



Concentration Behavior and Lattice Structure of 3D Surface Superconductivity in the Half Space

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Abstract

We study a 3D Ginzburg-Landau model in a half-space which is expected to capture the key features of surface superconductivity for applied magnetic fields between the second critical field H_{C_2} and the third critical field H_{C_3} . For the magnetic field in this regime, it is known from physics that superconductivity should be essentially restricted to a thin layer along the boundary of the sample. This leads to the introduction of a Ginzburg-Landau model on a half-space. We prove that the non-linear Ginzburg-Landau energy on the half-space with constant magnetic field is a decreasing function of the angle ν that the magnetic field makes with the boundary. In the case when the magnetic field is tangent to the boundary ($\nu = 0$), we show that the energy is determined to leading order by the minimization of a simplified 1D functional in the direction perpendicular to the boundary. For non-parallel applied fields, we also construct a periodic problem with vortex lattice minimizers reproducing the effective energy, which suggests that the order parameter of the full Ginzburg-Landau model will exhibit 3 dimensional vortex structure near the surface of the sample.

Keywords Ginzburg-Landau equations · Superconductivity · Surface superconductivity · Magnetic Schrodinger operator · Surface concentration · 3D vortices · Partial differential equations · Calculus of variation · Estimate of eigenvalue

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1 Introduction

1.1 The Ginzburg-Landau Model

The Ginzburg-Landau (GL) theory of superconductivity was first introduced in the '50s as a phenomenological macroscopic model [14]. It successfully describes the behavior of a superconductor subject to an external magnetic field and was later justified by Gor'kov [15] as emerging from the microscopic Bardeen-Cooper-Schrieffer (BCS) theory. This has recently been proved rigorously [13]. It has been widely used in the physics literature, for instance for successfully predicting the response of superconducting materials to an external magnetic field. Also, in the celebrated work of Abrikosov [1], this theory predicted the existence of type II superconductors - in particular, of vortex lattices - before they had been experimentally realized. We refer to [8] for a review of this topic for which A. Abrikosov was awarded the Nobel Prize, and the first discussion by Saint-James and de Gennes of the surface superconductivity phenomenon that is the subject of this paper.

In the GL theory, the superconducting state of a sample occupying a domain Ω in \mathbb{R}^3 is described by a complex-valued wave function $\psi : \Omega \rightarrow \mathbb{C}$ (the order parameter) and a vector field (magnetic potential) $\mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that the pair (ψ, \mathbf{A}) is a critical point of a specific energy functional. We shall use notation $\text{curl} \mathbf{A}$ to denote $\nabla \times \mathbf{A}$ for 3d vector fields. In a particular case where Ω is a cylinder of infinite height and with a cross section $D \subset \mathbb{R}^2$, then one may consider restriction of ψ on D and \mathbf{A} on \mathbb{R}^2 , and call the reduced model as *2D superconductivity*. For this reason, the model in 3D is called *3D superconductivity*.

The interpretation of ψ and \mathbf{A} is explained by the BCS theory as follows: $|\psi|^2$ is proportional to the relative density of superconducting particles (the so-called Cooper pairs) and $\kappa H \text{curl} \mathbf{A}$ measures the induced magnetic field inside the sample, with $\kappa > 0$ a physical characteristic of the material, and H measuring the intensity of the external magnetic field, that we assume to be constant throughout the sample. We shall be concerned with type-II superconductors, characterized by $\kappa > \frac{1}{\sqrt{2}}$, and more precisely with the limit $\kappa \rightarrow \infty$ (extreme type-II).

The modulus of the order parameter $|\psi|$ varies between 0 and 1: the vanishing of ψ in a certain region or point implies a loss of superconductivity there, due to the absence of Cooper pairs, whereas if $|\psi| = 1$ somewhere all the electrons are arranged in Cooper pairs and thus superconducting. The cases $|\psi| = 1$ and $|\psi| = 0$ everywhere in Ω correspond to the so-called perfectly superconducting and normal states, known to be preferred for small and large applied field respectively. When $|\psi|$ is not identically 0 nor 1, for intermediate values of the applied field, one says that the system is in a mixed state.

The behavior of a type-II superconductor is distinguished by three critical values of the intensity of the applied magnetic field which we denote by H_{C_1} , H_{C_2} and H_{C_3} . These critical fields may be described in terms of the wave function ψ as follows. When the external magnetic field strength H satisfies $H < H_{C_1}$, the material is in the superconducting phase, which corresponds to $|\psi| > 0$ everywhere. The sample stays in the superconducting state until the first critical field is reached. When $H_{C_1} < H < H_{C_2}$, the magnetic field penetrates the sample in quantized vortices. These

vortices correspond to zeros of ψ . In the 2D case, vortices are isolated points and their number increases with the increase of the strength of the external field κH . As H increases and gets close to H_{C_2} , vortices arrange themselves on a triangular lattice, the famous Abrikosov lattice, that survives until a second critical value of the field is reached. When $H_{C_2} < H < H_{C_3}$ superconductivity is confined to (part of) the surface of the sample. This means that $|\psi|$ is very small in the bulk. More precisely the GLorder parameter is exponentially decaying far from the boundary. This is the surface superconductivity regime. Finally, when $H > H_{C_3}$, superconductivity is lost, which is reflected by $\psi = 0$ everywhere, and the normal state becomes the global minimizer of the GLenergy.

In the last decades, much progress has been made towards establishing the aforementioned behavior of type-II superconductors by studying minimizers of the GLenergy. The monograph [24] and references therein contains an analysis of vortices and the critical field H_{C_1} . Concerning the analysis of the critical fields H_{C_2} and H_{C_3} we mention [9, 11] (and references therein). As one can see in [9, 24], the GLmodel has a rich mathematical structure whose analysis requires a diversity of techniques, many of which have been developed especially for the study of the model. While a detailed study of the GLmodel in a two dimensional domain has been the subject of numerous papers, the study of the model in a three dimensional domain is much less complete. We refer to [3, 9, 11, 12, 21] for some of the available results in 3D.

In order to state our results we briefly introduce the GL functional on a domain $\Omega \subset \mathbb{R}^3$ and in an external magnetic field of strength κH pointing in the direction \mathbf{e}_3 of the third coordinate axis.

$$\mathcal{G}_{\kappa,H}[\psi, \mathbf{A}] = \int_{\Omega} \left\{ |(-i\nabla + \kappa H \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right\} dx + (\kappa H)^2 \int_{\mathbb{R}^3} |\text{curl } \mathbf{A} - \mathbf{e}_3|^2 dx. \tag{1.1}$$

The ground state energy is the infimum of the functional over all order parameters ψ and all induced magnetic vector potentials \mathbf{A} .

$$E^{\text{GL}}(\kappa, H) := \inf_{(\psi, \mathbf{A})} \mathcal{G}_{\kappa,H}[\psi, \mathbf{A}]. \tag{1.2}$$

In the regime where the H is between the second and the third critical fields, i.e. H_{C_2} and H_{C_3} mentioned above, superconductivity is essentially localized to the boundary. This corresponds essentially to the parameters $\mathfrak{b}H = \kappa$ for $\mathfrak{b} \in]\Theta_0, 1[$, where $\Theta_0 \approx 0.59$ is a spectral constant which will be defined in Remark 1.1 below. In this regime the energy is known (see [12]) to behave to leading order in the limit $\kappa \rightarrow \infty$ as

$$E^{\text{GL}}(\kappa, H) \approx -\frac{\kappa^2}{2} \int_{\Omega} |\psi|^4 dx \approx \sqrt{\kappa H} \int_{\partial\Omega} e(\mathfrak{b}, \nu(x)) d\sigma, \tag{1.3}$$

where $\nu(x)$ denotes the angle between the tangent plane at $x \in \partial\Omega$ and \mathbf{e}_3 (the magnetic field). Here $e(\mathfrak{b}, \nu)$ is an explicitly constructed boundary energy density.

In the same paper also local versions are proved, so that one can think of $e(b, \nu)$ as measuring both the energy density and—through the connection to $|\psi|^4$ —the density of superconductivity. Our aim in the present paper is to study in more detail the function $e(b, \nu)$.

1.2 Objective of the Paper

Our study is motivated by the mathematical theory of the surface superconductivity of 3D samples. It is well understood [12, 20, 21] that in a suitable range of magnetic field strengths the solutions to the GLequations are localized near the boundary. We want to improve the understanding of this boundary layer. For this purpose, we examine the energy contribution of the order parameter in the vicinity of the domain boundary. We believe that in the surface superconducting state, the order parameters will exhibit a certain lattice structure in the vicinity of the domain boundary similar to the Abrikosov lattices of 2D samples. Understanding the lattice structure near the boundary will help us to understand the vortex lattices of superconductivity when the applied magnetic field decreases and approaches the second critical field H_{C_2} .

After rescaling and taking limits, the behavior of the solutions to the GLequations in a boundary layer can be understood from the limiting equations in \mathbb{R}_+^3 . So we consider only the problem in \mathbb{R}_+^3 . Notice, in particular, that this means that the boundary curvature has disappeared. This might appear surprising, but it turns out (see [11, 12]) that the curvature does not affect the energy to leading order—one might expect curvature effects to appear in more precise descriptions of the the surface superconductivity, as in the two-dimensional case [6, 7], however this is not the subject of the present paper. Furthermore, in this boundary layer the magnetic field can be considered constant. Again this is valid to leading order and follows from [11, 12]. One should merely see this as the fact that on short length scales any sufficiently smooth function is well approximated by a constant.

The important constant Θ_0 , often called the de Gennes constant, has already been mentioned in §1.1 and will appear in the statements. We refer to [9, 16, 18] for the mathematical analysis of Θ_0 .

Definition 1.1 (The spectral quantity Θ_0)

Consider the shifted harmonic oscillator $H(\xi)$ defined for all $\xi \in \mathbb{R}$ on the half-axis \mathbb{R}_+ as follows:

$$H(\xi) = -\frac{d^2}{dt^2} + (t - \xi)^2 \quad \text{in } L^2(\mathbb{R}_+), \quad (1.4)$$

with Neumann boundary condition $u'(0) = 0$. This operator has compact resolvent and it follows from Sturm-Liouville theory that its eigenvalues are simple. Let $\mu_1(\xi)$ denote the lowest eigenvalue of $H(\xi)$. The constant Θ_0 is defined as:

$$\Theta_0 = \inf_{\xi \in \mathbb{R}} \mu_1(\xi). \quad (1.5)$$

1.3 The 3D Surface Energy

In this paper, for all $m \in \{1, 2, 3\}$, we denote by x_1, \dots, x_m the coordinates of $x \in \mathbb{R}^m$. We define $\mathbb{R}_+^m = \{x \in \mathbb{R}^m : x_1 > 0\}$ and $\partial\mathbb{R}_+^m = \{x \in \mathbb{R}^m : x_1 = 0\}$ the boundary of \mathbb{R}_+^m .

Let $\nu \in [0, \frac{\pi}{2}]$, and $\ell > 0$. We introduce the set:

$$\mathcal{D}_\ell = (0, \infty) \times (-\ell, \ell) \times (-\ell, \ell), \tag{1.6}$$

and the magnetic potential \mathbf{A}_ν defined on \mathbb{R}_+^3 by

$$\mathbf{A} = \mathbf{A}_\nu = \begin{pmatrix} 0 \\ 0 \\ -x_1 \cos \nu + x_2 \sin \nu \end{pmatrix}, \tag{1.7}$$

for which the associated magnetic field is the constant unit vector that makes an angle ν with the x_2x_3 plane:

$$\mathbf{B} = \mathbf{B}_\nu = \nabla \times \mathbf{A}_\nu = \begin{pmatrix} \sin \nu \\ \cos \nu \\ 0 \end{pmatrix}. \tag{1.8}$$

Remark 1.2 By the standard gauge invariance arguments our energy (in particular, the energy functional $\mathcal{E}_{\mathbf{b}, \nu, \ell}$ defined below) depends on the magnetic field \mathbf{B}_ν but not on the specific choice of vector potential \mathbf{A}_ν with $\nabla \times \mathbf{A}_\nu = \mathbf{B}_\nu$. We only fix this choice for concreteness.

Definition 1.3 We consider the following reduced GLtype energy functional:

$$\mathcal{E}_{\mathbf{b}, \nu, \ell}(\varphi) = \int_{\mathcal{D}_\ell} \left\{ |(-i\nabla + \mathbf{A}_\nu)\varphi|^2 - \mathbf{b}|\varphi|^2 + \frac{\mathbf{b}}{2}|\varphi|^4 \right\} dx, \tag{1.9}$$

for φ in the space:

$$\mathcal{S}_\ell = \left\{ \varphi \in L^2(\mathcal{D}_\ell) : (-i\nabla + \mathbf{A}_\nu)\varphi \in L^2(\mathcal{D}_\ell, \mathbb{C}^3), \varphi = 0 \text{ on } \partial\mathcal{D}_\ell \setminus \{x_1 = 0\} \right\}. \tag{1.10}$$

Note that the Dirichlet boundary condition for the functions in \mathcal{S}_ℓ is given on the lateral sides $|x_2| = \ell$ and $|x_3| = \ell$, which is convenient for us to study the limit of the minimizers as $\ell \rightarrow \infty$. Furthermore, we define:

$$E(\mathbf{b}, \nu, \ell) = \inf_{\varphi \in \mathcal{S}_\ell} \mathcal{E}_{\mathbf{b}, \nu, \ell}(\varphi), \tag{1.11}$$

and, for those values of \mathbf{b} where the limit exists,

$$e(\mathbf{b}, \nu) = \lim_{\ell \rightarrow \infty} \frac{1}{4\ell^2} E(\mathbf{b}, \nu, \ell). \tag{1.12}$$

Remark 1.4 The existence of the limit (1.12) was proved under the restriction $\mathbf{b} \in [\Theta_0, 1]$ in [12]. More precisely, it has been proved that (see [12, Theorem 3.13]) for all $\nu \in [0, \frac{\pi}{2}]$, the function $\mathbf{b} \mapsto e(\mathbf{b}, \nu)$ is continuous and monotone decreasing, and that for all $\mathbf{b} \in [\Theta_0, \nu]$, the function $\nu \mapsto e(\mathbf{b}, \nu)$ is continuous. For $\mathbf{b} \leq \Theta_0$ it is

clear that $e(b, \nu) = 0$ and for $b > 1$ there is no boundary concentration so the limit in (1.12) does not exist. It is not stated explicitly in [12] but one can easily show that for all $\nu \in [0, \frac{\pi}{2}]$, the function $b \mapsto e(b, \nu)$ is also concave (as an infimum and a limit over affine functions).

1.4 Statement of the Results

For $\nu = 0$, we have a complete understanding of the limit $e(b, 0)$ in (1.12). For the similar problem in 2D it was proposed in [20] and proved in [5–7] that an Ansatz with separation of variables is correct for the ground state. We will see that the 3D case is completely analogous (with essentially the same proof).

Theorem 1.5 *For $\nu = 0$ and $b \in (\Theta_0, 1]$ we have*

$$e(b, \nu = 0) = E_0^{1D}, \quad (1.13)$$

where E_0^{1D} is defined by

$$E_0^{1D} = \inf_{\xi \in \mathbb{R}} \left(\inf_{f \in H^1(\mathbb{R}_+)} \mathcal{E}_{b,\xi}^{1D}(f) \right), \quad (1.14)$$

and where

$$\mathcal{E}_{b,\xi}^{1D}(f) := \int_0^\infty \left\{ |f'(t)|^2 + (t - \xi)^2 |f(t)|^2 - b |f(t)|^2 + \frac{b}{2} |f(t)|^4 \right\} dt. \quad (1.15)$$

We note that the infimum is both taken with respect to the function f and the real number ξ . Minimizing the 1D-functional (1.15) with respect to f , we obtain an energy $E_{b,\xi}^{1D}$ and a minimizer $f_{b,\xi} = f_\xi$. Then, minimizing $E_{b,\xi}^{1D}$ with respect to ξ gives a minimal energy $E_{b,\xi_0}^{1D} = E_0^{1D}$ and a minimizer $\xi_0 > 0$. The proof of Theorem 1.5 is similar to the 2D-case and is given in Section 3.

Theorem 1.6 *For all $b \in (\Theta_0, 1]$, the function $[0, \frac{\pi}{2}] \ni \nu \mapsto e(b, \nu)$ is monotone non-decreasing.*

The proof of this statement is given in Section 4. Theorem 1.6 complements the result [12, Theorem 3.13].

The monotonicity of the quantity $e(b, \nu)$ with respect to the angle ν has an interest in the theory of superconductivity. Indeed, the ground state energy is a function of the inclination of the magnetic field and the result gives that the energy density increases when the magnetic field tends to be perpendicular to the surface of the sample. This complements results from the linear analysis: It is well-known that superconductivity survives longest—when the exterior magnetic field is increased—in the boundary region where the magnetic field is parallel. Theorem 1.6 shows that also before this happens superconductivity is strongest where the magnetic field is parallel and monotonically decreases as the angle increases. This interpretation of

Theorem 1.6 follows from the correspondence between $e(b, \nu)$ and the density of superconductivity illustrated in (1.3).

Finally, in Section 5, we construct solutions with lattice structure to the (5.11), in the case when $\nu \neq 0$.

Proposition 1.7 *When $\nu \neq 0$, (5.11) has solutions with 3D lattice structure.*

This proposition will be proved (see the proof of Theorem 5.5) in Section 5. Proposition 1.7 suggests that, a bounded superconductor subjected to an applied magnetic field with strength between the second critical field H_{C_2} and the third critical field H_{C_3} , will be in a surface superconducting state, and vortex lattices will appear along some surface layers where the applied magnetic field is not tangential to the surface.

The construction of the lattice solutions is completely analogous to the Abrikosov solutions in 2D. The result we prove – Theorem 5.5, stated in Section 5.2 – implies an upper bound on the energies (1.11) and (1.12) defined in the first section. This situation complements Theorem 1.5: For $\nu = 0$ we have a separation of variables implying that the modulus of a minimizer is constant along the boundary. As soon as $\nu \neq 0$ we expect—still in analogy with the 2D-case—this continuous symmetry to be broken and be replaced by the discrete (lattice) symmetry. In particular, for $\nu \neq 0$ we expect minimizers to display a vortex lattice structure. However, just as this remains unproven in 2D, we also cannot give a rigorous proof of this expectation in the present 3D setting. The above discussion is also very much related to the Open Problem 3 in [9] considering the structure of general bounded solutions of the Euler-Lagrange equations of the Ginzburg-Landau functional.

In conclusion, the results of this paper demonstrate how the surface superconductivity varies along the boundary of a finite sample when the external magnetic field is between H_{C_2} and H_{C_3} . Superconductivity is strongest where the external magnetic field is parallel to the boundary and decreases monotonically with the angle. Also, where the field is parallel, the structure is precisely given in terms of a 1D-model by separation of variables. For non-zero angles this separation of variables seems to fail and is probably replaced by a lattice-type structure similar to the 2D-situation of Abrikosov lattices.

2 The Linear Problem

Before starting the analysis of the non-linear Ginzburg-Landau functional $\mathcal{E}_{b,\nu,\ell}$, we describe some of the linear spectral results that are needed.

In all of the paper, we will denote by $\sigma(\mathcal{L})$ the spectrum of any given operator \mathcal{L} .

First we consider the magnetic Schrödinger operator associated with a constant magnetic field in \mathbb{R}^3 .

Proposition 2.1 *Let the magnetic Schrödinger operator $(-i\nabla + \mathbf{A}_\nu)^2$ be defined as a self-adjoint operator on $L^2(\mathbb{R}^3)$ with form domain*

$$\left\{ \psi \in L^2(\mathbb{R}^3) : (-i\nabla + \mathbf{A}_\nu)\psi \in L^2(\mathbb{R}^3, \mathbb{C}^3) \right\}.$$

For all $\nu \in [0, \frac{\pi}{2}]$ we have

$$\inf \sigma \{(-i\nabla + \mathbf{A}_\nu)^2\} = 1.$$

Proof This is just the well known structure of the Landau bands for constant magnetic field in \mathbb{R}^3 . \square

Consider the Schrödinger operator of a particle moving in a 3-dimensional half-space \mathbb{R}_+^3 , subject to a constant magnetic field of unit strength having an angle ν to the boundary-plane $\partial\mathbb{R}_+^3$,

$$\mathcal{L}(\nu) = (-i\nabla + \mathbf{A}_\nu)^2 \quad \text{in } L^2(\mathbb{R}_+^3), \quad (2.1)$$

with domain

$$D(\mathcal{L}(\nu)) = \{u \in L^2(\mathbb{R}_+^3) : (-i\nabla + \mathbf{A}_\nu)u \in L^2(\mathbb{R}_+^3, \mathbb{C}^3), \\ (-i\nabla + \mathbf{A}_\nu)^2 u \in L^2(\mathbb{R}_+^3), \quad \partial_{x_1} u = 0 \text{ on } \partial\mathbb{R}_+^3\}. \quad (2.2)$$

The spectrum of the Schrödinger operator with (magnetic) Neumann boundary condition introduced in (2.1) has been the object of study of several works and is by now well understood. We denote by $\zeta(\nu)$ the lowest point in the spectrum of $\mathcal{L}(\nu)$,

$$\zeta(\nu) = \inf \sigma(\mathcal{L}(\nu)). \quad (2.3)$$

We collect below some properties concerning the quantity $\zeta(\nu)$ (see e.g. [9, Lemmas 7.2.1 & 7.2.2]).

In connection with the analysis of the operator $\mathcal{L}(\nu)$, we introduce the two-dimensional operator

$$L(\nu) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (-x_1 \cos \nu + x_2 \sin \nu)^2 \quad \text{in } H^2(\mathbb{R}_+^2),$$

whose domain $D(L(\nu))$ is

$$D(L(\nu)) = \{u \in H^2(\mathbb{R}_+^2) : (-x_1 \cos \nu + x_2 \sin \nu)^j u \in L^2(\mathbb{R}_+^2), j = 1, 2, \\ \partial_{x_1} u = 0 \text{ on } \partial\mathbb{R}_+^2\}.$$

Remark 2.2 It is elementary to verify that for $\nu \neq 0$, $L(\nu)$ is closely related to $\mathcal{L}(\nu)$. Indeed, one can carry out a partial Fourier transform in the third variable and—for $\nu \neq 0$ —make a translation in x_2 to absorb the Fourier parameter. The resulting operator is exactly $L(\nu)$.

Lemma 2.3 *Let Θ_0 be the universal constant introduced in (1.5). The function $[0, \pi/2] \ni \nu \mapsto \zeta(\nu)$ is continuous, monotonically non-decreasing, and we have that $\zeta(0) = \Theta_0$ and $\zeta(\pi/2) = 1$. Furthermore, for all $\nu \in (0, \pi/2)$ we have*

- (1) $\sigma(\mathcal{L}(\nu)) = \sigma(L(\nu))$;
- (2) $\sigma_{\text{ess}}(L(\nu)) = [1, \infty)$.

Conclusion (2) remains true for $\nu = 0, \pi/2$. The results of Lemma 2.3 (concerning the spectrum of $\mathcal{L}(\nu)$) have been obtained in [19, Theorem 3.1], before being

improved in [17]. Notice that the dimensional reduction in conclusion (2.3) in Lemma 2.3 is only valid for $\nu > 0$.

Remark 2.4 Suppose that $\nu \in (0, \pi/2)$. It results from Lemma 2.3 that $\zeta(\nu)$ is the lowest eigenvalue of $L(\nu)$. It is a simple eigenvalue by a positivity argument. Consequently, we can select a unique non-negative eigenfunction $\phi_\nu^{2D} \in L^2(\mathbb{R}_+^2)$, normalized and such that

$$\int_{\mathbb{R}_+^2} \left\{ |\nabla \phi_\nu^{2D}|^2 + |(-x_1 \cos \nu + x_2 \sin \nu) \phi_\nu^{2D}|^2 \right\} dx = \zeta(\nu). \tag{2.4}$$

The next result concerns the decay of the function ϕ_ν^{2D} that we will need. We refer to [23] for a stronger statement.

Proposition 2.5 [See [23]] *Let $\nu \in (0, \frac{\pi}{2})$. The ground state ϕ_ν^{2D} of the operator $L(\nu)$ belongs to the Schwartz class $\mathcal{S}(\overline{\mathbb{R}_+^2})$.*

For completeness, let us mention that other decay properties of the eigenfunction ϕ_ν^{2D} are established in [4, Theorem 1.1].

In [19, Theorem 4.2], it is proved that for all $\nu \in (0, \frac{\pi}{2})$, the dimension of the eigenspace associated with the lowest eigenvalue $\zeta(\nu)$ for the operator $\mathcal{L}(\nu)$ is infinite. Thus, we have that $\zeta(\nu)$ is not a discrete eigenvalue but belongs to the essential spectrum of the operator $\mathcal{L}(\nu)$. The following result gives the form of the $L^2(\mathbb{R}_+^3)$ eigenfunctions.

Lemma 2.6 *Let $\nu \in (0, \frac{\pi}{2})$. For all $f \in L^2(\mathbb{R})$, the function ϕ_ν^{3D} defined by*

$$\phi_\nu^{3D}(x) = \mathcal{F}^{-1} \left(\xi_3 \mapsto f(\xi_3) \phi_\nu^{2D} \left(x_1, x_2 - \frac{\xi_3}{\sin \nu} \right) \right), \tag{2.5}$$

(where \mathcal{F}^{-1} is the inverse Fourier transform in the ξ_3 variable) is an $L^2(\mathbb{R}_+^3)$ eigenfunction associated with the eigenvalue $\zeta(\nu)$ of the operator $\mathcal{L}(\nu)$ (in particular, the Neumann condition at the boundary of \mathbb{R}_+^3 is satisfied), and all the $L^2(\mathbb{R}_+^3)$ eigenfunctions associated with the eigenvalue $\zeta(\nu)$ are of this form.

What is more, for every $f \in \mathcal{C}_c^\infty(\mathbb{R})$ (the set of smooth functions with compact support), we have that the function ϕ_ν^{3D} defined by (2.5) belongs to the Schwartz class $\mathcal{S}(\overline{\mathbb{R}_+^3})$.

Proof To prove the first part of the statement, we note that the fact that the function ϕ_ν^{3D} is an eigenfunction associated with the eigenvalue $\zeta(\nu)$ of the operator $\mathcal{L}(\nu)$ comes from the fact that the function ϕ_ν^{2D} is an eigenfunction associated with the eigenvalue $\zeta(\nu)$ of the operator $L(\nu)$. We refer to the proof of Lemma 4.5 in [19] for the details.

We prove here the last assertion of the lemma. We will only prove the decay of ϕ_ν^{3D} since the same decay properties of the derivatives can be obtained in the same way, using that $\phi_\nu^{2D} \in \mathcal{S}(\overline{\mathbb{R}_+^2})$ and $f \in \mathcal{C}_c^\infty(\mathbb{R})$.

Since $\phi_v^{2D} \in \mathcal{S}(\overline{\mathbb{R}_+^2})$ and f belongs to $\mathcal{C}_c^\infty(\mathbb{R})$, we easily have that the function ϕ_v^{3D} defined by (2.5) belongs to $\mathcal{C}^\infty(\mathbb{R}^3)$.

It suffices to establish the decay in each variable individually. The decay in the x_1 variable (uniformly in x_2) is obvious since the function ϕ_v^{2D} belongs to $\mathcal{S}(\overline{\mathbb{R}_+^2})$ (in the x_1 and x_2 variables, see Proposition 2.5). The good estimate in the x_3 variable is also straightforward since the Fourier transform of a function in L^1 is bounded. We now deal with the decay in the x_2 variable. Let $k \in \mathbb{N}$. We are going to give an upper bound on the term

$$\left| x_2^k \int_{\mathbb{R}_{\xi_3}} \phi_v^{2D} \left(x_1, x_2 - \frac{\xi_3}{\sin v} \right) e^{ix_3 \xi_3} f(\xi_3) d\xi_3 \right|.$$

Let $\text{supp } f \subset [-M, M]$. For convenience, we perform a change of variable in the integral ($\eta_3 = x_2 - \frac{\xi_3}{\sin v}$) and estimate

$$\begin{aligned} & \left| x_2^k e^{ix_3 x_2 \sin v} \int_{\mathbb{R}_{\eta_3}} \phi_v^{2D}(x_1, \eta_3) e^{-ix_3 \eta_3 \sin v} f((x_2 - \eta_3) \sin v) d\eta_3 \right| \\ & \leq |x_2|^k \|f\|_\infty \int_{x_2 - M/\sin v}^{x_2 + M/\sin v} |\phi_v^{2D}(x_1, \eta_3)| d\eta_3 \\ & \leq \frac{2M}{\sin v} \|f\|_\infty \frac{|x_2|^k}{(1 + |x_2 - M/\sin v|^2)^N} \sup_{\mathbb{R}_+^2} \left((1 + \eta^2)^N |\phi_v^{2D}(x_1, \eta)| \right), \end{aligned}$$

which gives the desired bound upon choosing $N \geq k$. □

3 Parallel Field

In this section we study the case when $\nu = 0$ and prove Theorem 1.5. We will prove that the ‘thermodynamic’ limit—the limit in (1.12)—can be expressed through the 1D functional $\mathcal{E}_{\mathfrak{b}, \xi}^{1D}$ given in (1.15). The principal properties of the functional $\mathcal{E}_{\mathfrak{b}, \xi}^{1D}$ are well known (see [20, Section 3 and Appendix], [9, Section 14.2] and the references therein, and also [10]). For understanding and completeness, we recall in the following lemmas the underlying results that we will need.

Lemma 3.1 *For all $\xi \in \mathbb{R}$ and all $\mathfrak{b} \in \mathbb{R}_+$, the functional $\mathcal{E}_{\mathfrak{b}, \xi}^{1D}$ admits a non-negative minimizer $f_{\mathfrak{b}, \xi}$, in the space*

$$B^1(\mathbb{R}_+) = \left\{ f \in L^2(\mathbb{R}_+, \mathbb{R}) : t^p f^{(q)}(t) \in L^2(\mathbb{R}_+, \mathbb{R}), \forall p, q \in \mathbb{N}, p + q \leq 1 \right\}.$$

The minimizers satisfy the Euler-Lagrange equations

$$\begin{cases} -f''_{\mathfrak{b}, \xi} + (t - \xi)^2 f_{\mathfrak{b}, \xi} = \mathfrak{b}(1 - f_{\mathfrak{b}, \xi}^2) f_{\mathfrak{b}, \xi}, & t > 0, \\ f'_{\mathfrak{b}, \xi}(0) = 0. \end{cases} \tag{3.1}$$

Moreover, we have

$$\inf_{f \in B^1(\mathbb{R}_+)} \mathcal{E}_{\mathfrak{b}, \xi}^{1D}(f) = -\frac{\mathfrak{b}}{2} \|f_{\mathfrak{b}, \xi}\|_{L^4(\mathbb{R}_+)}^4, \tag{3.2}$$

and the inequality:

$$\|f_{\mathfrak{b}, \xi}\|_{L^\infty(\mathbb{R}_+)} \leq 1.$$

What is more, the equation:

$$\begin{cases} -f'' + (t - \xi)^2 f = \mathfrak{b}(1 - f^2)f, & t > 0, \\ f'(0) = 0. \end{cases} \tag{3.3}$$

admits non-trivial bounded solutions if and only if $\mu_1(\xi) < \mathfrak{b}$ (see Section 1.1 (above) for the definition of μ_1), and if $f \in L^\infty(\mathbb{R}_+)$ satisfies (3.3), then we have that

$$\|f\|_{L^\infty(\mathbb{R}_+)} \leq 1 \quad \text{and} \quad f \in L^2(\mathbb{R}_+).$$

If we have $\mu_1(\xi) < \mathfrak{b} < 1$, the non-negative minimizer $f_{\mathfrak{b}, \xi} = f_\xi$ of the functional $\mathcal{E}_{\mathfrak{b}, \xi}^{1D}$ is unique and strictly positive.

We should notice that our conventions are slightly different from the ones considered in [9] (see for instance (3.9) in [9] and (1.4) in the present paper) and the ones considered in [5] (see for instance the choice of $(\nabla + i\mathbf{A})$ for the linear part of the Ginzburg-Landau functional instead of the expression $(-i\nabla + \mathbf{A})$ we consider). The statements we give here have been adapted to our choices.

Notation 3.2 We recall that f_{ξ_0} is defined above Theorem 1.6 and we denote $f_0 = f_{\xi_0}$.

The proof of Theorem 1.5 directly follows the approach presented in [5] and is made in two steps consisting in obtaining an upper and a lower bound.

Proof of Theorem 1.5 We will only prove Theorem 1.5 for $\mathfrak{b} \in (\Theta_0, 1)$ —the extension to $\mathfrak{b} = 1$ following by continuity on both sides. The continuity of the map $[0, \frac{\pi}{2}] \ni \nu \mapsto e(\mathfrak{b}, \nu)$ is given in [12, Theorem 3.13] for all $\nu \in [0, \frac{\pi}{2}]$. The continuity of E_0^{1D} is easier and is left to the reader.

Upper Bound For $\nu = 0$ and $\mathfrak{b} \in (\Theta_0, 1]$ (here the endpoint $\mathfrak{b} = 1$ can easily be included), the estimate

$$E(\mathfrak{b}, \nu = 0, \ell) \leq 4\ell^2 E_0^{1D} + o(\ell) \tag{3.4}$$

is obtained by considering the following trial state

$$\varphi(x) = \varphi(x_1, x_2, x_3) = f_0(x_1)e^{i\xi_0 x_3}, \tag{3.5}$$

(defined for any $x \in \mathbb{R}_+^3$) suitably localized in order to satisfy the Dirichlet boundary condition. This trial state has been used in [5]. We omit the details.

Lower bound The lower bound results from an energy decoupling through a ground state representation. It relies on some preliminary results related to the 1D-functional $\mathcal{E}_{\mathfrak{b}, \xi_0}^{1D}$. In the following lemma we gather the preliminary properties that we need, see [20, Section 3], [9, Section 14.2] and [5, Section 3.2] for details.

Lemma 3.3 *For all $\mathfrak{b} \in (\Theta_0, 1)$, the following properties hold.*

- (i) *The function f_0 (introduced in Notation 3.2) is strictly positive everywhere in \mathbb{R}_+ .*
- (ii) *The function F_0 defined by*

$$F_0(x_1) = 2 \int_0^{x_1} (y - \xi_0) f_0^2(y) dy, \tag{3.6}$$

satisfies $F_0(+\infty) = 0$ and is negative for all $x_1 > 0$.

- (iii) *The ‘cost function’ K_0 defined as:*

$$K_0(x_1) = f_0^2(x_1) + F_0(x_1), \tag{3.7}$$

is positive on \mathbb{R}_+ .

In the particular case when $\nu = 0$, the functional $\mathcal{E}_{\mathfrak{b}, \nu, \ell}$ defined through (1.9) has the following form:

$$\mathcal{E}_{\mathfrak{b}, \nu=0, \ell}(\varphi) = \int_{[-\ell, \ell]^2} \left(\int_0^{+\infty} \left\{ |(-i\nabla - x_1 \mathbf{e}_3)\varphi|^2 - \mathfrak{b}|\varphi|^2 + \frac{\mathfrak{b}}{2}|\varphi|^4 \right\} dx_1 \right) dx_2 dx_3, \tag{3.8}$$

where \mathbf{e}_3 is the unit vector in the x_3 direction. Thanks to the property (i) of Lemma (ii), to any function φ we may associate a function v with the Ansatz:

$$\varphi(x_1, x_2, x_3) = f_0(x_1) e^{i\xi_0 x_3} v(x_1, x_2, x_3). \tag{3.9}$$

By density of the set of functions with compact support in \mathcal{S}_ℓ , it suffices to work with functions satisfying

$$\varphi(x_1, x_2, x_3) = 0, \quad \text{for } x_1 \text{ sufficiently large.} \tag{3.10}$$

Considering (3.10) and using the variational (3.2) for f_0 , integration by parts yields,

$$\mathcal{E}_{\mathfrak{b}, \nu=0, \ell}(\varphi) = 4\ell^2 E_0^{1D} + \mathcal{E}_0(v), \tag{3.11}$$

where $\mathcal{E}_0(v)$ is defined as

$$\mathcal{E}_0(v) = \int_{\mathcal{D}_\ell} f_0^2(x_1) \left\{ |\nabla v|^2 - 2(x_1 - \xi_0) \mathbf{e}_3 \cdot \mathbf{j}(v) + \frac{\mathfrak{b}}{2} f_0^2(x_1) (1 - |v|^2)^2 \right\} dx, \tag{3.12}$$

and where $\mathbf{j}(v) = (j_1, j_2, j_3)$ is given by

$$\mathbf{j}(v) = \frac{i}{2} (v \overline{\nabla v} - \overline{v} \nabla v). \tag{3.13}$$

The boundary terms vanish because the function f_0 satisfies the Neumann boundary condition at $x_1 = 0$ (and using (3.10)).

Therefore, it suffices to prove the positivity of the reduced functional \mathcal{E}_0 given in (3.12). We define the following field $\mathbf{F}_0 = (0, F_0, 0)$, with F_0 from (3.6). We notice that

$$\left(2(x_1 - \xi_0) f_0^2\right) \mathbf{e}_3 \cdot \mathbf{j}(v) = (\partial_{x_1} F_0)(v \overline{\partial_{x_3} v} - \overline{v} \partial_{x_3} v).$$

An integration by parts in the x_1 variable on the term involving $\mathbf{j}(v)$ (using (3.10)), yields

$$\int_{\mathcal{D}_\ell} -2(x_1 - \xi_0) f_0^2(x_1) \mathbf{e}_3 \cdot \mathbf{j}(v) \, dx = \int_{\mathcal{D}_\ell} F_0 \partial_{x_1} j_3(v) \, dx.$$

An integration by parts in the x_3 variable gives, for each fixed x_1 and x_2 (and using the Dirichlet boundary condition),

$$\frac{1}{2} \int_{-\ell}^\ell \partial_{x_1} (v \overline{\partial_{x_3} v} - \overline{v} \partial_{x_3} v) \, dx_3 = -i \int_{-\ell}^\ell \operatorname{curl} \mathbf{j}(v) \cdot \mathbf{e}_2 \, dx_3.$$

Therefore, we have finally obtained

$$\mathcal{E}_0(v) = \int_{\mathcal{D}_\ell} \left\{ f_0^2(x_1) |\nabla v(x)|^2 + F_0(x_1) \mu(v(x)) + \frac{b}{2} f_0^4(x_1) (1 - |v(x)|^2)^2 \right\} dx,$$

where $\mu(v) = \operatorname{curl} \mathbf{j}(v) \cdot \mathbf{e}_2$. Using that F_0 is negative (by Lemma 3.3, 3.3), and $|\mu(v)| \leq |\operatorname{curl} \mathbf{j}(v)| \leq |\nabla v|^2$, we get

$$\begin{aligned} \mathcal{E}_0(v) &\geq \int_{\mathcal{D}_\ell} \left(f_0^2(x_1) |\nabla v(x)|^2 + F_0(x_1) |\mu(v(x))| \right) dx \\ &\geq \int_{\mathcal{D}_\ell} \left(f_0^2(x_1) + F_0(x_1) \right) |\nabla v(x)|^2 \, dx \\ &\geq 0, \end{aligned} \tag{3.14}$$

where the last inequality follows from (iii) in Lemma 3.3. This finishes the proof of Theorem 1.5. □

4 Ground State Energy for General Direction of the Magnetic Field

In this section we will prove Theorem 1.6. As stated in Lemma 2.3 in Section 2, a similar monotonicity result is valid for the linear problem, i.e. for the spectral quantity $\zeta(v)$. The proof of the monotonicity of $v \mapsto \zeta(v)$ is rather easy once a clever change of variables is implemented. Our proof of Theorem 1.6 is very much inspired by the analysis of the linear problem and uses the same change of variables performed in the proof of the monotonicity of the lowest eigenvalue (see [19, Theorem 3.1] and [17]).

Let us briefly explain the strategy of the proof. Directly differentiating the functional in the angle v leads to terms that are not easy to understand. The change of variables from [17, 19] leads to the new functional $\mathcal{E}_{b,v,\alpha,L,L_3}$ given in (4.34) below. Here the parameter dependence is elementary, however the domain is changed. The change of variables replaces effectively the ‘cylinder’ \mathcal{D}_ℓ by a tilted cylinder, i.e. a cylinder having an axis which is not perpendicular to the boundary plane and also with a cross section being non-quadratic. However, from general experience with

‘thermodynamic’ limits, this slight change of geometry should not affect the boundary energy. That this is the case is the content of Lemma 4.4 below. Once this is established, the variation of the functional with respect to the angle becomes similar to the Feynman-Hellmann theorem from perturbation theory.

4.1 Generalization of the Initial Problem in 3D

Let $v \in [0, \frac{\pi}{2}]$, $\ell > 0$ and $\ell_3 > 0$. For $\beta \in [0, \frac{\pi}{2})$, we introduce the set

$$\mathcal{D}_{\ell, \ell_3, \beta} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, |x_2 - x_1 \tan \beta| < \ell, |x_3| < \ell_3\},$$

for which we have the following correspondence with the set defined in (1.6)

$$\mathcal{D}_\ell = \mathcal{D}_{\ell, \ell, 0}.$$

We consider the magnetic potential \mathbf{A}_v defined on \mathbb{R}_+^3 by (1.7), for which we recall that the associated magnetic field is the constant unit vector that makes an angle v with the x_2x_3 plane (see (1.8)).

Definition 4.1 We consider the following reduced Ginzburg-Landau type energy functional

$$\mathcal{E}_{\mathbf{b}, v, \beta, \ell, \ell_3}(\varphi) = \int_{\mathcal{D}_{\ell, \ell_3, \beta}} \left\{ |(-i\nabla + \mathbf{A}_v)\varphi|^2 - \mathbf{b}|\varphi|^2 + \frac{\mathbf{b}}{2}|\varphi|^4 \right\} dx, \tag{4.1}$$

for φ in the space

$$\mathcal{S}_{\ell, \ell_3, \beta} = \left\{ \varphi \in L^2(\mathcal{D}_{\ell, \ell_3, \beta}) : (-i\nabla + \mathbf{A}_v)\varphi \in L^2(\mathcal{D}_{\ell, \ell_3, \beta}, \mathbb{C}^3), \right. \\ \left. \varphi = 0 \text{ on } \partial(\mathcal{D}_{\ell, \ell_3, \beta}) \setminus \{x_1 = 0\} \right\}.$$

Furthermore, we define

$$E(\mathbf{b}, v, \beta, \ell, \ell_3) = \inf_{\varphi \in \mathcal{S}_{\ell, \ell_3, \beta}} \mathcal{E}_{\mathbf{b}, v, \beta, \ell, \ell_3}(\varphi). \tag{4.2}$$

Remark 4.2 We recall what we mentioned in Remark 1.4. The existence of the limit

$$\lim_{\ell \rightarrow \infty} \frac{1}{4\ell^2} E(\mathbf{b}, v, \beta = 0, \ell, \ell) = e(\mathbf{b}, v, \beta = 0), \tag{4.3}$$

was proved in [2] and [12] (see in particular [12, Theorem 3.9] with [11]). Following exactly the same steps of the proof of Theorem 3.9 in [12], we can actually easily show that for all sequence $\{(\ell^{(n)}, \ell_3^{(n)})\}_n$ satisfying

$$0 < \ell^{(n)} \xrightarrow{n \rightarrow +\infty} +\infty, \quad \text{and} \quad C^{-1} < \frac{\ell^{(n)}}{\ell_3^{(n)}} < C \quad \forall n, \tag{4.4}$$

with a fixed constant $C > 0$, we have that the following limit

$$e(\mathbf{b}, v, \beta = 0) := \lim_{n \rightarrow +\infty} \frac{1}{4\ell^{(n)}\ell_3^{(n)}} E(\mathbf{b}, v, \beta = 0, \ell^{(n)}, \ell_3^{(n)}) \tag{4.5}$$

exists, and is independent of the choice of sequence $\{(\ell^{(n)}, \ell_3^{(n)})\}_n$.

We will need the following result which gives in particular the existence of a minimizer and that all the minimizers have a good decay at infinity in the transverse variable x_1 .

Lemma 4.3 *Suppose that $\mathfrak{b} \in [\Theta_0, 1]$ and $\beta \in [0, \frac{\pi}{2})$ are fixed given constants. For all $\nu \in [0, \frac{\pi}{2}]$, $\ell > 0$ and $\ell_3 > 0$, the functional $\mathcal{E}_{\mathfrak{b}, \nu, \beta, \ell, \ell_3}$ in (4.1) has a minimizer, and any minimizer $\underline{\varphi} = \varphi_{\mathfrak{b}, \nu, \beta, \ell, \ell_3}$ satisfies*

$$\mathcal{E}_{\mathfrak{b}, \nu, \beta, \ell, \ell_3}(\underline{\varphi}) = E(\mathfrak{b}, \nu, \beta, \ell, \ell_3), \quad \|\underline{\varphi}\|_{L^\infty(\mathcal{D}_{\ell, \ell_3, \beta})} \leq 1. \tag{4.6}$$

and

$$\int_{\mathcal{D}_{\ell, \ell_3, \beta}} \left\{ |(-i\nabla + \mathbf{A}_\nu)\underline{\varphi}|^2 - \mathfrak{b}|\underline{\varphi}|^2 + \frac{\mathfrak{b}}{2}|\underline{\varphi}|^4 \right\} dx = -\frac{\mathfrak{b}}{2} \int_{\mathcal{D}_{\ell, \ell_3, \beta}} |\underline{\varphi}|^4 dx. \tag{4.7}$$

Furthermore, there exists a positive constant $C(\mathfrak{b}, \beta)$ such that if $\nu \in [0, \frac{\pi}{2}]$, $\ell > 0$ and $\ell_3 > 0$, any minimizer $\underline{\varphi}$ satisfies

$$\int_{\mathcal{D}_{\ell, \ell_3, \beta} \cap \{x_1 > 4\}} \frac{x_1}{(\ln x_1)^2} \left\{ |(-i\nabla + \mathbf{A}_\nu)\underline{\varphi}|^2 + |\underline{\varphi}|^2 + x_1^2|\underline{\varphi}|^4 \right\} dx \leq C(\mathfrak{b}, \beta)\ell\ell_3. \tag{4.8}$$

Proof This statement is proved in [12, Theorem 3.6] (see also [20]) for $\beta = 0$ and $\ell = \ell_3$. The fact that we have a different domain $\mathcal{D}_{\beta, \ell, \ell_3}$ depending on the two additional parameters β and ℓ_3 does not change anything because they are both fixed in the proof.

We omit the details. □

The following result is a key lemma which generalizes Theorem 3.9 in [12]. The proof of this result strongly relies on the generalized result presented in Remark 4.2. We refer to [12] (and [11]) for the technical details.

Lemma 4.4 *Let $C > 0$ and let $\{(\ell^{(n)}, \ell_3^{(n)})\}_n$ be a sequence satisfying (4.4). We have that*

$$e(\mathfrak{b}, \nu, \beta) := \lim_{n \rightarrow +\infty} \frac{1}{4\ell^{(n)}\ell_3^{(n)}} E(\mathfrak{b}, \nu, \beta, \ell^{(n)}, \ell_3^{(n)})$$

exists, is independent of the choice of sequence $\{(\ell^{(n)}, \ell_3^{(n)})\}_n$ and moreover is independent of β .

Thanks to Lemma 4.4, $e(\mathfrak{b}, \nu, \beta)$ is independent of β . Hence we can omit β from the notation and write

$$e(\mathfrak{b}, \nu) = e(\mathfrak{b}, \nu, \beta) = e(\mathfrak{b}, \nu, 0). \tag{4.9}$$

Proof of Lemma 4.4 We fix $C > 0$ and choose a sequence $\{(\ell^{(n)}, \ell_3^{(n)})\}_n$ such that (4.4) holds. For convenience, below we do not systematically indicate the dependence on n in the notation.

For all $\tilde{\ell} > 0$, we define the following box

$$\mathcal{D}_{\ell, \ell_3, \beta}^{\tilde{\ell}} = \{x = (x_1, x_2, x_3) \in \mathcal{D}_{\ell, \ell_3, \beta} : x_1 \in (0, \tilde{\ell})\},$$

and for all $\ell > 0$, $\ell_3 > 0$, $\nu \in (0, \frac{\pi}{2})$ and $\beta \in [0, \frac{\pi}{2})$, we denote by $E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, \beta, \ell, \ell_3)$ the minimal energy of the functional $\mathcal{E}_{\mathbf{b}, \nu, \beta, \ell, \ell_3}$ for which the set of test functions is given by all $\varphi \in \mathcal{S}_{\ell, \ell_3, \beta}$, such that φ satisfies the following Dirichlet condition

$$\varphi = 0 \text{ on } \partial \mathcal{D}_{\ell, \ell_3, \beta}^{\tilde{\ell}} \setminus \{x_1 = 0\}, \text{ and } \varphi \text{ extended by } 0 \text{ outside of } \mathcal{D}_{\ell, \ell_3, \beta}^{\tilde{\ell}}.$$

We are going to give upper bound and lower bounds on $E(\mathbf{b}, \nu, \beta, \ell, \ell_3)$ involving the quantity $E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, \beta, \ell, \ell_3)$.

The upper bound is obvious since, by inclusion of the variational spaces, we have

$$E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, \beta, \ell, \ell_3) \geq E(\mathbf{b}, \nu, \beta, \ell, \ell_3). \tag{4.10}$$

To obtain a lower bound on $E(\mathbf{b}, \nu, \beta, \ell, \ell_3)$, we start by choosing $\tilde{\ell}$ as follows

$$\tilde{\ell} = \ell^\varrho, \tag{4.11}$$

where $\varrho > 0$ is a constant which will be chosen later on. We consider a real number $\tilde{\ell}_-$ such that

$$4 \leq \tilde{\ell}_- < \tilde{\ell} \text{ with } |\tilde{\ell} - \tilde{\ell}_-| = R(\ell), \tag{4.12}$$

where $R(\ell)$ is a strictly positive quantity which can depend on ℓ and which will be given below. We consider two smooth functions χ_1 and χ_2 defined on \mathbb{R}_+^3 and constituting a partition of unity such that for all $x \in \mathbb{R}_+^3 \supset \mathcal{D}_{\ell, \ell_3, \beta}$:

$$\chi_1(x) = \chi_1(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 \in [0, \tilde{\ell}_-], \\ 0 & \text{if } x_1 \in [\tilde{\ell}, +\infty), \end{cases}$$

and

$$\chi_2(x) = \chi_2(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_1 \in [0, \tilde{\ell}_-], \\ 1 & \text{if } x_1 \in [\tilde{\ell}, +\infty), \end{cases}$$

with

$$\chi_1^2 + \chi_2^2 = 1 \text{ on } \mathbb{R}_+^3,$$

and

$$\sup_{x \in \mathbb{R}_+^3} |\nabla \chi_j(x)| \leq \frac{C}{R(\ell)}, \text{ for all } j \in \{1, 2\}. \tag{4.13}$$

For all $\varphi \in \mathcal{S}_{\ell, \ell_3, \beta}$, using the fact that

$$\int_{\mathcal{D}_{\ell, \ell_3, \beta}} (\chi_j^2(x) - \chi_j^4(x)) |\varphi(x)|^4 dx \geq 0$$

for all $j \in \{1, 2\}$ (since $0 \leq \chi_j(x) \leq 1$), we have the following standard (e.g. IMS) decomposition formula (see for instance [12, Section 5.1])

$$\mathcal{E}_{\mathbf{b},v,\beta,\ell,\ell_3}(\varphi) \geq \mathcal{E}_{\mathbf{b},v,\beta,\ell,\ell_3}(\chi_1\varphi) + \mathcal{E}_{\mathbf{b},v,\beta,\ell,\ell_3}(\chi_2\varphi) - \|\varphi|\nabla\chi_1\|_{L^2(\mathbb{R}_+^3)}^2 - \|\varphi|\nabla\chi_2\|_{L^2(\mathbb{R}_+^3)}^2. \tag{4.14}$$

To obtain a lower bound to $E(\mathbf{b}, v, \beta, \ell, \ell_3)$, we are going to use (4.14) by giving an estimate on each term of the right hand side of the inequality.

On one hand we have, by definition and using the property of χ_1 , that for all $\varphi \in \mathcal{S}_{\ell,\ell_3,\beta}$

$$\int_{\mathcal{D}_{\ell,\ell_3,\beta}} \left\{ |(-i\nabla + \mathbf{A}_v)\chi_1\varphi|^2 - \mathbf{b}|\chi_1\varphi|^2 + \frac{\mathbf{b}}{2}|\chi_1\varphi|^4 \right\} dx \geq E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, v, \beta, \ell, \ell_3). \tag{4.15}$$

On the other hand, using Proposition 2.1 (where $\chi_2\varphi$ is extended by zero to a regular function on all of $\mathbb{R}_+^3 \supset \mathcal{D}_{\ell,\ell_3,\beta}$), we have

$$\int_{\mathcal{D}_{\ell,\ell_3,\beta}} \left\{ |(-i\nabla + \mathbf{A}_v)\chi_2\varphi|^2 - \mathbf{b}|\chi_2\varphi|^2 + \frac{\mathbf{b}}{2}|\chi_2\varphi|^4 \right\} dx \geq 0, \tag{4.16}$$

since $\mathbf{b} \leq 1$.

It remains to give an estimate on the remainders of (4.14). Considering the support of the functions $|\nabla\chi_j|$ ($j \in \{1, 2\}$), and using (4.13), we have the following inequality

$$\sum_{j \in \{1,2\}} \|\varphi|\nabla\chi_j\|_{L^2(\mathbb{R}_+^3)}^2 \leq \frac{C^2}{R(\ell)^2} \int_{\tilde{\ell}_- \leq x_1 \leq \tilde{\ell}} |\varphi|^2 dx. \tag{4.17}$$

To estimate $\int_{\tilde{\ell}_- \leq x_1 \leq \tilde{\ell}} |\varphi|^2 dx$, we use Lemma 4.3 which gives in particular that for any $\gamma \in [1, 2)$ there exists a constant $C(\mathbf{b}, \beta, \gamma) > 0$ (which only depends on \mathbf{b}, β and γ) such that:

$$\int_{\mathcal{D}_{\ell,\ell_3,\beta} \cap \{x_1 > 4\}} x_1^\gamma |\varphi|^2 dx \leq C(\mathbf{b}, \beta, \gamma)\ell\ell_3. \tag{4.18}$$

Using (4.18) for γ in the form $\gamma = 1 - \epsilon$ with $\epsilon \in (0, 1)$, we have that, if R is bounded uniformly in ℓ and if ρ is fixed, then there exist a constant $C(\mathbf{b}, \beta) > 0$ (also depending the choice of ϵ) such that

$$\int_{\tilde{\ell}_- \leq x_1 \leq \tilde{\ell}} |\varphi|^2 dx \leq C(\mathbf{b}, \beta) \frac{\ell\ell_3}{\ell^{\varrho(1-\epsilon)}}. \tag{4.19}$$

We choose

$$0 < R(\ell) = R < 2 \quad \text{and} \quad \varrho = \frac{1}{2}, \tag{4.20}$$

where R is a fixed constant independent of ℓ and where we recall that ϱ is introduced in (4.11). Thus, we are lead to consider

$$\tilde{\ell} = \ell^{1/2}. \tag{4.21}$$

With the choices given by (4.20), we have $\tilde{\ell}_- = \tilde{\ell} + o(\tilde{\ell}) = \ell^{1/2} + o(\ell^{1/2})$. Therefore, we have with (4.19) that

$$\int_{\tilde{\ell}_- \leq x_1 \leq \tilde{\ell}} |\varphi|^2 dx \leq C(\mathbf{b}, \beta) \ell^{1/2 + \epsilon/2} \ell_3, \tag{4.22}$$

which finally gives

$$\sum_{j \in \{1, 2\}} \|\varphi | \nabla \chi_j \|_{L^2(\mathbb{R}_+^3)}^2 \leq C(\mathbf{b}, \beta) \ell^{1/2 - \epsilon/2} \ell_3, \tag{4.23}$$

Using (4.14) with a minimizer $\varphi = \varphi_{\mathbf{b}, \nu, \beta, \ell, \ell_3}$ of the functional $\mathcal{E}_{\mathbf{b}, \nu, \beta, \ell, \ell_3}$, and the estimates (4.15), (4.16) and (4.23), we obtain

$$E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, \beta, \ell, \ell_3) + o(\ell \ell_3) \leq E(\mathbf{b}, \nu, \beta, \ell, \ell_3). \tag{4.24}$$

Relying on (4.10) and (4.24), we can deal with $E(\mathbf{b}, \nu, \beta, \ell, \ell_3)$ using $E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, \beta, \ell, \ell_3)$ up to a remainder in $o(\ell \ell_3)$. We denote by ℓ_* the length of the segment joining the two points $(x_1 = \tilde{\ell}, x_2 = \ell, x_3 = 0)$ and $(x_1 = \tilde{\ell}, x_2 = \ell + \tilde{\ell} \tan \beta, x_3 = 0)$. The value of ℓ_* is given by

$$\ell_* = (\tan \beta) \tilde{\ell} = (\tan \beta) \ell^{1/2}. \tag{4.25}$$

For $\underline{\ell} = \ell - \ell_*$ and $\bar{\ell} = \ell + \ell_*$, we have the following geometrical inclusions

$$\mathcal{D}_{\underline{\ell}, \ell_3, 0}^{\tilde{\ell}} \subset \mathcal{D}_{\ell, \ell_3, \beta}^{\tilde{\ell}} \subset \mathcal{D}_{\bar{\ell}, \ell_3, 0}^{\tilde{\ell}}. \tag{4.26}$$

Using the inclusions (4.26), we have the following inequalities:

$$\frac{1}{4\ell \ell_3} E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, \beta, \ell, \ell_3) \leq \frac{\ell \ell_3}{\ell \ell_3} \left(\frac{1}{4\underline{\ell} \ell_3} E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, 0, \underline{\ell}, \ell_3) \right), \tag{4.27}$$

and

$$\frac{\bar{\ell} \ell_3}{\ell \ell_3} \left(\frac{1}{4\bar{\ell} \ell_3} E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, 0, \bar{\ell}, \ell_3) \right) \leq \frac{1}{4\ell \ell_3} E_{\tilde{\ell}}^{\text{Dir}}(\mathbf{b}, \nu, \beta, \ell, \ell_3). \tag{4.28}$$

Therefore, using (4.10), (4.24), we can get rid of the Dirichlet energy and obtain from (4.27) and (4.28) the following inequalities:

$$\frac{1}{4\ell \ell_3} E(\mathbf{b}, \nu, \beta, \ell, \ell_3) \leq \frac{\ell \ell_3}{\ell \ell_3} \left(\frac{1}{4\underline{\ell} \ell_3} (E(\mathbf{b}, \nu, 0, \underline{\ell}, \ell_3) + o(\ell \ell_3)) \right), \tag{4.29}$$

and

$$\frac{\bar{\ell} \ell_3}{\ell \ell_3} \left(\frac{1}{4\bar{\ell} \ell_3} E(\mathbf{b}, \nu, 0, \bar{\ell}, \ell_3) \right) \leq \frac{1}{4\ell \ell_3} (E(\mathbf{b}, \nu, \beta, \ell, \ell_3) + o(\ell \ell_3)). \tag{4.30}$$

As $\underline{\ell} = \ell + o(\ell)$ and $\bar{\ell} = \ell + o(\ell)$ (see (4.25)), we finally obtain (using the result given in Remark (4.2)) that

$$\limsup_{n \rightarrow +\infty} \frac{1}{4\ell^{(n)} \ell_3^{(n)}} E(\mathbf{b}, \nu, \beta, \ell^{(n)}, \ell_3^{(n)}) \leq \lim_{n \rightarrow +\infty} \frac{1}{4\ell^{(n)} \ell_3^{(n)}} E(\mathbf{b}, \nu, 0, \ell^{(n)}, \ell_3^{(n)}), \tag{4.31}$$

and

$$\lim_{n \rightarrow +\infty} \frac{1}{4\ell^{(n)}\ell_3^{(n)}} E(\mathbf{b}, \nu, 0, \ell^{(n)}, \ell_3^{(n)}) \leq \liminf_{n \rightarrow +\infty} \frac{1}{4\ell^{(n)}\ell_3^{(n)}} E(\mathbf{b}, \nu, \beta, \ell^{(n)}, \ell_3^{(n)}). \tag{4.32}$$

The result is obtained using (4.31) and (4.32). □

4.2 Proof of Theorem 1.6

In this subsection we will give the proof of Theorem 1.6 by comparing the energy of the functional $\mathcal{E}_{\mathbf{b}, \nu, \ell}$ on \mathcal{D}_ℓ to the energy of an auxiliary functional $\tilde{\mathcal{E}}$ on $\tilde{\mathcal{D}}$, where for $L, L_3 > 0$ and $\alpha \in (0, \pi)$ we define

$$\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_{L, L_3, \alpha} = \tilde{\mathcal{D}}_{L, \alpha} \times (-L_3, L_3),$$

with

$$\tilde{\mathcal{D}}_{L, \alpha} = \left\{ (v_1, v_2) \in \mathbb{R}^2 : v_1 > -v_2, \quad |v_2 - (\tan \alpha)v_1| \leq \frac{L}{\sqrt{2}}(1 + \tan \alpha) \right\} \tag{4.33}$$

To emphasize the different domains, we keep the notation (x_1, x_2, x_3) for the coordinates in \mathcal{D}_ℓ for the original problem, and use (v_1, v_2, v_3) to denote the coordinates in the set $\tilde{\mathcal{D}}$ for the auxiliary problem.

Remark 4.5 Notice that the area of $\tilde{\mathcal{D}}_{L, L_3, \alpha} \cap \{v_1 = -v_2\}$ is $4LL_3$.

Define furthermore the functional

$$\begin{aligned} \tilde{\mathcal{E}}(\tilde{\varphi}) &= \tilde{\mathcal{E}}_{\mathbf{b}, \nu, \alpha, L, L_3}(\tilde{\varphi}) \\ &= \int_{\tilde{\mathcal{D}}_{L, L_3, \alpha}} \left\{ |\partial_1 \tilde{\varphi}|^2 + \tan^2 \nu |\partial_2 \tilde{\varphi}|^2 + |(-i\partial_3 + v_1)\tilde{\varphi}|^2 - \mathbf{b}|\tilde{\varphi}|^2 + \frac{\mathbf{b}}{2} \tan \nu |\tilde{\varphi}|^4 \right\} dv_1 dv_2 dv_3, \end{aligned} \tag{4.34}$$

for $\tilde{\varphi}$ in the space

$$\begin{aligned} \tilde{\mathcal{S}}_{L, L_3, \alpha} &= \left\{ \tilde{\varphi} \in L^2(\tilde{\mathcal{D}}_{L, L_3, \alpha}) : \right. \\ &\quad \left. (|\partial_1 \tilde{\varphi}|^2 + \tan^2 \nu |\partial_2 \tilde{\varphi}|^2 + |(-i\partial_3 + v_1)\tilde{\varphi}|^2) \in L^2(\tilde{\mathcal{D}}_{L, L_3, \alpha}), \right. \\ &\quad \left. \tilde{\varphi} = 0 \text{ on } \partial\tilde{\mathcal{D}}_{L, L_3, \alpha} \setminus \{v_2 = -v_1\} \right\}. \end{aligned}$$

Also, introduce the following ground state energy,

$$\tilde{E} = \tilde{E}(\mathbf{b}, \nu, \alpha, L, L_3) = \inf_{\tilde{\varphi} \in \tilde{\mathcal{S}}_{L, L_3, \alpha}} \tilde{\mathcal{E}}(\tilde{\varphi}). \tag{4.35}$$

In the following lemma, we collect the needed results about $\tilde{\mathcal{E}}$.

Lemma 4.6 *Suppose that $\mathfrak{b} \in (\Theta_0, 1]$ and $\alpha \in (0, \pi)$. For all $\nu \in (0, \frac{\pi}{2})$, $L > 0$ and $L_3 > 0$, the functional $\tilde{\mathcal{E}}_{\mathfrak{b}, \nu, \alpha, L, L_3}$ defined in (4.34) has a minimizer, and any minimizer $\tilde{\varphi} = \varphi_{\mathfrak{b}, \nu, \alpha, L, L_3}$ satisfies*

$$\tilde{\mathcal{E}}_{\mathfrak{b}, \nu, \alpha, L, L_3}(\tilde{\varphi}) = \tilde{E}(\mathfrak{b}, \nu, \alpha, L, L_3), \quad \|\tilde{\varphi}\|_{L^\infty(\tilde{\mathcal{D}}_{L, L_3, \alpha})} \leq 1. \tag{4.36}$$

and the following Euler-Lagrange equation associated with the minimization problem (4.35)

$$\left(-\partial_{v_1}^2 - \tan^2 \nu \partial_{v_2}^2 + (-i\partial_{v_3} + v_1)^2\right)\tilde{\varphi} = \mathfrak{b}(1 - \tan \nu |\tilde{\varphi}|^2)\tilde{\varphi}, \tag{4.37}$$

and

$$\begin{aligned} \int_{\tilde{\mathcal{D}}_{L, L_3, \alpha}} \left\{ |\partial_1 \tilde{\varphi}|^2 + \tan^2 \nu |\partial_2 \tilde{\varphi}|^2 + |(-i\partial_3 + v_1)\tilde{\varphi}|^2 - \mathfrak{b}|\tilde{\varphi}|^2 + \frac{\mathfrak{b}}{2} \tan \nu |\tilde{\varphi}|^4 \right\} d\nu \\ = -\frac{\mathfrak{b}}{2} \tan \nu \int_{\tilde{\mathcal{D}}_{L, L_3, \alpha}} |\tilde{\varphi}|^4 d\nu \end{aligned} \tag{4.38}$$

Furthermore, we have the following conclusions,

(i) *In the case when*

$$\alpha = \arctan(\tan^2(\nu)), \quad L_3 = \ell, \quad L = \sqrt{2}\ell \sin(\nu), \tag{4.39}$$

we have the identity

$$\tilde{E}(\mathfrak{b}, \nu, \alpha, L, L_3) = E(\mathfrak{b}, \nu, \ell). \tag{4.40}$$

In particular, still with this special relation between the parameters,

$$\sqrt{2} \sin(\nu) \frac{\tilde{E}(\mathfrak{b}, \nu, \alpha, L, L_3)}{4LL_3} = \frac{E(\mathfrak{b}, \nu, \ell)}{4\ell^2}. \tag{4.41}$$

(ii) *Let $C > 0$ and let $\{(L^{(n)}, L_3^{(n)})\}_n$ be a sequence satisfying*

$$0 < L^{(n)} \xrightarrow{n \rightarrow +\infty} +\infty, \quad \text{and} \quad C^{-1} < \frac{L^{(n)}}{L_3^{(n)}} < C \quad \forall n, \tag{4.42}$$

with a fixed constant $C > 0$. The limit

$$\tilde{e} = \tilde{e}(\mathfrak{b}, \nu, \alpha) = \lim_{n \rightarrow +\infty} \frac{1}{4L^{(n)}L_3^{(n)}} \tilde{E}(\mathfrak{b}, \nu, \alpha, L^{(n)}, L_3^{(n)})$$

is finite and is independent of C and of the sequence $\{(L^{(n)}, L_3^{(n)})\}_n$.

(iii) *The quantity $\tilde{e}(\mathfrak{b}, \nu, \alpha)$ is independent of α .*

Proof Obtaining the Euler-Lagrange (4.37) is rather standard. We use (4.37) to obtain (4.38) by an integration by parts after multiplying by the conjugate of $\tilde{\varphi}$, and using the fact that the minimizer satisfies the zero Dirichlet boundary condition at the boundary portion where $v_2 = -v_1$, and satisfies the zero Neumann boundary condition at other part of the boundary, thus all boundary terms in the integral vanish.

The proof of the (i) is by a change of coordinates. Composing the two changes of variables

$$\begin{cases} x_1 = -u_1 \cos v - u_2 \sin v \\ x_2 = u_1 \sin v - u_2 \cos v \\ x_3 = u_3 \end{cases} \quad \text{and next} \quad \begin{cases} u_1 = -v_1 \\ u_2 = \frac{v_2}{-\tan v} \\ u_3 = v_3 \end{cases},$$

which are respectively a rotation and a dilatation in the second variable, changes the functional expression and maps the domain \mathcal{D}_ℓ onto the domain $\tilde{\mathcal{D}}_{\ell,v}$, where $\tilde{\mathcal{D}}_{\ell,v}$ is given by

$$\tilde{\mathcal{D}}_{\ell,v} = \left\{ (v_1, v_2) \in \mathbb{R}^2 : v_1 > -v_2, -\frac{\ell \tan v}{\cos v} < v_2 - v_1 \tan^2 v < \frac{\ell \tan v}{\cos v} \right\} \times (-\ell, \ell).$$

Combined with the change of function $\varphi = \sqrt{\tan v} \tilde{\varphi}$ we get

$$\tilde{\mathcal{E}}(\tilde{\varphi}) = \mathcal{E}_{b,v,\ell}(\varphi).$$

It is also clear that when α, L, L_3 satisfy (4.39) we have

$$\tilde{\mathcal{D}}_{L,L_3,\alpha} = \tilde{\mathcal{D}}_{\ell,v}.$$

Thus, (4.40) is proved. Furthermore, (4.41) follows immediately using Remark 4.5.

The second and the third assertions follow from the results of [12], as mentioned in Remark 4.2. □

Proof of Theorem 1.6 It suffices to prove the monotonicity of $e(b, v)$ in v restricted to $v \in (0, \frac{\pi}{2})$ since $e(b, \cdot)$ is continuous by [12, Theorem 3.13]. So in the remainder of the proof we work under this restriction.

By Lemma 4.6, we can define

$$\tilde{E}(b, v, L) = \tilde{E}(b, v, \alpha = \frac{\pi}{4}, L, L),$$

and (as in (4.9))

$$\tilde{e}(b, v) = \tilde{e}(b, v, \alpha = \frac{\pi}{4}). \tag{4.43}$$

Using the first point (i) of Lemma 4.6, we have the following correspondence

$$e(b, v) = \sqrt{2} \sin v \tilde{e}(b, v). \tag{4.44}$$

We consider

$$\Delta_{b,v}(\varepsilon) = e(b, v + \varepsilon) - e(b, v).$$

Using (4.44) and Lemma 4.6, we can write

$$\Delta_{b,v}(\varepsilon) = \sqrt{2} \lim_{L \rightarrow +\infty} \left(\sin(v + \varepsilon) \frac{\tilde{E}(b, v + \varepsilon, L)}{4L^2} - \sin(v) \frac{\tilde{E}(b, v, L)}{4L^2} \right). \tag{4.45}$$

We take $\varepsilon > 0$ and we are looking for a positive lower bound on $\Delta_{b,v}(\varepsilon)$ in order to prove the monotonicity. We will use a minimizer of the functional

$$\tilde{\mathcal{E}}_{b, v + \varepsilon, L} = \tilde{\mathcal{E}}_{b, v + \varepsilon, \alpha = \frac{\pi}{4}, L, L}.$$

This minimizer exists and will be denoted $\tilde{\varphi} = \tilde{\varphi}_{b, v + \varepsilon, L}$. Therefore we have

$$\tilde{\mathcal{E}}_{b, v, L}(\tilde{\varphi}) \geq \tilde{E}(b, v, L) \quad \text{and} \quad \tilde{\mathcal{E}}_{b, v + \varepsilon, L}(\tilde{\varphi}) = \tilde{E}(b, v + \varepsilon, L).$$

Upon inserting this in (4.45) we get

$$\Delta_{b,v}(\varepsilon) \geq \frac{\sqrt{2}}{4} \lim_{L \rightarrow +\infty} \frac{1}{L^2} \left(\sin(v + \varepsilon) \tilde{\mathcal{E}}_{b,v+\varepsilon,L}(\tilde{\varphi}) - \sin(v) \tilde{\mathcal{E}}_{b,v,L}(\tilde{\varphi}) \right).$$

Considering the integral expression given by (4.34) and writing

$$\begin{aligned} \sin(v + \varepsilon) \tilde{\mathcal{E}}_{b,v+\varepsilon,L}(\tilde{\varphi}) - \sin(v) \tilde{\mathcal{E}}_{b,v,L}(\tilde{\varphi}) &= \sin(v + \varepsilon) \tilde{\mathcal{E}}_{b,v+\varepsilon,L}(\tilde{\varphi}) \\ &\quad - \sin(v) \tilde{\mathcal{E}}_{b,v,L}(\tilde{\varphi}) - \sin(v) \tilde{\mathcal{E}}_{b,v+\varepsilon,L}(\tilde{\varphi}) \\ &\quad + \sin(v) \tilde{\mathcal{E}}_{b,v+\varepsilon,L}(\tilde{\varphi}), \end{aligned} \tag{4.46}$$

we have

$$\begin{aligned} \Delta_{b,v}(\varepsilon) \geq \frac{\sqrt{2}}{4} \lim_{L \rightarrow +\infty} \left(\frac{1}{L^2} (\sin(v + \varepsilon) - \sin(v)) \tilde{E}(b, v + \varepsilon, L) \right. \\ \left. + \frac{1}{L^2} \sin(v) (\tan^2(v + \varepsilon) - \tan^2(v)) \int_{\tilde{\mathcal{D}}_{L,L,\frac{\pi}{4}}} |\partial_2 \tilde{\varphi}|^2 dv \right. \\ \left. + \frac{1}{L^2} \sin(v) \frac{b}{2} (\tan(v + \varepsilon) - \tan(v)) \int_{\tilde{\mathcal{D}}_{L,L,\frac{\pi}{4}}} |\tilde{\varphi}|^4 dv \right). \end{aligned}$$

For $\varepsilon \geq 0$ small enough, we have that $\tan^2(v + \varepsilon) - \tan^2(v) \geq 0$, so that the term

$$\lim_{L \rightarrow +\infty} \frac{1}{4L^2} \int_{\tilde{\mathcal{D}}_{L,L,\frac{\pi}{4}}} \sin(v) (\tan^2(v + \varepsilon) - \tan^2(v)) |\partial_2 \tilde{\varphi}|^2 dv$$

is positive and we can discard it in the lower bound. Using the identity (4.38) of Lemma 4.6 we have

$$\tilde{E}(b, v + \varepsilon, L) = -\frac{b}{2} \tan(v + \varepsilon) \int_{\tilde{\mathcal{D}}_{L,L,\frac{\pi}{4}}} |\tilde{\varphi}|^4 dv.$$

Therefore, we get

$$\begin{aligned} \Delta_{b,v}(\varepsilon) \geq \frac{\sqrt{2}}{4} \frac{b}{2} \left(\sin(v) (\tan(v + \varepsilon) - \tan(v)) \right. \\ \left. - (\sin(v + \varepsilon) - \sin(v)) \tan(v + \varepsilon) \right) \lim_{L \rightarrow +\infty} \frac{1}{L^2} \int_{\tilde{\mathcal{D}}_{L,L,\frac{\pi}{4}}} |\tilde{\varphi}|^4 dv. \end{aligned}$$

To conclude, as $v \in (0, \frac{\pi}{2})$, it is easy to see that there exists $\varepsilon_0 > 0$ small enough such that for all $0 < \varepsilon < \varepsilon_0$

$$-\tan v (\sin(v + \varepsilon) - \sin(v)) + \sin(v + \varepsilon) (\tan(v + \varepsilon) - \tan v) \geq 0.$$

Indeed, by differentiation, we have as $\varepsilon \rightarrow 0$

$$\begin{aligned} &-\tan v (\sin(v + \varepsilon) - \sin(v)) + \sin(v + \varepsilon) (\tan(v + \varepsilon) - \tan v) \\ &= \varepsilon \left(-\tan v \cos(v) + \sin(v) (1 + \tan^2 v) \right) + O(\varepsilon^2) \\ &= \varepsilon \sin v \tan^2 v + O(\varepsilon^2). \end{aligned}$$

This finishes the proof. □

5 Abrikosov Structure on a Surface in 3D

In this section, we prove Theorem 5.5 stated in Section 5.2 below. We are interested in constructing bounded solutions with lattice structure, i.e. for which the physical quantities – the density of Cooper pairs $|\psi|^2$, the magnetic field \mathbf{B} and the magnetic current $\Re(\bar{\psi}(-i\nabla + \mathbf{A})\psi)$ are periodic. This corresponds to states ψ satisfying the magnetic periodic conditions given by (5.6) below.

Before stating the main result of this section, we should introduce some notation. First of all, we introduce for an interval $J \subset \mathbb{R}$

$$L^2_{\text{comp}}(J \times \mathbb{R}^2) := \{\psi \in L^2_{\text{loc}}(J \times \mathbb{R}^2) : \psi \in L^2(J \times K) \text{ for all compact } K \subset \mathbb{R}^2\}. \tag{5.1}$$

5.1 Spectral Problem with Periodic Conditions on a Parallelogram Cell

Let \mathbf{e}_2 and \mathbf{e}_3 denote the standard unit vectors $(0, 1, 0)$ and $(0, 0, 1)$ associated with the variables x_2 and x_3 (respectively).

Notation 5.1 Let $R > 0, R' > 0$ and $\theta \in (0, \pi)$. We denote by $\Lambda_{R,R',\theta}$ the lattice (in the x_2x_3 -plane) defined by

$$\Lambda_{R,R',\theta} = \text{span}_{\mathbb{Z}} \{s := R\mathbf{e}_2, t := R'\mathbf{e}\},$$

where $\mathbf{e} = \cos \theta \mathbf{e}_2 + \sin \theta \mathbf{e}_3$.

We will denote by $\mathcal{D}_{R,R',\theta}$ the following fundamental domain

$$\mathcal{D}_{R,R',\theta} = \left\{ (x_2, x_3) \in \mathbb{R}^2, \frac{x_3}{\tan \theta} \leq x_2 \leq \frac{x_3}{\tan \theta} + R, 0 \leq x_3 \leq R' \sin \theta \right\}.$$

Notice that for convenience of notation, we have assumed that the system of coordinates is chosen so that one of the ‘legs’ of the lattice is parallel to a coordinate axis. In general, there can be an angle τ between the lattice and the (projection onto the x_2x_3 -plane of the) magnetic field. Therefore, this choice necessitates a change of convention for the magnetic field compared to what has been used in the first part of the article. As a consequence, we introduce the following notation for the magnetic vector potential and field,

$$\mathbf{A} = \mathbf{A}_{\nu,\tau} = \begin{pmatrix} 0 \\ x_1 \cos \nu \sin \tau - \frac{1}{2}x_3 \sin \nu \\ -x_1 \cos \nu \cos \tau + \frac{1}{2}x_2 \sin \nu \end{pmatrix}, \tag{5.2}$$

and

$$\mathbf{B} = \mathbf{B}_{\nu,\tau} = \nabla \times \mathbf{A}_{\nu,\tau} = \begin{pmatrix} \sin \nu \\ \cos \nu \cos \tau \\ \cos \nu \sin \tau \end{pmatrix}. \tag{5.3}$$

The magnetic flux $\Phi_{\mathbf{A}_{\nu,\tau}}$ over the cell $\mathcal{D}_{R,R',\theta}$ is

$$\Phi_{\mathbf{A}_{\nu,\tau}} = \frac{1}{2\pi} \iint_{\mathcal{D}_{R,R',\theta}} \mathbf{B}_{\nu,\tau} \cdot \mathbf{e}_1 \, dx_2 dx_3 = \frac{RR' \sin \nu \sin \theta}{2\pi}. \tag{5.4}$$

The flux $\Phi_{\mathbf{A}_{v,\tau}}$ is obviously proportional to the area of the cell $RR' \sin \theta$. As we are interested in lattice states (i.e. satisfying a double-periodic condition over the lattice $\Lambda_{R,R',\theta}$), it is natural to impose the following flux quantization condition

$$RR' \sin v \sin \theta \in 2\pi\mathbb{Z}. \tag{5.5}$$

Under the flux quantization condition (5.5), we can introduce the following magnetic periodicity condition on wave functions:

$$\begin{cases} \psi(x_1, x_2 + R, x_3) = \psi(x) e^{-i \frac{R}{2} x_3 \sin v}, \\ \psi(x_1, x_2 + R' \cos \theta, x_3 + R' \sin \theta) = \psi(x) e^{-i \frac{R'}{2} \sin v (x_3 \cos \theta - x_2 \sin \theta)}. \end{cases} \tag{5.6}$$

The magnetic periodic conditions (5.6) can more compactly be written as

$$\psi(x + w) = \psi(x) e^{-i g_w(x)}, \quad \forall w \in \Lambda_{R,R',\theta}, \tag{5.7}$$

where g_w is the function defined as follows

$$g_w(x) = g_{js+kt}(x) = \frac{b}{2} ((js + kt) \wedge x + (js) \wedge (kt)), \tag{5.8}$$

for all $w = js + kt$ in $\Lambda_{R,R',\theta}$ ($j, k \in \mathbb{Z}$), with $b = \sin v$. Here, for $x, y \in \mathbb{R}^3$, we have used the definition

$$x \wedge y = x_2 y_3 - x_3 y_2.$$

Notice that because

$$\psi(x + js + kt) = \psi((x + js) + kt) = \psi((x + kt) + js),$$

when $\psi(x) \neq 0$ we are lead to the condition

$$e^{-i g_{js+kt}(x)} = e^{-i g_{js}(x+kt)} e^{-i g_{kt}(x)} \left(= e^{-i g_{kt}(x+js)} e^{-i g_{js}(x)} \right). \tag{5.9}$$

With our choice of g_w given by (5.8), (5.9) follows from (5.5). Thus the quantization of the flux is important for consistency of these periodic conditions.

We also state for later reference the following identity for wave functions ψ satisfying (5.6) for all $w \in \Lambda_{R,R',\theta}$,

$$(-i \nabla + \mathbf{A}_{v,\tau}) \psi \Big|_{x+w} = e^{-i g_w(x)} (-i \nabla + \mathbf{A}_{v,\tau}) \psi \Big|_x. \tag{5.10}$$

Under condition (5.5), we consider the following eigenvalue problem on the set $\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta}$

$$\begin{cases} (-i \nabla + \mathbf{A}_{v,\tau})^2 \psi = \lambda \psi & \text{in } \mathbb{R}_+ \times \mathcal{D}_{R,R',\theta}, \\ \frac{\partial \psi}{\partial x_1} = 0 & \text{on } \partial \mathbb{R}_+^3, \\ \psi \text{ satisfying (5.6)}. \end{cases} \tag{5.11}$$

Lemma 5.2 *We can associate with the spectral problem (5.11), a unique self-adjoint operator which corresponds to the Friedrichs extension of the following quadratic form*

$$\mathcal{Q}_{v,\theta,\tau}^{\text{per}}(\psi) = \int_{\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta}} |(-i \nabla + \mathbf{A}_{v,\tau}) \psi|^2 dx,$$

defined for all ψ in the form domain

$$H_\tau^1(\nu, R, R', \theta) = \{\psi \in L_{comp}^2(\mathbb{R}_+^3) : (-i\nabla + \mathbf{A}_{\nu,\tau})\psi \in L_{comp}^2(\mathbb{R}_+^3, \mathbb{C}^3), \psi \text{ satisfies (5.6)}\}. \tag{5.12}$$

Proof This is a standard result since $\mathcal{Q}_{\nu,\theta,\tau}^{per}$ defines a closed quadratic form on $H_\tau^1(\nu, R, R', \theta)$. □

Remark 5.3 Notice that if φ, ψ satisfy (5.6), $\partial_{x_1}\psi|_{x_1=0} = 0$ and are sufficiently regular, then

$$\mathcal{Q}_{\nu,\theta,\tau}^{per}(\varphi, \psi) = \langle \varphi, (-i\nabla + \mathbf{A}_{\nu,\tau})^2\psi \rangle. \tag{5.13}$$

This follows by integration by parts using that (by (5.7) and (5.10)),

$$\begin{aligned} \overline{\varphi}(-i\nabla + \mathbf{A}_{\nu,\tau})\psi|_{x+w} &= \overline{(e^{-ig_w(x)}\varphi(x)e^{-ig_w(x)}((-i\nabla + \mathbf{A}_{\nu,\tau})\psi)|_x)} \\ &= \overline{\varphi(x)}(-i\nabla + \mathbf{A}_{\nu,\tau})\psi(x), \end{aligned}$$

causing the boundary terms to cancel each other pairwise.

Notation 5.4 We denote by $\mathcal{L}_\tau^{per}(\nu, \theta)$ the linear self-adjoint operator associated with the eigenvalue problem (5.11), and by $\zeta_\tau(\nu, R, R', \theta)$ the following quantity

$$\zeta_\tau(\nu, R, R', \theta) = \inf \sigma(\mathcal{L}_\tau^{per}(\nu, \theta)).$$

5.2 Statement of the Result

We present here the statement of the result we prove in this section. Similarly to (1.9) and (4.1), we define (with the space $H_\tau^1(\nu, R, R', \theta)$ defined in (5.12).)

$$\begin{aligned} E_\tau^{per}(\mathfrak{b}, \nu, R, R', \theta) &= \inf_{\psi \in H_\tau^1(\nu, R, R', \theta)} \int_{\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta}} \left\{ |(-i\nabla + \mathbf{A}_{\nu,\tau})\psi|^2 - \mathfrak{b}|\psi|^2 + \frac{\mathfrak{b}}{2}|\psi|^4 \right\} dx. \tag{5.14} \end{aligned}$$

Theorem 5.5 Suppose that $\nu \in (0, \frac{\pi}{2})$, that the magnetic flux satisfies (5.5), and that the linear spectral quantity $\zeta(\nu)$ from (2.3) satisfies

$$\zeta(\nu) < \mathfrak{b} < 1. \tag{5.15}$$

Then

$$\zeta_\tau(\nu, R, R', \theta) = \zeta(\nu), \tag{5.16}$$

and the quantity $E_\tau^{per}(\mathfrak{b}, \nu, R, R', \theta)$ defined in (5.14) is achieved in $H_\tau^1(\nu, R, R', \theta)$ with

$$E_\tau^{per}(\mathfrak{b}, \nu, R, R', \theta) < 0. \tag{5.17}$$

In particular, minimizers of (5.14) exist and are non-trivial.

Remark 5.6 Theorem 5.5 states that the lowest eigenvalue of the spectral problem (5.11) is equal to the lowest eigenvalue of the following spectral problem on \mathbb{R}_+^3 ,

$$\begin{cases} (-i\nabla + \mathbf{A}_{v,\tau})^2 \psi = \lambda \psi & \text{in } \mathbb{R}_+^3, \\ \frac{\partial \psi}{\partial x_1} = 0 & \text{on } \partial \mathbb{R}_+^3, \end{cases} \quad (5.18)$$

which is $\zeta(v)$.

Remark 5.7 It is easy to see that ζ_τ is independent of τ . In fact, by a rotation by an angle τ with respect to the x_1 axis (which leaves the half-space \mathbb{R}_+^3 invariant), it is easy to see that the Schrödinger operator $\mathcal{L}(v)$ associated with the magnetic potential \mathbf{A}_v given in (1.7) with constant magnetic field on the half-space (defined in (2.1)) and the following self-adjoint operator

$$\mathcal{L}_\tau(v) = (-i\nabla + \mathbf{A}_{v,\tau})^2 \quad \text{in } L^2(\mathbb{R}_+^3), \quad (5.19)$$

with domain

$$D(\mathcal{L}_\tau(v)) = \{u \in L^2(\mathbb{R}_+^3) : (-i\nabla + \mathbf{A}_{v,\tau})u \in L^2(\mathbb{R}_+^3, \mathbb{C}^3), \\ (-i\nabla + \mathbf{A}_{v,\tau})^2 u \in L^2(\mathbb{R}_+^3), \partial_{x_1} u = 0 \text{ on } \partial \mathbb{R}_+^3\},$$

are unitarily equivalent. Indeed, the geometrical domain remains unchanged after the rotation so that the spectrum does not change either. Clearly, this rotation also maps the eigenfunctions of the operator $\mathcal{L}(v)$ to the eigenfunctions of the operator $\mathcal{L}_\tau(v)$.

Thus, denoting $\zeta_\tau(v)$ the bottom of the spectrum of the operator $\mathcal{L}_\tau(v)$, we have that $\zeta_\tau(v) = \zeta(v)$.

The rest of the section is devoted to the proof of Theorem 5.5.

5.3 Proof of Theorem 5.5

Proof of Theorem 5.5 By standard arguments, it suffices to prove (5.16). Indeed, if $\mathcal{L}_\tau^{\text{per}}(v, \theta)\psi = \zeta(v)\psi$, where as assumed $\zeta(v) < \mathfrak{b}$, we get (5.17) by using $\varepsilon\psi$ as a trial state in (5.14) for sufficiently small ε .

Now (5.16) follows upon combining Lemma 5.10 and Lemma 5.13 below. \square

So we proceed to establish the spectral estimates of Lemma 5.10 and Lemma 5.13 below.

We consider ϕ_v^{3D} defined in (2.5), with $f \in \mathcal{C}_c^\infty(\mathbb{R})$ (for concreteness, let us consider the same $f \in \mathcal{C}_c^\infty(\mathbb{R})$ as the one specified in the proof of Lemma 2.6).

Notation 5.8 Related to the mapping mentioned above, we denote by $\phi_{v,\tau}^{3D}$ the eigenfunction of the operator $\mathcal{L}_\tau(v)$. This eigenfunction is obviously associated with the eigenvalue $\zeta(v)$.

We define with g_w from (5.8), the following function on \mathbb{R}_+^3

$$\psi(x) = \sum_{w \in \Lambda} \phi_{v,\tau}^{3D}(x+w)e^{ig_w(x)} = \sum_{j,k \in \mathbb{Z}} \phi_{v,\tau}^{3D}(x+js+kt)e^{ig_{js+kt}(x)}. \tag{5.20}$$

Lemma 5.9 For the function ψ introduced in (5.20), which is defined on \mathbb{R}_+^3 , the following properties hold.

- (1) The functions ψ , $(-i\nabla + \mathbf{A}_{v,\tau})\psi$, $(-i\nabla + \mathbf{A}_{v,\tau})^2\psi$ belong to $L^2(\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta})$ and $(-i\nabla + \mathbf{A}_{v,\tau})^2\psi = \zeta(v)\psi$.
- (2) The function ψ satisfies the magnetic periodic condition (5.6) and the Neumann boundary condition $\frac{\partial \psi}{\partial x_1} = 0$ on the set $\partial \mathbb{R}_+^3$.

Proof We prove (1). We first show that $\psi \in L^2(\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta})$. It suffices to show that

$$\sum_{(j,k) \in \mathbb{Z}^2} a_{j,k} < +\infty, \tag{5.21}$$

with $a_{j,k} = \|\phi_{v,\tau}^{3D}(x+js+kt)\|_{L^2(\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta})}^2$.

Using the last assertion of Lemma 2.6, there exists a positive constant $C > 0$ such that for all $x \in \mathbb{R}_+^3$

$$(1+x_1^2)(1+x_2^2)(1+x_3^2)|\phi_{v,\tau}^{3D}(x_1, x_2, x_3)| \leq C. \tag{5.22}$$

Inequality (5.21) follows from (5.22). We omit the technical details.

Now $(-i\nabla + \mathbf{A}_{v,\tau})^2\psi \in L^2(\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta})$ since $(-i\nabla + \mathbf{A}_{v,\tau})^2\psi = \zeta(v)\psi$ (in the sense of distributions). We can obtain that the function $(-i\nabla + \mathbf{A}_{v,\tau})\psi$ belongs to $L^2(\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta}, \mathbb{C}^3)$ proceeding in the same way as for the function ψ . Indeed, thanks to the last assertion of Lemma 2.6, we have in particular that all the first derivatives of ψ satisfy an analogous inequality as in (5.22), which is the central argument. This finishes the proof of 1.

We prove (2). The Neumann boundary condition at $x_1 = 0$ is satisfied for each term in the sum and is therefore clear. We prove the magnetic periodicity in the form of (5.7), by calculating

$$\begin{aligned} & e^{ig_{j_0s+k_0t}(x)} \psi(x+j_0s+k_0t) \\ &= \sum_{j,k \in \mathbb{Z}} \phi_{v,\tau}^{3D}(x+(j+j_0)s+(k+k_0)t)e^{ig_{js+kt}(x+j_0s+k_0t)} e^{ig_{j_0s+k_0t}(x)}. \end{aligned} \tag{5.23}$$

We need to verify that

$$e^{ig_{js+kt}(x+j_0s+k_0t)} e^{ig_{j_0s+k_0t}(x)} = e^{ig_{(j+j_0)s+(k+k_0)t}(x)}. \tag{5.24}$$

But it follows by direct calculation using the definition (5.8) of g_w that

$$e^{i g_{js+kt}(x+j_0s+k_0t)} e^{i g_{j_0s+k_0t}(x)} e^{-i g_{(j+j_0)s+(k+k_0)t}(x)} = e^{ib((kt)\wedge(j_0s)+(js)\wedge(k_0t))} = 1, \tag{5.25}$$

where the last equality uses the flux quantization (5.5). This finishes the proof of (2). \square

Lemma 5.10 *We have the following estimate:*

$$\zeta_\tau(v, R, R', \theta) \leq \zeta(v). \tag{5.26}$$

Proof By Lemma 5.9, ψ is an eigenfunction of $\mathcal{L}_\tau^{\text{per}}(v, \theta)$ with eigenvalue $\zeta(v)$. \square

Recall the expression of the magnetic potential $\mathbf{A}_{v,\tau}$ defined in (5.2). Thanks to the translation invariance in the third variable x_3 , and to the fact that the magnetic field $\mathbf{B}_{v,\tau}$ (see (5.3)) does not have any growth at infinity (it is constant), we can easily understand that the operator $\mathcal{L}_\tau^{\text{per}}(v, \theta)$ has essential spectrum. We will use Lemma 5.10 to prove the following result

Lemma 5.11 *The operator $\mathcal{L}_\tau^{\text{per}}(v, \theta)$ has an eigenvalue strictly less than the infimum of its essential spectrum.*

Proof Step 1. To prove that the operator $\mathcal{L}_\tau^{\text{per}}(v, \theta)$ has an eigenvalue below the threshold of its essential spectrum, it is enough to prove that

$$\inf \sigma_{\text{ess}}(\mathcal{L}_\tau^{\text{per}}(v, \theta)) \geq 1. \tag{5.27}$$

Indeed, as we know that $\zeta(v) < 1$ for all $v \in (0, \frac{\pi}{2})$ (see Lemma 2.3), Lemma 5.10 therefore implies that $\zeta_\tau(v, R, R', \theta) < 1$ for all $v \in (0, \frac{\pi}{2})$ (see Notation 5.4). Establishing (5.27) will give that the quantity $\zeta_\tau(v, R, R', \theta)$ belongs to the discrete spectrum of the operator $\mathcal{L}_\tau^{\text{per}}(v, \theta)$, giving in particular that it is an eigenvalue.

Step 2. We introduce the following self-adjoint operator \mathcal{L} , having formally the same expression as the operator $\mathcal{L}_\tau^{\text{per}}(v, \theta)$, but considered on the whole space \mathbb{R}^3 . By rotational invariance of the operator \mathcal{L} , it is well known that its spectrum does not depend on v . This operator corresponds to the magnetic Laplacian with constant magnetic field and we know that (see Proposition 2.1)

$$\inf \sigma(\mathcal{L}) = \inf \sigma_{\text{ess}}(\mathcal{L}) = 1. \tag{5.28}$$

We choose $L > 0$ (to be chosen large at the end of the proof) and we denote by $\mathcal{L}_{\tau,L}^{\text{per,Dir}}(v, \theta)$ the self-adjoint realization of the linear operator $\mathcal{L}_\tau^{\text{per}}(v, \theta)$ on $H_\tau^1(v, R, R', \theta)$, on the subset of functions vanishing on $x_1 = L$. Namely, the domain of $\mathcal{L}_{\tau,L}^{\text{per,Dir}}(v, \theta)$ is the set

$$\{\psi \in L_{\text{comp}}^2((L, +\infty) \times \mathbb{R}^2) : (-i\nabla + \mathbf{A}_{v,\tau})\psi \in L_{\text{comp}}^2((L, +\infty) \times \mathbb{R}^2, \mathbb{C}^3), (-i\nabla + \mathbf{A}_{v,\tau})^2\psi \in L_{\text{comp}}^2((L, +\infty) \times \mathbb{R}^2), \psi \text{ satisfies (5.6), } \psi = 0 \text{ on } x_1 = L\}.$$

where we recall that $H_\tau^1(v, R, R', \theta)$ has been introduced in (5.12).

Using Persson’s Theorem ([22], see also [9, Theorem B.1.1]), we have

$$\inf \sigma_{\text{ess}} (\mathcal{L}_\tau^{\text{per}}(v, \theta)) = \lim_{L \rightarrow +\infty} \inf \sigma \left(\mathcal{L}_{\tau,L}^{\text{per,Dir}}(v, \theta) \right). \tag{5.29}$$

By inclusion of the domains, we have

$$\lim_{L \rightarrow +\infty} \inf \sigma \left(\mathcal{L}_{\tau,L}^{\text{per,Dir}}(v, \theta) \right) \geq \inf \sigma \left(\mathcal{L}_{\tau,0}^{\text{per,Dir}}(v, \theta) \right), \tag{5.30}$$

where $\mathcal{L}_{\tau,0}^{\text{per,Dir}}(v, \theta)$ is the operator on \mathbb{R}_+^3 with the same expression as $\mathcal{L}_\tau^{\text{per}}(v, \theta)$, and with Dirichlet boundary condition at $\partial\mathbb{R}_+^3$.

Step 3. Using (5.28), (5.29) and (5.30), we need to prove that

$$\lambda_0 := \inf \sigma \left(\mathcal{L}_{\tau,0}^{\text{per,Dir}}(v, \theta) \right) \geq \inf \sigma(\mathcal{L}) = 1. \tag{5.31}$$

To prove (5.31) we construct a sequence (Ψ_n) of functions in the domain of the operator \mathcal{L} such that

$$\lim_{n \rightarrow +\infty} \frac{\mathcal{Q}(\Psi_n)}{\|\Psi_n\|_{L^2(\mathbb{R}_+^3)}^2} = \lambda_0, \tag{5.32}$$

where \mathcal{Q} is the quadratic form associated with the operator \mathcal{L} . We know that there exists a minimizing sequence $(\psi_n)_{n \in \mathbb{N}^*}$ such that

$$\lim_{n \rightarrow +\infty} \frac{\mathcal{Q}_{v,\theta,\tau}^{\text{per,Dir}}(\psi_n)}{\|\psi_n\|_{L^2(\mathbb{R}_+^3)}^2} = \lambda_0, \tag{5.33}$$

where $\mathcal{Q}_{v,\theta,\tau}^{\text{per,Dir}}$ is the quadratic form associated with the operator $\mathcal{L}_{\tau,0}^{\text{per,Dir}}(v, \theta)$. By density of the set of smooth functions with support bounded with respect to x_1 , we can assume without loss of generality that ψ_n is smooth and for each n there exists k such that

$$\text{supp}(\psi_n) \subset \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| \leq k\}. \tag{5.34}$$

We let $k(n)$ be the smallest such integer k . For all $n \in \mathbb{N}^*$, we denote by $\tilde{\psi}_n$ the function ψ_n extended by 0 to the complement of \mathbb{R}_+^3 so that for all $n \in \mathbb{N}^*$, the function $\tilde{\psi}_n$ is defined on the whole space \mathbb{R}^3 .

We consider a sequence of cutoff functions $(\chi_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}_c^\infty(\mathbb{R})$ satisfying

$$0 \leq \chi_n \leq 1 \text{ on } \mathbb{R}, \quad \chi_n = 1 \text{ on } [-n, n], \quad \chi_n = 0 \text{ on } \mathbb{R} \setminus [-(n+1), n+1], \tag{5.35}$$

and

$$|\nabla \chi_n| \leq C, \tag{5.36}$$

for all $n \in \mathbb{N}^*$ and where $C > 0$ is a fixed constant. Using these cutoff functions, we build the following sequence of real valued functions η_n in $\mathcal{C}_c^\infty(\mathbb{R}^3)$ defined (for all $n \in \mathbb{N}^*$) on \mathbb{R}^3 as follows

$$\eta_n(x) = \chi_{k(n)}(x_1) \chi_n \left(\frac{x_2}{R} \right) \chi_n \left(\frac{x_3}{R' \sin \theta} \right). \tag{5.37}$$

Notice that the cutoff function η_n defined in (5.37) depends on θ, R, R' and the function k but we have chosen not to indicate this dependence to simplify the notations.

For all $n \in \mathbb{N}^*$, we define Ψ_n as

$$\Psi_n = \eta_n \tilde{\psi}_n.$$

Thanks to the cutoff, the functions $\eta_n \tilde{\psi}_n$ belong to the domain of the operator \mathcal{L} , for all $n \in \mathbb{N}^*$. What is more, thanks the choice of $k(n)$ we have

$$\eta_n(x) \tilde{\psi}_n(x) = \chi_n \left(\frac{x_2}{R} \right) \chi_n \left(\frac{x_3}{R' \sin \theta} \right) \tilde{\psi}_n(x). \tag{5.38}$$

We introduce the following sets (for $l \in \mathbb{N}$)

$$\mathcal{C}^0 = (0, +\infty) \times \mathcal{D}_{R,R',\theta},$$

$$\mathcal{C}_l = (0, +\infty) \times \{(x_2, x_3) \in \mathbb{R}^2 : |x_2| < lR, \quad |x_3| < lR' \sin \theta\},$$

$$\mathcal{K}_l = (0, +\infty) \times \{(x_2, x_3) \in \mathbb{R}^2 : lR < |x_2| < (l+1)R, \quad \text{or} \\ lR' \sin \theta < |x_3| < (l+1)R' \sin \theta\}.$$

Notice that these sets depend on θ, R and R' but we have chosen not to encumber the notations.

Define

$$I_l := \{w \in \Lambda_{R,R',\theta} : \mathcal{C}^0 + w \subset \mathcal{C}_l\}, \\ \tilde{I}_l := \{w \in \Lambda_{R,R',\theta} : (\mathcal{C}^0 + w) \cap \mathcal{K}_l \neq \emptyset\}.$$

We will use that

$$|\tilde{I}_l|/|I_l| \rightarrow 0, \quad \text{as } l \rightarrow \infty, \tag{5.39}$$

which easily follows by an area consideration.

We now calculate/estimate using the periodicity of $\tilde{\psi}_n$ and the definition of η_n

$$\left| \int |(-i\nabla + \mathbf{A}_{v,\tau})\Psi_n|^2 dx - |I_n| \int_{\mathcal{C}^0} |(-i\nabla + \mathbf{A}_{v,\tau})\tilde{\psi}_n|^2 dx \right| \\ \leq \int_{\mathbb{R}_+^3 \setminus (\cup_{w \in I_n} \mathcal{C}^0 + w)} |(-i\nabla + \mathbf{A}_{v,\tau})\Psi_n|^2 dx \\ \leq \int_{\cup_{w \in \tilde{I}_n} (\mathcal{C}^0 + w)} |(-i\nabla + \mathbf{A}_{v,\tau})\Psi_n|^2 dx \\ \leq C |\tilde{I}_n| \int_{\mathcal{C}^0} |(-i\nabla + \mathbf{A}_{v,\tau})\tilde{\psi}_n|^2 + |\tilde{\psi}_n|^2 dx. \tag{5.40}$$

Similarly,

$$\left| \int |\Psi_n|^2 dx - |I_n| \int_{\mathcal{C}^0} |\tilde{\psi}_n|^2 dx \right| \leq |\tilde{I}_n| \int_{\mathcal{C}^0} |\tilde{\psi}_n|^2 dx. \tag{5.41}$$

From (5.41) we get using (5.39) that

$$\left| \frac{\int |\Psi_n|^2 dx}{|I_n| \int_{\mathcal{C}^0} |\tilde{\psi}_n|^2 dx} - 1 \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{5.42}$$

Similarly, from (5.40) we find using (5.39) and (5.33) that

$$\frac{\int |(-i\nabla + \mathbf{A}_{v,\tau})\Psi_n|^2 dx}{|I_n| \int_{\mathcal{C}^0} |\tilde{\psi}_n|^2 dx} \rightarrow \lambda_0. \tag{5.43}$$

Combining (5.43) and (5.42) we conclude (5.32) which finishes the proof. □

Since we know by Lemma 5.11 that $\zeta_\tau(v, R, R', \theta)$ is an eigenvalue strictly below the essential spectrum we have the following classical result (see for example [9, Theorem B.5.1]).

Lemma 5.12 *Let $\psi_{v,\theta,\tau}^{R,R'}$ be a normalized eigenfunction of the operator $\mathcal{L}_\tau^{\text{per}}(v, \theta)$, associated with the lowest eigenvalue $\zeta_\tau(v, R, R', \theta)$. There exist constants $C, \alpha > 0$ such that:*

$$\int_{\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta}} e^{\alpha x_1} \left\{ |\psi_{v,\theta,\tau}^{R,R'}(x)|^2 + |(-i\nabla + \mathbf{A}_{v,\tau})\psi_{v,\theta,\tau}^{R,R'}(x)|^2 \right\} dx < C.$$

Lemma 5.13 *We have*

$$\zeta_\tau(v, R, R', \theta) \geq \zeta(v).$$

Proof To prove (5.13) it suffices to construct a sequence (Ψ_n) of functions in the domain of the operator $\mathcal{L}(v)$, such that

$$\lim_{n \rightarrow +\infty} \frac{\mathcal{Q}_v(\Psi_n)}{\|\Psi_n\|_{L^2(\mathbb{R}_+^3)}^2} = \zeta_\tau(v, R, R', \theta), \tag{5.44}$$

where \mathcal{Q}_v is the quadratic form associated with the operator $\mathcal{L}(v)$.

Let $\psi_{v,\theta,\tau}^{R,R'} \in H_\tau^1(v, R, R', \theta)$ be an eigenfunction of the operator $\mathcal{L}_\tau^{\text{per}}(v, \theta)$, associated with the lowest eigenvalue $\zeta_\tau(v, R, R', \theta)$ (see Lemma 5.11). In particular,

$$\frac{\mathcal{Q}_{v,\theta,\tau}^{\text{per}}(\psi_{v,\theta,\tau}^{R,R'})}{\|\psi_{v,\theta,\tau}^{R,R'}\|_{L^2(\mathbb{R}_+ \times \mathcal{D}_{R,R',\theta})}^2} = \zeta_\tau(v, R, R', \theta). \tag{5.45}$$

We consider a sequence $(\chi_n)_{n \in \mathbb{N}^*}$ of cutoff functions $(\chi_n) \subset \mathcal{C}_c^\infty(\mathbb{R})$ satisfying (5.35) and (5.36). Using these cutoff functions, we define η_n in $\mathcal{C}_c^\infty(\mathbb{R}_+^3)$ by

$$\eta_n(x) = \chi_n(x_2) \chi_n(x_3).$$

Thanks to the cutoff in the variables x_2 and x_3 , and the decay in the variable x_1 , the functions $\Psi_n := \eta_n \tilde{\psi}_{v,\theta,\tau}^{R,R'}$ belong to the domain of the operator $\mathcal{L}(v)$, for all $n \in \mathbb{N}^*$.

One can now prove that (Ψ_n) satisfies (5.44) using the exact same arguments as in the proof of Lemma 5.11. We omit the details. \square

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