathcal{I}^{pqr}_{M_c\star} = \mathcal{I}^{pqr}_{M_c}[[\nu]]$. $\mathcal{I}_{M_c\star}$ and $\mathcal{I}^{pqr}_{M_c\star}$ are $U\mathfrak{g}^{\mathcal{F}}$-$*_\star$-submodules.

This leads to the following

Proposition 3 *For all $c \in \mathcal{D}_f$ $\mathcal{Q}^\bullet_{M_c\star} := \mathcal{A}'^\bullet_\star/\mathcal{I}_{M_c\star}$ defines a $U\mathfrak{g}^{\mathcal{F}}$-module $*_\star$-algebra, which we shall name twisted differential calculus algebra on M_c; taking the quotient commutes with deforming the product:*

$$\mathcal{Q}^\bullet_{M_c\star} := \mathcal{A}'^\bullet_\star/\mathcal{I}_{M_c\star} = (\mathcal{A}'^\bullet/\mathcal{I}_{M_c})_\star. \tag{104}$$

All components $\mathcal{Q}^{pqr}_{M_c\star} := \mathcal{A}'^{pqr}_\star/\mathcal{I}^{pqr}_{M_c\star}$ $(p,q,r \in \mathbb{N}_0)$ are $U\mathfrak{g}^{\mathcal{F}}$-$*_\star$-submodules. $\mathcal{Q}^\bullet_{M_c\star}$ is graded by p, filtered by both q,r, and

$$\mathcal{Q}^\bullet_{M_c\star} = \bigoplus_{p=0}^{n} \biguplus_{q=0}^{\infty} \biguplus_{r=0}^{\infty} \mathcal{Q}^{pqr}_{M_c\star}, \qquad \mathcal{Q}^{pqr}_{M_c\star} \star \mathcal{Q}^{p'q'r'}_{M_c\star} \subseteq \mathcal{Q}^{(p+p')(q+q')(r+r')}_{\star}. \tag{105}$$

$\mathcal{Q}^\bullet_{M_c\star} = \mathcal{Q}^\bullet_{M_c}[[\nu]]$ *and*

$$\mathcal{Q}^{pqr}_{M_c\star} = \mathcal{Q}^{pqr}_{M_c}[[\nu]] \tag{106}$$

hold for all $p,q,r \in \mathbb{N}_0$ as equalities of $\mathbb{C}[[\nu]]$-modules. The set of characterizing polynomial relations $f^J_c(a) = 0$ is equivalent to the set of relations $\hat{f}^J_c(a\star) = 0$ consisting of (90) and other relations of the same degrees in x^i, ξ^i, e_α $(\alpha \leq B)$ as their undeformed counterparts. From any basis $\mathcal{B}^{pqr}_{M_c}$ of $\mathcal{Q}^{pqr}_{M_c}$ consisting of polynomials in x^i, ξ^i, e_α one can obtain a basis $\mathcal{B}^{pqr}_{M_c\star}$ of $\mathcal{Q}^{pqr}_{M_c\star}$ consisting of $\star$-polynomials of the same degrees. If $\mathcal{F}$ is either real or unitary, $\mathcal{Q}^\bullet_{M_c\star}$ is a $U\mathfrak{g}^{\mathcal{F}}$-module $*$-algebra with the $\mathcal{Q}^{pqr}_{M_c\star}$ as $*$-submodules. If $\mathcal{F}$ is real the involution is undeformed $*$. If $\mathcal{F}$ is unitary the involution is given by (29), i.e. on ξ^i, x^i $*_\star$ acts as in (94), while $L^{*_\star}_{ij} = -\tau^{ih}\left(\beta_{(1)}\right)\tau^{jk}\left(\beta_{(2)}\right)L_{hk}$ (this differs from $L^*_{ij} = -L_{ij}$).

These results are the strict analogue of their undeformed counterparts. Relation (106) is much stronger than the equality of infinite-dimensional $\mathbb{C}[[\nu]]$-modules $\mathcal{Q}^\bullet_{M_c\star} = \mathcal{Q}^\bullet_{M_c}[[\nu]]$; it implies $\dim\big(\mathcal{Q}^{pqr}_{M_c\star}\big) = \dim\big(\mathcal{Q}^{pqr}_{M_c}\big)$ over $\mathbb{C}[[\nu]]$, so that the Hilbert-Poincaré series of $\mathcal{Q}^\bullet_{M_c\star}$, $\mathcal{Q}^\bullet_{M_c}[[\nu]]$ coincide. In particular, setting $p=r=0$, we find $\dim\big(\widetilde{\mathcal{X}}^q_\star\big) = \dim\big(\widetilde{\mathcal{X}}^q\big)$.

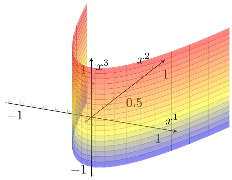
In Section 4 we explictly determine all of the relations $\hat{f}^J_c(a\star) = 0$ in the specific case of some deformed quadrics in $\mathbb{R}^3$.

4 The Quadrics in $\mathbb{R}^3$

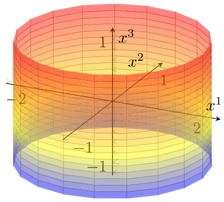
Using the notions and results presented in the previous sections, here we study in detail twist deformations of the quadric surfaces in $\mathbb{R}^3$. As usual, we identify two quadric surfaces if they can be translated into each other via an Euclidean transformation. This leads to nine classes of quadrics, identified by their equations in canonical (i.e. simplest) form. These are summarized in Fig. 1, together with their rank, the associated symmetry Lie algebra $\mathfrak{g}$, and the type of twist deformation we perform. A plot of each class is given in Fig. 2. These classes make up 7 families of submanifolds, differing by the value of c. In fact classes (f), (g), (h) altogether give a single family: (f) consists of connected manifolds, the 1-sheeted hyperboloids; (g), (h) of two-component manifolds, the 2-sheeted hyperboloids and the cone, which has two nappes separated by the apex (a singular point); all are closed, except the cone. For all families, except (i) (consisting of ellipsoids), we succeed in building $U\mathfrak{g}$-based Drinfel'd twists of either abelian (30) or Jordanian (31) type (depending on the coefficients of the normal form) and through the latter in creating explicit twist deformations. Those twists are the simplest ones resp. based on an abelian or "$ax+b$" Lie subalgebra of the symmetry Lie algebras. Note that there are other choices of Drinfel'd twists on the "$ax+b$"-Lie algebra. In particular we like to mention the twist of Theorem 2.10 of [34], which is the real (i.e. $\mathcal{F}^{*\otimes *} = (S\otimes S)[\mathcal{F}_{21}]$) counterpart of the unitary Jordanian twist we utilize; both twists lead to the same commutation relations. Since we are especially interested in describing the deformed spaces in terms of deformed generators and relations, i.e. we intend to explicitly calculate $\star$-commutators and the twisted Hopf algebra structures, we use abelian and Jordanian twists, which admit an explicit exponential formulation. Furthermore, all of the considered symmetry Lie algebras (except the one of the ellipsoids) contain an abelian or "$ax+b$" Lie subalgebra, which allows us to perform a homogeneous deformation approach for all quadric surfaces. We devote a subsection to each of the remaining six families of quadrics, and a proposition to each twist deformation; propositions are proved in the appendix. Throughout this section the star product $X \star h$ of a vector field X by a function h from the right is understood in the $\mathcal{Q}_\star$, $\mathcal{Q}_{M_c\star}$ sense (see Section 3.1) $X \star h = X \triangleleft_\star h + X_\star(h) \equiv (\overline{\mathcal{R}}_1 \triangleright h) \star (\overline{\mathcal{R}}_2 \triangleright X) + X_\star(h)$.

	a_1	a_2	a_3	a_{03}	a_{00}	r	quadric	$\mathfrak{g} \simeq$	Abelian	Jordanian
(a)	+	0	0	−		3	parabolic cylinder	$\mathfrak{h}(1)$	Yes	No
(b)	+	+	0	−		4	elliptic paraboloid	$\mathfrak{so}(2) \ltimes \mathbb{R}^2$	Yes	No
(c)	+	+	0	0	−	3	elliptic cylinder	$\mathfrak{so}(2) \ltimes \mathbb{R}^2$ $\mathfrak{so}(2) \times \mathbb{R}$	Yes Yes	No No
(d)	+	−	0	−		4	hyperbolic paraboloid	$\mathfrak{so}(1,1) \ltimes \mathbb{R}^2$	Yes	Yes
(e)	+	−	0	0	−	3	hyperbolic cylinder	$\mathfrak{so}(1,1) \ltimes \mathbb{R}^2$ $\mathfrak{so}(1,1) \times \mathbb{R}$	Yes Yes	Yes No
(f)	+	+	−	0	−	4	1-sheet hyperboloid	$\mathfrak{so}(2,1)$	No	Yes
(g)	+	+	−	0	+	4	2-sheet hyperboloid	$\mathfrak{so}(2,1)$	No	Yes
(h)	+	+	−	0	0	3	elliptic cone†	$\mathfrak{so}(2,1) \times \mathbb{R}$	Yes†	Yes
(i)	+	+	+	0	−	4	ellipsoid	$\mathfrak{so}(3)$	No	No

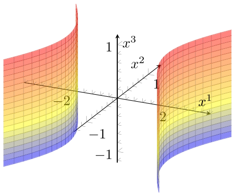
Fig. 1 Overview of the quadrics in $\mathbb{R}^3$: signs of the coefficients of the equations in canonical form (if not specified, all $a_{00} \in \mathbb{R}$ are possible), rank, associated symmetry Lie algebra $\mathfrak{g}$, type of twist deformation; $\mathfrak{h}(1)$ stands for the Heisenberg algebra. For fixed a_i each class gives a family of submanifolds M_c parametrized by c, except classes (f), (g), (h), which altogether give a single family; so there are 7 families of submanifolds. We can always make $a_1 = 1$ by a rescaling of f. The † reminds that the cone (e) is not a single closed manifold, due to the singularity in the apex; we build an abelian twist for it using also the generator of dilatations



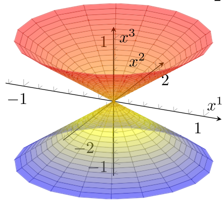
(a) Parabolic cylinder with $a_{02} = -1$



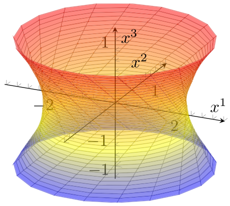
(b) Elliptic cylinder with $a_1 = \frac{1}{2}$, $a_2 = 2$



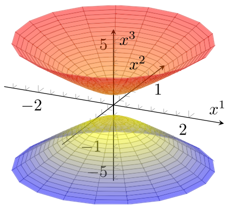
(c) Hyperbolic cylinder with $a_1 = \frac{1}{2}$, $a_2 = -2$



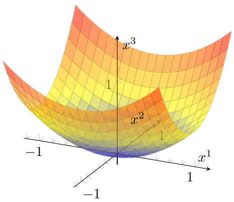
(d) Elliptic cone with $a_1 = -a_3 = 2$, $a_2 = \frac{1}{2}$



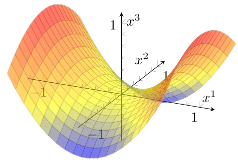
(e) One-sheet elliptic hyperboloid with $a_1 = \frac{1}{2}$, $a_2 = -a_3 = 2$



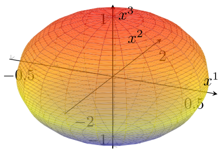
(f) Two-sheet elliptic hyperboloid with $a_1 = 8$, $a_2 = 32$, $a_3 = -2$



(g) Elliptic paraboloid with $a_1 = 1$, $a_2 = 2$



(h) Hyperbolic paraboloid with $a_1 = 2$, $a_2 = -1$



(i) Ellipsoid with $a_1 = 8$, $a_2 = \frac{1}{2}$, $a_3 = 2$

Fig. 2 The irreducible quadric surfaces of $\mathbb{R}^3$

4.1 (a) Family Of Parabolic Cylinders: $a_2 = a_3 = a_{01} = a_{02} = 0$

Their equations in canonical form are parametrized by $c,\ b \equiv a_{03} \in \mathbb{R}$ and read

$$f_c(x) := \frac{1}{2}(x^1)^2 - bx^3 - c = 0. \tag{107}$$

For every fixed b, $\{M_c\}_{c\in\mathbb{R}}$ is a foliation of $\mathbb{R}^3$. The Lie algebra $\mathfrak{g}$ is spanned by the vector fields $L_{12}=x^1\partial_2$, $L_{13}=x^1\partial_3+b\partial_1$, $L_{23}=b\partial_2$, which fulfill

$$[L_{23},\mathfrak{g}]=0, \qquad [L_{13},L_{12}]=L_{23}. \tag{108}$$

Clearly, $\mathfrak{g}\simeq\mathfrak{h}(1)$, the Heisenberg algebra. The actions of the L_{ij} on the x^h,ξ^h,∂_h are

$$\begin{aligned}
&L_{12}\triangleright x^i=\delta^i_2x^1, \quad L_{13}\triangleright x^i=\delta^i_1b+\delta^i_3x^1, \; L_{23}\triangleright x^i=\delta^i_2b,\\
&L_{12}\triangleright \xi^i=\delta^i_2\xi^1, \quad L_{13}\triangleright \xi^i=\delta^i_3\xi^1, \qquad L_{23}\triangleright \xi^i=0,\\
&L_{12}\triangleright \partial_i=-\delta_{i1}\partial_2, \; L_{13}\triangleright \partial_i=-\delta_{i1}\partial_3, \qquad L_{23}\triangleright \partial_i=0;
\end{aligned} \tag{109}$$

the commutation relations $[L_{ij},x^h]=L_{ij}\triangleright x^h$, $[L_{ij},\partial_h]=L_{ij}\triangleright\partial_h$, $[L_{ij},\xi^h]=0$ hold in $\mathcal{Q}^\bullet$.

Proposition 4 *$\mathcal{F}=\exp(i\nu L_{13}\otimes L_{23})$ is a unitary abelian twist inducing the following twisted deformations of $U\mathfrak{g}$, of $\mathcal{Q}^\bullet$ on $\mathbb{R}^3$ and of $\mathcal{Q}^\bullet_{M_c}$ on the parabolic cylinders (107). The $U\mathfrak{g}^{\mathcal{F}}$ counit, coproduct, antipode on the $\{L_{ij}\}_{1\le i<j\le 3}$ coincide with the undeformed ones, except*

$$\Delta_{\mathcal{F}}(L_{12})=L_{12}\otimes\mathbf{1}+\mathbf{1}\otimes L_{12}+i\nu L_{23}\otimes L_{23}, \qquad S_{\mathcal{F}}(L_{12})=-L_{23}+i\nu L_{23}^2. \tag{110}$$

The twisted star products and Lie brackets of the L_{ij} coincide with the untwisted ones. The twisted star products of the L_{ij} with the $x^i,\xi^i\equiv dx^i,\partial_i$, and those among the latter, equal their undeformed counterparts, except $L_{12}\star x^2=L_{12}x^2-i\nu b\,L_{23}$,

$$\begin{aligned}
&x^1\star x^2=x^1x^2-i\nu b^2, \quad x^3\star x^2=x^2x^3-i\nu bx^1,\\
&\xi^3\star x^2=\xi^3x^2-i\nu b\,\xi^1, \; \partial_1\star x^2=\partial_1x^2+i\nu b\partial_3.
\end{aligned}$$

Hence the $\star$-commutation relations of the $U\mathfrak{g}^{\mathcal{F}}$-equivariant $*$-algebra $\mathcal{Q}^\bullet_\star$ read

$$\begin{aligned}
&x^2\star x^1=x^1\star x^2+i\nu b^2, \qquad x^3\star x^1=x^1\star x^3, \qquad x^3\star x^2=x^2\star x^3-i\nu b\,x^1,\\
&x^i\star\xi^j=\xi^j\star x^i+\delta^i_2\,\delta^j_3\,i\nu\,b\,\xi^1, \qquad \xi^i\star\xi^j+\xi^j\star\xi^i=0, \qquad \partial_i\star\xi^j=\xi^j\star\partial_i,\\
&\partial_j\star x^i=\delta^i_j\mathbf{1}+x^i\star\partial_j+\delta_{1j}\delta^i_2\,i\nu b\,\partial_3, \qquad \partial_i\star\partial_j=\partial_j\star\partial_i,\\
&L_{12}\star x^2=x^2\star L_{12}-i\nu b\,L_{23}, \qquad L_{ij}\star x^h=x^h\star L_{ij}+L_{ij}\triangleright x^h \quad \textit{otherwise},\\
&L_{ij}\star\partial_h=\partial_h\star L_{ij}+L_{ij}\triangleright\partial_h, \qquad L_{ij}\star\xi^h=L_{ij}\star\xi^h.
\end{aligned} \tag{111}$$

In terms o star products $L_{12}=x^1\star\partial_2$, $L_{13}=x^1\star\partial_3+b\partial_1$, $L_{23}=b\partial_2$. *Also the relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-equivariant $*$-algebra $\mathcal{Q}^\bullet_{M_c\star}$, i.e. Equation (107), its differential and the linear dependence relations, keep the same form:*

$$f_c(x)\equiv\frac{1}{2}x^1\star x^1-bx^3-c=0, \qquad df_c\equiv x^1\star\xi^1-b\,\xi^3=0, \qquad \epsilon^{ijk}f_i\star L_{jk}=0. \tag{112}$$

The $*$-structures on $U\mathfrak{g}^{\mathcal{F}}$, $\mathcal{Q}^\bullet_\star$, $\mathcal{Q}^\bullet_{M_c\star}$ remain undeformed.

Alternatively, one could twist everything by the unitary abelian twist $\mathcal{F} = \exp(i\nu L_{12} \otimes L_{23})$.

4.2 (b) Family of Elliptic Paraboloids: $a_2 > 0$, $a_3 = 0$, $a_{03} < 0$

Their equations in canonical form are parametrized by $a = a_2, c = -a_{00} \in \mathbb{R}$, $b = -a_3 \in \mathbb{R}^+$ and read

$$f_c(x) := \frac{1}{2}\big[(x^1)^2 + a(x^2)^2\big] - bx^3 - c = 0. \tag{113}$$

For every fixed a, b, $\{M_c\}_{c\in\mathbb{R}}$ is a foliation of $\mathbb{R}^3$. The vector fields $L_{12} = x^1\partial_2 - ax^2\partial_1$, $L_{13} = x^1\partial_3 + b\partial_1$, $L_{23} = ax^2\partial_3 + b\partial_2$ fulfill

$$[L_{12}, L_{13}] = -L_{23}, \qquad [L_{12}, L_{23}] = aL_{13}, \qquad [L_{13}, L_{23}] = 0. \tag{114}$$

Clearly, $\mathfrak{g} \simeq \mathfrak{so}(2)\ltimes\mathbb{R}^2$. The actions of the L_{ij} on the x^h, ξ^h, ∂_h are given by

$$L_{12} \triangleright \partial_i = \delta_{i2}a\,\partial_1 - \delta_{i1}\partial_2, \qquad L_{12} \triangleright u^i = \delta^i_2 u^1 - \delta^i_1 a\,u^2, \qquad \text{for } u^i = x^i, \xi^i, \tag{115}$$

$$\begin{aligned} &L_{13} \triangleright \partial_i = -\delta_{i1}\partial_3, \qquad L_{13} \triangleright x^i = \delta^i_3 x^1 + b\delta^i_1, \qquad L_{13} \triangleright \xi^i = \delta^i_3\xi^1, \\ &L_{23} \triangleright \partial_i = -\delta_{i2}a\partial_3, \qquad L_{23} \triangleright x^i = \delta^i_3 ax^2 + b\delta^i_2 \qquad L_{23} \triangleright \xi^i = \delta^i_3 a\xi^2; \end{aligned} \tag{116}$$

the commutation relations $[L_{ij}, x^h] = L_{ij} \triangleright x^h$, $[L_{ij}, \partial_h] = L_{ij} \triangleright \partial_h$, $[L_{ij}, \xi^h] = 0$ hold in $\mathcal{Q}^\bullet$.

Proposition 5 $\mathcal{F} = \exp(i\nu L_{13} \otimes L_{23})$ *is a unitary abelian twist inducing the following twisted deformation of* $U\mathfrak{g}$, *of* $\mathcal{Q}^\bullet$ *on* $\mathbb{R}^3$ *and of* $\mathcal{Q}^\bullet_{M_c}$ *on the elliptic paraboloids* (113). *The* $U\mathfrak{g}^{\mathcal{F}}$ *counit, coproduct, antipode on the* $\{L_{ij}\}_{1\le i<j\le 3}$ *coincide with the undeformed ones, except*

$$\begin{aligned} &\Delta_{\mathcal{F}}(L_{12}) = L_{12} \otimes \mathbf{1} + \mathbf{1} \otimes L_{12} + i\nu\,(L_{23} \otimes L_{23} - aL_{13} \otimes L_{13})\,, \\ &S_{\mathcal{F}}(L_{12}) = -L_{12} + i\nu\left(L_{23}^2 - aL_{13}^2\right). \end{aligned} \tag{117}$$

The twisted star products and Lie brackets of the $\{L_{ij}\}_{1\le i<j\le 3}$ *coincide with the untwisted ones except* $L_{12} \star L_{12} = L_{12}^2 + i\nu a L_{23}L_{13}$. *The twisted star products of*

the L_{ij} with the x^i, $\xi^i \equiv dx^i$, ∂_i, and those among the latter, equal their undeformed counterparts, except

$$\begin{aligned}
&L_{12} \star u^3 = L_{12}u^3 - i\nu a\, L_{23}u^2, && u^3 \star L_{12} = u^3 L_{12} + i\nu a\, u^1 L_{13},\\
&L_{12} \star x^2 = L_{12}x^2 - i\nu b L_{23} && x^1 \star L_{12} = x^1 L_{12} + i\nu ab L_{13},\\
&L_{12} \star \partial_2 = L_{12}\partial_2 + i\nu a\, L_{23}\partial_3, && \partial_1 \star L_{12} = \partial_1 L_{12} - i\nu a\, \partial_3 L_{13},\\
&x^1 \star x^2 = x^1x^2 - i\nu b^2, && x^1 \star x^3 = x^1x^3 - i\nu ab x^2,\\
&x^3 \star x^3 = x^3x^3 - i\nu a x^1x^2 - ab^2\tfrac{\nu^2}{2}, && x^3 \star x^2 = x^3x^2 - i\nu b x^1,\\
&x^1 \star \xi^3 = x^1\xi^3 - i\nu ab\xi^2, && x^3 \star \xi^3 = x^3\xi^3 - i\nu a\, x^1\xi^2,\\
&\xi^3 \star x^2 = \xi^3x^2 - i\nu b\xi^1, && \xi^3 \star x^3 = \xi^3x^3 - i\nu a\, \xi^1x^2,\\
&\xi^3 \star \xi^3 = -i\nu a\xi^1\xi^2, && \partial_1 \star \partial_2 = \partial_1\partial_2 - i\nu a\, \partial_3\partial_3,\\
&x^1 \star \partial_2 = x^1\partial_2 + i\nu ab\partial_3, && x^3 \star \partial_2 = x^3\partial_2 + i\nu a\, x^1\partial_3,\\
&\partial_1 \star x^2 = \partial_1 x^2 + i\nu b\partial_3, && \partial_1 \star x^3 = \partial_1 x^3 + i\nu a\, \partial_3 x^2,
\end{aligned} \tag{118}$$

where $u^i = x^i, \xi^i$. Hence the $\star$-commutation relations of the $U\mathfrak{g}^{\mathcal{F}}$-equivariant algebra $\mathcal{Q}_\star$ read

$$\begin{aligned}
&x^1 \star x^2 = x^2 \star x^1 - i\nu b^2, \quad x^1 \star x^3 = x^3 \star x^1 - i\nu ab\, x^2, \quad x^2 \star x^3 = x^3 \star x^2 + i\nu b\, x^1,\\
&x^i \star \xi^j = \xi^j \star x^i + i\nu\delta^j_3\left(\xi^1 \star (a\delta^i_3 x^2 + b\delta^i_2) - \xi^2 \star (\delta^i_3 x^1 + b\delta^i_1)a\right),\\
&\xi^i \star \xi^j + \xi^j \star \xi^i = -\delta^j_3\,\delta^i_3\, i2\nu a\, \xi^1 \star \xi^2, \qquad \partial_i \star \partial_j = \partial_j \star \partial_i - \delta_{1i}\,\delta_{2j}\, i\nu a\, \partial_3 \star \partial_3,\\
&\partial_j \star x^i = \delta^i_j \boldsymbol{1} + x^i \star \partial_j + i\nu\left(\delta_{j1}(a\delta^i_3 x^2 + b\delta^i_2) - a\delta_{j2}(\delta^i_3 x^1 + b\delta^i_1)\right) \star \partial_3,\\
&\partial_i \star \xi^j = \xi^j \star \partial_i + \delta^i_3 i\nu a\, (\delta_{1j}\xi^2 - \delta_{2j}\xi^1) \star \partial_3,
\end{aligned} \tag{119}$$

while those among the tangent vectors L_{ij} and the generators x^i, ξ^i, ∂_i read

$$\begin{aligned}
&L_{12} \star x^i = L_{12} \triangleright x^i + x^i \star L_{12} - i\nu b\, (a\delta^i_1 L_{13} + \delta^i_2 L_{23} + a\delta^i_3) - i\nu a\delta^i_3\, (x^1 \star L_{13} + x^2 \star L_{23}),\\
&L_{12} \star \xi^i = \xi^i \star L_{12} - i\nu a\delta^i_3\, (\xi^1 \star L_{13} + \xi^2 \star L_{23}),\\
&L_{12} \star \partial_i = L_{12} \triangleright \partial_i + \partial_i \star L_{12} + i\nu a\, \partial_3 \star L_{i3},\\
&L_{j3} \star x^i = L_{j3} \triangleright x^i + x^i \star L_{j3}, \qquad L_{j3} \star \xi^i = \xi^i \star L_{j3}, \qquad j = 1, 2,\\
&L_{j3} \star \partial_i = L_{j3} \triangleright \partial_i + \partial_i \star L_{j3}, \qquad j = 1, 2.
\end{aligned} \tag{120}$$

In terms of star products $L_{12} = \partial_2 \star x^1 - ax^2 \star \partial_1$, $L_{13} = x^1 \star \partial_3 + b\partial_1$, $L_{23} = ax^2 \star \partial_3 + b\partial_2$. Also the relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-equivariant $*$-algebra $\mathcal{Q}^\bullet_{M_c\star}$, i.e. Equation (113), its differential and the linear dependence relations keep the same form

$$f_c(x) \equiv \frac{1}{2}(x^1 \star x^1 + a x^2 \star x^2) - bx^3 - c = 0, df \equiv \xi^1 \star x^1 + a\, \xi^2 \star x^2 - b\xi^3 = 0, \epsilon^{ijk} f_i \star L_{jk} = 0. \tag{121}$$

The $*$-structures on $U\mathfrak{g}^{\mathcal{F}}$, $\mathcal{Q}^\bullet_\star$, $\mathcal{Q}^\bullet_{M_c\star}$ remain undeformed.

4.3 (c) Family of Elliptic Cylinders: $a_2 > 0$, $a_3 = a_{0i} = 0$, $a_{00} < 0$

Their equations in canonical form are parametrized by $c,\ a \equiv a_2 \in \mathbb{R}^+$ and read

$$f_c(x) := \frac{1}{2}\big[(x^1)^2 + a(x^2)^2\big] - c = 0. \tag{122}$$

For every $a > 0$, $\{M_c\}_{c\in\mathbb{R}^+}$ is a foliation of $\mathbb{R}^3 \setminus \vec{z}$, where $\vec{z}$ is the axis $x^1 = x^2 = 0$. Equation (122) can be obtained from the one (113) characterizing the elliptic paraboloids (b) setting $b = 0$. Hence also the tangent vector fields L_{ij}, their commutation relations, their actions on the x^h, ξ^h, ∂_h, the commutation relations of the L_{ij} with the x^h, ξ^h, ∂_h can be obtained from the ones of case (b) by setting $b = 0$. The L_{ij} fulfill again (114), so that $\mathfrak{g} \simeq \mathfrak{so}(2){\ltimes}\mathbb{R}^2$. Hence we can deform all objects with the same abelian twist as in (b), and obtain the corresponding results:

Proposition 6 $\mathcal{F} = \exp(i\nu L_{13} \otimes L_{23})$ *is a unitary abelian twist inducing the twisted deformation of* $U\mathfrak{g}$, *of* $\mathcal{Q}^\bullet$ *on* $\mathbb{R}^3$ *and of* $\mathcal{Q}^\bullet_{M_c}$ *on the elliptic cylinders* (122) *which is obtained by setting* $b = 0$ *in Proposition 5.*

This is essentially the same as Proposition 15 in [32]. Alternatively, as a complete set in Ξ_t instead of $\{L_{12}, L_{13}, L_{23}\}$ we can use $S_t = \{L_{12}, \partial_3\}$, which is actually a basis of Ξ_t; the Lie algebra $\mathfrak{g} \simeq \mathfrak{so}(2) \times \mathbb{R}$ generated by the latter is abelian; the relevant relations are $(115)_{b=0}$,

$$L_{12} \triangleright \partial_i = \delta_{i2} a\, \partial_1 - \delta_{i1}\partial_2, \qquad L_{12} \triangleright u^i = \delta^i_2 u^1 - \delta^i_1 a\, u^2, \qquad \text{for } u^i \in \{x^i, \xi^i\}, \tag{115$_{b=0}$}$$

and

$$\partial_3 \triangleright x^i \equiv \partial_3(x^i) = \delta^i_3 \mathbf{1}, \qquad \partial_3 \triangleright \partial_i = [\partial_3, \partial_i] = 0, \qquad \partial_3 \triangleright L_{12} = [\partial_3, L_{12}] = 0. \tag{123}$$

We correspondingly adopt the unitary abelian twist $\mathcal{F} = \exp(i\nu\partial_3 \otimes L_{12})$.

Proposition 16 in [32] . $\mathcal{F} = \exp(i\nu\partial_3 \otimes L_{12})$ is a unitary abelian twist inducing the following twist deformation of $U\mathfrak{g}$, of $\mathcal{Q}^\bullet$ on $\mathbb{R}^3$ and of $\mathcal{Q}^\bullet_{M_c}$ on the elliptic cylinders (122). The $U\mathfrak{g}^{\mathcal{F}}$ counit, coproduct, antipode on $\{\partial_3, L_{12}\}$ coincide with the undeformed ones. The twisted star products and Lie brackets of $\{\partial_3, L_{12}\}$ coincide with the untwisted ones. The twisted star products of ∂_3, L_{12} with $x^i, \xi^i \equiv dx^i, \partial_i$, and those among the latter, equal the untwisted ones, except

$$\begin{aligned} x^3 \star x^1 &= x^1 x^3 + i\nu a x^2, & x^3 \star x^2 &= x^2 x^3 - i\nu x^1,\\ x^3 \star \xi^1 &= x^3 \xi^1 + i\nu a \xi^2, & x^3 \star \xi^2 &= x^3 \xi^2 - i\nu \xi^1,\\ x^3 \star \partial_1 &= x^3 \partial_1 + i\nu \partial_2, & x^3 \star \partial_2 &= x^3 \partial_2 - i\nu a \partial_1. \end{aligned}$$

Hence the $\star$-commutation relations of the $U\mathfrak{g}^{\mathcal{F}}$-equivariant algebra $\mathcal{Q}_\star$ read

$$\begin{aligned} x^i \star x^j &= x^j \star x^i + i\nu\delta^i_3(\delta^j_1 a x^2 - \delta^j_2 x^1) - i\nu\delta^j_3(\delta^i_1 a x^2 - \delta^i_2 x^1),\\ x^i \star \xi^j &= \xi^j \star x^i + i\nu\delta^i_3(\delta^j_1 a \xi^2 - \delta^j_2 \xi^1),\\ x^i \star \partial_j &= -\delta^i_j \mathbf{1} + \partial_j \star x^i + i\nu\delta^i_3(\delta^j_1 \partial_2 - \delta^j_2 a \partial_1),\\ \xi^i \star \xi^j &= -\xi^j \star \xi^i,\\ \xi^i \star \partial_j &= \partial_j \star \xi^i,\\ \partial_i \star \partial_j &= \partial_j \star \partial_i. \end{aligned} \tag{124}$$

In terms of star products $L_{12}=x^1\star\partial_2-ax^2\star\partial_1$. Also the relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-equivariant $*$-algebra $\mathcal{Q}^\bullet_{M_c\star}$, i.e. Equation (122), its differential and (76), keep the same form:

$$f_c(x)\equiv\frac{1}{2}(x^1\star x^1+ax^2\star x^2)-c=0,\quad df_c\equiv\xi^1\star x^1+a\,\xi^2\star x^2=0,\quad \epsilon^{ijk}f_i\star L_{jk}=0. \tag{125}$$

The $*$-structures on $U\mathfrak{g}^{\mathcal{F}}$, $\mathcal{Q}^\bullet_\star$, $\mathcal{Q}^\bullet_{M_c\star}$ remain undeformed.

4.3.1 Circular Cylinders Embedded in Euclidean $\mathbb{R}^3$

If $a_1=a_2=1$, i.e. $f_c(x)=\frac{1}{2}\big[(x^1)^2+(x^2)^2\big]-c=0$ and we endow $\mathbb{R}^3$ with the Euclidean metric (circular cylinder of radius $R=\sqrt{2c}$), then $S:=\{L,\partial_3,N_\perp\}$ is an orthonormal basis of Ξ alternative to $S':=\{\partial_1,\partial_2,\partial_3\}$ and such that $S_t:=\{L,\partial_3\}$, $S_\perp:=\{N_\perp\}$ are orthonormal bases of Ξ_t, $\Xi_\perp$ respectively; here $L:=L_{12}/R$, $N_\perp=f^i\partial_i/R=(x^1\partial_1+x^2\partial_2)/R$ (*outward* normal). The Killing Lie algebra $\mathfrak{k}$ is abelian and spanned (over $\mathbb{R}$) by S_t. $\nabla_XY=0$ for all $X,Y\in S'$, whereas the only non-zero∇_XY, with $X,Y\in S$ are

$$\nabla_LL=-\frac{1}{R}N_\perp,\qquad \nabla_LN_\perp=\frac{1}{R}L,\qquad \nabla_{N_\perp}L=\frac{1}{R}L,\qquad \nabla_{N_\perp}N_\perp=\frac{1}{R}N_\perp. \tag{126}$$

The second fundamental form $II(X,Y)=(\nabla_XY)_\perp$, $X,Y\in\Xi_t$, is thus explicitly given by

$$II(X,Y)=-\frac{\tilde{X}\,\tilde{Y}}{R}N_\perp; \tag{127}$$

here we are using the decomposition $Z=\tilde{Z}L+Z^3\partial_3$ of a generic $Z\in\Xi_t$. Thus II is diagonal in the basis S_t, with diagonal elements (i.e. principal curvatures) $\kappa_1=0,\kappa_2=-1/R$. Hence the Gauss (i.e. intrinsic) curvature $K=\kappa_1\kappa_2$ vanishes; $\mathsf{R}_t=0$ easily follows also from $\mathsf{R}=0$ using the Gauss theorem. The mean (i.e. extrinsic) curvature is $H=(\kappa_1+\kappa_2)/2=-1/2R$. The Levi-Civita covariant derivative ∇_t on M_c is the tangent projection of ∇

$$\nabla_{t,X}Y=\mathrm{pr}_t(\nabla_XY)=\nabla_XY-II(X,Y)=\nabla_XY+\tilde{X}\,\tilde{Y}\,N_\perp/R.$$

The deformation via the abelian twist $\mathcal{F}=\exp(i\nu\partial_3\otimes L_{12})\in U\mathfrak{k}\otimes U\mathfrak{k}[[\nu]]$ yields

$$\nabla^{\mathcal{F}}_X=\nabla_X\qquad\qquad \forall\,X\in S\cup S'=\{\partial_1,\partial_2,\partial_3,L,N_\perp\}, \tag{128}$$

$$\nabla^{\mathcal{F}}_{t,X}Y=\mathrm{pr}_t(\nabla_XY)=\nabla_{t,X}Y\qquad \forall\,X,Y\in S_t=\{\partial_3,L\}, \tag{129}$$

because ∂_3 commutes with all such X, so that $\overline{\mathcal{F}}_1\triangleright X\otimes\overline{\mathcal{F}}_2=X\otimes\mathbf{1}$, and the projections $\mathrm{pr}_\perp,\mathrm{pr}_t$, stay undeformed, as shown in Proposition 7. Equations (128-129) determine $\nabla^{\mathcal{F}}_XY$ for all $X,Y\in\Xi_\star$ and $\nabla^{\mathcal{F}}_{t,X}Y=\nabla_{t,X}Y$ for all $X,Y\in\Xi_{t\star}$ via the function left $\star$-linearity in X and the deformed Leibniz rule for Y. The twisted curvatures $\mathsf{R}^{\mathcal{F}}$, $\mathsf{R}^{\mathcal{F}}_t$ vanish, by Theorem 7 in [2]. Furthermore,

$$II^{\mathcal{F}}_\star(X,Y)\overset{(67)}{=}II(\mathcal{F}_1^{-1}\triangleright X,\mathcal{F}_2^{-1}\triangleright Y)N_\perp=II(X,Y) \tag{130}$$

for all $X,Y\in S_t$, leading to the same principal curvatures $\kappa_1=0,\kappa_2=1/R$, Gauss and mean curvatures as in the undeformed case.

4.4 (d) Family of Hyperbolic Paraboloids: $a_2, a_{03} < 0, a_3 = 0$

Their equations in canonical form are parametrized by $a = -a_2, b = -a_{03} > 0$, $c = -a_{00} \in \mathbb{R}$ and read

$$f_c(x) := \frac{1}{2}\big[(x^1)^2 - a(x^2)^2\big] - bx^3 - c = 0. \tag{131}$$

For all fixed $a, b > 0$, $\{M_c\}_{c\in\mathbb{R}}$ is a foliation of $\mathbb{R}^3$. The Lie algebra $\mathfrak{g}$ is spanned by the vector fields $L_{12} = x^1\partial_2 + ax^2\partial_1$, $L_{13} = x^1\partial_3 + b\partial_1$, $L_{23} = b\partial_2 - ax^2\partial_3$, which fulfill

$$[L_{12}, L_{13}] = -L_{23}, \qquad [L_{12}, L_{23}] = -aL_{13}, \qquad [L_{13}, L_{23}] = 0, \tag{132}$$

whence $\mathfrak{g} \simeq \mathfrak{so}(1,1) \ltimes \mathbb{R}^2$. The abelian twist deformation is entirely similar to the one of (b): just replace a by $-a$ in the equations of Proposition 5.

In addition, there is also a Jordanian twist deformation on the hyperbolic paraboloid which we are going to discuss in detail. The tangent vector fields $H = -\frac{2}{\sqrt{a}}L_{12}$, $E = L_{13} + \frac{1}{\sqrt{a}}L_{23}$, $E' = L_{13} - \frac{1}{\sqrt{a}}L_{23}$ fulfill the commutation relations

$$[H, E] = 2E, \quad [H, E'] = -2E', \quad [E, E'] = 0. \tag{133}$$

To compute the action of $\mathcal{F}$ on functions it is convenient to adopt the eigenvectors of H

$$y^1 = x^1 - \sqrt{a}x^2, \qquad y^2 = x^1 + \sqrt{a}x^2, \qquad y^3 = x^3, \tag{134}$$

as new coordinates. In fact, $H \triangleright y^i = \lambda_i y^i$ with $\lambda_1 = 2, \lambda_2 = -2$ and $\lambda_3 = 0$. Abbreviating $\tilde{\partial}_i = \frac{\partial}{\partial y^i}$, the inverse coordinate and the partial derivatives transformations read

$$\begin{aligned} &x^1 = \tfrac{1}{2}(y^1 + y^2), && \tilde{\partial}_1 = \tfrac{1}{2}(\partial_1 - \tfrac{1}{\sqrt{a}}\partial_2), && \partial_1 = \tilde{\partial}_1 + \tilde{\partial}_2,\\ &x^2 = \tfrac{1}{2\sqrt{a}}(y^2 - y^1), && \tilde{\partial}_2 = \tfrac{1}{2}(\partial_1 + \tfrac{1}{\sqrt{a}}\partial_2), && \partial_2 = \sqrt{a}(\tilde{\partial}_2 - \tilde{\partial}_1),\\ &x^3 = y^3, && \tilde{\partial}_3 = \partial_3, && \partial_3 = \tilde{\partial}_3. \end{aligned} \tag{135}$$

In the new coordinates $f_c(y) = \frac{1}{2}y^1y^2 - by^3 - c$ and

$$H = 2(y^1\tilde{\partial}_1 - y^2\tilde{\partial}_2), \qquad E = y^1\tilde{\partial}_3 + 2b\tilde{\partial}_2, \qquad E' = y^2\tilde{\partial}_3 + 2b\tilde{\partial}_1.$$

The actions of H, E, E' on coordinate functions, differential forms $\eta^i = dy^i$ and vector fields are given by for all $1 \le i \le 3$

$$\begin{aligned} &H \triangleright y^i = \lambda_i y^i, && E \triangleright y^i = \delta^i_3 y^1 + 2b\delta^i_2, && E' \triangleright y^i = \delta^i_3 y^2 + 2b\delta^i_1,\\ &H \triangleright \eta^i = \lambda_i \eta^i, && E \triangleright \eta^i = \delta^i_3 \eta^1, && E' \triangleright \eta^i = \delta^i_3 \eta^2,\\ &H \triangleright \tilde{\partial}_i = -\lambda_i \tilde{\partial}_i, && E \triangleright \tilde{\partial}_i = -\delta_{i1}\tilde{\partial}_3, && E' \triangleright \tilde{\partial}_i = -\delta_{i2}\tilde{\partial}_3. \end{aligned} \tag{136}$$

Proposition 7 $\mathcal{F} = \exp[H/2 \otimes \log(\mathbf{1} + i\nu E)]$ *is a unitary Jordanian twist inducing the following twisted deformation of* $U\mathfrak{g}$, *of* $\mathcal{Q}^\bullet$ *on* $\mathbb{R}^3$ *and of* $\mathcal{Q}^\bullet_{M_c}$ *on the hyperbolic* *paraboloid. The* $U\mathfrak{g}^{\mathcal{F}}$ *coproduct, antipode on* $\{H, E, E'\}$ *read*

$$\begin{aligned}
\Delta_{\mathcal{F}}(H) &= \Delta(H) - i\nu\, H \otimes \frac{E}{\mathbf{1} + i\nu E}, & S_{\mathcal{F}}(H) &= S(H) - i\nu HE, \\
\Delta_{\mathcal{F}}(E) &= \Delta(E) + i\nu E \otimes E, & S_{\mathcal{F}}(E) &= \frac{S(E)}{\mathbf{1} + i\nu E}, \\
\Delta_{\mathcal{F}}(E') &= \Delta(E') - i\nu E' \otimes \frac{E}{\mathbf{1} + i\nu E}, & S_{\mathcal{F}}(E') &= S(E') - i\nu EE'.
\end{aligned} \tag{137}$$

The $*$*-structures on* $U\mathfrak{g}^{\mathcal{F}}$, $\mathcal{Q}^\bullet_\star$, $\mathcal{Q}^\bullet_{M_c\star}$ *remain undeformed apart from* $(y^2)^{*_\star} = y^2 + 2i\nu b$ *and* $(\tilde{\partial}_1)^{*_\star} = -\tilde{\partial}_1 + i\nu\tilde{\partial}_3$. *The twisted star products of* $\{H, E, E'\}$ *coincide with the untwisted ones, except*

$$E \star H = EH + 2i\nu E^2, \qquad E' \star H = E'H + 2i\nu E^2. \tag{138}$$

The twisted star products of H, E, E' *with* $y^i, \eta^i, \tilde{\partial}_i$ *equal the untwisted ones, except*

$$\begin{aligned}
E \star y^i &= Ey^i - i\nu E(2b\delta^i_2 + y^1\delta^i_3), \\
E \star \eta^3 &= E\eta^3 - i\nu E\eta^1, \\
E \star \tilde{\partial}_1 &= E\tilde{\partial}_1 + i\nu E\tilde{\partial}_3, \\
E' \star y^i &= E'y^i + i\nu E'(2b\delta^i_2 + y^1\delta^i_3), \\
E' \star \eta^3 &= E'\eta^3 + i\nu E'\eta^1, \\
E' \star \tilde{\partial}_1 &= E'\tilde{\partial}_1 - i\nu E'\tilde{\partial}_3, \\
y^i \star H &= y^i H - 2i\nu(\delta^i_2 - \delta^i_1)y^i E, \\
\eta^i \star H &= \eta^i H - 2i\nu(\delta^i_2 - \delta^i_1)\eta^i E, \\
\tilde{\partial}_i \star H &= \tilde{\partial}_i H - 2i\nu(\delta_{i1} - \delta_{i2})\tilde{\partial}_i E;
\end{aligned} \tag{139}$$

the twisted star products among $y^i, \eta^i, \tilde{\partial}_i$ *equal the untwisted ones, except*

$$\begin{aligned}
y^i \star y^j &= y^i y^j + i\nu(\delta^i_2 - \delta^i_1)y^i(2b\delta^j_2 + \delta^j_3 y^1), \\
y^i \star \tilde{\partial}_1 &= y^i\tilde{\partial}_1 + i\nu(\delta^i_1 - \delta^i_2)y^i\tilde{\partial}_3, \\
\tilde{\partial}_i \star y^j &= \tilde{\partial}_i y^j + i\nu(\delta^1_i - \delta^2_i)\tilde{\partial}_i(2b\delta^j_2 + \delta^j_3 y^1), \\
\tilde{\partial}_1 \star \tilde{\partial}_1 &= \tilde{\partial}_1\tilde{\partial}_1 - i\nu\tilde{\partial}_1\tilde{\partial}_3, \\
\eta^2 \star \eta^3 &= \eta^2\eta^3 + i\nu\eta^2\eta^1, \\
y^i \star \eta^3 &= y^i\eta^3 + i\nu(\delta^i_2 - \delta^i_1)y^i\eta^1, \\
\eta^i \star \tilde{\partial}_1 &= \eta^i\tilde{\partial}_1 + i\nu(\delta^i_1 - \delta^i_2)\eta^i\tilde{\partial}_3, \\
\tilde{\partial}_i \star \eta^3 &= \tilde{\partial}_i\eta^3 + i\nu(\delta_{i1} - \delta_{i2})\tilde{\partial}_i\eta^1, \\
\tilde{\partial}_2 \star \tilde{\partial}_1 &= \tilde{\partial}_1\tilde{\partial}_2 + i\nu\tilde{\partial}_2\tilde{\partial}_3, \\
\eta^i \star y^j &= \eta^i y^j + i\nu(\delta^i_2 - \delta^i_1)\eta^i(2b\delta^j_2 + \delta^j_3 y^1).
\end{aligned} \tag{140}$$

Hence the $\star$-commutation relations of the $U\mathfrak{g}^{\mathcal{F}}$-equivariant algebra $\mathcal{Q}_\star$ read

$$
\begin{aligned}
y^1 \star y^2 &= y^2 \star y^1 - 2bi\nu y^1,\\
y^i \star y^3 &= y^3 \star y^i + i\nu(\delta^i_2 - \delta^i_1)\, y^i \star y^1, \qquad \textit{for } i = 1,2\\
y^1 \star \eta^j &= \eta^j \star y^1 - i\nu\delta^j_3 \eta^1 \star y^1,\\
y^2 \star \eta^j &= \eta^j \star y^2 + 2i\nu b(\delta^j_1 - \delta^j_2)\eta^j + i\nu\delta^j_3\eta^1 \star (y^2 + 2i\nu b\boldsymbol{1}),\\
y^3 \star \eta^j &= \eta^j \star y^3 + i\nu(\delta^j_1 - \delta^j_2)\eta^j \star y^1,\\
\tilde{\partial}_i \star y^1 &= \delta^1_i\boldsymbol{1} + y^1 \star \tilde{\partial}_i - i\nu\delta^1_i y^1 \star \tilde{\partial}_3,\\
\tilde{\partial}_i \star y^2 &= \delta^2_i\boldsymbol{1} + y^2 \star \tilde{\partial}_i + i\nu\delta^1_i y^2 \star \tilde{\partial}_3 + 2i\nu b(\delta_{i1} - \delta_{i2})\tilde{\partial}_i,\\
\tilde{\partial}_i \star y^3 &= \delta^3_i\boldsymbol{1} + y^3 \star \tilde{\partial}_i + i\nu(\delta^1_i - \delta^2_i)\, y^1 \star \tilde{\partial}_i + i\nu\delta^1_i + \nu^2\delta^1_i\, y^1 \star \tilde{\partial}_3,\\
\eta^i \star \eta^j &= -\eta^j \star \eta^i + i\nu(\delta^i_3\delta^j_1 - \delta^j_3\delta^i_1)\eta^1 \star \eta^2,\\
\tilde{\partial}_j \star \eta^i &= \eta^i \star \tilde{\partial}_j + i\nu[\delta^1_j(\delta^i_2 - \delta^i_1)\eta^i \star \tilde{\partial}_3 + \delta^i_3(\delta^1_j - \delta^2_j)\eta^1 \star \tilde{\partial}_j - i\nu\delta^i_3\delta^1_j\, \eta^1 \star \tilde{\partial}_3]\\
\tilde{\partial}_i \star \tilde{\partial}_j &= \tilde{\partial}_j \star \tilde{\partial}_i + i\nu(\delta_{i2}\delta_{j1} - \delta_{j2}\delta_{i1})\tilde{\partial}_2 \star \tilde{\partial}_3
\end{aligned}
\tag{141}
$$

and

$$
\begin{aligned}
H \star y^i &= y^i \star H + \lambda_i y^i + 2i\nu(\delta^i_2 - \delta^i_1)\, y^i \star E,\\
H \star \eta^i &= \eta^i \star H + 2i\nu(\delta^i_2 - \delta^i_1)\, \eta^i \star E,\\
H \star \tilde{\partial}_i &= \tilde{\partial}_i \star H - \lambda_i\tilde{\partial}_i + 2i\nu(\delta^1_i - \delta^2_i)\, \tilde{\partial}_i \star E,\\
E \star y^i &= E \triangleright y^i + y^i \star E - i\nu(2b\delta^i_2 + y^1\delta^i_3) \star E,\\
E \star \eta^i &= \eta^i \star E - i\nu\delta^i_3\, \eta^1 \star E,\\
E \star \tilde{\partial}_i &= E \triangleright \tilde{\partial}_i + \tilde{\partial}_i \star E + i\nu\delta^1_i\, \tilde{\partial}_3 \star E,\\
E' \star y^i &= E' \triangleright y^i + y^i \star E' + i\nu[(2b\delta^i_2 + y^1\delta^i_3) \star E' + 2b\delta^i_3],\\
E' \star \eta^i &= \eta^i \star E' + i\nu\delta^i_3\, \eta^1 \star E',\\
E' \star \tilde{\partial}_i &= E' \triangleright \tilde{\partial}_i + \tilde{\partial}_i \star E' - i\nu\delta^1_i\, \tilde{\partial}_3 \star E'.
\end{aligned}
\tag{142}
$$

In terms of star products

$$
H = 2\big(y^1 \star \tilde{\partial}_1 - y^2 \star \tilde{\partial}_2 - i\nu y^1 \star \tilde{\partial}_3\big), \qquad E = y^1 \star \tilde{\partial}_3 + 2b\tilde{\partial}_2, \qquad E' = y^2 \star \tilde{\partial}_3 + 2b\tilde{\partial}_1
$$

and the relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-equivariant $*$-algebra $\mathcal{Q}^\bullet_{M_c\star}$ become

$$
f(y) = \frac{1}{2}y^2\star y^1 - by^3, \qquad \mathrm{d}f = \frac{1}{2}(y^2\star\eta^1 + y^1\star\eta^2), \qquad \epsilon^{ijk} f_i\star L_{jk} = 0. \tag{143}
$$

4.5 (e) Family of Hyperbolic Cylinders: $a_2 < 0$, $a_3 = a_{0\mu} = 0$

Their equations in canonical form are parametrized by c, $a \equiv -a_2 \in \mathbb{R}^+$ and read

$$f_c(x) := \frac{1}{2}\big[(x^1)^2 - a(x^2)^2\big] - c = 0. \tag{144}$$

For every $a > 0$, this equation with $c = 0$ singles out a variety π consisting of two planes intersecting along the $\vec{z}$-axis; $\{M_c\}_{c\in\mathbb{R}^+}$ is a foliation of $\mathbb{R}^3\setminus\pi$. The case $c < 0$ is reduced to the case $c > 0$ by a $\pi/2$ rotation around the $\vec{z}$-axis. Equation (144) can be obtained from the one (131) characterizing the hyperbolic paraboloids (d) setting $b = 0$. Hence also the tangent vector fields L_{ij} (or equivalently H, E, E'), their commutation relations, their actions on the x^h, ξ^h, ∂_h (or equivalently on the $y^h, \eta^h = dy^h, \tilde{\partial}_h$ defined by (134-135)), the commutation relations of the L_{ij} with the x^h, ξ^h, ∂_h can be obtained from the ones of case (d) by setting $b = 0$. The L_{ij} fulfill again (132), or equivalently (133), so that $\mathfrak{g} \simeq \mathfrak{so}(1,1) \ltimes \mathbb{R}^2$.

Proposition 8 $\mathcal{F} = \exp(i\nu L_{13} \otimes L_{23})$ *is a unitary abelian twist inducing the twisted deformation of* $U\mathfrak{g}$*, of* $\mathcal{Q}^\bullet$ *on* $\mathbb{R}^3$ *and of* $\mathcal{Q}^\bullet_{M_c}$ *on the hyperbolic cylinders* (144) *that is obtained by replacing* $a \mapsto -a$ *in Proposition 16 in* [32], Section 4.3.

We can also deform everything with the same Jordanian twist as in (d). We find

Proposition 9 *Setting* $b = 0$ *in Proposition 7 one obtains the deformed* $U\mathfrak{g}$, $\mathcal{Q}^\bullet$ *on* $\mathbb{R}^3$ *and* $\mathcal{Q}^\bullet_{M_c}$ *on the hyperbolic cylinders* (144) *induced by the unitary twist* $\mathcal{F} = \exp\big[\frac{H}{2} \otimes \log(\boldsymbol{1}+i\nu E)\big]$.

4.6 (f-g-h) Family of Hyperboloids and Cone: $a_2, -a_3 > 0$

Their equations in canonical form are parametrized by $a = a_2, b = -a_3 > 0, c = -a_{00}$ ($c > 0$, $c < 0$ resp. for the 1-sheet and the 2-sheet hyperboloids, $c = 0$ for the cone) and read

$$f_c(x) := \tfrac{1}{2}[(x^1)^2 + a(x^2)^2 - b(x^3)^2] - c = 0. \tag{145}$$

For all $a, b > 0$, $\{M_c\}_{c\in\mathbb{R}\setminus\{0\}}$ is a foliation of $\mathbb{R}^3 \setminus M_0$, where M_0 is the cone of equation $f_0 = 0$ (see Section 4.6.2). The Lie algebra $\mathfrak{g}$ is spanned by $L_{12} = x^1\partial_2 - ax^2\partial_1$, $L_{13} = x^1\partial_3 + bx^3\partial_1$, $L_{23} = ax^2\partial_3 + bx^3\partial_2$, which fulfill $[L_{12}, L_{13}] = -L_{23}$, $[L_{12}, L_{23}] = aL_{13}$, $[L_{13}, L_{23}] = bL_{12}$. Setting $H := \frac{2}{\sqrt{b}}L_{13}$, $E := \frac{1}{\sqrt{a}}L_{12} + \frac{1}{\sqrt{ab}}L_{23}$ and $E' := \frac{1}{\sqrt{a}}L_{12} - \frac{1}{\sqrt{ab}}L_{23}$, we obtain

$$[H, E] = 2E, \qquad [H, E'] = -2E', \qquad [E, E'] = -H, \tag{146}$$

showing that the corresponding symmetry Lie algebra is $\mathfrak{g} \simeq \mathfrak{so}(2,1)$. The commutation relations $[L_{ij}, x^h] = L_{ij} \triangleright x^h$, $[L_{ij}, \partial_h] = L_{ij} \triangleright \partial_h$, $[L_{ij}, \xi^h] = 0$ hold in $\mathcal{Q}^\bullet$. To compute the action of $\mathcal{F}$ on functions it is convenient to adopt the eigenvectors of H

$$y^1 = x^1 + \sqrt{b}x^3, \qquad y^2 = x^2, \qquad y^3 = x^1 - \sqrt{b}x^3, \tag{147}$$

as new coordinates; the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 0$ and $\lambda_3 = -2$. Abbreviating

$$\eta^i := dy^i, \qquad \tilde{\partial}_i := \partial/\partial y^i, \qquad \tilde{\partial}^2 := \tilde{\partial}_2, \quad \tilde{\partial}^1 := 2a\,\tilde{\partial}_3, \quad \tilde{\partial}^3 := 2a\,\tilde{\partial}_1$$

the inverse coordinate and the partial derivative transformations read

$$\begin{array}{lll} x^1 = \frac{1}{2}(y^1 + y^3), & \tilde{\partial}_1 = \frac{1}{2}\left(\partial_1 + \frac{1}{\sqrt{b}}\partial_3\right) = \frac{1}{2a}\tilde{\partial}^3, & \partial_1 = \tilde{\partial}_1 + \tilde{\partial}_3, \\ x^2 = y^2, & \tilde{\partial}_2 = \partial_2 = \tilde{\partial}^2, & \partial_2 = \tilde{\partial}_2, \\ x^3 = \frac{1}{2}\frac{1}{\sqrt{b}}(y^1 - y^3), & \tilde{\partial}_3 = \frac{1}{2}\left(\partial_1 - \frac{1}{\sqrt{b}}\partial_3\right) = \frac{1}{2a}\tilde{\partial}^1, & \partial_3 = \sqrt{b}\left(\tilde{\partial}_1 - \tilde{\partial}_3\right). \end{array} \tag{148}$$

In the new coordinates, $\left(\tilde{\partial}_i\right)^* = -\tilde{\partial}_i$, $f_c(y) = \frac{1}{2}y^1y^3 + \frac{a}{2}(y^2)^2 - c$ and

$$H = 2y^1\tilde{\partial}_1 - 2y^3\tilde{\partial}_3, \quad E = \frac{1}{\sqrt{a}}y^1\tilde{\partial}_2 - 2\sqrt{a}y^2\tilde{\partial}_3, \quad E' = \frac{1}{\sqrt{a}}y^3\tilde{\partial}_2 - 2\sqrt{a}y^2\tilde{\partial}_1. \tag{149}$$

The actions of H, E, E' on any $u^i \in \{y^i, \tilde{\partial}^i, \eta^i\}$ read

$$H \triangleright u^i = \lambda_i u^i, \qquad E \triangleright u^i = \delta^i_2\frac{1}{\sqrt{a}}u^1 - 2\delta^i_3\sqrt{a}u^2, \quad E' \triangleright u^i = \delta^i_2\frac{1}{\sqrt{a}}u^3 - 2\delta^i_1\sqrt{a}u^2. \tag{150}$$

Proposition 17 in [32] . $\mathcal{F} = \exp(H/2 \otimes \log(\mathbf{1} + i\nu E))$ is a unitary twist inducing the following twisted deformation of $U\mathfrak{g}$, of $\mathcal{Q}^\bullet$ on $\mathbb{R}^3$ and of $\mathcal{Q}^\bullet_{M_c}$ on the hyperboloids or cone (145). The $U\mathfrak{g}^{\mathcal{F}}$ coproduct, antipode on $\{H, E, E'\}$ are given by

$$\begin{array}{l} \Delta_{\mathcal{F}}(E) = \Delta(E) + i\nu E \otimes E, \qquad \Delta_{\mathcal{F}}(H) = \Delta(H) - i\nu H \otimes \dfrac{E}{\mathbf{1} + i\nu E}, \\ \Delta_{\mathcal{F}}(E') = \Delta(E') - \dfrac{i\nu}{2}H \otimes \left(H + \dfrac{i\nu E}{\mathbf{1}+i\nu E}\right)\dfrac{\mathbf{1}}{\mathbf{1}+i\nu E} \\ \qquad\qquad -i\nu E' \otimes \dfrac{E}{\mathbf{1}+i\nu E} - \dfrac{\nu^2}{4}H^2 \otimes \dfrac{E}{(\mathbf{1}+i\nu E)^2}, \end{array} \tag{151}$$

$$\begin{array}{rl} S_{\mathcal{F}}(H) = & S(H)(\mathbf{1} + i\nu E), \qquad\qquad S_{\mathcal{F}}(E) = \frac{S(E)}{\mathbf{1}+i\nu E}, \\ S_{\mathcal{F}}(E') = & S(E')(\mathbf{1}+i\nu E) - \frac{i\nu}{2}H(\mathbf{1}+i\nu E)\left(H + \frac{i\nu E}{\mathbf{1}+i\nu E}\right) + \frac{\nu^2}{4}H(\mathbf{1}+i\nu E)HE. \end{array} \tag{152}$$

The twisted star products of $\{H, E, E'\}$ coincide with the untwisted ones, except

$$\begin{array}{rlrl} E \star H = & EH + 2i\nu E^2, & E' \star H = & E'H - 2i\nu E'E, \\ E \star E' = & EE' + i\nu EH - 2\nu^2E^2, & E' \star E' = & (E')^2 - i\nu E'H. \end{array} \tag{153}$$

The twisted star products of $u^i = y^i, \eta^i, \tilde{\partial}^i$ with $v^j = y^j, \eta^j, \tilde{\partial}^j$ and with H, E, E' are given by

$$\begin{array}{rl} u^i \star v^j = & u^iv^j + i\nu(\delta^i_3 - \delta^i_1)u^i\left(\frac{1}{\sqrt{a}}\delta^j_2v^1 - 2\sqrt{a}\delta^j_3v^2\right) + \delta^i_1\delta^j_3 2\nu^2u^1v^1, \\ H \star u^i = & Hu^i, \qquad u^i \star H = u^iH + 2i\nu\left(\delta^i_1 - \delta^i_3\right)u^iE, \\ u^i \star E = & u^iE, \qquad E \star u^i = Eu^i + i\nu E\left(2\delta^i_3\sqrt{a}u^2 - \frac{1}{\sqrt{a}}\delta^i_2u^1\right) + 2\nu^2\delta^i_3Eu^1, \\ E' \star u^i = & E'u^i + i\nu\left(\frac{1}{\sqrt{a}}\delta^i_2E'u^1 - 2\sqrt{a}\delta^i_3E'u^2\right), \\ u^i \star E' = & u^iE' + i\nu\left(\delta^i_1 - \delta^i_3\right)u^iH - 2i\nu\delta^i_1u^1E. \end{array} \tag{154}$$

Hence the $\star$-commutation relations of the $U\mathfrak{g}^{\mathcal{F}}$-equivariant algebra $\mathcal{Q}_\star$ read as follows:

$$\begin{aligned}
&u^1\star u^2 = u^2\star u^1 - \tfrac{i\nu}{\sqrt{a}}u^1\star u^1, \qquad u^1\star u^3 = u^3\star u^1 + 2i\nu\sqrt{a}\,u^2\star u^1 + 2\nu^2 u^1\star u^1,\\
&u^2\star u^3 = u^3\star u^2 - \tfrac{i\nu}{\sqrt{a}}u^3\star u^1, \qquad u^1\star\eta^1 = \eta^1\star u^1, \quad u^1\star\eta^2 = \eta^2\star u^1 - \tfrac{i\nu}{\sqrt{a}}\eta^1\star u^1,\\
&u^1\star\eta^3 = \eta^3\star u^1 + 2i\nu\sqrt{a}\,\eta^2\star u^1 + 2\nu^2\eta^1\star u^1, \qquad u^2\star\eta^1 = \eta^1\star u^2 + \tfrac{i\nu}{\sqrt{a}}\eta^1\star u^1,\\
&u^2\star\eta^2 = \eta^2\star u^2, \quad u^2\star\eta^3 = \eta^3\star u^2 - \tfrac{i\nu}{\sqrt{a}}\eta^3\star u^1, \quad u^3\star\eta^1 = \eta^1\star u^3 - 2i\nu\sqrt{a}\,\eta^1\star u^2,\\
&u^3\star\eta^2 = \eta^2\star u^3 + \tfrac{i\nu}{\sqrt{a}}\eta^1\star u^3 + 2\nu^2\eta^1\star u^2,\\
&u^3\star\eta^3 = \eta^3\star u^3 + 2i\nu\sqrt{a}\big(\eta^3\star u^2 - \eta^2\star u^3\big) + 2\nu^2\,\eta^3\star u^1
\end{aligned}\tag{155}$$

for $u^i = y^i, \tilde{\partial}^i$; the twisted Leibniz rule for the derivatives read

$$\begin{aligned}
&\tilde\partial^1\star y^1 = y^1\star\tilde\partial^1, \quad \tilde\partial^2\star y^1 = y^1\star\tilde\partial^2 + \tfrac{i\nu}{\sqrt{a}}y^1\star\tilde\partial^1, \quad \tilde\partial^3\star y^1 = 2a + y^1\star\tilde\partial^3 - i2\nu\sqrt{a}y^1\star\tilde\partial^2,\\
&\tilde\partial^1\star y^2 = y^2\star\tilde\partial^1 - \tfrac{i\nu}{\sqrt{a}}y^1\star\tilde\partial^1, \qquad \tilde\partial^3\star y^2 = y^2\star\tilde\partial^3 + i2\nu\sqrt{a} + \tfrac{i\nu}{\sqrt{a}}y^1\star\tilde\partial^3 + 2\nu^2 y^1\star\tilde\partial^2,\\
&\tilde\partial^2\star y^2 = 1 + y^2\star\tilde\partial^2, \qquad \tilde\partial^1\star y^3 = 2a + y^3\star\tilde\partial^1 + i2\nu\sqrt{a}\,y^2\star\tilde\partial^1 + 2\nu^2 y^1\star\tilde\partial^1,\\
&\tilde\partial^2\star y^3 = y^3\star\tilde\partial^2 - \tfrac{i\nu}{\sqrt{a}}y^3\star\tilde\partial^1, \qquad \tilde\partial^3\star y^3 = y^3\star\tilde\partial^3 + i2\nu\sqrt{a}\big(y^3\star\tilde\partial^2 - y^2\star\tilde\partial^3\big) + 2\nu^2\,y^3\star\tilde\partial^1,
\end{aligned}\tag{156}$$

while the twisted wedge products fulfill

$$\begin{aligned}
&\eta^1\star\eta^1 = 0, \qquad \eta^2\star\eta^2 = 0, \qquad \eta^3\star\eta^3 = 2i\nu\sqrt{a}\,\eta^2\star\eta^3,\\
&\eta^1\star\eta^2 + \eta^2\star\eta^1 = 0, \; \eta^1\star\eta^3 + \eta^3\star\eta^1 = 2i\nu\sqrt{a}\,\eta^1\star\eta^2, \; \eta^2\star\eta^3 + \eta^3\star\eta^2 = \tfrac{i\nu}{\sqrt{a}}\eta^3\star\eta^1.
\end{aligned}\tag{157}$$

The $\star$-commutation relations between generators of $\mathcal{Q}_\star$ and the tangent vectors H, E, E' are

$$\begin{aligned}
&u^i\star H = H\star u^i - \vartheta\lambda_i u^i + 2i\nu\left(\delta^i_1 - \delta^i_3\right)u^i\star E,\\
&u^1\star E = E\star u^1, \qquad u^2\star E = E\star u^2 - \tfrac{\vartheta}{\sqrt{a}}u^1 + \tfrac{i\nu}{\sqrt{a}}E\star u^1,\\
&u^3\star E = E\star u^3 + 2\vartheta\sqrt{a}u^2 - 2i\nu\sqrt{a}E\star u^2,\\
&u^1\star E' = E'\star u^1 + 2\vartheta\big(\sqrt{a}u^2 - i\nu u^1\big) + i\nu H\star u^1 - 2i\nu E\star u^1\\
&u^2\star E' = E'\star u^2 - \tfrac{\vartheta}{\sqrt{a}}u^3 - \tfrac{i\nu}{\sqrt{a}}E'\star u^1,\\
&u^3\star E' = E'\star u^3 - 2\vartheta i\nu u^3 + 2i\nu\sqrt{a}E'\star u^2 - i\nu H\star u^3 + 2\nu^2 E'\star u^1,
\end{aligned}\tag{158}$$

where $\vartheta = 1$ if $u^i = y^i, \tilde\partial^i$, $\vartheta = 0$ if $u^i = \eta^i$. In terms of star products

$$\begin{aligned}
H &= 2(\tilde\partial_1\star y^1 - 1 - y^3\star\tilde\partial_3),\\
E &= \frac{1}{\sqrt{a}}\tilde\partial_2\star y^1 - 2\sqrt{a}y^2\star\tilde\partial_3,\\
E' &= \frac{1}{\sqrt{a}}\tilde\partial_2\star y^3 - 2\sqrt{a}y^2\star\tilde\partial_1.
\end{aligned}\tag{159}$$

The relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-equivariant $*$-algebra $\mathcal{Q}^{\bullet}_{M_c\star}$ become

$$\begin{aligned} 0 &= f_c(y) \equiv \tfrac{1}{2}y^3\star y^1+\tfrac{a}{2}y^2\star y^2-c,\\ 0 &= \mathrm{d}f_c=\tfrac{1}{2}(y^3\star\eta^1+\eta^3\star y^1)+ay^2\star\eta^2,\\ 0 &= y^3\star E-y^1\star E'-\sqrt{a}\,y^2\star H+i\nu y^1\star H-2i\nu(1+i\nu)y^1\star E. \end{aligned} \tag{160}$$

The $*$-structures on $U\mathfrak{g}^{\mathcal{F}}$, $\mathcal{Q}^{\bullet}_{\star}$, $\mathcal{Q}^{\bullet}_{M_c\star}$ remain undeformed except $(u^3)^{*_\star}=(u^3)^*-2i\nu\sqrt{a}(u^2)^*$ for $u^i=y^i,\eta^i,\tilde{\partial}^i$.

4.6.1 Circular Hyperboloids and Cone Embedded in Minkowski $\mathbb{R}^3$

We now focus on the case $1=a_1=a=b$, i.e. $f_c(x)=\frac{1}{2}[(x^1)^2+(x^2)^2-(x^3)^2]-c$. This covers the circular cone and hyperboloids of one and two sheets. We endow $\mathbb{R}^3$ with the Minkowski metric $\mathbf{g}:=\eta_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j=\mathrm{d}x^1\otimes\mathrm{d}x^1+\mathrm{d}x^2\otimes\mathrm{d}x^2-\mathrm{d}x^3\otimes\mathrm{d}x^3$, whence $\mathbf{g}(\partial_i,\partial_j)=\eta_{ij}$. $\mathbf{g}$ is equivariant with respect to $U\mathfrak{g}$, where $\mathfrak{g}\simeq\mathfrak{so}(2,1)$ is the Lie *-algebra spanned by the vector fields L_{ij}, tangent to $M_c=f_c^{-1}(\{0\})$. The first fundamental form $\mathbf{g}_t:=\mathbf{g}\circ(\mathrm{pr}_t\otimes\mathrm{pr}_t)$ makes M_c Riemannian if $c<0$, Lorentzian if $c>0$, whereas is degenerate on the cone M_0. Moreover,

$$II(X,Y)=-\frac{1}{2c}\,\mathbf{g}(X,Y)\,V_\perp \qquad \forall X,Y\in\Xi_t \tag{161}$$

where $V_\perp=f_j\eta^{ji}\partial_i=x^i\partial_i$ (*outward* normal); in particular, this implies the proportionality relation $II(v_\alpha,v_\beta)=-\frac{1}{2c}\,g_{\alpha\beta}\,V_\perp$ (here $g_{\alpha\beta}:=\mathbf{g}(v_\alpha,v_\beta)$) between the matrix elements of II, $\mathbf{g}_t$ in any basis $S_t:=\{v_1,v_2\}$ of Ξ_t, and, applying the Gauss theorem, one finds the following components of the curvature and Ricci tensors, Ricci scalar (or *Gauss curvature*) on M_c:

$$\mathsf{R}_t{}^{\delta}_{\ \alpha\beta\gamma}=\frac{g_{\alpha\gamma}\delta^\delta_\beta-g_{\beta\gamma}\delta^\delta_\alpha}{2c},\qquad \mathsf{Ric}_{t\,\beta\gamma}=\mathsf{R}_t{}^{\alpha}_{\ \alpha\beta\gamma}=-\frac{g_{\beta\gamma}}{2c},\qquad \mathfrak{R}_t=\mathsf{Ric}_t{}^{\beta}_{\ \beta}=-\frac{1}{c} \tag{162}$$

[we recall that by the Bianchi identity one can express the whole curvature tensor on a (pseudo)Riemanian surface in terms of the Ricci scalar in this way, and that $\mathsf{R}_t{}^{\delta}_{\ \alpha\beta\gamma}v_\delta=\mathsf{R}_t(v_\alpha,v_\beta,v_\gamma)$]. All diverge as $c\to 0$ (i.e. in the cone M_0 limit). M_c is therefore de Sitter space dS_2 if $c>0$, the union of two copies of anti-de Sitter space AdS_2 (the hyperbolic plane) if $c<0$. In Appendix B.7.2 we recall how these results can be derived. In terms of the y^i coordinates and the tangent vector fields H,E,E' (145), (76) become the linear dependence relations $y^1y^3+(y^2)^2=2c$ and $y^3E-y^1E'-y^2H=0$, i.e. Equation (160) for $a=1$, $\nu=0$. At all points of M_c at least two out of E,E',H are non-zero (in the case $c=0$ we have already excluded the only point where this does not occur, the apex) and make up another basis $S'_t=\{\epsilon_1,\epsilon_2\}$ of Ξ_t. More precisely, we can choose $\epsilon_1:=E$, $\epsilon_2:=E'$ in a chart where $y^2\neq 0$, $\epsilon_1:=E$, $\epsilon_2:=H$ in a chart where $y^1\neq 0$, $\epsilon_1:=E'$, $\epsilon_2:=H$ in a chart where $y^3\neq 0$. One can use (161), (162) with each basis S'_t; $g_{\alpha\beta}$ stands for $g_{\alpha\beta}\equiv\mathbf{g}(\epsilon_\alpha,\epsilon_\beta)$, and these matrix elements are given in (181). Alternatively, we can use the complete set $S^c_t=\{E,E',H\}$ on all of M_c, keeping in mind the mentioned linear dependence relations.

We now analyze the effects on the geometry of the twist deformation of Proposition 17 in [32] restated above. The curvature (and Ricci) tensor on $\mathbb{R}^3$ remain zero. Moreover, (66), (67) apply; namely, on M_c the first and second fundamental forms, as well as the curvature and Ricci tensor, remain undeformed as elements of the corresponding tensor spaces; only the associated multilinear maps of twisted tensor products $\mathbf{g}_{t\star} : \Xi_{t\star} \otimes_\star \Xi_{t\star} \to \mathcal{X}_\star$, ..., 'feel' the twist (compare also to [2] Theorem 7 and eq. 6.138). Also the Ricci scalar (or Gauss curvature) $\mathfrak{R}_t^{\mathcal{F}}$ remains the undeformed one $-1/c$. By (67) the twisted counterpart of (161) becomes

$$II_\star^{\mathcal{F}}(X,Y) = -\frac{1}{2c}\,\mathbf{g}_{t\star}(X,Y)\,V_\perp = -\frac{1}{2c}\,\mathbf{g}_{t\star}(X,Y)\star V_\perp; \tag{163}$$

the second equality holds because $V_\perp$ is $U\mathfrak{k}$-invariant. Similarly, by (67), (55)

$$\mathsf{R}_{t\star}^{\mathcal{F}}(X,Y,Z) = \frac{\left(\overline{\mathcal{R}}_1\triangleright Y\right) \triangleleft_\star \mathbf{g}_{t\star}\left(\overline{\mathcal{R}}_2\triangleright X,Z\right) - X \triangleleft_\star \mathbf{g}_{t\star}(Y,Z)}{2c}, \qquad \mathsf{Ric}_{t\star}^{\mathcal{F}}(Y,Z) = -\frac{\mathbf{g}_{t\star}(Y,Z)}{2c} \tag{164}$$

for all $X, Y, Z \in \Xi_{t\star}$; the twisted counterpart of (162) is obtained choosing $(X,Y,Z) = (v_\alpha, v_\beta, v_\gamma)$. Hence the matrix elements of $II_\star^{\mathcal{F}}$, $\mathsf{R}_{t\star}^{\mathcal{F}}$, $\mathsf{Ric}_{t\star}^{\mathcal{F}}$ in any basis S_t' are obtained from those of the twisted metric $\mathbf{g}_{t\star}$ on M_c. In the appendix we sketchily prove that on E, E', H

$$\begin{aligned}
&\mathbf{g}_{t\star}(H,H) = -8y^1y^3, \qquad \mathbf{g}_{t\star}(H,E) = -2y^1y^2, \qquad \mathbf{g}_{t\star}(H,E') = -2y^2y^3\\
&\mathbf{g}_{t\star}(E,E) = (y^1)^2, \qquad \mathbf{g}_{t\star}(E,E') = 2c + (y^2)^2 - 2i\nu y^1y^2 - 2\nu^2(y^1)^2,\\
&\mathbf{g}_{t\star}(E,H) = -2y^1y^2 + 2i\nu(y^1)^2, \qquad \mathbf{g}_{t\star}(E',E) = 2c + (y^2)^2,\\
&\mathbf{g}_{t\star}(E',E') = (y^3)^2, \qquad \mathbf{g}_{t\star}(E',H) = -2y^2y^3 - 2i\nu[2c + (y^2)^2] + 2i\nu y^2y^3.
\end{aligned} \tag{165}$$

Finally, we also show that the twisted Levi-Civita connection on E, E', H gives

$$\begin{aligned}
&\nabla_E^{\mathcal{F}}E = -2y^1\tilde{\partial}_3, \qquad \nabla_E^{\mathcal{F}}E' = -2y^1\tilde{\partial}_1 - 2y^2\tilde{\partial}_2 + 4i\nu\tilde{\partial}_3 + 4\nu^2y^1\tilde{\partial}_3,\\
&\nabla_E^{\mathcal{F}}H = 4y^2\tilde{\partial}_3 - 4i\nu y^1\tilde{\partial}_3, \qquad \nabla_{E'}^{\mathcal{F}}E = -2y^3\tilde{\partial}_3 - 2y^2\tilde{\partial}_2,\\
&\nabla_{E'}^{\mathcal{F}}E' = -2y^3\tilde{\partial}_1 + 4i\nu y^2\tilde{\partial}_1, \qquad \nabla_{E'}^{\mathcal{F}}H = -4y^2\tilde{\partial}_1 + 4i\nu(y^2\tilde{\partial}_2 + y^3\tilde{\partial}_3),\\
&\nabla_H^{\mathcal{F}}E = 2y^1\tilde{\partial}_2, \qquad \nabla_H^{\mathcal{F}}E' = -2y^3\tilde{\partial}_2, \qquad \nabla_H^{\mathcal{F}}H = 4y^1\tilde{\partial}_1 + 4y^3\tilde{\partial}_3.
\end{aligned} \tag{166}$$

We recall that a sheet of the hyperboloid M_c, $c < 0$, is equivalent to a hyperbolic plane. Other deformation quantizations of the latter have been done, in particular that of [8] in the framework [7, 52] (cf. the introduction). However, while the $\star$-product [8] is $U\mathfrak{k}$-equivariant, i.e. relation (16) (which is the 'infinitesimal' version of the invariance property (10) in [8] or (1) of [7]) holds, our $\star$-product is $U\mathfrak{k}^{\mathcal{F}}$-equivariant i.e. relation (21) holds.

4.6.2 (h) Additional Twist Deformation of the Cone

The equation of the cone M_0 in canonical form is (145) with $c = 0$. In addition to the tangent vector fields L_{ij} or H, E, E' fulfilling (146) also the generator $\mathsf{D} = x^i\partial_i = y^i\tilde{\partial}_i$ of dilatations is tangent to M_0 (only), $\mathsf{D} \in \Xi_{M_0}$, since $\mathsf{D}(f) = 2f$; furthermore it commutes with all L_{ij}. Hence the anti-Hermitian elements H, E, E', D span a Lie algebra $\mathfrak{g} \simeq \mathfrak{so}(2,1) \times \mathbb{R}$. The actions of H, E, E' on $\mathcal{Q}_M$ are as in cases (e-f), while that of D is determined by

$$\mathsf{D} \triangleright y^i := [\mathsf{D}, y^i] = y^i, \qquad \mathsf{D} \triangleright \eta^i := d(\mathsf{D} \triangleright y^i) = \eta^i, \qquad \mathsf{D} \triangleright \tilde{\partial}_i := [\mathsf{D}, \tilde{\partial}_i] = -\tilde{\partial}_i. \tag{167}$$

Therefore, we can build also abelian twist deformations of M_0 of the form $\mathcal{F} = \exp(i\nu\mathsf{D} \otimes g)$, $g \in \mathfrak{g}$. Here we choose $g = \frac{L_{13}}{\sqrt{b}} = \frac{H}{2}$, i.e. $\mathcal{F} = \exp(i\nu\mathsf{D} \otimes \frac{H}{2})$. The cases with L_{23}, L_{12} are similar. Setting $\mu_1 = 1 = -\mu_3$ and $\mu_2 = 0$, for $u^i, v^i \in \{y^i, \eta^i\}$ we find

$$\overline{\mathcal{F}}(\triangleright\otimes\triangleright)(u^i \otimes v^j) = e^{-i\nu\mu_j} u^i \otimes v^j, \qquad \overline{\mathcal{F}}(\triangleright\otimes\triangleright)(u^i \otimes \tilde{\partial}_j) = e^{i\nu\mu_j} u^i \otimes \tilde{\partial}_j,$$

$$\overline{\mathcal{F}}(\triangleright\otimes\triangleright)(\tilde{\partial}_i \otimes u^j) = e^{i\nu\mu_j} \tilde{\partial}_i \otimes u^j, \qquad \overline{\mathcal{F}}(\triangleright\otimes\triangleright)(\tilde{\partial}_i \otimes \tilde{\partial}_j) = e^{-i\nu\mu_j} \tilde{\partial}_i \otimes \tilde{\partial}_j.$$

Having this in mind, in the appendix we easily determine the twist deformed structures.

Proposition 10 *$\mathcal{F} = \exp(i\nu\mathsf{D} \otimes H/2)$ is a unitary abelian twist inducing the following twisted deformation of $U\mathfrak{g}$, of $\mathcal{Q}^\bullet$ on $\mathbb{R}^3$ and of $\mathcal{Q}^\bullet_{M_c}$ on the cone M_0. The $U\mathfrak{g}^{\mathcal{F}}$ counit, coproduct, antipode on $\{\mathsf{D}, H, E, E'\}$ coincide with the undeformed ones, except*

$$\begin{aligned} \Delta_{\mathcal{F}}(E) &= E \otimes \mathbf{1} + \exp(i\nu\mathsf{D}) \otimes E, & S_{\mathcal{F}}(E) &= -E\exp(-i\nu\mathsf{D}), \\ \Delta_{\mathcal{F}}(E') &= E' \otimes \mathbf{1} + \exp(-i\nu\mathsf{D}) \otimes E', & S_{\mathcal{F}}(E') &= -E'\exp(i\nu\mathsf{D}). \end{aligned} \tag{168}$$

The twisted star products among D, L_{ij} coincide with the untwisted ones. The twisted star products of D, L_{ij} with $u^i \in \{y^i, \eta^i\}$, $\tilde{\partial}_i$ coincide with the untwisted ones, except

$$\begin{aligned} u^i \star E &= e^{-i\nu} u^i E, & u^i \star E' &= e^{i\nu} u^i E', \\ \tilde{\partial}_i \star E &= e^{i\nu} \tilde{\partial}_i E, & \tilde{\partial}_i \star E' &= e^{-i\nu} \tilde{\partial}_i E'. \end{aligned} \tag{169}$$

The twisted star products among $y^i, \eta^i, \tilde{\partial}_i$ read

$$\begin{aligned} u^i \star v^j &= e^{-i\nu\mu_j} u^i v^j, & \tilde{\partial}_i \star \tilde{\partial}_j &= e^{-i\nu\mu_j} \tilde{\partial}_i \tilde{\partial}_j, \\ u^i \star \tilde{\partial}_j &= e^{i\nu\mu_j} u^i \tilde{\partial}_j, & \tilde{\partial}_i \star u^j &= e^{i\nu\mu_j} \tilde{\partial}_i u^j, \end{aligned} \tag{170}$$

with $u^i, v^i \in \{y^i, \eta^i\}$. Hence the $\star$-commutation relations of the $U\mathfrak{g}^{\mathcal{F}}$-equivariant algebra $\mathcal{Q}_\star$ are

$$\begin{aligned} y^i \star y^j &= e^{i\nu(\mu_i-\mu_j)} y^j \star y^i, & \eta^i \star \eta^j &= -e^{i\nu(\mu_i-\mu_j)} \eta^j \star \eta^i, \\ y^i \star \eta^j &= e^{i\nu(\mu_i-\mu_j)} \eta^j \star y^i, & \eta^i \star \tilde{\partial}_j &= e^{-i\nu(\mu_i-\mu_j)} \tilde{\partial}_j \star y^i, \\ \tilde{\partial}_j \star y^i &= e^{i\nu\mu_i} \delta^i_j \mathbf{1} + e^{i\nu(\mu_i-\mu_j)} y^i \star \tilde{\partial}_j, & \tilde{\partial}_i \star \tilde{\partial}_j &= e^{i\nu(\mu_i-\mu_j)} \tilde{\partial}_j \star \tilde{\partial}_i. \end{aligned} \tag{171}$$

The $*$-structures on $U\mathfrak{g}^{\mathcal{F}}$, $\mathcal{Q}^\bullet_\star$, $\mathcal{Q}_{M\star}$ are undeformed, except

$$(\tilde{\partial}_i)^{*_\star} = -e^{-i\nu\mu_i} \tilde{\partial}_i, \qquad (u^i)^{*_\star} = e^{-i\nu\mu_i} u^i, \quad u^i = y^i, \eta^i,$$

which are nontrivial for $i = 1$ and $i = 3$. In terms of star products $\mathsf{D} = \sum_{i=1}^3 e^{-i\nu\mu_i} y^i \star \tilde{\partial}_i$, $H = 2(e^{-i\nu} y^1 \star \tilde{\partial}_1 - e^{i\nu} y^3 \star \tilde{\partial}_3)$, $E = \frac{e^{-i\nu}}{\sqrt{a}} y^1 \star \tilde{\partial}_2 - 2\sqrt{a} y^2 \star \tilde{\partial}_3$, $E' = \frac{e^{i\nu}}{\sqrt{a}} y^3 \star \tilde{\partial}_2 - 2\sqrt{a} y^2 \star \tilde{\partial}_1$, and the relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-equivariant

$*$-algebra $\mathcal{Q}^\bullet_{M_c\star}$, i.e. Equation (145)$_{c=0}$, its differential and the linear dependence relations become

$$
\begin{aligned}
f(y) &\equiv \frac{1}{2}e^{-i\nu}y^1 \star y^3 + \frac{a}{2}y^2 \star y^2 = 0,\\
\mathrm{d}f &\equiv \frac{1}{2}(e^{i\nu}y^3 \star \eta^1 + e^{-i\nu}y^1 \star \eta^3) + ay^2 \star \eta^2 = 0,\\
\epsilon^{ijk}L_{jk} \star f_i &= 0.
\end{aligned}
\tag{172}
$$

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Appendix A: Real Nullstellensatz

First of all, we recall some basic notions and notation in algebraic geometry that we are using in this subsection. In what follows we fix a ground field $\mathbb{K}$ of any characteristic (even though we work only over real and complex fields, all the notions and definitions we are going to review hold true in a much wider generality).

1. (Algebraic Sets [38, p. 2]) A subset of $\mathbb{K}^n$ is an *algebraic set* if it is defined as the set of common solutions of a system of polynomial equations. By Hilbert basis theorem [46, Theorem 3.3], algebraic sets can be also defined as
$$Z(I) := \{x \in \mathbb{K}^n \mid \; P(x) = 0, \; \forall P \in I\},$$
where I denotes an ideal of the polynomial ring $\mathbb{K}[x^1, \dots, x^n]$.
2. (Zariski topology [38, p. 2]) The affine space $\mathbb{K}^n$ can be endowed with a topology, the so called Zariski topology, where closed sets coincide with algebraic sets. In this section we will equip algebraic sets with the induced topology.
3. (Algebraic, or affine, varieties) An algebraic variety is an *irreducible* algebraic set, i.e. an algebraic set which is not the union of two proper (i.e. strictly contained) closed subsets. It turns out [38, Exercise I.1.6] that a non-empty open set of an affine variety is irreducible.
4. (Decomposition of algebraic sets) An algebraic set M can be expressed uniquely as a union of varieties, no one containing another [38, Corollary 1.6]. Such varieties are called *irreducible components* of M.

5. (Radicals) For any ideal $I \leq \mathbb{K}[x^1, \ldots, x^n]$ the *radical* of I [46, p. 3], [33, Section 1.3] is the ideal defined as

$$\mathrm{Rad}(I) := \{P \in \mathbb{K}[x^1, \ldots, x^n] : \ \exists k > 0 \ \mid \ P^k \in I\}.$$

A *radical ideal* is an ideal I s.t. $I = \mathrm{Rad}(I)$. By the very definition of *prime ideal* [46, p. 2], any such an ideal is radical.
6. (Correspondence among varieties and prime ideals) An affine algebraic set is a variety if and only if its ideal is a prime ideal [38, Corollary 1.4].
7. (Associated primes) Consider an algebraic set $M = Z(I)$. An *associated prime* is a prime ideal of $\mathbb{K}[x^1, \ldots, x^n]$ which is the annihilator $ann(x)$ of some element $x \in \frac{\mathbb{K}[x^1,\ldots,x^n]}{I}$. It turns out [46, Section 6] that there are two kinds of associated primes: *minimal associated primes* (they are in one to one correspondence with the irreducible components of M), and *embedded associated primes* (they have NOT a simple geometric interpretation).
8. (Hilbert's Nullstellensatz [46, Section 5]) Assume now $\mathbb{K}$ algebraically closed and define

$$\mathcal{I}(S) := \{P \in \mathbb{K}[x^1, \ldots, x^n] \mid \ P\mid_S \equiv 0\},$$

for any subset $S \subseteq \mathbb{K}^n$. Then we have

$$\mathcal{I}(Z(I)) = \mathrm{Rad}(I), \quad \forall I \leq \mathbb{K}[x^1, \ldots, x^n].$$

A weak form of this result says that $Z(I) \neq \emptyset$, for any proper ideal $I \leq \mathbb{K}[x^1, \ldots, x^n]$ [33, Section 1.7].
9. (Regular sequences) A set of polynomials $P_1, \ldots, P_k$ form a *regular sequence* in $\frac{\mathbb{K}[x^1,\ldots,x^n]}{I}$ [46, Section 16], if every P_i is not a zero divisor in $\frac{\mathbb{K}[x^1,\ldots,x^n]}{(I,P_1,\ldots,P_{i-1})}$.
10. (Cohen-Macaulay property [46, Section 17]) An affine variety $M = Z(I)$, such that $\dim M = m$, is said *Cohen-Macaulay* at $x \in M$ if there is a regular sequence $P_1, \ldots, P_m$ in $\frac{\mathbb{K}[x^1,\ldots,x^n]}{I}$, such that $P_i(x) = 0, \forall i$. An affine variety $M = Z(I)$ is said Cohen-Macaulay if it is Cohen-Macaulay at any point.

Now consider an algebraic submanifold, i.e. a smooth algebraic variety, $M \subseteq \mathbb{R}^n$, defined by a system of polynomial (1), with $f^1, \ldots, f^k \in \mathbb{R}[x^1, \ldots, x^n]$. Assume that $\dim M = n - k$. Then, the hypersurfaces defined by each of the equations in (1) meet transversally et each point of M; in other words, the Jacobian matrix is of rank k at each point of M. Consider a further polynomial $Q \in \mathbb{R}[x^1, \ldots, x^n]$ and assume that

$$Q\mid_M \equiv 0.$$

One may wonder whether the irreducibility in $\mathbb{R}[x^1, \ldots, x^n]$ of each polynomial $f^1, \ldots, f^k$ is a sufficient condition in order that Q lies in $(f^1, \ldots, f^k)$, the ideal generated by $f^1, \ldots, f^k$. The following example answers in the negative.

Example 11 Consider in $\mathbb{R}^3$ the variety defined by the system

$$\begin{cases} 2x^3 - y^3 = 1, \\ y = 1, \end{cases} \tag{173}$$

where the first equation represents a cubic cylinder C. Since the curve defined by

$$2x^3 - y^3 - z^3 = 0$$

is smooth in $\mathbb{P}^2_{\mathbb{C}}$, the cylinder C is smooth and the polynomial $2x^3 - y^3 - 1$ is irreducible in $\mathbb{R}[x, y, z]$ (the same conclusion is obvious for $y - 1$). The real variety defined by (173) is the line

$$l := \{(1, 1, t) : \ t \in \mathbb{R}\},$$

which is obviously smooth. Furthermore, the equation of the tangent plane to the cylinder C at the point $(1, 1, t) \in l$ is $2(x-1)-(y-1) = 0$, hence the intersection is transversal at each point of l. On the other hand, the plane π defined by $x+y-2 = 0$ contains l but

$$x + y - 2 \notin (2x^3 - y^3 - 1, y - 1),$$

since both $2x^3 - y^3 - 1$ and $y - 1$ do vanish at the points $(\exp \frac{2}{3}\pi i, 1, t)$, $\forall t \in \mathbb{R}$, and conversely $x + y - 2$ does not. In view of the previous example, it is interesting to ask for some sufficient condition in order that $Q \in (f^1, \ldots, f^k)$. An answer is provided by Theorem 1, which we now prove.

Proof of Theorem 1 Denote by

$$I := (f^1, \ldots, f^k) \cdot \mathbb{C}[x^1, \ldots, x^n] \leq \mathbb{C}[x^1, \ldots, x^n]$$

the ideal of $\mathbb{C}[x^1, \ldots, x^n]$ generated by $f^1, \ldots, f^k$. Since we are assuming that the zero locus of I is irreducible, there is only one minimal prime associated to the ideal I. The hypothesis that the hypersurfaces corresponding to the generators of I meet transversally at M imply that $f^1, \ldots, f^k$ form a *regular sequence* in $\mathbb{C}[x^1, \ldots, x^n]$ [46, Section 16], hence the zero locus $Z(I)$ of I is a complete intersection, i.e. an affine variety defined as the intersection of as many hypersurfaces as its codimension. This implies that $Z(I)$ is Cohen-Macaulay, hence there is no embedded associated prime [46, Theorem 17.3] and the ideal I is primary, i.e. there is only one associated prime. Again, the hypothesis that the hypersurfaces defined by the equations in (1) meet transversally et each point of M imply that I is a prime ideal in $\mathbb{C}[x^1, \ldots, x^n]$.

On the other hand, by Hilbert's Nullstellensatz [5, Exercise 7.14], [33, Section 1.7], [46, Theorem 5.4], the hypothesis $Q \mid_M \equiv 0$ amounts to

$$Q \in \mathrm{Rad}(I) := \{P \in \mathbb{C}[x^1, \ldots, x^n] : \ \exists n > 0 \ \mid \ P^n \in I\} = I,$$

where $\mathrm{Rad}(I)$ denotes the *radical* of I [5, Exercise 1.12], [33, Section 1.3] and where the last equality follows because I is prime. This shows that $Q \in I \cap \mathbb{R}[x^1, \ldots, x^n] = (f^1, \ldots, f^k)$. Finally, for a complex-valued $h = Q_1 + i Q_2$ vanishing on M both Q_1, Q_2 do, and therefore h belongs to the complexification of $(f^1, \ldots, f^k)$.

As for the last statement, the projective closure $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ of the zero locus of (1) in $\mathbb{P}^n_{\mathbb{C}}$ has degree at least s. On the other hand, s is the maximum degree so X is a complete intersection in $\mathbb{P}^n_{\mathbb{C}}$. Then, there cannot be other components and the variety defined by (1) is irreducible in $\mathbb{P}^n_{\mathbb{C}}$. The statement follows at once, since a non-empty open set of an irreducible variety is irreducible [38, Exercise I.1.6]. □

Remark 12 Consider an algebraic smooth hypersurface M, defined by a single equation $f(x) = 0$, with $d := \deg f$. By the result above, in order that any polynomial h, such that $h\,|_M \equiv 0$, is a multiple of f it suffices that there exists a line meeting M in d points.

Appendix B: Proofs of Sections 2, 3, 4

B.1 Proof of Proposition 2

Using the definition $\beta := \mathcal{F}_1 \cdot S(\mathcal{F}_2)$, $a' \star a = (\overline{\mathcal{R}}_1 \triangleright a) \star (\overline{\mathcal{R}}_2 \triangleright a')$ and the relation

$$(S_{\mathcal{F}} \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{21} = (\mathrm{id} \otimes S_{\mathcal{F}}^{-1})(\mathcal{R}). \qquad \Rightarrow \qquad (S_{\mathcal{F}} \otimes S_{\mathcal{F}})(\mathcal{R}) = \mathcal{R}, \tag{174}$$

valid for all triangular Hopf algebras, one can prove relations (90–91) as follows:

$$\begin{aligned}
x^i \star x^j &= \left(\mathcal{R}_2 \triangleright x^j\right) \star \left(\mathcal{R}_1 \triangleright x^i\right) = x^\mu \star x^\nu \,\check{\tau}^{\mu j}\,(\mathcal{R}_2)\,\check{\tau}^{\nu i}\,(\mathcal{R}_1) = R^{\nu\mu}_{ij} x^h \star x^k,\\
x^i \star \xi^j &= \left(\mathcal{R}_2 \triangleright \xi^j\right) \star \left(\mathcal{R}_1 \triangleright x^i\right) = \xi^h \star x^k \,\tau^{hj}\,(\mathcal{R}_2)\,\tau^{ki}\,(\mathcal{R}_1) = R^{kh}_{ij} \xi^h \star x^k,\\
\xi^i \star \xi^j &= -R^{kh}_{ij} \xi^h \star \xi^k \qquad \text{obtained from the previous one applyingd}\\
\partial'_i \star \partial'_j &= \left[\mathcal{R}_2 \triangleright \partial'_j\right] \star \left[\mathcal{R}_1 \triangleright \partial'_i\right] = \tau^{jh}[S_{\mathcal{F}}(\mathcal{R}_2)]\,\tau^{ik}[S_{\mathcal{F}}(\mathcal{R}_1)]\,\partial'_h \star \partial'_k\\
&\overset{(174)}{=} \tau^{jh}\,[\mathcal{R}_2]\,\tau^{ik}\,[\mathcal{R}_1]\,\partial'_h \star \partial'_k = R^{ij}_{kh} \partial'_h \star \partial'_k,\\
\partial'_i \star x^j &\overset{(10)}{=} \left[\overline{\mathcal{F}}_1 \triangleright \partial'_i\right]\left[\overline{\mathcal{F}}_2 \triangleright x^j\right] = \tau^{ik}\big[\beta S\big(\overline{\mathcal{F}}_1\big)\big]\,\check{\tau}^{\mu j}\big[\overline{\mathcal{F}}_2\big]\,\partial_k x^\mu\\
&= \tau^{ik}\big[\beta S\big(\overline{\mathcal{F}}_1\big)\big]\left[\check{\tau}^{0j}\big(\overline{\mathcal{F}}_2\big)\,\mathbf{1}\partial_k + \check{\tau}^{hj}\big(\overline{\mathcal{F}}_2\big)\,(\delta^h_k \mathbf{1} + x^h \partial_k)\right]\\
&= \tau^{ik}\big[\beta S\big(\overline{\mathcal{F}}_1\big)\big]\left[\mathbf{1}\tau^{kj}\big(\overline{\mathcal{F}}_2\big) + \check{\tau}^{\mu j}\big(\overline{\mathcal{F}}_2\big)\,x^\mu \partial_k\right]\\
&\overset{(10)}{=} \tau^{ij}\big[\beta S\big(\overline{\mathcal{F}}_1\big)\,\overline{\mathcal{F}}_2\big]\mathbf{1} + \tau^{ik}\big[\beta S\big(\overline{\mathcal{F}}_1\big)\big]\,\check{\tau}^{\mu j}\big(\overline{\mathcal{F}}_2\big)\,\big(\mathcal{F}_1 \triangleright x^\mu\big) \star (\mathcal{F}_2 \triangleright \partial_k)\\
&= \tau^{ij}\big[\beta S\big(\overline{\mathcal{F}}_1\big)\,\overline{\mathcal{F}}_2\big]\mathbf{1} + \check{\tau}^{\mu j}\big[\mathcal{F}_{1'}\overline{\mathcal{F}}_2\big]\,\tau^{ik}\big[\beta S\big(\mathcal{F}_{2'}\overline{\mathcal{F}}_1\big)\big]\,x^\mu \star \partial_k\\
&= \delta^i_j \mathbf{1} + \check{\tau}^{\mu j}[\mathcal{R}_2]\,\tau^{ik}[\beta S(\mathcal{R}_1)]\,x^\mu \star \partial_k = \delta^i_j \mathbf{1} + \check{\tau}^{\mu j}[\mathcal{R}_2]\,\tau^{ik}[S_{\mathcal{F}}(\mathcal{R}_1)\beta]\,x^\mu \star \partial_k\\
&= \delta^i_j \mathbf{1} + \check{\tau}^{\mu j}[\mathcal{R}_2]\,\tau^{ik}[S_{\mathcal{F}}(\mathcal{R}_1)]\,x^\mu \star \partial'_k \overset{(174)}{=} \delta^i_j \mathbf{1} + \check{\tau}^{\mu j}[\mathcal{R}_1]\,\tau^{ik}[\mathcal{R}_2]\,x^\mu \star \partial'_k\\
&= \delta^i_j \mathbf{1} + R^{\mu i}_{jk} x^\mu \star \partial'_k.
\end{aligned}$$

By (85) the action of either leg $\mathcal{F}_1, \mathcal{F}_2$ of the twist, or $\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2$ of its inverse, as well as of any tensor factor in the (iterated) coproducts of $\mathcal{F}_1, \mathcal{F}_2, \overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2$, maps every homogeneous polynomial in ξ^i or ∂_i into another one of the same degree, and every polynomial in x^i into another one of the same degree: hence (92), (93) follow. Finally, the relations $g \triangleright \partial'_i = \tau^{ij}[S_{\mathcal{F}}(g)]\partial'_j$, (94) are straightforward consequences of (19), (29), (85):

$$\begin{aligned}
g \triangleright \partial'_i &= gS(\beta) \triangleright \partial_i = \tau^{ij}[\beta S(g)]\partial_j = \tau^{ij}\,[S_{\mathcal{F}}(g)\beta]\,\partial_j = \tau^{ij}\,[S_{\mathcal{F}}(g)]\,\partial'_j\\
x^{i\,*_\star} &= S(\beta) \triangleright x^{i\,*} = S(\beta) \triangleright x^i = x^\mu \check{\tau}^{\mu i}\,[S(\beta)]\,,\\
\xi^{i\,*_\star} &= S(\beta) \triangleright \xi^{i\,*} = S(\beta) \triangleright \xi^i = \xi^k \tau^{ki}\,[S(\beta)]\,,\\
\partial_i'^{*_\star} &= S(\beta) \triangleright [S(\beta) \triangleright \partial_i]^* = S(\beta)\beta^* \triangleright \partial_i^* = -S(\beta)S(\beta^{-1}) \triangleright \partial_i = -\partial_i = -\tau^{ik}\left(\beta^{-1}\right)\partial'_k.
\end{aligned}$$

B.2 Proof of Proposition 3

All statements up to (106) and the statement that the $\star$-polynomials $\hat{f}_c^J(a\star)$ have the same degrees in x^i, ξ^i, e_α as the polynomials $f_c^J(a)$ are straightforward consequences of (85) and of what precedes the proposition. Under $U\mathfrak{g}$ the f_i transform as the ∂_i; in fact, since $g\triangleright f = \varepsilon(g)f$, we find $g \triangleright f_i = g \triangleright (\partial_i f - f\partial_i) = (g \triangleright \partial_i)f - f(g \triangleright \partial_i) = \tau^{ih}(Sg)(\partial_h f - f\partial_h) = \tau^{ih}(Sg)f_h$. Hence

$$\begin{aligned} g \triangleright L_{ij} &= g\triangleright(f_i\partial_j - f_j\partial_i) = \tau^{ih}(Sg_{(1)})\tau^{jk}(Sg_{(2)})[f_h\partial_k - f_k\partial_h] = \tau^{ih}(Sg_{(1)})\tau^{jk}(Sg_{(2)})L_{hk}, \\ L_{ij}^{*\star} &= -S(\beta)\triangleright L_{ij} = -\tau^{ih}(\beta_{(1)})\tau^{jk}(\beta_{(2)})L_{hk}; \end{aligned}$$

this can be computed more explicitly using the relation (see e.g. Equation (126) in [29])

$$\Delta(\beta) = \mathcal{F}^{-1}(\beta\otimes\beta)[(S\otimes S)\mathcal{F}_{21}^{-1}] = \mathcal{F}_{21}^{-1}(\beta\otimes\beta)[(S\otimes S)\mathcal{F}^{-1}]. \tag{175}$$

B.3 (a) Family of Parabolic Cylinders

Proof of Proposition 4

Since L_{13} and L_{23} are commuting anti-Hermitian vector fields it follows that $\mathcal{F}$ is a unitary abelian twist on $U\mathfrak{g}$. We find $S(\beta) = \exp(-i\nu L_{12}L_{23})$, and

$$L_{13}^n L_{12} = L_{12}L_{13}^n + nL_{23}L_{13}^{n-1}$$

for all $n > 0$, since $[L_{13}, L_{12}] = L_{23}$, $[L_{13}, L_{23}] = 0$. This implies

$$\begin{aligned} \mathcal{F}(L_{12}\otimes\mathbf{1}) &= \sum_{n=0}^{\infty}\frac{(i\nu)^n}{n!}L_{13}^n L_{12}\otimes L_{23}^n \\ &= L_{12}\otimes\mathbf{1} + \sum_{n=1}^{\infty}\frac{(i\nu)^n}{n!}L_{12}L_{13}^n\otimes L_{23}^n + \sum_{n=1}^{\infty}\frac{(i\nu)^n}{n!}nL_{23}L_{13}^{n-1}\otimes L_{23}^n \\ &= (L_{12}\otimes\mathbf{1})\mathcal{F} + i\nu(L_{23}\otimes L_{23})\sum_{n=1}^{\infty}\frac{(i\nu)^{n-1}}{(n-1)!}L_{13}^{n-1}\otimes L_{23}^{n-1} \\ &= (L_{12}\otimes\mathbf{1})\mathcal{F} + i\nu(L_{23}\otimes L_{23})\mathcal{F}, \end{aligned}$$

$$\Delta_{\mathcal{F}}(L_{12}) = \mathcal{F}(L_{12}\otimes\mathbf{1} + \mathbf{1}\otimes L_{12})\mathcal{F}^{-1} = L_{12}\otimes\mathbf{1} + \mathbf{1}\otimes L_{12} + i\nu L_{23}\otimes L_{23},$$

where in the last equation we use $\mathcal{F}(\mathbf{1}\otimes L_{12}) = (\mathbf{1}\otimes L_{12})\mathcal{F}$ since the second leg of the twist is central. Moreover $\mathcal{F}(L_{13}\otimes\mathbf{1}) = \sum_{n=0}^{\infty}\frac{(i\nu)^n}{n!}L_{13}^{n+1}\otimes L_{23}^n = (L_{13}\otimes\mathbf{1})\mathcal{F}$ and $\mathcal{F}(L_{23}\otimes\mathbf{1}) = (L_{23}\otimes\mathbf{1})\mathcal{F}$ show that $\Delta_{\mathcal{F}}(L_{13}) = \Delta(L_{13})$ and $\Delta_{\mathcal{F}}(L_{23}) = \Delta(L_{23})$. We have thus proved the claimed coproducts $\Delta_{\mathcal{F}}(L_{ij})$. Next, the latter and the antipode property $\mu[(S_{\mathcal{F}}\otimes\mathrm{id})\Delta_{\mathcal{F}}(g)] = \epsilon(L_{ij})\mathbf{1} = 0$ easily determine the twisted antipode $S_{\mathcal{F}}(L_{12})$ as in (110), and other ones $S_{\mathcal{F}}(L_{ij}) = S(L_{ij}) = -L_{ij}$. Furthermore, since the L_{23} contained in the second leg of the twist commutes with

$L_{k\ell}$ we conclude that $L_{ij} \star L_{k\ell} = L_{ij}L_{k\ell}$ and $[L_{ij}, L_{k\ell}]_\star = [L_{ij}, L_{k\ell}]$ for all $1 \leq i < j \leq 3$ and $1 \leq k < \ell \leq 3$. For the same reason one gets for all $1 \leq i, j, k \leq 3$

$$x^i \star L_{jk} = x^i L_{jk}, \quad \xi^i \star L_{jk} = \xi^i L_{jk}, \quad \partial_i \star L_{jk} = \partial_i L_{jk}.$$

On the other hand by (109) we obtain

$$\begin{aligned}
L_{ij} \star x^k &= L_{ij}x^k - i\nu b\epsilon_{ij3}\delta_2^k L_{23}, \quad L_{ij} \star \xi^k = L_{ij}\xi^k, \quad L_{ij} \star \partial_k = L_{ij}\partial_k,\\
x^i \star x^j &= x^i x^j - i\nu b\delta_2^j(\delta_1^i b + \delta_3^i x^1), \quad \partial_i \star \partial_j = \partial_i\partial_j, \quad \xi^i \star \xi^j = \xi^i\xi^j,\\
\partial_i \star x^j &= x^j\partial_i + i\nu b\delta_{i1}\delta_2^j\partial_3, \qquad x^i \star \partial_j = x^i\partial_j, \qquad \xi^i \star \partial_j = \xi^i\partial_j,\\
\xi^i \star x^j &= x^j\xi^i - i\nu b\delta_3^i\delta_2^j\xi^1, \qquad x^i \star \xi^j = x^i\xi^j, \qquad \partial_i \star \xi^j = \partial_i\xi^j,
\end{aligned}$$

for all $1 \leq i, j, k \leq 3$. The commutation relations respectively follow. Furthermore this means we can express the generators of the Lie algebra in terms of the twisted module action, namely $L_{12} = x^1\partial_2 = x^1 \star \partial_2$, $L_{13} = x^1\partial_3 + b\partial_1 = x^1 \star \partial_3 + b\partial_1$ and $L_{23} = b\partial_2$, while $f_c(x) = \frac{1}{2}(x^1)^2 - bx^3 - c = \frac{1}{2}x^1 \star x^1 - bx^3 - c$, $\mathrm{d}f_c = x^1\xi^1 - b\xi^3 = x^1 \star \xi^1 - b\xi^3$ and

$$\epsilon^{ijk} f_i L_{jk} = 2(x^1 L_{23} - bL_{12}) = 2(x^1 \star L_{23} - bL_{12})$$

hold. Again by (109) $L_{12}L_{23} \triangleright x^i = 0$, $L_{12}L_{23} \triangleright \xi^i = 0$, $L_{12}L_{23} \triangleright \partial_i = 0$ for all $i = 1, 2, 3$, which implies that the $*$-structure on $\mathcal{Q}_\star^\bullet$ remains the same as on $\mathcal{Q}_{M_c}^\bullet$: $*_\star = *$.

B.4 (b) Family of Elliptic Paraboloids

Proof of Proposition 5

Since L_{13}, L_{23} are commuting anti-Hermitian vector fields $\mathcal{F}$ is a unitary abelian twist on $U\mathfrak{g}$. By a direct calculation one finds $\beta = S(\beta) = \mathcal{F}_2 S(\mathcal{F}_1) = \exp(-i\nu L_{13}L_{23})$. The commutation relations (114) also imply $\mathcal{F}\Delta(L_{13}) = \Delta(L_{13})\mathcal{F}$, $\mathcal{F}\Delta(L_{23}) = \Delta(L_{23})\mathcal{F}$, resulting in $\Delta_\mathcal{F}(L_{13}) = \Delta(L_{13})$ and $\Delta_\mathcal{F}(L_{13}) = \Delta(L_{13})$, respectively. Moreover,

$$L_{13}^n L_{12} = L_{12}L_{13}^n + nL_{23}L_{13}^{n-1}, \quad \text{and} \quad L_{23}^n L_{12} = L_{12}L_{23}^n - naL_{13}L_{23}^{n-1}$$

for $n > 0$, which follow by iteratively applying (114). Then

$$\begin{aligned}
\mathcal{F}(L_{12} \otimes \mathbf{1}) &= \sum_{n=0}^\infty \frac{(i\nu)^n}{n!} L_{13}^n L_{12} \otimes L_{23}^n\\
&= L_{12} \otimes \mathbf{1} + \sum_{n=1}^\infty \frac{(i\nu)^n}{n!} L_{12}L_{13}^n \otimes L_{23}^n + \sum_{n=1}^\infty \frac{(i\nu)^n}{n!} nL_{23}L_{13}^{n-1} \otimes L_{23}^n\\
&= (L_{12} \otimes \mathbf{1})\mathcal{F} + i\nu(L_{23} \otimes L_{23})\mathcal{F},
\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\mathbf{1}\otimes L_{12}) &= \mathbf{1}\otimes L_{12}+\sum_{n=1}^{\infty}\frac{(i\nu)^n}{n!}L_{13}^n\otimes L_{23}^n L_{12}\\ &= \mathbf{1}\otimes L_{12}+\sum_{n=1}^{\infty}\frac{(i\nu)^n}{n!}L_{13}^n\otimes L_{12}L_{23}^n - a\sum_{n=1}^{\infty}\frac{(i\nu)^n}{n!}nL_{13}^n\otimes L_{13}L_{23}^{n-1}\\ &= (\mathbf{1}\otimes L_{12})\mathcal{F}-i\nu a(L_{13}\otimes L_{13})\mathcal{F}\end{aligned}$$

imply $(117)_1$. The twisted antipodes $(117)_2$ follow using the properties $\mu\circ(S_{\mathcal{F}}\otimes \mathrm{id})\circ\Delta_{\mathcal{F}}=\eta\circ\epsilon=\mu\circ(\mathrm{id}\otimes S_{\mathcal{F}})\circ\Delta_{\mathcal{F}}$. The twisted tensor and star products coincide with the untwisted ones as soon as one of the factors is L_{13} or L_{23}. This is because the latter commute with both legs of the twist. Among all star products of generators of $\mathfrak{g}$ only the one

$$L_{12}\star L_{12}=\sum_{n=0}^{\infty}\frac{(-i\nu)^n}{n!}(L_{13}^n\triangleright L_{12})(L_{23}^n\triangleright L_{12})=L_{12}L_{12}+i\nu aL_{23}L_{13}$$

is different. By a similar direct calculation one can prove (118). The latter imply (119-120) and that the submanifold constraints coincide with their twisted analogues, namely (121) holds. The twisted star involutions coincide with the untwisted ones, since $L_{13}L_{23}\triangleright x^i=L_{13}L_{23}\triangleright\xi^i=L_{13}L_{23}\triangleright\partial_i=0$. This concludes the proof of the proposition.

B.5 (c) Family of Elliptic Cylinders

Proof of Proposition 16 in [32], see Section 4.3

Since $[\partial_3, L_{12}]=0$ and ∂_3, L_{12} are anti-Hermitian, $\mathcal{F}=\exp(i\nu\partial_3\otimes L_{12})$ is a unitary abelian twist on $U\mathfrak{g}$. As ∂_3, L_{12} commute with both legs of the twist, $\Delta_{\mathcal{F}}(\partial_3)=\Delta(\partial_3)$, $\Delta_{\mathcal{F}}(L_{12})=\Delta(L_{12})$, $S_{\mathcal{F}}(\partial_3)=S(\partial_3)$, $S_{\mathcal{F}}(L_{12})=S(L_{12})$ and all twisted tensor and star products as well as Lie brackets where one of the factors is ∂_3, L_{12} coincide with the untwisted ones. Furthermore the star products of $x^i,\xi^j,\partial^k,L_{12}$ coincide with the classical ones unless the first leg of the twist acts on x^3. Consequently, $(115)_{b=0}$ implies (124) and the equations (125) coincide with their classical analogues. The twisted star involutions are trivial since $\partial_3L_{12}\triangleright x^i=\partial_3L_{12}\triangleright\xi^i=\partial_3L_{12}\triangleright\partial_i=0$. This concludes the proof.

B.6 (d) Family of Hyperbolic Paraboloids

Proof of Proposition 7

The anti-Hermitian vector fields H and E satisfy (133), which implies that $\mathcal{F}=\exp(H/2\otimes\log(\mathbf{1}+i\nu E))$ is a unitary Jordanian twist on $U\mathfrak{g}$. We note that

$$E^mH=HE^m-2mE^m$$

for all $m > 0$, which follows by iteratively applying (133). In particular this implies

$$\log(\mathbf{1}+i\nu E)H = -\sum_{m=1}^{\infty}\frac{(-i\nu)^m}{m}E^m H = -H\sum_{m=1}^{\infty}\frac{(-i\nu)^m}{m}E^m + 2\sum_{m=1}^{\infty}\frac{(-i\nu)^m}{m}mE^m$$
$$= H\log(\mathbf{1}+i\nu E) + 2(-i\nu)E\sum_{m=1}^{\infty}(-i\nu E)^{m-1} = H\log(\mathbf{1}+i\nu E) - 2i\nu E\frac{\mathbf{1}}{\mathbf{1}+i\nu E},$$

where we have made use of the expansions

$$\log(\mathbf{1}+i\nu E) = -\sum_{m=1}^{\infty}\frac{(-i\nu E)^m}{m}, \qquad \frac{\mathbf{1}}{\mathbf{1}+i\nu E} = \sum_{m=0}^{\infty}(-i\nu E)^m.$$

Both are well-defined in the ν-adic topology. Applying this result iteratively we obtain

$$[\log(\mathbf{1}+i\nu E)]^n H = H[\log(\mathbf{1}+i\nu E)]^n - 2n\frac{i\nu E}{\mathbf{1}+i\nu E}[\log(\mathbf{1}+i\nu E)]^{n-1}$$

for all $n > 0$, whence

$$\begin{aligned}\mathcal{F}(\mathbf{1}\otimes H) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{H}{2}\right)^n \otimes [\log(\mathbf{1}+i\nu E)]^n H \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{H}{2}\right)^n \otimes H[\log(\mathbf{1}+i\nu E)]^n - 2\sum_{n=1}^{\infty}\frac{n}{n!}\left(\frac{H}{2}\right)^n \otimes \frac{i\nu E}{\mathbf{1}+i\nu E}[\log(\mathbf{1}+i\nu E)]^{n-1} \\ &= (\mathbf{1}\otimes H)\mathcal{F} - \left(H\otimes\frac{i\nu E}{\mathbf{1}+i\nu E}\right)\mathcal{F}.\end{aligned}$$

This and $\mathcal{F}(H\otimes\mathbf{1}) = (H\otimes\mathbf{1})\mathcal{F}$ determine $\Delta_{\mathcal{F}}(H)$ as in (137). For the twisted coproduct of E we first remark that $\left(\frac{H}{2}\right)^n E = E\left(\frac{H}{2}+\mathbf{1}\right)^n$ for all $n \geq 0$, which is proven by induction. Then

$$\begin{aligned}\mathcal{F}(E\otimes\mathbf{1}) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{H}{2}\right)^n E\otimes\log(\mathbf{1}+i\nu E)^n \\ &= (E\otimes\mathbf{1})\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{H}{2}+\mathbf{1}\right)^n\otimes\log(\mathbf{1}+i\nu E)^n \\ &= (E\otimes\mathbf{1})\exp\left(\left(\frac{H}{2}+\mathbf{1}\right)\otimes\log(\mathbf{1}+i\nu E)\right) \\ &= (E\otimes\mathbf{1})\exp\left(\mathbf{1}\otimes\log(\mathbf{1}+i\nu E)\right)\exp\left(\frac{H}{2}\otimes\log(\mathbf{1}+i\nu E)\right) \\ &= [E\otimes(\mathbf{1}+i\nu E)]\mathcal{F}.\end{aligned}$$

This and $\mathcal{F}(\mathbf{1}\otimes E) = (\mathbf{1}\otimes E)\mathcal{F}$ determine $\Delta_{\mathcal{F}}(E)$ as in (137). Similarly one proves

$$\begin{aligned}\mathcal{F}(E'\otimes\mathbf{1}) &= (E'\otimes\mathbf{1})\exp\left(-\mathbf{1}\otimes\log(\mathbf{1}+i\nu E)\right)\exp\left(\frac{H}{2}\otimes\log(\mathbf{1}+i\nu E)\right) \\ &= \left(E'\otimes\frac{\mathbf{1}}{\mathbf{1}+i\nu E}\right)\mathcal{F}\end{aligned} \tag{176}$$

and $\mathcal{F}(\mathbf{1}\otimes E')=(\mathbf{1}\otimes E')\mathcal{F}$, which determine $\Delta_{\mathcal{F}}(E')$ as in (137). Next, it is straightforward to check that the coproducts $\Delta_{\mathcal{F}}(g)$, with $g=H,E,E'$, and the antipode property $\mu[(S_{\mathcal{F}}\otimes\mathrm{id})\Delta_{\mathcal{F}}(g)]=\epsilon(g)\mathbf{1}=0$ determine the twisted antipodes $S_{\mathcal{F}}(g)$ as in (137). To compute the twisted tensor and star products we first make only the first leg $\overline{\mathcal{F}}_1$ of $\overline{\mathcal{F}}$ to act on its eigenvectors H,E,E',u^i,∂_i (generators of $U\mathfrak{g}$ and of $\mathcal{Q}$), and find

$$\begin{aligned}
\overline{\mathcal{F}}_1\triangleright H\otimes\overline{\mathcal{F}}_2 &= H\otimes\mathbf{1},\\
\overline{\mathcal{F}}_1\triangleright E\otimes\overline{\mathcal{F}}_2 &= E\otimes\exp\left(-\log(\mathbf{1}+i\nu E)\right)=E\otimes(\mathbf{1}+i\nu E)^{-1},\\
\overline{\mathcal{F}}_1\triangleright E'\otimes\overline{\mathcal{F}}_2 &= E'\otimes\exp\left(\log(\mathbf{1}+i\nu E)\right)=E'\otimes(\mathbf{1}+i\nu E),\\
\overline{\mathcal{F}}_1\triangleright u^i\otimes\overline{\mathcal{F}}_2 &= u^i\otimes\exp\left(-\frac{\lambda_i}{2}\log(\mathbf{1}+i\nu E)\right)=u^i\otimes(\mathbf{1}+i\nu E)^{-\frac{\lambda_i}{2}},\\
\overline{\mathcal{F}}_1\triangleright \partial_i\otimes\overline{\mathcal{F}}_2 &= \partial_i\otimes\exp\left(\frac{\lambda_i}{2}\log(\mathbf{1}+i\nu E)\right)=\partial_i\otimes(\mathbf{1}+i\nu E)^{\frac{\lambda_i}{2}},
\end{aligned}$$

for all $u^i\in\{y^i,\eta^i\}$, $1\le i\le 3$; note that the exponents $\pm\lambda_i/2$ take the values $\pm1,0$. This simplifies the computation of the action of the second leg $\overline{\mathcal{F}}_2$ on the second factor; using (133) and (136) and noting that only the terms of degree lower than two in the power expansion of $\mathbf{1}/(\mathbf{1}+i\nu E)$ contribute to its action on the H,E,E',u^i,∂_i, by a direct computation one thus finds the star products (138-140). In particular the twisted tensor and star products are trivial if H, u^3 or $\tilde{\partial}_3$ appears in the first factor. The twisted commutation relations (141)-(142) and the twisted submanifold constraints (143) follow. For the twisted star involution note that

$$\begin{aligned}
S(\beta)\triangleright y^i &= \sum_{n=0}^{\infty}\frac{1}{n!}[\log(\mathbf{1}+i\nu E)]^n\left(-\frac{H}{2}\right)^n\triangleright y^i=(\mathbf{1}+i\nu E)^{-\frac{\lambda_i}{2}}\triangleright y^i=y^i+i\nu 2b\delta^i_2\\
S(\beta)\triangleright \tilde{\partial}_i &= \sum_{n=0}^{\infty}\frac{1}{n!}[\log(\mathbf{1}+i\nu E)]^n\left(-\frac{H}{2}\right)^n\triangleright \tilde{\partial}_i=(\mathbf{1}+i\nu E)^{\frac{\lambda_i}{2}}\triangleright \tilde{\partial}_i=\tilde{\partial}_i-i\nu\delta_{i1}\tilde{\partial}_3,
\end{aligned}$$

since $E\triangleright y^i=\delta^i_3 y^1+2b\delta^i_2$ and $\lambda_3=0$, $\lambda_2=-1$, while $E\triangleright\tilde{\partial}_i=-\delta_{i1}\tilde{\partial}_3$, and $\lambda_1=1$.

B.7 (d-e-f) Elliptic Cone and Hyperboloids

B.7.1 Proof of Proposition 17 in [32], see Section 4.6

From the anti-Hermiticity of the vector fields H,E and from $[H,E]=2E$ it follows that $\mathcal{F}=\exp(H/2\otimes\log(\mathbf{1}+i\nu E))$ is a unitary Jordanian twist on $U\mathfrak{g}$ and that the coproducts and antipodes of H,E are exactly as in case (d). Similarly, (176) holds, because it is only based on the relation $[H,E']=-2E'$.

To compute $\Delta_{\mathcal{F}}(E')$ we first determine $\mathcal{F}(\mathbf{1}\otimes E')$. We use $[H, E] = 2E$, $EE' = E'E - H$, and find by induction first that $E^n H = (H - 2n)E^n$, then

$$\begin{aligned} E^n E' &= E^{n-1}E'E - E^{n-1}H = E^{n-2}E'E^2 - E^{n-2}HE - E^{n-1}H \\ &= \dots \\ &= E'E^n - HE^{n-1} - EHE^{n-2} - \dots - E^{n-2}HE - E^{n-1}H \\ &= E'E^n + [-nH + 2(1+2+\dots+n-1)]E^{n-1} \\ &= E'E^n - nHE^{n-1} + n(n-1)E^{n-1} \end{aligned}$$

for all $n \geq 0$, by the "little Gauss" $\sum_{h=1}^{n-1} h = \frac{n^2-n}{2}$. Consequently, using the series expansions

$$\log(\mathbf{1}+i\nu E) = -\sum_{n=1}^{\infty}\frac{(-i\nu E)^n}{n}, \quad \frac{\mathbf{1}}{\mathbf{1}+i\nu E} = \sum_{n=0}^{\infty}(-i\nu E)^n, \quad \frac{\mathbf{1}}{(\mathbf{1}+i\nu E)^2} = \sum_{n=1}^{\infty} n(-i\nu E)^{n-1},$$

we obtain

$$\begin{aligned} \log(\mathbf{1}+i\nu E)H &= -\sum_{n=1}^{\infty}\frac{(-i\nu E)^n}{n}H = -\sum_{n=1}^{\infty}(H-2n)\frac{(-i\nu E)^n}{n} = H\log(\mathbf{1}+i\nu E) - \frac{2i\nu E}{1+i\nu E}, \\ \log(\mathbf{1}+i\nu E)E' &= -\sum_{n=1}^{\infty}\frac{(-i\nu)^n}{n}E^nE' \\ &= -E'\sum_{n=1}^{\infty}\frac{(-i\nu)^n}{n}E^n - i\nu H\sum_{n=1}^{\infty}(-i\nu E)^{n-1} + i\nu\sum_{n=2}^{\infty}(n-1)(-i\nu E)^{n-1} \\ &= E'\log(\mathbf{1}+i\nu E) + \frac{\nu^2 E}{(\mathbf{1}+i\nu E)^2} - H\frac{i\nu}{\mathbf{1}+i\nu E}, \end{aligned}$$

and in turn

$$\begin{aligned} [\log(\mathbf{1}+i\nu E)]^n H &= [\log(\mathbf{1}+i\nu E)]^{n-1}H\log(\mathbf{1}+i\nu E) - \frac{2i\nu E}{\mathbf{1}+i\nu E}[\log(\mathbf{1}+i\nu E)]^{n-1} \\ &= \dots = H[\log(\mathbf{1}+i\nu E)]^n - 2n\frac{i\nu E}{\mathbf{1}+i\nu E}[\log(\mathbf{1}+i\nu E)]^{n-1}, \end{aligned}$$

$$\begin{aligned} [\log(\mathbf{1}+i\nu E)]^n E' &= [\log(\mathbf{1}+i\nu E)]^{n-1}\left\{E'\log(\mathbf{1}+i\nu E) + \frac{\nu^2 E}{(\mathbf{1}+i\nu E)^2} - H\frac{i\nu}{\mathbf{1}+i\nu E}\right\} \\ &= [\log(\mathbf{1}+i\nu E)]^{n-1}E'\log(\mathbf{1}+i\nu E) + \frac{\nu^2 E[\log(\mathbf{1}+i\nu E)]^{n-1}}{(\mathbf{1}+i\nu E)^2} \\ &\quad -i\nu H\frac{[\log(\mathbf{1}+i\nu E)]^{n-1}}{\mathbf{1}+i\nu E} - 2(n-1)\frac{\nu^2 E}{(\mathbf{1}+i\nu E)^2}[\log(\mathbf{1}+i\nu E)]^{n-2} \\ &= \dots = E'\log(\mathbf{1}+i\nu E)]^n + n\frac{\nu^2 E[\log(\mathbf{1}+i\nu E)]^{n-1}}{(\mathbf{1}+i\nu E)^2} - ni\nu H\frac{[\log(\mathbf{1}+i\nu E)]^{n-1}}{\mathbf{1}+i\nu E} \\ &\quad -2[(n-1)+(n-2)+\dots+1]\frac{\nu^2 E[\log(\mathbf{1}+i\nu E)]^{n-2}}{(\mathbf{1}+i\nu E)^2} \end{aligned}$$

$$= E'\log(\mathbf{1}+i\nu E)]^n + n\left[\frac{\nu^2 E}{\mathbf{1}+i\nu E} - i\nu H\right]\frac{[\log(\mathbf{1}+i\nu E)]^{n-1}}{\mathbf{1}+i\nu E} - n(n-1)\frac{\nu^2 E[\log(\mathbf{1}+i\nu E)]^{n-2}}{(\mathbf{1}+i\nu E)^2}.$$

Hence

$$
\begin{aligned}
\mathcal{F}(\mathbf{1}\otimes E') &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{H}{2}\right)^n \otimes [\log(\mathbf{1}+i\nu E)]^n E' \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{H}{2}\right)^n \otimes \left\{ E'\log(\mathbf{1}+i\nu E)]^n + n\left[\frac{\nu^2 E}{\mathbf{1}+i\nu E} - i\nu H\right]\frac{[\log(\mathbf{1}+i\nu E)]^{n-1}}{\mathbf{1}+i\nu E}\right. \\
&\qquad\left. -n(n-1)\frac{\nu^2 E\log\,(\mathbf{1}+i\nu E)]^{n-2}}{(\mathbf{1}+i\nu E)^2}\right\} \\
&= \left\{\mathbf{1}\otimes E' + \frac{H}{2}\otimes\left[\frac{\nu^2 E}{\mathbf{1}+i\nu E} - i\nu H\right]\frac{\mathbf{1}}{\mathbf{1}+i\nu E} - \left(\frac{H}{2}\right)^2\otimes\frac{\nu^2 E}{(\mathbf{1}+i\nu E)^2}\right\}\mathcal{F}.
\end{aligned}
$$

On the other hand, using $\left(\frac{H}{2}\right)^n E' = E'\left(\frac{H}{2}-\mathbf{1}\right)^n$ we obtain

$$
\begin{aligned}
\mathcal{F}(E'\otimes\mathbf{1}) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{H}{2}\right)^n E'\otimes[\log(\mathbf{1}+i\nu E)]^n = \sum_{n=0}^{\infty}\frac{1}{n!}E'\left(\frac{H}{2}-\mathbf{1}\right)^n\otimes[\log(\mathbf{1}+i\nu E)]^n \\
&= (E'\otimes\mathbf{1})\exp\left[\left(\frac{H}{2}-\mathbf{1}\right)\otimes\log(\mathbf{1}+i\nu E)\right] = \left(E'\otimes\frac{\mathbf{1}}{\mathbf{1}+i\nu E}\right)\mathcal{F}.
\end{aligned}
$$

Summing the last two equations we find that $\Delta_{\mathcal{F}}(E')$ is as in (151). The antipode $S_{\mathcal{F}}(E')$ follows from the antipode property $\mu[(S_{\mathcal{F}}\otimes\mathrm{id})\Delta_{\mathcal{F}}(E')] = 0$.

To compute the twisted tensor and star products we first make only the first leg $\overline{\mathcal{F}}_1$ of $\overline{\mathcal{F}}$ to act on its eigenvectors H, E, E', u^i (generators of $U\mathfrak{g}$ and of $\mathcal{Q}$), and find

$$
\begin{aligned}
\overline{\mathcal{F}}_1 \triangleright H\otimes\overline{\mathcal{F}}_2 &= H\otimes\mathbf{1}, \\
\overline{\mathcal{F}}_1 \triangleright E\otimes\overline{\mathcal{F}}_2 &= E\otimes\exp\left[-\log(\mathbf{1}+i\nu E)\right] = E\otimes(\mathbf{1}+i\nu E)^{-1}, \\
\overline{\mathcal{F}}_1 \triangleright E'\otimes\overline{\mathcal{F}}_2 &= E'\otimes\exp\left[\log(\mathbf{1}+i\nu E)\right] = E'\otimes(\mathbf{1}+i\nu E), \\
\overline{\mathcal{F}}_1 \triangleright u^i\otimes\overline{\mathcal{F}}_2 &= u^i\otimes\exp\left[-\tfrac{\lambda_i}{2}\log(\mathbf{1}+i\nu E)\right] = u^i\otimes(\mathbf{1}+i\nu E)^{-\frac{\lambda_i}{2}}, \qquad (177)\\
&= \begin{cases} u^1\otimes(\mathbf{1}+i\nu E)^{-1} & \text{if } i=1,\\ u^2\otimes\mathbf{1} & \text{if } i=2,\\ u^3\otimes(\mathbf{1}+i\nu E) & \text{if } i=3,\end{cases}
\end{aligned}
$$

where $u^i \in \{y^i, \eta^i, \tilde{\partial}^i\}$, $1\le i\le 3$; note that the exponents $\pm\lambda_i/2$ take the values $\pm 1, 0$. By the first relation the twisted tensor or star products are trivial if H or some u^2 is the first factor. The following two imply (153). Moreover, for all $u^i, v^i \in \{y^i, \eta^i, \tilde{\partial}^i\}$, we find

$$
u^i\star v^j = \begin{cases} u^1\left(\mathbf{1}-i\nu E-\nu^2E^2\right)\triangleright v^j \\ u^2 v^j \\ u^3(\mathbf{1}+i\nu E)\triangleright v^j\end{cases} = \begin{cases} u^1\left[v^j - i\nu\left(\frac{\delta_2^j}{\sqrt{a}}v^1 - 2\delta_3^j\sqrt{a}v^2\right) + 2\nu^2\delta_3^j v^1\right] & \text{if } i=1,\\ u^2 v^j & \text{if } i=2,\\ u^3\left[v^j + i\nu\left(\delta_2^j\frac{1}{\sqrt{a}}v^1 - 2\delta_3^j\sqrt{a}v^2\right)\right] & \text{if } i=3.\end{cases}
$$

By explicit calculations these imply relations (154-159), as well as (160), once one notes that

$$
\frac{2}{\sqrt{ab}}\varepsilon^{ijk}f_iL_{jk} = y^3E - y^1E' - \sqrt{a}y^2H = y^3\star E - y^1\star E' - \sqrt{a}\,y^2\star H + i\nu y^1\star H - 2i\nu(1+i\nu)y^1\star E.
$$

To determine $u^{i*_\star} = S(\beta) \triangleright u^{i*}$ recall that $\beta = \mathcal{F}_1 S(\mathcal{F}_2)$. Then

$$S(\beta) \triangleright u^i = (\mathcal{F}_2 S(\mathcal{F}_1)) \triangleright u^i = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \log(\mathbf{1} + i\nu E)^n \left(-\frac{H}{2}\right)^n\right) \triangleright u^i$$
$$= \left(\sum_{n=0}^{\infty} \frac{1}{n!} \log(\mathbf{1} + i\nu E)^n \left(-\frac{\lambda_i}{2}\right)^n\right) \triangleright u^i = (\mathbf{1} + i\nu E)^{-\frac{\lambda_i}{2}} \triangleright u^i$$
$$= \begin{cases} \sum_{n=0}^{\infty}(-i\nu E)^n \triangleright u^1 \\ u^2 \\ (\mathbf{1} + i\nu E) \triangleright u^3 \end{cases} = \begin{cases} u^1 & \text{if } i = 1, \\ u^2 & \text{if } i = 2, \\ u^3 - 2i\nu\sqrt{a}u^2 & \text{if } i = 3. \end{cases}$$

B.7.2 Metric and Principal Curvatures on the Circular Hyperboloids and Cone

One can easily check the statements of the first paragraph of Section 4.6.1 using e.g. the basis $S := \{v_1, v_2, v_3\}$ of Ξ, where

$$v_1 := L_{12}, \qquad v_2 := \rho^2 \partial_3 + x^3(x^1\partial_1 + x^2\partial_2), \qquad v_3 := V_\perp = f_j \eta^{ji}\partial_i = x^i\partial_i; \tag{178}$$

we have abbreviated $\rho^2 \equiv (x^1)^2 + (x^2)^2$. S is orthogonal with respect to $\mathbf{g}$, while $S_t := \{v_1, v_2\}$ is an orthogonal basis of Ξ_t with respect to $\mathbf{g}_t$, since, by an easy computation,

$$\mathbf{g}(v_i, v_j) = \begin{cases} \rho^2 & \text{if } i = j = 1, \\ -\mathsf{E}\rho^2 & \text{if } i = j = 2, \\ \mathsf{E} & \text{if } i = j = 3, \\ 0 & \text{otherwise,} \end{cases} \tag{179}$$

where $\mathsf{E}(x) \equiv f^i(x) f_i(x) = x^i x_i$. Since $\mathsf{E} = 2c$ on M_c, M_c is indeed Riemannian if $c < 0$, Lorentzian if $c > 0$, whereas the metric induced on the cone M_0 is degenerate. One can easily check (161), (162) on such a S_t by explicit computations. The dual basis consists of

$$\vartheta^1 = \frac{1}{\rho^2}(x^1\xi^2 - x^2\xi^1), \qquad \vartheta^2 = \frac{1}{\mathsf{E}\rho^2}[x^3(x^1\xi^1 + x^2\xi^2) - \rho^2\xi^3], \qquad \vartheta^3 = \frac{1}{\mathsf{E}} df. \tag{180}$$

The principal curvatures on M_c are indeed $-\text{sign}(c)/\sqrt{|2c|}$, $1/\sqrt{|2c|}$, because in the 'orthonormal' basis $S := \{e_1, e_2, e_3\}$, with $e_1 := \frac{1}{\rho} v_1$, $e_2 := \frac{1}{\rho\sqrt{|\mathsf{E}|}} v_2$, $e_3 := N_\perp = \frac{1}{\sqrt{|\mathsf{E}|}} V_\perp$, one finds

$$\mathbf{g}(e_i, e_j) = \begin{cases} 1 & \text{if } i = j = 1, \\ -\text{sign}(\mathsf{E}) & \text{if } i = j = 2, \\ \text{sign}(\mathsf{E}) & \text{if } i = j = 3, \\ 0 & \text{otherwise,} \end{cases} \qquad II(e_\alpha, e_\beta) = -\frac{\text{sign}(\mathsf{E})}{\sqrt{|\mathsf{E}|}} \mathbf{g}(e_\alpha, e_\beta) e_3. \tag{181}$$

On H, E, E' the metric gives

$$\begin{aligned} &\mathbf{g}(E, E) = (y^1)^2, && \mathbf{g}(E', E') = (y^3)^2, && \mathbf{g}(H, H) = -8y^1y^3, \\ &\mathbf{g}(E, E') = \mathsf{E} + (y^2)^2 && \mathbf{g}(E, H) = -2y^1y^2, && \mathbf{g}(E', H) = -2y^2y^3, \end{aligned} \tag{182}$$

and the same results in the last line if we flip the arguments. To prove (165) we use (67), (56), (177), the $\mathcal{X}$-linearity of $\mathbf{g}$. To prove (163) we use (67), (177). The undeformed version of (164) follows from (162) by $\mathcal{X}$-linearity. We prove (164) using (67), (18), the definition of $\overline{\mathcal{R}}$:

$$\mathsf{R}^{\mathcal{F}}_{t\star}(X,Y,Z)=\mathsf{R}_t\big(\overline{\mathcal{F}}_1\triangleright X,\overline{\mathcal{F}}_2\triangleright Y,\overline{\mathcal{F}}_3\triangleright Z\big)=(A-B)/2c,\qquad \text{where}$$

$$\begin{aligned}
A &:= \left(\overline{\mathcal{F}}_2\triangleright Y\right)\mathbf{g}\left(\overline{\mathcal{F}}_1\triangleright X,\overline{\mathcal{F}}_3\triangleright Z\right)=\Big(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2\triangleright Y\Big)\mathbf{g}\Big(\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1\triangleright X,\overline{\mathcal{F}}_2\triangleright Z\Big)\\
&= \Big(\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}''_1\mathcal{F}'''_1\overline{\mathcal{F}}'_2\triangleright Y\Big)\mathbf{g}\Big(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}''_2\mathcal{F}'''_2\overline{\mathcal{F}}'_1\triangleright X,\overline{\mathcal{F}}_2\triangleright Z\Big)\\
&= \Big(\overline{\mathcal{F}}_1\mathcal{F}'''_1\overline{\mathcal{F}}'_2\triangleright Y\Big)\mathbf{g}\Big(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1\mathcal{F}'''_2\overline{\mathcal{F}}'_1\triangleright X,\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2\triangleright Z\Big)\\
&= \left(\overline{\mathcal{F}}_1\overline{\mathcal{R}}_1\triangleright Y\right)\mathbf{g}\Big(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}_1\overline{\mathcal{R}}_2\triangleright X,\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2\triangleright Z\Big)=\left(\overline{\mathcal{F}}_1\overline{\mathcal{R}}_1\triangleright Y\right)\overline{\mathcal{F}}_2\triangleright\mathbf{g}\Big(\overline{\mathcal{F}}_1\overline{\mathcal{R}}_2\triangleright X,\overline{\mathcal{F}}''_2\triangleright Z\Big)\\
&= \left(\overline{\mathcal{R}}_1\triangleright Y\right)\star\mathbf{g}_{t\star}\left(\overline{\mathcal{R}}_2\triangleright X,Z\right),\\
B &:= \left(\overline{\mathcal{F}}_1\triangleright X\right)\mathbf{g}\left(\overline{\mathcal{F}}_2\triangleright Y,\overline{\mathcal{F}}_3\triangleright Z\right)=\left(\overline{\mathcal{F}}_1\triangleright X\right)\mathbf{g}\Big(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1\triangleright Y,\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2\triangleright Z\Big)\\
&= \left(\overline{\mathcal{F}}_1\triangleright X\right)\overline{\mathcal{F}}_2\triangleright\mathbf{g}\Big(\overline{\mathcal{F}}'_1\triangleright Y,\overline{\mathcal{F}}'_2\triangleright Z\Big)=X\star\mathbf{g}_{t\star}(Y,Z).
\end{aligned}$$

To prove (166) we note that classically ∇ is the Levi-Civita covariant derivative $\nabla_{X^i\partial_i}(Y^i\partial_i)=X^i\partial_i(Y^j)\partial_j$ and $\overline{\mathcal{F}}(\triangleright\otimes\triangleright)(H\otimes X)=H\otimes X$, $\overline{\mathcal{F}}(\triangleright\otimes\triangleright)(X\otimes E)=X\otimes E$ for all $X\in\Xi$, while by (177), (146) $\overline{\mathcal{F}}(\triangleright\otimes\triangleright)(E\otimes H)=E\otimes H+2i\nu E\otimes E$, $\overline{\mathcal{F}}(\triangleright\otimes\triangleright)(E\otimes E')=E\otimes E'+i\nu E\otimes H-2\nu^2E\otimes E$, $\overline{\mathcal{F}}(\triangleright\otimes\triangleright)(E'\otimes H)=E'\otimes H-2i\nu E'\otimes E$ and $\overline{\mathcal{F}}(\triangleright\otimes\triangleright)(E'\otimes E')=E'\otimes E'-i\nu E'\otimes H$.

Proof of Proposition 10

$[D,\mathfrak{g}]=0$ implies $\mathcal{F}(g\otimes\mathbf{1})=(g\otimes\mathbf{1})\mathcal{F}$ for all $g\in\mathfrak{g}$. Moreover, since $\mathcal{F}(\mathbf{1}\otimes H)=(\mathbf{1}\otimes H)\mathcal{F}$, $\mathcal{F}(\mathbf{1}\otimes D)=(\mathbf{1}\otimes D)\mathcal{F}$, it follows that $\Delta_{\mathcal{F}}(H)=\Delta(H)$, $S_{\mathcal{F}}(H)=S(H)=-H$, $\Delta_{\mathcal{F}}(D)=\Delta(D)$, $S_{\mathcal{F}}(D)=S(D)=-D$. On the other hand, $HE=E(H+2)$, $HE'=E'(H-2)$ imply

$$\begin{aligned}
&\mathcal{F}(\mathbf{1}\otimes E)=\exp(i\nu D\otimes imesH/2)(\mathbf{1}\otimes E)=(\mathbf{1}\otimes E)\exp[i\nu D\otimes(H/2+\mathbf{1})]=(e^{i\nu D}\otimes E)\mathcal{F},\\
&\mathcal{F}(\mathbf{1}\otimes E')=\exp(i\nu D\otimes H/2)(\mathbf{1}\otimes E')=(\mathbf{1}\otimes E')\exp[i\nu D\otimes(H/2-\mathbf{1})]=(e^{-i\nu D}\otimes E')\mathcal{F},
\end{aligned}$$

which together with $\mathcal{F}(g\otimes\mathbf{1})=(g\otimes\mathbf{1})\mathcal{F}$ and $\mu[(S_{\mathcal{F}}\otimes\mathrm{id})\Delta_{\mathcal{F}}(g)]=0$ for $g=E,E'$ imply (168).

$[D,\mathfrak{g}]=0$ also implies $\overline{\mathcal{F}}_1\triangleright g\otimes\overline{\mathcal{F}}_2=g\otimes\mathbf{1}$ for all $g\in\mathfrak{g}$, whence $g\star\alpha=g\alpha$ for all $\alpha\in U\mathfrak{g},\mathcal{Q}$, in particular for the α appearing in the formulas of the proposition. $D\triangleright\tilde{\partial}_i=-\tilde{\partial}_i$, $D\triangleright u^i=u^i$ for $u^i=y^i,\eta^i$ imply $\overline{\mathcal{F}}_1\triangleright\tilde{\partial}_i\otimes\overline{\mathcal{F}}_2=\tilde{\partial}_i\otimes e^{i\nu H/2}$, $\overline{\mathcal{F}}_1\triangleright u^i\otimes\overline{\mathcal{F}}_2=u^i\otimes e^{-i\nu H/2}$, whence $\tilde{\partial}_i\star\alpha=\tilde{\partial}_i(e^{i\nu H/2}\triangleright\alpha)$, $u^i\star\alpha=u^i(e^{-i\nu H/2}\triangleright\alpha)$. Since D,H,E,E' and $y^i,\eta^i,\tilde{\partial}_i$ (generators of $\mathcal{Q}$) are all eigenvectors of $H\triangleright$, choosing α as each of them, we immediately find the remaining formulae in (169-170). One finds the involution $*_\star$ using the following results:

$$\begin{aligned}
&S(\beta)=\mathcal{F}_2S(\mathcal{F}_1)=\textstyle\sum_{n=0}^{\infty}\frac{(-i\nu H)^n}{2}\frac{D^n}{n!}=e^{-\frac{i}{2}\nu HD}\qquad\Rightarrow\\
&S(\beta)\triangleright u^i=e^{-\frac{i}{2}\nu H}\triangleright u^i=e^{-i\nu\mu_i}u^i,\qquad S(\beta)\triangleright\tilde{\partial}_i=e^{\frac{i}{2}\nu H}\triangleright\tilde{\partial}_i=e^{-i\nu\mu_i}\tilde{\partial}_i.
\end{aligned}$$

The commutation relations (171), the realization of D, H, E, E' as combinations of $y^i \star \tilde{\partial}_i$, and the relations (172) characterizing $\mathcal{Q}_{M_c\star}$ follow from (169-170) by direct computations.

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