



The Induced Semigroup of Schwarz Maps to the Space of Hilbert-Schmidt Operators

George Androulakis¹ · Alexander Wiedemann^{1,2}  · Matthew Ziemke³

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Abstract

We prove that for every semigroup of Schwarz maps on the von Neumann algebra of all bounded linear operators on a Hilbert space which has a subinvariant faithful normal state there exists an associated semigroup of contractions on the space of Hilbert-Schmidt operators of the Hilbert space. Moreover, we show that if the original semigroup is weak* continuous then the associated semigroup is strongly continuous. We introduce the notion of the extended generator of a semigroup on the bounded operators of a Hilbert space with respect to an orthonormal basis of the Hilbert space. We describe the form of the generator of a quantum Markov semigroup on the von Neumann algebra of all bounded linear operators on a Hilbert space which has an invariant faithful normal state under the assumption that the generator of the associated semigroup has compact resolvent.

Keywords Schwarz maps · Strongly continuous semigroups · Generator of a semigroup · Form generator · Faithful normal states · Moore-Penrose inverse

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✉ George Androulakis
giorgis@math.sc.edu

Alexander Wiedemann
alwiedemann@davidson.edu

Matthew Ziemke
mjz55@drexel.edu

¹ Department of Mathematics, University of South Carolina, Columbia, South Carolina

² Present address: Department of Mathematics and Computer Science, Davidson College, Davidson, NC, USA

³ Department of Mathematics, Drexel University, Philadelphia, PA, USA

1 Introduction

It is known that, under certain assumptions, semigroups on von Neumann algebras or their preduals give rise to associated semigroups on Hilbert spaces. Moreover, these associated semigroups often have stronger continuity properties than the original semigroups. For example, in [35, Equation (2.1)] it is stated that if $(T_t)_{t \geq 0}$ is a quantum Markov semigroup on a von Neumann algebra \mathcal{A} which has an invariant faithful normal state, and if $(\mathcal{K}, \pi, \Omega)$ is the GNS triple associated to that state, then there exists a strongly continuous semigroup $(\overline{T}_t)_{t \geq 0}$ of contractions on \mathcal{K} such that

$$\overline{T}_t(\pi(A)\Omega) = \pi(T_t(A))\Omega \quad \text{for all } A \in \mathcal{A} \text{ and } t \geq 0. \quad (1)$$

Since the proof of this statement is not included in [35] we provide a proof here (see Remarks 2 and 9). Other results which give rise to semigroups on Hilbert spaces starting from semigroups defined on spaces of operators can be found in literature. For example, in [34, Footnote of Theorem 6] it is proved that every strongly continuous semigroup $(T_t)_{t \geq 0}$ of positive isometries on the real Banach space of self-adjoint trace-class operators on a Hilbert space gives rise to a strongly continuous semigroup $(V_t)_{t \geq 0}$ of isometries on the Hilbert space such that T_t is given as a conjugation by V_t for all $t \geq 0$. In [19, Theorem 3] it is proved that, under appropriate assumptions, weakly continuous semigroups on $\mathcal{B}(\mathcal{H})$ (where \mathcal{H} is a separable Hilbert space) give rise to corresponding semigroups of unitaries on some associated Hilbert space. In [31, Theorem 3.3.7] the author produces a strongly continuous group of unitaries associated with a norm continuous semigroup on the space of trace-class operators on a related Hilbert space.

In this work we prove a result similar to the result stated above in (1) (Theorem 2). More precisely, we prove that every semigroup of Schwarz maps on $\mathcal{B}(\mathcal{H})$ (where \mathcal{H} is a Hilbert space) which has an invariant faithful state gives rise to an associated semigroup $(\widetilde{T}_t)_{t \geq 0}$ of contractions on the space of Hilbert-Schmidt operators on \mathcal{H} . Our map is “more symmetric” than the one provided by (1) (see the comments following Remark 2). We introduce the notion of the extended generator of a semigroup on bounded operators on a Hilbert space with respect to an orthonormal basis of the Hilbert space, and we explicitly describe how the generators of $(T_t)_{t \geq 0}$ and $(\widetilde{T}_t)_{t \geq 0}$ and the extended generator of $(T_t)_{t \geq 0}$ are related. We apply these descriptions to a quantum Markov semigroup $(T_t)_{t \geq 0}$ having an invariant faithful normal state under the assumption that the generator of $(\widetilde{T}_t)_{t \geq 0}$ has compact resolvent, which allows us to describe the form of the extended generator (and thereby the generator) of the semigroup $(T_t)_{t \geq 0}$ with respect to an orthonormal basis, and thereby the generator itself (see Theorem 4).

1.1 Structure

- In Section 2 we establish formal notation and definitions, and give some historical notes on the terminology.
- In Section 3 we consider several constructions arising from faithful, positive, normal functionals. In particular, in Section 3.1 we prove that every faithful positive normal functional on $\mathcal{B}(\mathcal{H})$ induces a canonical bounded linear map from $\mathcal{B}(\mathcal{H})$

to $\mathcal{S}_2(\mathcal{H})$. This map is used in Theorem 1 to prove that for every bounded linear Schwarz map on $\mathcal{B}(\mathcal{H})$, which has a subinvariant faithful positive functional, there exists a corresponding contraction on $\mathcal{S}_2(\mathcal{H})$. In Section 3.2 we consider an alternate construction for such induced maps using the GNS construction, and then compare and contrast the two methods.

- In Section 4 we recall the basic notions of continuity for semigroups, as well as formalize the definition of a semigroup’s generator and its generator’s domain. In Section 4.1 we introduce the notion of an extended generator, which can be defined on a larger domain while still agreeing with the usual generator on all finite subspaces. Theorem 2 relates the domains and actions of the generator, the extended generator, and the generator of the semigroup induced on $\mathcal{S}_2(\mathcal{H})$.
- In Section 5 we investigate the applications of Theorem 2 in the study of quantum Markov semigroups (QMSs), for which the exact form of the generator is known if the generator is bounded (see [28] and [36]). In Section 5.1, we describe the form of a QMS generator in the case that the generator of the semigroup induced on $\mathcal{S}_2(\mathcal{H})$ has compact resolvent.

2 Preliminaries

We first fix some notation. If \mathcal{H} is a Hilbert space, let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ denote the space of all bounded linear operators on \mathcal{H} . For $1 \leq p < \infty$, let $(\mathcal{S}_p(\mathcal{H}), \|\cdot\|_p)$ denote the Schatten- p space of operators. In particular, $(\mathcal{S}_2(\mathcal{H}), \|\cdot\|_2)$ denotes the space of Hilbert-Schmidt operators on \mathcal{H} and $(\mathcal{S}_1(\mathcal{H}), \|\cdot\|_1)$ denotes the space of trace-class operators on \mathcal{H} . Let $\langle \cdot, \cdot \rangle_{\mathcal{S}_2(\mathcal{H})}$ denote the inner product in $\mathcal{S}_2(\mathcal{H})$. If L is a linear operator which is not necessarily bounded, then $D(L)$ will denote the domain of L .

We adopt the convention that **functional** will always mean bounded linear functional. Usually the functionals that we will consider will be faithful, positive, and normal, so this convention will help us to cut down the number of adjectives.

We would like to recall the Schwarz inequality and define the Schwarz maps. The classical Cauchy-Schwarz inequality states that $|\langle y, x \rangle| \leq \|y\| \|x\|$ for all vectors x, y in a Hilbert space. This inequality is extended to $|\phi(y^*x)| \leq \sqrt{\phi(y^*y)} \sqrt{\phi(x^*x)}$ for all x, y in a C^* -algebra \mathcal{A} , where ϕ is a positive functional on \mathcal{A} (see [33, Theorem 4.3.1]). The last inequality can be further extended to $(T(y^*x))^* T(y^*x) \leq \|T(y^*y)\| T(x^*x)$ if T is a completely positive map from a C^* -algebra \mathcal{A} to the C^* -algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space \mathcal{H} (see [5, Lemma 2.6]). If in the last inequality one assumes that \mathcal{A} is unital and T is unital, then by replacing y by the unit we obtain

$$T(x)^* T(x) \leq T(x^*x) \quad \text{for all } x \in \mathcal{A}. \tag{2}$$

A similar inequality was proved by Choi [11, Corollary 2.8] who proved that if \mathcal{A} is a unital C^* -algebra and T is a 2-positive unital map from \mathcal{A} to \mathcal{A} then $T(x^*)T(x) \leq T(x^*x)$ for all $x \in \mathcal{A}$. Choi calls the last inequality ‘‘Schwarz inequality’’. Similar inequalities appear in [32, Theorem 1] and [40, Theorem 7.4]. Since a positive linear map T on a C^* -algebra \mathcal{A} satisfies $T(x^*) = T(x)^*$ for all $x \in \mathcal{A}$, the last inequality is equivalent to (2). Following [41, page 14], we say that a bounded linear operator T

on a C^* -algebra \mathcal{A} is a **Schwarz map** if Inequality (2) is satisfied. The advantage of Inequality (2), versus the inequality proved by Choi, is that Inequality (2) implies that T is positive. Be warned that Inequality (2) is not homogeneous for T , and therefore by scaling the operator T by a positive constant the above inequality is affected.

Next we recall the definition of invariant functionals and we define the notion of subinvariant positive functionals on a C^* -algebra. If X is a Banach space, $T : X \rightarrow X$ is a bounded linear operator, and ω is a functional on X , then ω is called **invariant for T** if

$$\omega(Tx) = \omega(x) \quad \text{for all } x \in X.$$

If \mathcal{A} is a C^* -algebra, $T : \mathcal{A} \rightarrow \mathcal{A}$ is a positive bounded linear operator, and ω is a positive functional on \mathcal{A} , then we will say that ω is **subinvariant for T** if

$$\omega(Ta) \leq \omega(a) \quad \text{for all } a \in \mathcal{A} \text{ with } a \geq 0.$$

If \mathcal{H} is a Hilbert space, then a functional ω on $\mathcal{B}(\mathcal{H})$ is called **normal** if and only if it is positive and continuous in the w^* topology. This is equivalent to the fact that there exists a unique positive operator $\rho \in \mathcal{S}_1(\mathcal{H})$ such that

$$\omega(x) = \text{Tr}(\rho x) \quad \text{for all } x \in \mathcal{B}(\mathcal{H}) \tag{3}$$

where Tr denotes the trace. The positive functional ω associated to the positive trace-class operator ρ via (3) is denoted by ω_ρ . If ω is a state (i.e. unital positive functional) on $\mathcal{B}(\mathcal{H})$ then ω is normal if and only if the positive trace-class operator ρ which satisfies (3) has trace equal to 1. Note that if \mathcal{H} is a Hilbert space and $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a bounded linear operator, then a normal positive functional ω_ρ (for some positive trace-class operator ρ) is invariant for T if and only if

$$T^\dagger(\rho) = \rho,$$

where T^\dagger denotes the Banach dual operator of T restricted to $\mathcal{S}_1(\mathcal{H})$ (viewed as a subspace of the dual of $\mathcal{B}(\mathcal{H})$). Also, if \mathcal{H} is a Hilbert space and $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a positive bounded linear operator, then a normal positive functional ω_ρ (for some positive trace-class operator ρ) is subinvariant for T if and only if

$$T^\dagger(\rho) \leq \rho.$$

If \mathcal{H} is a Hilbert space, recall that a positive functional ω on $\mathcal{B}(\mathcal{H})$ is **faithful** provided that $\omega(x) > 0$ for all $x > 0$. It is worth noting that $\mathcal{B}(\mathcal{H})$ has a faithful normal functional if and only if \mathcal{H} is separable (see [6, Example 2.5.5]).

3 Constructions Using Faithful, Positive, Normal Functionals

We extensively use the next proposition, so we want to give it along with a proof.

Proposition 1 *Let \mathcal{H} be a Hilbert space and $\rho \in \mathcal{S}_1(\mathcal{H})$ be positive. Then the following are equivalent:*

- (i) *the positive normal functional ω_ρ is faithful,*
- (ii) *the operator ρ is injective,*
- (iii) *the operator ρ has dense range.*

Proof [(i) ⇒ (ii)]. Suppose ω_ρ is faithful. Let h be a nonzero element of \mathcal{H} and P_h be the orthogonal projection onto the span of h . Then P_h is a positive non-zero operator on \mathcal{H} . Hence, since ω_ρ is faithful,

$$0 < \omega_\rho(P_h) = \text{Tr}(\rho P_h) = \text{Tr}(\rho^{1/2} P_h P_h \rho^{1/2}) = \|P_h \rho^{1/2}\|_2^2 = \frac{1}{\|h\|^2} \|\rho^{1/2} h\|^2,$$

and so $\rho^{1/2} h \neq 0$. By using the same argument with h replaced by $\rho^{1/2} h$, we have that $\rho h \neq 0$. Thus, ρ is injective.

[(ii) ⇒ (iii)]. Assume that ρ does not have dense range and let P be the orthogonal projection to $\text{Range}(\rho)^\perp$. Then P is a positive non-zero operator on \mathcal{H} , and so $\omega_\rho(P) > 0$. However, $P\rho = 0$, and so

$$\omega_\rho(P) = \text{Tr}(\rho P) = \text{Tr}(P\rho) = \text{Tr}(0) = 0$$

which is a contradiction. Thus, ρ has dense range.

[(iii) ⇒ (i)]. Let $A \in \mathcal{B}(\mathcal{H})$ and suppose $\omega_\rho(A^*A) = 0$. Then

$$0 = \omega_\rho(A^*A) = \text{Tr}(\rho A^*A) = \text{Tr}(\rho^{1/2} A^* A \rho^{1/2}) = \|A \rho^{1/2}\|_2^2. \tag{4}$$

Hence $A \rho^{1/2} = 0$, and therefore $A\rho = 0$. Since ρ has dense range, this implies $A = 0$. Thus, ω_ρ is faithful.

[(ii) ⇒ (i)]. Assume that ρ is injective and let $A \in \mathcal{B}(\mathcal{H})$ such that $\omega_\rho(A^*A) = 0$. Equation (4) implies that $A \rho^{1/2} = 0$ and hence $\rho^{1/2} A^* = 0$, thus $\rho A^* = 0$. This implies $\rho A^* x = 0$ for any $x \in \mathcal{H}$, and since ρ is injective we have that $A^* x = 0$ for all $x \in \mathcal{H}$, and so $A = 0$. Thus, ρ is faithful. □

Remark 1 Note that in the proof of [(i) ⇒ (ii)] of the above proposition, we proved that (i) implies that $\rho^{1/2}$ is injective. Since $\rho^{1/2} = \rho^{1/4} \rho^{1/4}$ we immediately obtain that $\rho^{1/4}$ is injective. Since $\rho^{3/4} = \rho^{1/2} \rho^{1/4}$ we obtain that $\rho^{3/4}$ is injective as it is a composition of two injective maps. Further, since an operator is injective if and only if its adjoint has dense range, and $\rho^{1/4}$, $\rho^{1/2}$, and $\rho^{3/4}$ are self-adjoint, we have that $\rho^{1/4}$, $\rho^{1/2}$, and $\rho^{3/4}$ have dense range.

3.1 Inducing Maps on $S_2(\mathcal{H})$

Let \mathcal{H} be a Hilbert space and fix $\rho \in S_1(\mathcal{H})$ which is positive. Define

$$i_\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \quad \text{by} \quad i_\rho(x) = \rho^{1/4} x \rho^{1/4}.$$

The next proposition summarizes the properties of the map i_ρ . It is useful to first recall that for any Hilbert space \mathcal{H} the following set inclusions hold:

$$S_1(\mathcal{H}) \subseteq S_2(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}).$$

Proposition 2 *Let $\rho \in S_1(\mathcal{H})$ be positive such that ω_ρ is a faithful positive functional. Then the following statements are valid:*

- (a) *The map i_ρ is injective.*
- (b) *The map i_ρ preserves positivity.*

- (c) The restriction $i_\rho|_{\mathcal{S}_2(\mathcal{H})}$ of i_ρ to $\mathcal{S}_2(\mathcal{H})$ is a contraction from $\mathcal{S}_2(\mathcal{H})$ into $\mathcal{S}_1(\mathcal{H})$.
- (d) The map i_ρ is a contraction from $\mathcal{B}(\mathcal{H})$ onto a dense subset of $\mathcal{S}_2(\mathcal{H})$.

Proof To prove (a), let $x \in \mathcal{B}(\mathcal{H})$ and suppose $i_\rho(x) = 0$. By Remark 1 we have that $\rho^{1/4}$ is injective. Therefore, since $\rho^{1/4}x\rho^{1/4} = 0$, we obtain that $x\rho^{1/4} = 0$. Further, since $\rho^{1/4}$ has dense range, (by Remark 1 again), we obtain that $x = 0$. Thus i_ρ is injective.

To prove (b), let $x \in \mathcal{B}(\mathcal{H})$ where $x \geq 0$. Let $h \in \mathcal{H}$. Then

$$\langle h, i_\rho(x)h \rangle = \langle h, \rho^{1/4}x\rho^{1/4}h \rangle = \langle \rho^{1/4}h, x\rho^{1/4}h \rangle \geq 0$$

since $x \geq 0$. Thus i_ρ maps positive operators to positive operators.

To prove (c), first note that for $p, q, r \geq 1$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and for $x \in \mathcal{S}_p(\mathcal{H})$, $y \in \mathcal{S}_q(\mathcal{H})$, and $z \in \mathcal{S}_r(\mathcal{H})$, two applications of Holder’s inequality give that $\|xyz\|_1 \leq \|x\|_p \|y\|_q \|z\|_r$. From this we obtain that for $y \in \mathcal{S}_2(\mathcal{H})$ with $\|y\|_2 \leq 1$ we have

$$\|i_\rho(y)\|_1 = \|\rho^{1/4}y\rho^{1/4}\|_1 \leq \|\rho^{1/4}\|_4 \|y\|_2 \|\rho^{1/4}\|_4 = \|\rho\|_1^{1/4} \|y\|_2 \|\rho\|_1^{1/4} \leq \|\rho\|_1^{1/4}.$$

To prove (d), first notice that $i_\rho(x) \in \mathcal{S}_2(\mathcal{H})$ for all $x \in \mathcal{B}(\mathcal{H})$ since

$$\begin{aligned} \|i_\rho(x)\|_2^2 &= \|\rho^{1/4}x\rho^{1/4}\|_2^2 = \text{Tr}(\rho^{1/2}x^*\rho^{1/2}x) \\ &\leq \|\rho^{1/2}x^*\rho^{1/2}x\|_1 \leq \|\rho^{1/2}x^*\|_2 \|\rho^{1/2}x\|_2 \\ &= \text{Tr}((\rho^{1/2}x^*)^*(\rho^{1/2}x^*))^{1/2} \text{Tr}((\rho^{1/2}x)^*(\rho^{1/2}x))^{1/2} \\ &= \text{Tr}(x\rho x^*)^{1/2} \text{Tr}(x^*\rho x)^{1/2} < \infty. \end{aligned}$$

Let $y \in \mathcal{S}_2(\mathcal{H})$ such that $y \perp i_\rho(x)$ for all $x \in \mathcal{B}(\mathcal{H})$, (where the orthogonality is taken with respect to the Hilbert-Schmidt inner product). Then, for all $x \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} 0 &= \langle i_\rho(x), y \rangle_{\mathcal{S}_2(\mathcal{H})} = \langle \rho^{1/4}x\rho^{1/4}, y \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \text{Tr}(\rho^{1/4}x^*\rho^{1/4}y) = \langle x, \rho^{1/4}y\rho^{1/4} \rangle_{\mathcal{S}_2(\mathcal{H})}. \end{aligned}$$

Therefore $\rho^{1/4}y\rho^{1/4} = 0$. Since $\rho^{1/4}$ is injective, we have that $y\rho^{1/4} = 0$ and, since $\rho^{1/4}$ has dense range, we have that $y = 0$. Therefore i_ρ has dense range.

To see that $\|i_\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H})\| \leq 1$, let $x \in \mathcal{B}(\mathcal{H})$ and notice that

$$\begin{aligned} \|i_\rho(x)\|_2 &= \sup_{\substack{y \in \mathcal{S}_2(\mathcal{H}) \\ \|y\|_2 \leq 1}} |\langle i_\rho(x), y \rangle_{\mathcal{S}_2(\mathcal{H})}| = \sup_{\substack{y \in \mathcal{S}_2(\mathcal{H}) \\ \|y\|_2 \leq 1}} |\text{Tr}(i_\rho(x)^*y)| \\ &= \sup_{\substack{y \in \mathcal{S}_2(\mathcal{H}) \\ \|y\|_2 \leq 1}} |\text{Tr}(\rho^{1/4}y\rho^{1/4}x^*)| \leq \sup_{\substack{y \in \mathcal{S}_2(\mathcal{H}) \\ \|y\|_2 \leq 1}} \|\rho^{1/4}y\rho^{1/4}\|_1 \|x\|_\infty \\ &= \|i_\rho|_{\mathcal{S}_2(\mathcal{H})} : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_1(\mathcal{H})\| \|x\|_\infty \leq \|x\|_\infty, \end{aligned}$$

where we used part (c) for the last inequality. □

Definition 1 Let \mathcal{H} be a Hilbert space and $\rho \in \mathcal{S}_1(\mathcal{H})$ be a positive operator. If $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a bounded linear operator, we define the operator $\tilde{T} : i_\rho(\mathcal{B}(\mathcal{H})) \rightarrow i_\rho(\mathcal{B}(\mathcal{H}))$ by

$$\tilde{T}(\rho^{1/4}x\rho^{1/4}) = \rho^{1/4}T(x)\rho^{1/4} \quad \text{for all } x \in \mathcal{B}(\mathcal{H}).$$

Note that \tilde{T} depends on ρ but, for simplicity, we chose notation which does not reflect this dependence.

The following theorem was first proven in [7]. For the convenience of the reader we provide a proof of it here.

Theorem 1 Suppose \mathcal{H} is a Hilbert space and $\rho \in \mathcal{S}_1(\mathcal{H})$ be a positive operator such that ω_ρ is a faithful positive functional on $\mathcal{B}(\mathcal{H})$. Let $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a bounded linear operator which is a Schwarz map such that ω_ρ is a subinvariant functional for T . Then the corresponding operator \tilde{T} can be extended to all of $\mathcal{S}_2(\mathcal{H})$ as a contraction from $\mathcal{S}_2(\mathcal{H})$ to $\mathcal{S}_2(\mathcal{H})$.

Proof Since ω_ρ is a faithful normal functional on $\mathcal{B}(\mathcal{H})$, we have that \mathcal{H} must be separable (see the comment above Proposition 1), so let $(e_k)_{k \geq 0}$ be an orthonormal basis for \mathcal{H} which diagonalizes ρ and let $P_n = \sum_{k=0}^n |e_k\rangle\langle e_k|$. Note that ρ and its positive powers commute with each P_n . Define the linear subspace $\mathcal{A} = \{x\rho^{1/2} : x \in \mathcal{B}(\mathcal{H})\}$ and the map $\hat{T} : \mathcal{A} \rightarrow \mathcal{A}$ by $\hat{T}(x\rho^{1/2}) = T(x)\rho^{1/2}$. Further, for $n \in \mathbb{N}$, define the map $\Delta_n : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H})$ by

$$\Delta_n(x) = P_n\rho^{1/2}x\rho^{-1/2}P_n \quad \text{for all } x \in \mathcal{S}_2(\mathcal{H})$$

(note that $\rho^{1/2}$ is not invertible but $\rho^{-1/2}P_n$ is a bounded operator). Then, for any $x \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} \|\tilde{T}(i_\rho(x))\|_2^2 &= \|\rho^{1/4}T(x)\rho^{1/4}\|_2^2 = \text{Tr}\left(\rho^{1/4}T(x)^*\rho^{1/2}T(x)\rho^{1/4}\right) \\ &= \lim_{n \rightarrow \infty} \text{Tr}\left(\rho^{1/2}T(x)^*P_n\rho^{1/2}T(x)\rho^{1/2}\rho^{-1/2}P_n\right) \\ &= \lim_{n \rightarrow \infty} \left\langle T(x)\rho^{1/2}, \Delta_n(T(x)\rho^{1/2}) \right\rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \lim_{n \rightarrow \infty} \left\langle \hat{T}(x\rho^{1/2}), \Delta_n\hat{T}(x\rho^{1/2}) \right\rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \lim_{n \rightarrow \infty} \left\langle x\rho^{1/2}, \hat{T}^*\Delta_n\hat{T}(x\rho^{1/2}) \right\rangle_{\mathcal{S}_2(\mathcal{H})} \end{aligned} \tag{5}$$

where we will see later on why \hat{T}^* is well-defined.

Define $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ by $\Delta(x\rho^{1/2}) = \rho^{1/2}x$, which is well-defined since $\rho^{1/2}$ has dense range (hence, for $x, y \in \mathcal{B}(\mathcal{H})$, $x\rho^{1/2} = y\rho^{1/2}$ implies $x = y$). Let $\mathcal{B} = \{x\rho : x \in \mathcal{B}(\mathcal{H})\}$. We make the following three claims:

- (i) \hat{T} is a contraction on \mathcal{A} . Therefore \hat{T} can be extended to a contraction on $\mathcal{S}_2(\mathcal{H})$ since \mathcal{A} is dense in $\mathcal{S}_2(\mathcal{H})$.

(ii) Δ_n^2 is positive. Therefore, by [37, Lemma 1.2], we have

$$\widehat{T}^* \Delta_n \widehat{T} \leq (\widehat{T}^* \Delta_n^2 \widehat{T})^{1/2}. \tag{6}$$

(iii) $\widehat{T}^* \Delta_n^2 \widehat{T} \leq \Delta^2$ on \mathcal{B} . Thus,

$$(\widehat{T}^* \Delta_n^2 \widehat{T})^{1/2} \leq (\Delta^2)^{1/2} = \Delta. \tag{7}$$

Hence, by combining (6) and (7), we obtain $\widehat{T}^* \Delta_n \widehat{T} \leq \Delta$ on \mathcal{B} .

Assume for the moment that the above claims (i), (ii), and (iii) are true. By replacing x by $x\rho^{1/2}$, in (5) we obtain that

$$\begin{aligned} \|\widetilde{T}(i_\rho(x\rho^{1/2}))\|_2^2 &= \lim_{n \rightarrow \infty} \langle x\rho, \widehat{T}^* \Delta_n \widehat{T}(x\rho) \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &\leq \langle x\rho, \Delta(x\rho) \rangle_{\mathcal{S}_2(\mathcal{H})} = \langle x\rho, \rho^{1/2}x\rho^{1/2} \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \text{Tr}(\rho x^* \rho^{1/2} x \rho^{1/2}) = \text{Tr}(\rho^{3/4} x^* \rho^{1/4} \rho^{1/4} x \rho^{3/4}) \\ &= \langle \rho^{1/4} x \rho^{3/4}, \rho^{1/4} x \rho^{3/4} \rangle_{\mathcal{S}_2(\mathcal{H})} = \|i_\rho(x\rho^{1/2})\|_2^2 \end{aligned}$$

and so \widetilde{T} is a contraction on $i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$. We now show that $i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$ is dense in $\mathcal{S}_2(\mathcal{H})$. Let $y \in \mathcal{S}_2(\mathcal{H})$ such that $y \perp i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$. Then, for any $x \in \mathcal{B}(\mathcal{H})$ we have that

$$\begin{aligned} 0 &= \langle i_\rho(x\rho^{1/2}), y \rangle_{\mathcal{S}_2(\mathcal{H})} = \text{Tr}(i_\rho(x\rho^{1/2})^* y) \\ &= \text{Tr}(\rho^{1/4} \rho^{1/2} x^* \rho^{1/4} y) = \langle x, \rho^{1/4} y \rho^{3/4} \rangle_{\mathcal{S}_2(\mathcal{H})} \end{aligned}$$

and hence $\rho^{1/4} y \rho^{3/4} = 0$. Since $\rho^{1/4}$ is injective, we then have that $y \rho^{3/4} = 0$ and, since $\rho^{3/4}$ has dense range, we obtain that $y = 0$. Therefore $i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$ is dense in $\mathcal{S}_2(\mathcal{H})$. Since \widetilde{T} is a contraction on $i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$, we can extend it to a contraction on $\mathcal{S}_2(\mathcal{H})$. This finishes the proof of the theorem pending verification of claims (i), (ii), and (iii), as well as the fact that \widehat{T}^* is well-defined.

First, we prove claim (i), i.e., that \widehat{T} is a contraction on \mathcal{A} . Let $x \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned} \|\widehat{T}(x\rho^{1/2})\|_2^2 &= \|T(x)\rho^{1/2}\|_2^2 = \langle T(x)\rho^{1/2}, T(x)\rho^{1/2} \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \text{Tr}(\rho^{1/2} T(x)^* T(x) \rho^{1/2}) \leq \text{Tr}(\rho^{1/2} T(x^* x) \rho^{1/2}) \end{aligned}$$

since T is a Schwarz map. Further,

$$\begin{aligned} \text{Tr}(\rho^{1/2} T(x^* x) \rho^{1/2}) &= \text{Tr}(\rho T(x^* x)) \leq \text{Tr}(\rho x^* x) = \text{Tr}(\rho^{1/2} x^* x \rho^{1/2}) \\ &= \langle x\rho^{1/2}, x\rho^{1/2} \rangle_{\mathcal{S}_2(\mathcal{H})} = \|x\rho^{1/2}\|_2^2. \end{aligned}$$

Therefore $\|\widehat{T}(x\rho^{1/2})\|_2^2 \leq \|x\rho^{1/2}\|_2^2$, and so \widehat{T} is a contraction on \mathcal{A} . Hence, \widehat{T} can be extended to a contraction on $\mathcal{S}_2(\mathcal{H})$ since \mathcal{A} is dense in $\mathcal{S}_2(\mathcal{H})$ (this also shows that \widehat{T}^* is well-defined).

For claim (ii), i.e., that Δ_n^2 is positive, first note that since ρ commutes with P_n we have

$$\Delta_n^2 x = P_n \rho x P_n \rho^{-1} P_n \quad \text{for all } x \in \mathcal{S}_2(\mathcal{H})$$

(note that ρ is not invertible but $\rho^{-1}P_n$ is a bounded operator). Indeed, if $x \in \mathcal{S}_2(\mathcal{H})$ then

$$\begin{aligned} \langle x, \Delta_n^2 x \rangle_{\mathcal{S}_2(\mathcal{H})} &= \langle x, P_n \rho x P_n \rho^{-1} P_n \rangle_{\mathcal{S}_2(\mathcal{H})} = \text{Tr} \left(x^* P_n \rho x P_n \rho^{-1} P_n \right) \\ &= \text{Tr} \left(\rho^{1/2} P_n x P_n \rho^{-1/2} P_n P_n \rho^{-1/2} P_n x^* P_n \rho^{1/2} \right) \\ &= \text{Tr} \left((\rho^{1/2} P_n x P_n \rho^{-1/2} P_n) (\rho^{1/2} P_n x P_n \rho^{-1/2} P_n)^* \right) \geq 0, \end{aligned}$$

and so Δ_n^2 is positive. By [37, Lemma 1.2], we then have that

$$\widehat{T}^* \Delta_n \widehat{T} \leq (\widehat{T}^* \Delta_n^2 \widehat{T})^{1/2}. \tag{8}$$

It is left to prove claim (iii), i.e., that $\widehat{T}^* \Delta_n^2 \widehat{T} \leq \Delta^2$ on \mathcal{B} . Indeed,

$$\begin{aligned} \langle x\rho, \widehat{T}^* \Delta_n^2 \widehat{T}(x\rho) \rangle_{\mathcal{S}_2(\mathcal{H})} &= \langle T(x\rho^{1/2})\rho^{1/2}, \Delta_n^2 T(x\rho^{1/2})\rho^{1/2} \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \langle T(x\rho^{1/2})\rho^{1/2}, P_n \rho T(x\rho^{1/2})\rho^{1/2} P_n \rho^{-1} P_n \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \text{Tr} \left(\rho^{1/2} T(x\rho^{1/2})^* P_n \rho T(x\rho^{1/2})\rho^{1/2} P_n \rho^{-1} P_n \right) \\ &= \text{Tr} \left(\rho T(x\rho^{1/2}) P_n T(x\rho^{1/2})^* P_n \right) \\ &\leq \text{Tr} \left(\rho T(x\rho^{1/2}) T(x\rho^{1/2})^* \right) \quad (\text{see below}) \tag{9} \\ &\leq \text{Tr} \left(\rho T \left((x\rho^{1/2})(x\rho^{1/2})^* \right) \right) \quad (T \text{ is a Schwarz map}) \\ &\leq \text{Tr} \left(\rho (x\rho^{1/2})(x\rho^{1/2})^* \right) \quad (\omega_\rho \text{ is subinvariant for } T) \\ &= \text{Tr} \left(\rho x \rho x^* \right) = \text{Tr} \left(\rho x (x\rho)^* \right) = \text{Tr} \left((x\rho)^* \rho x \right) \\ &= \text{Tr} \left((x\rho)^* \Delta^2(x\rho) \right) \quad (\text{since } \Delta^2(x\rho) = \rho x) \\ &= \langle x\rho, \Delta^2(x\rho) \rangle_{\mathcal{S}_2(\mathcal{H})}. \end{aligned}$$

This completes the proof as long as we justify the inequality (9). Indeed, we have that the inequality $\text{Tr}(P_n A^* P_n A) \leq \text{Tr}(A^* A)$ holds in general for any $A \in \mathcal{S}_2(\mathcal{H})$, since if $(e_k)_{k \geq 1}$ is the orthonormal basis of \mathcal{H} used to define each P_n , then

$$\begin{aligned} \text{Tr}(P_n A^* P_n A) &= \sum_{k=1}^{\infty} \langle e_k, P_n A^* P_n^2 A e_k \rangle \\ &= \sum_{k=1}^{\infty} \langle P_n A P_n e_k, P_n A e_k \rangle = \sum_{k=1}^n \langle P_n A e_k, P_n A e_k \rangle, \end{aligned}$$

and further

$$\begin{aligned} \sum_{k=1}^n \langle P_n A e_k, P_n A e_k \rangle &= \sum_{k=1}^n \|P_n A e_k\|^2 \leq \sum_{k=1}^n \|P_n\|^2 \|A e_k\|^2 \\ &\leq \sum_{k=1}^{\infty} \|A e_k\|^2 = \text{Tr}(A^* A). \end{aligned}$$

□

3.2 An Alternate Construction

There is another situation where a bounded operator on a C^* -algebra gives rise to a corresponding operator on a Hilbert space, and we would like to mention this in the next remark.

Remark 2 Let \mathcal{A} be a unital C^* -algebra and ω be a faithful state on \mathcal{A} . Consider the GNS construction of \mathcal{A} associated with ω . Let \mathcal{K} be the Hilbert space associated with the GNS construction, $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$ be the $*$ -representation of \mathcal{A} into the C^* -algebra of all bounded operators on \mathcal{K} , and Ω denote the cyclic element of the Hilbert space \mathcal{K} for the representation π , (i.e. the subspace $\{\pi(a)(\Omega) : a \in \mathcal{A}\}$ is norm dense in \mathcal{K}) which is equal to the unit of \mathcal{A} viewed as an element of \mathcal{K} . Let T be a bounded operator on \mathcal{A} which is a Schwarz map. Assume that ω is subinvariant for T . Define an operator \bar{T} on the dense subspace $\{\pi(a)(\Omega) : a \in \mathcal{A}\}$ of \mathcal{K} with values in \mathcal{K} by

$$\bar{T}(\pi(a)(\Omega)) = \pi(T(a))(\Omega) \quad \text{for all } a \in \mathcal{A}.$$

Then \bar{T} is a contraction (hence it extends to \mathcal{K}).

Proof Since ω is faithful, the quotient that is usually associated with the GNS construction does not take place, and the elements of \mathcal{A} belong to \mathcal{K} . Let $\langle \cdot, \cdot \rangle_\omega$ denote the inner product in \mathcal{K} and $\| \cdot \|_\omega$ denote the norm of \mathcal{K} . Then since ω is faithful, we have that for $a, b \in \mathcal{A}$, $\langle a, b \rangle_\omega = \omega(a^*b)$ and hence $\|(\pi(a))(\Omega)\|_\omega^2 = \omega(a^*a)$.

For every $a \in \mathcal{A}$ we have

$$\begin{aligned} \|\bar{T}(\pi(a)(\Omega))\|_\omega^2 &= \|\pi(T(a))(\Omega)\|_\omega^2 = \omega(T(a)^*T(a)) \leq \omega(T(a^*a)) \\ &\quad (\text{since } \omega \text{ is positive and } T \text{ is a Schwarz map}) \\ &\leq \omega(a^*a) \quad (T \geq 0 \text{ is a Schwarz map; } \omega \text{ is subinvariant for } T) \\ &= \|\pi(a)(\Omega)\|_\omega^2, \end{aligned}$$

which finishes the proof. □

Notice the similarities between Theorem 1 and Remark 2. Both refer to a bounded operator on some C^* -algebra where a positive linear functional is fixed, and they each conclude the existence of an associated contraction on some Hilbert space. But there are three key differences between Theorem 1 and Remark 2. First, Theorem 1 refers to an operator on $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} (which is necessarily separable since $\mathcal{B}(\mathcal{H})$ is assumed to admit a faithful normal state), while Remark 2 assumes

that the operator is defined on a general C^* -algebra. Second, the state ω_ρ which is mentioned in Theorem 1 is normal since it is defined via the trace-class operator ρ , while there is no such assumption in Remark 2 (the normality of the state ω in Remark 2 does not make sense in general since \mathcal{A} is simply assumed to be a C^* -algebra and not a von Neumann algebra as it is assumed in [35, Equation (2.1)]). Third, the Hilbert space that is used in Theorem 1 is the space $\mathcal{S}_2(\mathcal{H})$ which does not depend on the positive linear functional, while the map i_ρ which maps $\mathcal{B}(\mathcal{H})$ to $\mathcal{S}_2(\mathcal{H})$, does depend on the positive linear functional. On the other hand, the Hilbert space that is used in Remark 2 (i.e. the GNS construction associated to the faithful state ω of the C^* -algebra \mathcal{A}) depends on the state, while the $*$ -representation π of the von Neumann algebra which is associated with the GNS construction does not depend on the state. Notice also that the combinations of the Hilbert spaces with the representations in Theorem 1 and Remark 2 are very similar. More precisely, for $a, b \in \mathcal{B}(\mathcal{H})$ we have that $i_\rho(a), i_\rho(b) \in \mathcal{S}_2(\mathcal{H})$ hence

$$\begin{aligned} \langle i_\rho(a), i_\rho(b) \rangle_{\mathcal{S}_2(\mathcal{H})} &= \text{Tr}(i_\rho(a)^* i_\rho(b)) \\ &= \text{Tr}(\rho^{1/4} a^* \rho^{1/4} \rho^{1/4} b \rho^{1/4}) = \text{Tr}(a^* \rho^{1/2} b \rho^{1/2}). \end{aligned}$$

On the other hand, if we assume for the moment that the C^* -algebra \mathcal{A} that appears in Remark 2 is equal to $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} , and the faithful state ω on the C^* -algebra \mathcal{A} is given by $\omega(a) = \text{Tr}(\rho a)$ for some positive trace-class operator ρ on \mathcal{H} , then the inner product of two elements $a, b \in \mathcal{A}$ via the GNS construction is given by

$$\langle a, b \rangle_\omega = \omega(a^* b) = \text{Tr}(\rho a^* b).$$

Thus the combination of the inner product with the representation that is used in Theorem 1 is slightly more “symmetric” than the combination of the inner product with the representation that is used in Remark 2. The reader of course will notice the difference between the complexity of the proof of Theorem 1 and that of Remark 2. The extra intricacies in the proof of Theorem 1 is the price we pay in order to achieve the extra symmetry in the combination of the inner product and the representation as discussed above.

Remark 3 The assumption that “ ω is subinvariant for T ” cannot be omitted in Remark 2.

An example where ω is not a subinvariant functional for T but all the other assumptions of Remark 2 are valid is presented in Remark 11.7 of [43].

Remark 4 Note that if \mathcal{H} is a Hilbert space, $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a bounded positive linear operator, and ω is a subinvariant positive faithful functional for T , then $\omega/\omega(1)$ is a subinvariant faithful state for T (here 1 denotes the identity operator on \mathcal{H}). Thus, instead of assuming the existence of subinvariant positive faithful functionals, we henceforth simply assume the existence of subinvariant faithful states. Our subsequent results thus remain valid if the assumptions of the existence of subinvariant faithful states are replaced by the assumptions of the existence of subinvariant positive faithful functionals.

4 Semigroups of Schwarz Maps

We first recall some basic definitions about semigroups.

Definition 2 Let X be a Banach space. A one-parameter family $(T_t)_{t \geq 0}$ of bounded operators on X is a **semigroup** on X if $T_{t+s} = T_t T_s$ for all $t, s \geq 0$, and $T_0 = I$ where I is the identity operator on X . We say the semigroup $(T_t)_{t \geq 0}$ on a Banach space X is

- **uniformly continuous** if the map $t \mapsto T_t$ is continuous with respect to the operator norm.
- **strongly continuous** if for all $x \in X$ the map $t \mapsto T_t x$ is continuous with respect to the norm on X .
- **weakly continuous** if for all $x \in X$ and all $x^* \in X^*$ the map $t \mapsto x^*(T_t x)$ is continuous.
- **weak* continuous** if X is a dual Banach space $X = Y^*$ and for all $y \in Y$ and $x \in X$ the map $t \mapsto (T_t(x))(y)$ is continuous.

If \mathcal{H} is a Hilbert space and $X = \mathcal{B}(\mathcal{H})$ then the semigroup $(T_t)_{t \geq 0}$ on the Banach space X is **WOT continuous** (where this acronym stands as usually for the weak operator topology) if for all $h_1, h_2 \in \mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$ we have that the map $t \mapsto \langle h_1, T_t(x)h_2 \rangle$ is continuous.

It can be shown that a semigroup on a Banach space is strongly continuous if and only if it is weakly continuous (see [4, Thm. 3.31]). If $(T_t)_{t \geq 0}$ is a uniformly continuous semigroup on a Banach space X then its **generator** is defined as the operator norm limit

$$L = \lim_{t \rightarrow 0} \frac{T_t - I}{t}.$$

This limit exists and it defines a bounded operator on X . If we do not assume the uniform continuity of the semigroup, then the definition of the generator is given next:

Definition 3 Let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup (resp. weakly continuous, resp. weak* continuous), on a Banach space X (of course, when we assume that the semigroup is weak* continuous we assume that X is a dual Banach space). We say an element $x \in X$ belongs to the **domain** $D(L)$ of the generator L of $(T_t)_{t \geq 0}$, if

$$\lim_{t \rightarrow 0} \frac{T_t(x) - x}{t} \tag{10}$$

converges in norm (resp. weakly, resp. weak*) and, in this case, define the **generator** to be the generally unbounded operator L such that

$$L(x) = \lim_{t \rightarrow 0} \frac{T_t(x) - x}{t} \quad \text{for all } x \in D(L) \tag{11}$$

where the last limit is taken in the norm (resp. weak, resp. weak*) topology of X .

Since a semigroup on a Banach space is strongly continuous if and only if it is weakly continuous, it is natural to ask whether the limits (10) and (11) can be replaced by weak limits and end up with the same $D(L)$ and L . It turns out that this is indeed the case (see [4, Proposition 3.36]). We will make use of this fact in the proof of Theorem 2.

4.1 The Extended Generator $L_{(h_n)}$ of $(T_t)_{t \geq 0}$

We now wish to extend the definition of the generator to include some cases where the limit (11) does not exist. We first require the following notation:

Definition 4 Let \mathcal{H} be a Hilbert space and $(h_n)_{n \in N}$ be an (countable or uncountable) orthonormal basis of \mathcal{H} . We let $M_N^{(h_n)}$ denote the set of all complex $N \times N$ matrices with rows and columns indexed by N . We view a matrix $L \in M_N^{(h_n)}$ as a linear map $L : D(L) \rightarrow \mathbb{C}^N$ acting on \mathcal{H} as follows: denote $L = (L_{n,m})_{n,m \in N}$, and define $D(L) \subset \mathcal{H}$ as the set of all vectors $h = \sum_{m \in N} \langle h_m, h \rangle h_m \in \mathcal{H}$ such that the series $\sum_{m \in N} L_{n,m} \langle h_m, h \rangle$ converges for all $n \in N$. Then

$$L(h) = \left(\sum_{m \in N} L_{n,m} \langle h_m, h \rangle \right)_{n \in N} .$$

This is in particular the natural matrix multiplication of L against h written as a column vector.

The following definition is given as the minimal requirements for the outputs of L to be considered as linear maps in the sense given above for a fixed orthonormal basis $(h_n)_{n \in N}$ of \mathcal{H} .

Definition 5 Let \mathcal{H} be a Hilbert space and $(h_n)_{n \in N}$ be a (countable or uncountable) orthonormal basis of \mathcal{H} . Let $(T_t)_{t \geq 0}$ be a semigroup of bounded operators on $\mathcal{B}(\mathcal{H})$. To define the extended generator $L_{(h_n)}$ of $(T_t)_{t \geq 0}$ with respect to the basis $(h_n)_{n \in N}$ we first define its domain as the linear subspace of all $x \in \mathcal{B}(\mathcal{H})$ such that the function

$$[0, \infty) \ni t \mapsto \langle h_n, T_t(x)h_m \rangle$$

is differentiable at 0 for every $n, m \in N$; that is, $D(L_{(h_n)})$ is the linear subspace of all $x \in \mathcal{B}(\mathcal{H})$ such that the limit

$$\lim_{t \rightarrow 0} \langle h_n, \frac{T_t(x) - x}{t} h_m \rangle$$

exists for every $n, m \in N$. In general $D(L_{(h_n)})$ can be the zero subspace, but if the semigroup is WOT continuous then $D(L_{(h_n)})$ is WOT dense in $\mathcal{B}(\mathcal{H})$. Define the **extended generator $L_{(h_n)}$ of $(T_t)_{t \geq 0}$ (with respect to the orthonormal basis**

$(h_n)_{n \in \mathbb{N}}$) to be the map with domain $D(L_{(h_n)})$ whose range elements are matrices $L_{(h_i)}(x) \in M_N^{(h_n)}$ with entries given by

$$[L_{(h_i)}(x)]_{n,m} = \lim_{t \rightarrow 0} \left\langle h_n, \frac{T_t(x) - x}{t} h_m \right\rangle.$$

Next we want to compare the generator of a semigroup on $\mathcal{B}(\mathcal{H})$ with respect to an orthonormal basis of \mathcal{H} to the usual generator. Since the definition of the generator depends on the continuity of the semigroup, in the next remark we will consider a weak* continuous semigroup on $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} . The reason that we choose the weak* continuity versus any other continuity assumption is because it is the weakest and the most natural among all continuity assumptions that appear in Definition 2.

Remark 5 Let \mathcal{H} be a Hilbert space, $(T_t)_{t \geq 0}$ be a weak* continuous semigroup of bounded operators on $\mathcal{B}(\mathcal{H})$, and let L denote its generator. Let $(h_n)_{n \in \mathbb{N}}$ be a (countable or uncountable) orthonormal basis of \mathcal{H} , and let $L_{(h_n)}$ denote the generator of $(T_t)_{t \geq 0}$ with respect to $(h_n)_{n \in \mathbb{N}}$. Then $D(L) \subseteq D(L_{(h_n)})$, and for every $x \in D(L)$ we have $L(x) = L_{(h_n)}(x)$, by which we mean the matrix of $L(x)$ with respect to $(h_n)_{n \in \mathbb{N}}$ and the matrix $L_{(h_n)}(x)$ are equal.

Indeed, for fixed $x \in D(L)$ and every $h, h' \in \mathcal{H}$ we have that

$$\left\langle h, \frac{T_t(x) - x}{t} h' \right\rangle \rightarrow \langle h, L(x)h' \rangle \quad \text{as } t \rightarrow 0. \tag{12}$$

In particular,

$$\lim_{t \rightarrow 0} \left\langle h_n, \frac{T_t(x) - x}{t} h_m \right\rangle = \langle h_n, L(x)h_m \rangle$$

for every $n, m \in N$. Thus $x \in D(L_{(h_n)})$.

Notation: If N is a nonempty set, then we denote by $\Pi_{\text{fin}}(N)$ the set of all finite subsets of N .

Notation: Let \mathcal{H} and \mathcal{K} be Hilbert spaces with $\mathcal{H} \subseteq \mathcal{K}$ and let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. We shall denote by

$$A = \text{pr}_{\mathcal{H}}(B)$$

the fact that

$$A = P_{\mathcal{H}}B|_{\mathcal{H}}$$

where $|_{\mathcal{H}}$ denotes the restriction to \mathcal{H} and $P_{\mathcal{H}} : \mathcal{K} \rightarrow \mathcal{H}$ denotes the orthogonal projection from \mathcal{K} onto \mathcal{H} . The operator B is called a **dilation** of the operator A and the operator A is called a **compression** of the operator B .

Remark 6 Let \mathcal{H} be a Hilbert space with (countable or uncountable) dimension N , $(h_n)_{n \in \mathbb{N}}$ be an orthonormal basis of \mathcal{H} , $(T_t)_{t \geq 0}$ be a semigroup of bounded

operators on $\mathcal{B}(\mathcal{H})$, and let $L_{(h_n)}$ denote its generator with respect to $(h_n)_{n \in N}$. For $x \in D(L_{(h_n)})$ and $F \in \Pi_{\text{fin}}(N)$ there exists a unique operator

$$L_{(h_n)}(x)_F : \text{Span}(h_n)_{n \in F} \rightarrow \text{Span}(h_n)_{n \in F}$$

satisfying

$$\lim_{t \rightarrow 0} \left\langle h, \frac{T_t(x) - x}{t} h' \right\rangle = \langle h, L_{(h_n)}(x)_F h' \rangle \quad \text{for all } h, h' \in \text{Span}(h_n)_{n \in F}, \quad (13)$$

or equivalently

$$\left\| \text{pr}_{\text{Span}(h_n)_{n \in F}} \left(\frac{T_t(x) - x}{t} \right) - L_{(h_n)}(x)_F \right\|_{\mathcal{B}(\text{Span}(h_n)_{n \in F})} \rightarrow 0 \quad \text{as } t \rightarrow 0. \quad (14)$$

Indeed, fix $F \in \Pi_{\text{fin}}(N)$. From Definition 5, $L_{(h_n)}(x)_F : \text{Span}(x_n)_{n \in F} \rightarrow \text{Span}(x_n)_{n \in F}$ is uniquely defined by

$$[L_{(h_i)}(x)_F](h) = \sum_{n, m \in F} \lim_{t \rightarrow 0} \langle h_n, \frac{T_t(x) - x}{t} h_m \rangle \langle h_m, h \rangle h_n \quad \text{if } h = \sum_{m \in F} \langle h_m, h \rangle h_m. \quad (15)$$

Then (13) is obvious from Definition 5 and (15). The equivalence of (13) and (14) then follows for any finite subset F of N , since all linear Hausdorff topologies on the space of linear operators on $\text{Span}(h_n)_{n \in F}$ are equivalent. Thus the WOT on $\text{Span}(h_n)_{n \in F}$ in (13) can be replaced by the $\mathcal{B}(\text{Span}(h_n)_{n \in F})$ topology.

Remark 7 Let \mathcal{H} be a Hilbert space with (countable or uncountable) dimension N , $(h_n)_{n \in N}$ be an orthonormal basis of \mathcal{H} , $(T_t)_{t \geq 0}$ be a semigroup of bounded operators on $\mathcal{B}(\mathcal{H})$, and let $L_{(h_n)}$ denote its generator with respect to $(h_n)_{n \in N}$. Fix $x \in D(L_{(h_n)})$. Then the family $(L_{(h_n)}(x)_F)_{F \in \Pi_{\text{fin}}(N)}$ is compatible in the following sense: If $G \subset F$ are two finite subsets of N then $\text{pr}_{\text{Span}(h_n)_{n \in G}}(L_{(h_n)}(x)_F) = L_{(h_n)}(x)_G$.

Indeed, this is obvious from (15).

Remark 8 The generator of a semigroup with respect to an orthonormal basis that we defined above is related to the **form generator** which was defined by Davies [13] and was further studied in [3, 9, 10, 20, 29, 30, 39], and [38]. If $(T_t)_{t \geq 0}$ is a weak* continuous semigroup on $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} , then a form generator is the map $\phi : \mathcal{K} \times \mathcal{B}(\mathcal{H}) \times \mathcal{K} \rightarrow \mathbb{C}$ where \mathcal{K} is a dense linear subspace of \mathcal{H} , defined by

$$\phi(h, x, h') = \left\langle h, \lim_{t \rightarrow 0} \frac{T_t(x) - x}{t} h' \right\rangle \quad \text{for every } h, h' \in \mathcal{K} \text{ and every } x \in \mathcal{B}(\mathcal{H}).$$

Note that if $(h_n)_{n \in N}$ is an orthonormal basis of \mathcal{H} and \mathcal{K} denotes the linear span of $(h_n)_{n \in N}$ then the form generator coincides with the generator with respect to $(h_n)_{n \in N}$ if the domain of the generator with respect to $(h_n)_{n \in N}$ is equal to $\mathcal{B}(\mathcal{H})$. Here we assume that the domain of the generator with respect to an orthonormal basis is a linear subspace of $\mathcal{B}(\mathcal{H})$, not necessarily equal to $\mathcal{B}(\mathcal{H})$.

We require a few more definitions in order to state the next result.

Definition 6 Let \mathcal{H} be a Hilbert space, ω be a state on $\mathcal{B}(\mathcal{H})$ and $(T_t)_{t \geq 0}$ be a semigroup of positive operators on $\mathcal{B}(\mathcal{H})$. We say that ω is a subinvariant state for the semigroup $(T_t)_{t \geq 0}$, if and only if ω is subinvariant for T_t for every $t \geq 0$.

Definition 7 The **Moore-Penrose inverse** or **pseudoinverse** $x^{(-1)}$ of $x \in \mathcal{B}(\mathcal{H})$ is defined as the unique linear extension of $(x|_{\mathcal{N}(x)^\perp})^{-1}$, the inverse as a function, to

$$D(x^{(-1)}) := \mathcal{R}(x) + \mathcal{R}(x)^\perp$$

with $\mathcal{N}(x^{(-1)}) = \mathcal{R}(x)^\perp$, where $\mathcal{N}(x)$ and $\mathcal{R}(x)$ denote the nullspace and range of x , respectively. Letting P and Q denote the orthogonal projections onto $\mathcal{N}(x)$ and $\overline{\mathcal{R}(x)}$, respectively, it can be shown (see e.g. [17]) that $x^{(-1)}$ is uniquely determined by the relations

$$x^{(-1)}x = I - P \quad \text{and} \quad xx^{(-1)} = Q|_{D(x^{(-1)})}.$$

Notation By $i_{\rho^{(-1)}}$ we mean the map from $\mathcal{B}(\mathcal{H})$ to the space of linear maps on \mathcal{H} defined via

$$i_{\rho^{(-1)}}(x) = (\rho^{1/4})^{(-1)}x(\rho^{1/4})^{(-1)}.$$

Now we are ready to state the next result.

Theorem 2 Let \mathcal{H} be a Hilbert space, $(T_t)_{t \geq 0}$ be a semigroup of Schwarz maps on $\mathcal{B}(\mathcal{H})$, and let $\rho \in \mathcal{S}_1(\mathcal{H})$ be such that ω_ρ is a faithful state on $\mathcal{B}(\mathcal{H})$ which is subinvariant for the semigroup $(T_t)_{t \geq 0}$. Then there exists a unique semigroup $(\tilde{T}_t)_{t \geq 0}$ of contractions on $\mathcal{S}_2(\mathcal{H})$ such that

$$\tilde{T}_t(i_\rho(x)) = i_\rho(T_t(x)) \quad \text{for all } x \in \mathcal{B}(\mathcal{H}). \tag{16}$$

Moreover, if $(T_t)_{t \geq 0}$ is weak*-continuous then $(\tilde{T}_t)_{t \geq 0}$ is strongly continuous. Let L denote the generator of $(T_t)_{t \geq 0}$, let \tilde{L} denote the generator of $(\tilde{T}_t)_{t \geq 0}$, and let $L_{(h_n)}$ denote the generator of $(T_t)_{t \geq 0}$ with respect to $(h_n)_{n \in \mathbb{N}}$, where $(h_n)_{n \in \mathbb{N}}$ is an orthonormal basis of \mathcal{H} consisting of eigenvectors of ρ . Then $x \in D(L)$ implies $i_\rho(x) \in D(\tilde{L})$, and moreover

$$\tilde{L}(i_\rho(x)) = i_\rho(L(x));$$

conversely, $i_\rho(x) \in D(\tilde{L})$ implies $x \in D(L_{(h_n)})$, and moreover

$$L_{(h_n)}(x) = i_{\rho^{(-1)}}(\tilde{L}(i_\rho(x))). \tag{17}$$

Proof The operators \tilde{T}_t are well-defined by Theorem 1. Uniqueness comes from (16) and the fact that $i_\rho(\mathcal{B}(\mathcal{H}))$ is dense in $\mathcal{S}_2(\mathcal{H})$. It is easy to see that $\tilde{T}_{t+s} = \tilde{T}_t\tilde{T}_s$ and that $\tilde{T}_0 = 1$ on $i_\rho(\mathcal{B}(\mathcal{H}))$, and the density of $i_\rho(\mathcal{B}(\mathcal{H}))$ implies these hold on all of $\mathcal{S}_2(\mathcal{H})$.

For the continuity statement, it suffices to assume that $(T_t)_{t \geq 0}$ is weak*-continuous and show $(\tilde{T}_t)_{t \geq 0}$ is strongly continuous on $i_\rho(\mathcal{B}(\mathcal{H}))$, since $i_\rho(\mathcal{B}(\mathcal{H}))$ is dense in $\mathcal{S}_2(\mathcal{H})$ and T_t is a contraction on $\mathcal{S}_2(\mathcal{H})$ for all $t \geq 0$. To this end, let $x \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned} \|\tilde{T}_t(i_\rho(x)) - i_\rho(x)\|_2^2 &= \|\rho^{1/4}T_t(x)\rho^{1/4} - \rho^{1/4}x\rho^{1/4}\|_2^2 \\ &= \|\rho^{1/4}T_t(x)\rho^{1/4}\|_2^2 + \|\rho^{1/4}x\rho^{1/4}\|_2^2 \\ &\quad - \langle \rho^{1/4}x\rho^{1/4}, \rho^{1/4}T_t(x)\rho^{1/4} \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &\quad - \langle \rho^{1/4}T_t(x)\rho^{1/4}, \rho^{1/4}x\rho^{1/4} \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \|\tilde{T}_t(i_\rho(x))\|_2^2 + \|i_\rho(x)\|_2^2 \\ &\quad - 2\Re \langle \rho^{1/4}x\rho^{1/4}, \rho^{1/4}T_t(x)\rho^{1/4} \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &\leq 2\|i_\rho(x)\|_2^2 - 2\Re \left(\text{tr}(\rho^{1/4}x^*\rho^{1/4}\rho^{1/4}T_t(x)\rho^{1/4}) \right) \\ &= 2\Re \left(\text{tr}(\rho^{1/4}x^*\rho^{1/4}\rho^{1/4}x\rho^{1/4} - \rho^{1/4}x^*\rho^{1/4}\rho^{1/4}T_t(x)\rho^{1/4}) \right) \\ &= 2\Re \left(\text{tr} \left(\rho^{1/2}x^*\rho^{1/2}(x - T_t(x)) \right) \right) \rightarrow 0 \end{aligned}$$

since $\rho^{1/2}x^*\rho^{1/2}$ is trace-class. Therefore $(\tilde{T}_t)_{t \geq 0}$ is a strongly continuous semigroup of contractions on $\mathcal{S}_2(\mathcal{H})$.

To prove the final statement, first assume that $x \in D(L)$. Then

$$\text{weak}^* - \lim_{t \rightarrow 0} \frac{T_t(x) - x}{t} = L(x). \tag{18}$$

Notice that for every $y \in \mathcal{S}_2(\mathcal{H})$ we obtain, by Proposition 2(c), that $\rho^{1/4}y^*\rho^{1/4} \in \mathcal{S}_1(\mathcal{H})$ and therefore the map $\mathcal{B}(\mathcal{H}) \ni z \mapsto \text{Tr}(z\rho^{1/4}y^*\rho^{1/4}) \in \mathbb{C}$ is weak* continuous. Thus (18) implies

$$\text{Tr} \left(\rho^{1/4}y^*\rho^{1/4} \frac{T_t(x) - x}{t} \right) \xrightarrow{t \rightarrow 0} \text{Tr} \left(\rho^{1/4}y^*\rho^{1/4}L(x) \right);$$

that is,

$$\left\langle y, \rho^{1/4} \frac{T_t(x) - x}{t} \rho^{1/4} \right\rangle_{\mathcal{S}_2(\mathcal{H})} \xrightarrow{t \rightarrow 0} \left\langle y, \rho^{1/4}L(x)\rho^{1/4} \right\rangle_{\mathcal{S}_2(\mathcal{H})},$$

and hence,

$$\left\langle y, \frac{\tilde{T}_t(\rho^{1/4}x\rho^{1/4}) - \rho^{1/4}x\rho^{1/4}}{t} \right\rangle_{\mathcal{S}_2(\mathcal{H})} \xrightarrow{t \rightarrow 0} \left\langle y, \rho^{1/4}L(x)\rho^{1/4} \right\rangle_{\mathcal{S}_2(\mathcal{H})}. \tag{19}$$

By [4, Proposition 3.36], we obtain that $\rho^{1/4}x\rho^{1/4} \in D(\tilde{L})$ and $\tilde{L}(\rho^{1/4}x\rho^{1/4}) = \rho^{1/4}L(x)\rho^{1/4}$.

Conversely, by the Spectral Theorem there exists an orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of \mathcal{H} formed by eigenvectors of ρ . Let $L_{(h_n)}$ denote the generator of $(T_t)_{t \geq 0}$ with respect to $(h_n)_{n \in \mathbb{N}}$. Let $x \in \mathcal{B}(\mathcal{H})$ and assume that $\rho^{1/4}x\rho^{1/4} \in D(\tilde{L})$. Then we have that

$$\frac{\tilde{T}_t(\rho^{1/4}x\rho^{1/4}) - \rho^{1/4}x\rho^{1/4}}{t} \xrightarrow{t \rightarrow 0} \tilde{L}(\rho^{1/4}x\rho^{1/4}) \quad \text{in } \mathcal{S}_2(\mathcal{H}),$$

and hence

$$\rho^{1/4} \frac{T_t(x) - x}{t} \rho^{1/4} \xrightarrow{t \rightarrow 0} \tilde{L}(\rho^{1/4} x \rho^{1/4}) \quad \text{in } \mathcal{S}_2(\mathcal{H}). \tag{20}$$

We will prove that $x \in D(L_{(h_n)})$. Indeed, we have that

$$\left\langle h, \rho^{1/4} \frac{T_t(x) - x}{t} \rho^{1/4} h' \right\rangle \xrightarrow{t \rightarrow 0} \langle h, \tilde{L}(\rho^{1/4} x \rho^{1/4}) h' \rangle$$

for all $h, h' \in \mathcal{H}$, so for any $n, m \in \mathbb{N}$ we may set $h = (\rho^{1/4})^{(-1)} h_n$ and $h' = (\rho^{1/4})^{(-1)} h_m$ to obtain

$$\begin{aligned} & \left\langle (\rho^{1/4})^{(-1)} h_n, \rho^{1/4} \frac{T_t(x) - x}{t} \rho^{1/4} (\rho^{1/4})^{(-1)} h_m \right\rangle \\ & \xrightarrow{t \rightarrow 0} \langle (\rho^{1/4})^{(-1)} h_n, \tilde{L}(\rho^{1/4} x \rho^{1/4}) (\rho^{1/4})^{(-1)} h_m \rangle. \end{aligned}$$

Noting that $(\rho^{1/4})^* = \rho^{1/4}$, $((\rho^{1/4})^{(-1)})^* = (\rho^{1/4})^{(-1)}$, and $\rho^{1/4} (\rho^{1/4})^{(-1)} h_k = h_k$ for all $k \in \mathbb{N}$, this implies

$$\left\langle h_n, \frac{T_t(x) - x}{t} h_m \right\rangle \xrightarrow{t \rightarrow 0} \langle h_n, (\rho^{1/4})^{(-1)} \tilde{L}(\rho^{1/4} x \rho^{1/4}) (\rho^{1/4})^{(-1)} h_m \rangle.$$

Because this limit exists for all $n, m \in \mathbb{N}$ we have $x \in D(L_{(h_n)})$, and moreover

$$L_{(h_n)}(x) = (\rho^{1/4})^{(-1)} \tilde{L}(\rho^{1/4} x \rho^{1/4}) (\rho^{1/4})^{(-1)}.$$

□

Remark 9 Since the proof of (1) is not included in [35], we want to mention that its proof follows from our Remark 2 in a similar way that our Theorem 2 followed from our Theorem 1 (even the proof of the strong continuity of the semigroup $(\overline{T}_t)_{t \geq 0}$ follows the exact same argument as the proof of the strong continuity of the semigroup $(\tilde{T}_t)_{t \geq 0}$ that appeared in Theorem 2). Moreover, the assumptions that the faithful state is normal and invariant for the semigroup and that the operators of the semigroup are completely positive that are mentioned in [35] for (1) are not needed for its proof, because such assumptions were not used in Remark 2. Instead, for the validity of (1), one merely needs to assume that the faithful state is subinvariant for the semigroup of Schwarz maps. Note also that, unlike (1), Theorem 2 relates the generators of the two semigroups.

5 Applications to Quantum Markov Semigroups and Their Generators

Since quantum Markov semigroups (QMSs) are semigroups of completely positive maps on von Neumann algebras (and hence 2-positive maps and hence Schwarz maps), we naturally obtain applications of Theorem 2 in the study of QMSs. The existence of invariant normal states for QMSs has been discussed in [22] and [24]. Sufficient conditions for a semigroup to be decomposable into a sequence of irreducible semigroups each of them having an invariant normal state are given in [42] (see top half of page 608, Theorem 5 on page 608, and Proposition 5 on page 609).

There are many results in the literature of semigroups which depend on the existence of invariant faithful normal states (for example, see [21, 25–27], and [8]) and this assumption is often taken for granted as being physically reasonable. QMSs have been extensively studied since the 1970s with the exact form for the generators being one of the topics which has garnered a fair amount of attention. See for example [1, 3, 12, 14, 28, 30, 36], and [38]. The generator of a QMS is a generally unbounded operator defined on a weak* dense linear subspace of $\mathcal{B}(\mathcal{H})$. If the generator is bounded then the semigroup is uniformly continuous and the exact form of the generator was found in [28] and [36]. In this section, given a Hilbert space \mathcal{H} and a QMS on $\mathcal{B}(\mathcal{H})$ having an invariant faithful normal state we study the associated semigroup of contractions on $\mathcal{S}_2(\mathcal{H})$. In particular, in Theorem 4 we describe the extended generator (and hence generator) of such a QMS under the assumption that the generator of the associated semigroup on $\mathcal{S}_2(\mathcal{H})$ has compact resolvent.

Definition 8 A **quantum Markov semigroup (QMS)** on $\mathcal{B}(\mathcal{H})$, (for some Hilbert space \mathcal{H}), is a weak*-continuous one-parameter semigroup of bounded linear operators acting on $\mathcal{B}(\mathcal{H})$, such that each member of the semigroup is completely positive and identity preserving.

Remark 10 If \mathcal{H} is a Hilbert space and $(T_t)_{t \geq 0}$ is a QMS on $\mathcal{B}(\mathcal{H})$ which has a subinvariant normal state ω_ρ for some $\rho \in \mathcal{S}_1(\mathcal{H})$, then ω_ρ is in fact an invariant state for $(T_t)_{t \geq 0}$. Indeed for every $t \geq 0$,

$$\text{Tr}(T_t^\dagger(\rho)) = \text{Tr}(T_t^\dagger(\rho)1) = \text{Tr}(\rho T_t(1)) = \text{Tr}(\rho 1) = \text{Tr}(\rho),$$

which together with $T_t^\dagger(\rho) \leq \rho$ implies that $T_t^\dagger(\rho) = \rho$.

Usually the notion of complete positivity applies to maps on C^* -algebras. In particular, if the C^* -algebra is equal to $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} , then the notion of complete positivity becomes equivalent to the following: A map $\mathcal{T} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is completely positive if and only if for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathcal{B}(\mathcal{H})$ and $h_1, \dots, h_n \in \mathcal{H}$,

$$\sum_{i,j=1}^n \langle h_i, \mathcal{T}(x_i^* x_j) h_j \rangle \geq 0. \tag{21}$$

Note that (21) makes perfect sense even if the map \mathcal{T} is not defined on a C^* -algebra, as long as \mathcal{T} is defined on a Banach *-algebra \mathcal{S} of operators on a Hilbert space \mathcal{H} . For example, \mathcal{S} can be equal to $\mathcal{S}_2(\mathcal{H})$ and \mathcal{T} can be a bounded linear operator from \mathcal{S} to \mathcal{S} . We make this extension of the notion of complete positivity in the next definition.

Definition 9 Let \mathcal{H} be a Hilbert space and \mathcal{S} be a Banach *-algebra of bounded linear operators on \mathcal{H} . A bounded linear map $\mathcal{T} : \mathcal{S} \rightarrow \mathcal{S}$ will be called **completely positive** if for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathcal{S}$ and $h_1, \dots, h_n \in \mathcal{H}$, (21) holds.

This terminology will be used in the next result.

Proposition 3 *Let $(T_t)_{t \geq 0}$ be a weak*-continuous semigroup of Schwarz maps on $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} which possesses an invariant faithful normal state ω_ρ for some $\rho \in \mathcal{S}_1(\mathcal{H})$. Then the operators T_t are completely positive for all $t \geq 0$ if and only if the operators \tilde{T}_t constructed in Theorem 2 are completely positive for all $t \geq 0$.*

Proof First, assume T_t is completely positive for $t \geq 0$, and let $x_1, x_2, \dots, x_n \in \mathcal{B}(\mathcal{H})$ and $h_1, h_2, \dots, h_n \in \mathcal{H}$. Then

$$\begin{aligned} & \sum_{i,j=1}^n \left\langle h_i, \tilde{T}_t \left((\rho^{1/4} x_i \rho^{1/4})^* (\rho^{1/4} x_j \rho^{1/4}) \right) h_j \right\rangle \\ &= \sum_{i,j=1}^n \left\langle \rho^{1/4} h_i, T_t \left((\rho^{1/4} x_i)^* (\rho^{1/4} x_j) \right) \rho^{1/4} h_j \right\rangle \geq 0 \end{aligned}$$

since T_t is completely positive. Further, since the map i_ρ from Proposition 2 has dense range, \tilde{T}_t is completely positive on $\mathcal{S}_2(\mathcal{H})$.

Conversely, assume \tilde{T}_t is completely positive for $t \geq 0$, and let $t \geq 0$, $x_1, x_2, \dots, x_n \in \mathcal{B}(\mathcal{H})$ and $h_1, h_2, \dots, h_n \in \mathcal{H}$. Then

$$\sum_{i,j=1}^n \left\langle \rho^{1/4} h_i, T_t(x_i^* x_j) \rho^{1/4} h_j \right\rangle = \sum_{i,j=1}^n \left\langle h_j, \tilde{T}_t((x_i \rho^{1/4})^* (x_j \rho^{1/4})) h_j \right\rangle \geq 0$$

since \tilde{T}_t is completely positive. Because the map $\rho^{1/4}$ has dense range, this is sufficient to show T_t is completely positive. □

For the next result, recall the notion of conditionally completely positive maps introduced by Lindblad in [36]. A linear operator $L : D(L) (\subseteq \mathcal{B}(\mathcal{H})) \rightarrow \mathcal{B}(\mathcal{H})$ is called **conditionally completely positive** if for all $n \in \mathbb{N}$, for all $a_1, a_2, \dots, a_n \in \mathcal{B}(\mathcal{H})$ such that $a_i^* a_j \in D(L)$ for all $i, j = 1, 2, \dots, n$, that for all $h_1, h_2, \dots, h_n \in \mathcal{H}$ with $\sum_{i=1}^n a_i(h_i) = 0$, we have that

$$\sum_{i,j=1}^n \langle h_i, L(a_i^* a_j) h_j \rangle \geq 0.$$

The next result is known for uniformly continuous semigroups. For example, see [20, Proposition 3.12 and Lemma 3.13], or see [18, Proposition 2.9]. In fact the known proof works for a more general setting as the next result indicates.

Theorem 3 *Let \mathcal{S} be a Banach *-algebra of operators acting on a Hilbert space \mathcal{H} .*

1. *Let $(T_t)_{t \geq 0}$ be a WOT continuous semigroup on \mathcal{S} and let \mathcal{L} be its generator. If T_t is completely positive for all $t \geq 0$ then $\mathcal{L}(a^*) = \mathcal{L}(a)^*$ for all $a \in D(\mathcal{L})$ and \mathcal{L} is conditionally completely positive.*

2. Let $(T_t)_{t \geq 0}$ be a uniformly continuous semigroup on \mathcal{S} with generator \mathcal{L} . If $\mathcal{L}(a^*) = \mathcal{L}(a)^*$ for all $a \in \mathcal{S}$ and \mathcal{L} is conditionally completely positive, then \tilde{T}_t is completely positive for all $t \geq 0$.

Proof The proof of (2) immediately follows from [20, Proposition 3.12 and Lemma 3.13]. To prove (1), suppose $a_1, a_2, \dots, a_n \in \mathcal{S}$ such that $a_i^* a_j \in D(\mathcal{L})$ for all $i, j = 1, \dots, n$ and $h_1, h_2, \dots, h_n \in \mathcal{H}$ such that $\sum_{i=1}^n a_i(h_i) = 0$. Then,

$$\begin{aligned} \sum_{i,j=1}^n \langle h_i, \mathcal{L}(a_i^* a_j) h_j \rangle &= \lim_{t \rightarrow 0^+} \sum_{i,j=1}^n \frac{1}{t} \langle h_i, (T_t - 1)(a_i^* a_j) h_j \rangle \\ &= \lim_{t \rightarrow 0^+} \sum_{i,j=1}^n \frac{1}{t} \langle h_i, T_t(a_i^* a_j) h_j \rangle \quad \left(\text{since } \sum_{i=1}^n a_i(h_i) = 0 \right) \\ &\geq 0 \end{aligned}$$

since T_t is completely positive for all $t \geq 0$. □

Corollary 1 Let \mathcal{H} be a Hilbert space and $(T_t)_{t \geq 0}$ be a QMS on $\mathcal{B}(\mathcal{H})$ which possesses an invariant faithful normal state ω_ρ for some $\rho \in \mathcal{S}_1(\mathcal{H})$. Let \tilde{L} be the generator of the strongly continuous semigroup $(\tilde{T}_t)_{t \geq 0}$ of contractions on $\mathcal{S}_2(\mathcal{H})$ defined in Theorem 2. Then $\tilde{L}(a^*) = \tilde{L}(a)^*$ for all $a \in D(\tilde{L})$ and \tilde{L} is conditionally completely positive.

Proof The proof follows immediately from Proposition 3 and Theorem 3(1). □

5.1 The Form of $L_{(h_n)}$ when the Resolvent of \tilde{L} is Compact

In this subsection we consider the form of the extended generator $L_{(h_n)}$ when the resolvent of \tilde{L} is compact, by which we mean that $(\tilde{L} - \lambda)^{-1}$ is compact for some λ in the resolvent set of \tilde{L} (equivalently all λ in the resolvent set, by the resolvent identity). Notably, this assumption implies \tilde{L} is necessarily unbounded if \mathcal{H} is infinite dimensional, since the composition of bounded with compact is compact but $I = (\tilde{L} - \lambda)(\tilde{L} - \lambda)^{-1}$ is not compact. More information about operators on a Hilbert space with compact resolvent can be found in Theorems XIII.4.1 and XIII.4.2 of [15]. Examples of such operators are the diagonal operator with eigenvalues $1, 2, 3, \dots$, or the Laplacian on a bounded domain of \mathbb{R}^d .

We will make use of the following two notations:

Notation: Let \mathcal{H} be a Hilbert space and $w, z \in \mathcal{S}_2(\mathcal{H})$. Define $M_w \otimes z : \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$ by

$$M_w \otimes z \left(\sum_{i=1}^k x_i \otimes h_i \right) = \sum_{i=1}^k x_i w \otimes z(h_i).$$

Notation: Let \mathcal{H} be a Hilbert space and $e \in \mathcal{H}$ such that $\|e\| = 1$. Define $T_e : \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$ by

$$T_e \left(\sum_{i=1}^k x_i \otimes h_i \right) = \sum_{i=1}^k |x_i \langle h_i, e \rangle| \langle e \rangle \otimes e.$$

We are now ready to state the main result:

Theorem 4 *Let \mathcal{H} be a Hilbert space, $(T_t)_{t \geq 0}$ be a QMS on $\mathcal{B}(\mathcal{H})$ having an invariant faithful normal state ω_ρ for some $\rho \in \mathcal{S}_1(\mathcal{H})$, and L be the generator of $(T_t)_{t \geq 0}$. Let $(\tilde{T}_t)_{t \geq 0}$ be the strongly continuous semigroup of contractions on $\tilde{\mathcal{S}}_2(\mathcal{H})$ defined in Theorem 2 and let \tilde{L} be its generator. Assume that the generator \tilde{L} has compact resolvent. Then the following assertions are valid:*

- (a) *There exist complete orthonormal families $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ of self-adjoint elements in $\mathcal{S}_2(\mathcal{H})$ and a sequence of positive scalars $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ (if \mathcal{H} is infinite dimensional) such that*

$$\tilde{L} = I + \sum_{n=1}^{\infty} \lambda_n |a_n \rangle \langle b_n| \tag{22}$$

where the sum converges in the SOT (if it is infinite), i.e. for every $x \in D(\tilde{L})$ we have that $\tilde{L}(x) = x + \sum_{n=1}^{\infty} \lambda_n \langle b_n, x \rangle a_n$ with $\sum_n |\lambda_n \langle b_n, x \rangle|^2 < \infty$.

- (b) *By the Spectral Theorem there is an orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of \mathcal{H} which consists of eigenvectors of ρ . Let $L_{(h_n)}$ denote the generator of $(T_t)_{t \geq 0}$ with respect to $(h_n)_{n \in \mathbb{N}}$. Then*

$$L_{(h_n)} = I + \sum_{n=1}^{\infty} \lambda_n |i_{\rho^{(-1)}}(a_n) \rangle \langle i_{\rho}(b_n)| \tag{23}$$

where the sum converges in the SOT (if it is infinite). We note that $|i_{\rho^{(-1)}}(a_n) \rangle \langle i_{\rho}(b_n)|$ has domain $\mathcal{B}(\mathcal{H})$ for all n , since

$$|i_{\rho^{(-1)}}(a_n) \rangle \langle i_{\rho}(b_n)|x = \langle i_{\rho}(b_n), x \rangle i_{\rho^{(-1)}}(a_n) = \langle b_n, i_{\rho}(x) \rangle i_{\rho^{(-1)}}(a_n)$$

and $b_n, i_{\rho}(x) \in \mathcal{S}_2(\mathcal{H})$. Explicitly, for every $x \in D(L_{(h_n)})$ and every $i, j \in \mathbb{N}$ we have that

$$\langle h_i, [L_{(h_n)}(x)]h_j \rangle = \langle h_i, xh_j \rangle + \sum_{n=1}^{\infty} \lambda_n \langle b_n, i_{\rho}(x) \rangle \langle h_i, i_{\rho^{(-1)}}(a_n)h_j \rangle.$$

- (c) *We have*

$$I = - \sum_{n=1}^{\infty} \lambda_n \langle b_n, \rho^{1/2} \rangle i_{\rho^{(-1)}}(a_n),$$

where the sum converges in the SOT (if it is infinite).

(d) For all $e \in \mathcal{H}$ with $\|e\| = 1$ we have that the operator $\tilde{L}_{\otimes, e} : S_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow S_2(\mathcal{H}) \otimes \mathcal{H}$ is positive, where the operator $\tilde{L}_{\otimes, e}$ is defined by

$$\tilde{L}_{\otimes, e} = (Id + T_e^*) \left(\sum_{n=1}^{\infty} \lambda_n M_{b_n} \otimes a_n \right) (Id + T_e) \tag{24}$$

where Id stands for the identity operator on $S_2(\mathcal{H}) \otimes \mathcal{H}$ and the sum converges in the SOT (if it is infinite).

We note that the sum in (22) is finite if and only if \mathcal{H} is finite dimensional. Indeed, if \tilde{L} is bounded with compact resolvent then \mathcal{H} is finite dimensional, as remarked in the preamble of this subsection. The proof of Theorem 4 is at the end of this subsection, after the following three results:

Lemma 1 *Let \mathcal{H} be a separable Hilbert space and A be an invertible linear operator on $S_2(\mathcal{H})$ with dense domain which is closed under adjoints. If A satisfies $A(a^*) = (A(a))^*$ for all $a \in D(A)$, then $D(A^\dagger)$ and $D(A^{-1})$ are closed under adjoints, $A^\dagger(b^*) = (A^\dagger(b))^*$ for all $b \in D(A^\dagger)$, and $A^{-1}(c^*) = (A^{-1}(c))^*$ for all $c \in D(A^{-1})$.*

Proof Let $a \in D(A)$ and $b \in D(A^\dagger)$. Then

$$\begin{aligned} |\langle A(a), b^* \rangle| &= |\langle (A(a^*))^*, b^* \rangle| = |\langle b, A(a^*) \rangle| \\ &= |\langle A^\dagger(b), a^* \rangle| = |\langle a, (A^\dagger(b))^* \rangle| \leq \|a\| \| (A^\dagger(b))^* \|, \end{aligned}$$

and so $b^* \in D(A^\dagger)$ by definition. As before,

$$\langle a, A^\dagger(b^*) \rangle = \langle A(a), b^* \rangle = \langle a, (A^\dagger(b))^* \rangle,$$

and since $D(A)$ is dense this implies $A^\dagger(b^*) = (A^\dagger(b))^*$ for all $b \in D(A^\dagger)$. Further, for every $c \in D(A^{-1})$ there exists an $a \in D(A)$ such that $A(a) = c$. Since A is star-preserving we have that $A(a^*) = c^*$. Then, by definition, $(A^{-1}(c))^* = a^* = A^{-1}(c^*)$. □

Lemma 2 *Let \mathcal{H} be a Hilbert space and A be a compact and self-adjoint linear operator on $S_2(\mathcal{H})$. Then A satisfies $A(a^*) = (A(a))^*$ for all $a \in S_2(\mathcal{H})$ if and only if*

$$A = \sum_{n=1}^{\infty} \lambda_n |x_n\rangle \langle x_n|$$

with $(\lambda_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ and $(x_n)_{n=1}^{\infty}$ an orthonormal basis of $S_2(\mathcal{H})$ consisting of self-adjoint operators.

Proof If A is compact and self-adjoint, then the Spectral Theorem implies there is an eigensystem decomposition

$$A = \sum_{n=1}^{\infty} \lambda_n |y_n\rangle \langle y_n|,$$

with $(\lambda_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ and $(y_n)_{n=1}^{\infty}$ an orthonormal basis of $S_2(\mathcal{H})$. Because A is self-adjoint and star-preserving, we have that $A(y_n) = \lambda_n y_n$ implies $A(y_n^*) = \lambda_n y_n^*$. Thus, every eigenspace of A is self-adjoint. Let E be an eigenspace of A corresponding to a fixed eigenvalues, and consider the orthonormal basis $(y_{n_j})_{j=1}^N \subseteq (y_n)_{n=1}^{\infty}$ of E . Because E is self-adjoint, from $\langle y_{n_j}, y_{n_k} \rangle = \langle y_{n_j}^*, y_{n_k}^* \rangle = \delta_{jk}$ it follows that $(y_{n_j}^*)_{j=1}^N$ is an orthonormal basis of E . Define self-adjoint operators $a_j = y_{n_j} + y_{n_j}^*$ and $a_{N+j} = i(y_{n_j} - y_{n_j}^*)$ for each $1 \leq j \leq N$ so that $E = \text{Span}(a_j)_{j=1}^{2N}$. We follow the Gram-Schmidt process and set $b_1 = a_1$ and recursively define

$$b_k = a_k - \sum_{j=1}^{k-1} \frac{\langle b_j, a_k \rangle}{\langle b_j, b_j \rangle} b_j$$

to produce a sequence of N many orthogonal operators which span E (the remaining N many operators produced by the Gram-Schmidt process become zero). Straight forward calculation reveals that $\langle a_j, a_k \rangle$ is real for every $1 \leq j, k \leq 2N$, and hence $\langle b_j, a_k \rangle$ is real for every $1 \leq j, k \leq 2N$. Each b_k is thus self-adjoint as a real combination of self-adjoint operators, and the set $(b_k)_{k=1}^N$ can therefore be normalized to a set of self-adjoint orthonormal operators $(x_j)_{j=1}^N$ which span E . Replacing y_n with x_n in the original eigensystem decomposition for each eigenspace E , we have

$$A = \sum_{n=1}^{\infty} \lambda_n |x_n\rangle \langle x_n|,$$

as desired. □

Lemma 3 *Let \mathcal{H} be a Hilbert space and \tilde{L} be a bounded linear operator on $S_2(\mathcal{H})$ which has the form (22). Then \tilde{L} is conditionally completely positive if and only if for some (equivalently all) normalized vector $e \in \mathcal{H}$, the operator $\tilde{L}_{\otimes, e}$ defined on the Hilbert space $S_2(\mathcal{H}) \otimes \mathcal{H}$, by (24), is positive.*

Proof First note that $I + A$ is conditionally completely positive if and only if A is (as is easily verified), so for simplicity we may assume instead that $\tilde{L} = \sum_{n=1}^{\infty} \lambda_n |a_n\rangle \langle b_n|$.

We will start with the forward direction and suppose \tilde{L} is conditionally completely positive. Let $e \in \mathcal{H}$ with $\|e\| = 1$. Since $W = \{\sum_{i=1}^k y_i \otimes h'_i : y_i \in S_2(\mathcal{H}), h'_i \in \mathcal{H}\}$ is dense in $S_2(\mathcal{H}) \otimes \mathcal{H}$, in order to verify that $\tilde{L}_{\otimes, e} \geq 0$ it is enough to consider an element $w = \sum_{i=1}^k y_i \otimes h'_i \in W$ and verify that $\langle w, \tilde{L}_{\otimes, e} w \rangle_{\otimes} \geq 0$, where $\langle \cdot, \cdot \rangle_{\otimes}$ will denote the inner product of $S_2(\mathcal{H}) \otimes \mathcal{H}$. (The reason that we chose h'_i to denote a generic element of \mathcal{H} is because we have used h_n to denote the orthonormal eigenvectors of ρ in the statement of Theorem 4). We will denote the inner product

of \mathcal{H} by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Fix $w = \sum_{i=1}^k y_i \otimes h'_i \in W$ and let $v = -\sum_{i=1}^k y_i \langle h'_i \rangle$. Define $y_{k+1} = |v\rangle\langle e|$ and $h'_{k+1} = e$. Then $\sum_{i=1}^{k+1} y_i \langle h'_i \rangle = 0$ and, since \tilde{L} is conditionally completely positive, we have that

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^{k+1} \langle h'_i, \tilde{L}(y_i^* y_j) h'_j \rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^{k+1} \sum_{n=1}^{\infty} \lambda_n \text{Tr}(y_i^* y_j b_n) \langle h'_i, a_n(h'_j) \rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^{k+1} \sum_{n=1}^{\infty} \lambda_n \langle y_i \otimes h'_i, y_j b_n \otimes a_n(h'_j) \rangle_{\otimes} \\ &= \sum_{i,j=1}^{k+1} \sum_{n=1}^{\infty} \left\langle y_i \otimes h'_i, \lambda_n M_{b_n} \otimes a_n(y_j \otimes h'_j) \right\rangle_{\otimes} \\ &= \left\langle \sum_{i=1}^{k+1} y_i \otimes h'_i, \left(\sum_{n=1}^{\infty} \lambda_n M_{b_n} \otimes a_n \right) \left(\sum_{j=1}^{k+1} y_j \otimes h'_j \right) \right\rangle_{\otimes}. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{i=1}^{k+1} y_i \otimes h'_i &= \sum_{i=1}^k y_i \otimes h'_i + y_{k+1} \otimes h'_{k+1} = w - \sum_{i=1}^k |y_i \langle h'_i \rangle \langle e| \otimes e \\ &= w - T_e \left(\sum_{i=1}^k y_i \otimes h'_i \right) = (Id - T_e)(w) \end{aligned}$$

where Id denotes the identity operator on $\mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$, which finishes the proof of the forward direction.

For the other direction, suppose that $\tilde{L}_{\otimes, e} \geq 0$ for some $e \in \mathcal{H}$ with $\|e\| = 1$. Let $k \in \mathbb{N}$, $y_1, \dots, y_k \in \mathcal{S}_2(\mathcal{H})$ and $h'_1, \dots, h'_k \in \mathcal{H}$ such that $\sum_{i=1}^k y_i \langle h'_i \rangle = 0$. Let $w = \sum_{i=1}^k y_i \otimes h'_i \in \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$. Then,

$$0 \leq \langle w, \tilde{L}_{\otimes, e}(w) \rangle_{\otimes} \tag{25}$$

$$= \left\langle w, (Id - T_e)^* \left(\sum_{n \in N} \lambda_n M_{b_n} \otimes a_n \right) (Id - T_e)(w) \right\rangle_{\otimes}$$

$$= \left\langle (Id - T_e)w, \left(\sum_{n \in N} \lambda_n M_{b_n} \otimes a_n \right) (Id - T_e)(w) \right\rangle_{\otimes}. \tag{26}$$

Notice that

$$T_e(w) = T_e \left(\sum_{i=1}^k y_i \otimes h'_i \right) = \left| \sum_{i=1}^k y_i \langle h'_i \rangle \right\rangle \langle e| \otimes e = |0\rangle \langle e| \otimes e = 0.$$

Hence Inequality (25) gives

$$\begin{aligned}
 0 &\leq \left\langle Id(w), \left(\sum_{n \in N} \lambda_n M_{b_n} \otimes a_n \right) Id(w) \right\rangle_{\otimes} \\
 &= \sum_{i,j=1}^k \sum_{n \in N} \left\langle y_i \otimes h'_i, \lambda_n M_{b_n} \otimes a_n (y_j \otimes h'_j) \right\rangle_{\otimes} \\
 &= \sum_{i,j=1}^k \sum_{n \in N} \lambda_n \langle y_i \otimes h'_i, y_j b_n \otimes a_n (h'_j) \rangle_{\otimes} \\
 &= \sum_{i,j=1}^k \sum_{n \in N} \lambda_n \text{Tr}(y_i^* y_j b_n) \langle h'_i, a_n (h'_j) \rangle_{\mathcal{H}} \\
 &= \sum_{i,j=1}^{k+1} \langle h_i, \tilde{L}(y_i^* y_j) h'_j \rangle_{\mathcal{H}}.
 \end{aligned}$$

Therefore \tilde{L} is conditionally completely positive. This completes the proof. □

The proof of Lemma 3 reveals the following:

Remark 11 Let $\mathcal{A} = \{ \sum_{i=1}^k y_i \otimes h'_i \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{H} : \sum_{i=1}^k y_i (h'_i) = 0 \}$. Then

- For every $w = \sum_{i=1}^k y_i \otimes h'_i \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{H}$ there exists $y_{k+1} \in \mathcal{B}(\mathcal{H})$ and $h'_{k+1} \in \mathcal{H}$ such that $\sum_{i=1}^{k+1} y_i \otimes h'_i \in \mathcal{A}$ and $(Id - T_{h'_{k+1}})(w) = \sum_{i=1}^{k+1} y_i \otimes h'_i$.
- If a bounded operator \tilde{L} on \mathcal{H} has form (22) then \tilde{L} is completely positive if and only if the operator $\sum_{n=1}^{\infty} \lambda_n M_{b_n} \otimes a_n : S_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow S_2(\mathcal{H}) \otimes \mathcal{H}$ is positive.
- For every $e \in \mathcal{H}$ we have $\mathcal{A} \subseteq \ker T_e$.

We are now ready to present the

Proof of Theorem 4 Since \tilde{L} generates a strongly continuous semigroup of contractions, we have that $\lambda \in \rho(\tilde{L})$ for all $\lambda \geq 0$ by the Hille-Yosida Generation Theorem (e.g. Theorem 3.5 of [16]). Further, $D(\tilde{L})$ is dense in $S_2(\mathcal{H})$ by Theorem 3.1.16 of [6] and \tilde{L} is star-preserving by Corollary 1, and so $K := (\tilde{L} - I)^{-1}$ is star-preserving by Lemma 1 as the inverse of a star-preserving map with dense domain. Because L has compact resolvent by assumption, we have that K is furthermore compact. Thus, $K^\dagger K$ is compact, self-adjoint, and star-preserving, and so Lemma 2 implies

$$K^\dagger K = \sum_{n=1}^{\infty} \sigma_n^2 |v_n\rangle \langle v_n|,$$

where $\{\sigma_n^2\}_{n \in \mathbb{N}}$ are the nonzero eigenvalues of $K^\dagger K$ corresponding to the system $\{v_n\}_{n \in \mathbb{N}}$ of self-adjoint orthonormal eigenoperators. This notation is chosen so that, following Section 2.2 of [17], the singular value expansion of K can be written

$$K = \sum_{n=1}^{\infty} \sigma_n |u_n\rangle\langle v_n|,$$

where $\{u_n\}_{n \in \mathbb{N}}$ are self-adjoint orthonormal eigenoperators of KK^\dagger given by the relation $\sigma_n u_n := K v_n$. By Theorem 2.8 of [17] we have that

$$\tilde{L} - I = K^{(-1)} = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} |v_n\rangle\langle u_n|,$$

and hence

$$\tilde{L} = I + \sum_{n=1}^{\infty} \frac{1}{\sigma_n} |v_n\rangle\langle u_n|,$$

proving (22). Equation (23) follows from (22) and (17). Part (c) is valid since $L(I) = 0$, hence $L_{(h_n)}(I) = 0$, and hence

$$I + \sum_{n=1}^{\infty} \lambda_n \langle h_n, \rho^{1/2} \rangle i_{\rho^{(-1)}}(a_n) = 0.$$

Part (d) follows from Lemma 3; indeed, $(\tilde{T}_t)_{t \geq 0}$ is a completely positive semigroup by Proposition 3, and so \tilde{L} is conditionally completely positive by Theorem 3. \square

Remark 12 Let \mathcal{H} be a finite dimensional Hilbert space, and let $(T_t)_{t \geq 0}$ be a QMS on $\mathcal{B}(\mathcal{H})$. Then $(T_t)_{t \geq 0}$ possesses an invariant faithful normal state [23, Theorem 4.2] and so Theorem 4 applies. The sum in (23) is finite, and hence by linearity/conjugate linearity we can write

$$\begin{aligned} L_{(h_n)} &= I + \sum_n \lambda_n |i_{\rho^{(-1)}}(a_n^+ - a_n^-)\rangle\langle i_{\rho}(b_n^+ - b_n^-)| \\ &= I + \sum_m \lambda'_m |i_{\rho^{(-1)}}(c_m)\rangle\langle i_{\rho}(d_m)|, \end{aligned}$$

where $c_m \in \{a_n^+, a_n^- : n\}$, $d_m \in \{b_n^+, b_n^- : n\}$, and $\lambda'_m \in \{\pm \lambda_n : n\}$. In particular, $c_m, d_m \geq 0$ and $\lambda'_m \in \mathbb{R}$. Thus, for every $x \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} |i_{\rho^{(-1)}}(c_m)\rangle\langle i_{\rho}(d_m)|x &= \langle i_{\rho}(d_m), x \rangle i_{\rho^{(-1)}}(c_m) \\ &= \langle \rho^{1/4} d_m \rho^{1/4}, x \rangle \rho^{-1/4} c_m \rho^{-1/4} \\ &= \rho^{-1/4} \sqrt{c_m} \operatorname{Tr} \left(\rho^{1/4} d_m \rho^{1/4} x \right) \sqrt{c_m} \rho^{-1/4} \\ &= \rho^{-1/4} \sqrt{c_m} \operatorname{Tr} \left(\sqrt{d_m} \rho^{1/4} x \rho^{1/4} \sqrt{d_m} \right) \sqrt{c_m} \rho^{-1/4}. \end{aligned}$$

Now, fix any orthonormal basis $(E_k)_k$ of $\mathcal{B}(\mathcal{H})$. Then for every $A \in \mathcal{B}(\mathcal{H})$ we have

$$\operatorname{Tr}(A) = \sum_k E_k A E_k^*.$$

Thus,

$$\mathrm{Tr} \left(\sqrt{d_m} \rho^{1/4} x \rho^{1/4} \sqrt{d_m} \right) = \sum_k E_k \sqrt{d_m} \rho^{1/4} x \rho^{1/4} \sqrt{d_m} E_k^*,$$

and so

$$L_{(h_n)}(x) = x + \sum_{m,k} \lambda'_m \rho^{-1/4} \sqrt{c_m} E_k \sqrt{d_m} \rho^{1/4} x \rho^{1/4} \sqrt{d_m} E_k^* \sqrt{c_m} \rho^{-1/4}.$$

By defining $y_\ell = \rho^{-1/4} \sqrt{c_m} E_k \sqrt{d_m} \rho^{1/4}$ and λ'_ℓ to be the corresponding λ'_m , we obtain

$$L_{(h_n)}(x) = x + \sum_\ell \lambda'_\ell y_\ell x y_\ell^*.$$

Moreover, since $L_{(h_n)}(I) = L(I) = 0$ we obtain as in the proof of Theorem 4(c) that

$$I + \sum_\ell \lambda_\ell y_\ell y_\ell^* = 0.$$

Thus

$$L_{(h_n)}(x) = \frac{1}{2} I x + \frac{1}{2} x I + \sum_\ell \lambda'_\ell y_\ell x y_\ell^* = \sum_\ell \lambda'_\ell \left(y_\ell x y_\ell^* - \frac{1}{2} \{y_\ell y_\ell^*, x\} \right)$$

which resembles the standard GKSL form developed in [28] and [36], except with Hamiltonian part zero and without the demand that the jump operators y_ℓ are traceless. A comparison of such GKSL form to the standard GKSL form, including conversion between the two, is discussed in section 2.1 of [2].

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Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflict of interest.

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