Math Phys Anal Geom (2019) 22: 7
https://doi.org/10.1007/s11040-019-9305-9

Harnack Inequalities for Simple Heat Equations on Riemannian Manifolds

Li Ma[1] · Yang Liu[2]

Received: 24 July 2018 / Accepted: 13 February 2019 / Published online: 26 February 2019

Abstract

In this paper, we consider Harnack inequalities (the gradient estimates) of positive solutions for two different heat equations via the use of the maximum principle. In the first part, we obtain the gradient estimate for positive solutions to the following nonlinear heat equation problem

$$\partial_t u = \Delta u + au \log u + Vu, \quad u > 0$$

on the compact Riemannian manifold (M, g) of dimension n and with $Ric(M) \geq -K$. Here $a > 0$ and K are some constants and V is a given smooth positive function on M. Similar results are showed to be true in case when the manifold (M, g) has compact convex boundary or (M, g) is a complete non-compact Riemannian manifold. In the second part, we study Harnack inequality (gradient estimate) for positive solution to the following linear heat equation on a compact Riemannian manifold with non-negative Ricci curvature:

$$\partial_t u = \Delta u + \sum W_i u_i + Vu,$$

where W_i and V only depend on the space variable $x \in M$. The novelties of our paper are the refined global gradient estimates for the corresponding evolution equations, which are not previously considered by other authors such as Yau (Math. Res. Lett. **2**(4), 387–399, 1995), Ma (J. Funct. Anal. **241**(1), 374–382, 2006), Cao et al. (J. Funct. Anal. **265**, 2312–2330, 2013), Qian (Nonlinear Anal. **73**, 1538–1542, 2010).

Keywords Positive solution · Nonlinear heat equation · Gradient estimate · Harnack inequality

Mathematics Subject Classification (2010) 53C44 · 53C21 · 35K05

The research of L.Ma is partially supported by the National Natural Science Foundation of China (No. 11771124 and No.1127111) and the Dingxin research grant of USTB

✉ Li Ma
lma17@ustb.edu.cn

Extended author information available on the last page of the article.

1 Introduction

In this paper, we study Harnack inequalities (also called the gradient estimates) for positive solutions to heat equations of Schrodinger type on Riemannian manifolds. In the first part, we obtain the gradient estimate for positive solutions to the following nonlinear heat equation problem

$$\partial_t u = \Delta u + au \log u + Vu. \tag{1.1}$$

on the compact Riemannian manifold (M^n, g) of dimension n and with $Ric(M) \geq -K$. Here $a > 0$ and K are some uniform constants and V is a given smooth positive function on M. In recent years there has been increasing interest in the study of the nonlinear heat equations (1.1) [14]. The special case when $V = b$ is a constant, that is, a nonlinear heat equation

$$\partial_t u = \Delta u + au \log u + bu. \tag{1.2}$$

where a and b are real constants, had been studied by Ma [17], X. Cao et al. [5] and others. See also [4, 25, 30], and [20] for related works. The motivation for studying such a nonlinear heat equation is from the soliton solutions of Ricci flow introduced by Hamilton [10]. L. Ma proved local gradient estimates of positive solutions to the following elliptic equation

$$\Delta u + au \log u + bu = 0, \quad in \;\; (M, g) \tag{1.3}$$

where $a < 0$ and b are real constants, (M, g) is a complete noncompact Riemannian manifolds with Ricci curvature locally bounded below. In 2008 [29], Yang generalized Ma's result and derived a local gradient estimates for positive solution to the simple parabolic equation (1.2). Since then, it is an interesting question to find gradient estimate for positive solutions the heat equation (1.1) with non-trivial potential function V. In [18], the author proved that the local gradient estimates holds true for positive solution to the heat equation (1.1). In particular, L. Ma derived the following result.

Theorem 1.1 *(L. Ma). Assume that the compact n-dimensional Riemannian manifold (M, g) has non-negative Ricci curvature. Assume that $a \leq 0$ and V is a given smooth function on M such that $-\Delta V \leq A$ on M for some constant $A \geq 0$. Let $u > 0$ be a positive smooth solution to the nonlinear equation (1.1). Let $f = -\log u$. Then we have, for all $t > 0$*

$$\Delta f - At - \frac{n}{2t} \leq 0.$$

In [22], B. Qian obtained an uniform bound for the positive solutions to a simple nonlinear equation on Riemannian manifolds with Ricci curvature bounded below by extending Yang's result. Qian's result can be stated as below.

Theorem 1.2 *(B. Qian). Let (M, g) be a complete Riemannian manifold without boundary. Suppose that the Ricci curvature of M is bounded below $-K$, $K \geq 0$. If u is a positive solution to the elliptic equation*

$$\Delta u + au \log u = 0, \quad on\ M,$$

then we have, for $\frac{2}{3} < \delta < 1$,

$$\begin{cases} u(x) \geq e^{-\frac{5n}{4}+\frac{5nK}{2n}}, & \text{if } \ a < 0, \\ u(x) \leq e^{\frac{Kn}{2a(2\delta-1)}+\sqrt{\frac{2Knd}{a(2\delta-1)}}}, & \text{if } \ a > 0, \end{cases}$$

where $d = \frac{na}{\sqrt{8(1-\delta)(2\delta-1)}} \bigvee \frac{4n^2K^2}{(2\delta-1)^2}$.

B. Qian [22] obtained the uniform estimates above based on the method of S.-T. Yau [27] and proved the above result by the global gradient estimates for the corresponding nonlinear heat equations. Cao et al. [2], Huang and Ma [11], Huang-Huang-Li [12], Chen and Chen [8], Ma [19], Qian [22], Souplet and Zhang [23], Wu [24] and others also found more gradient estimates to related heat equations with possible drifting terms.

In this work, we prove the following new results for the nonlinear heat equation (1.1).

Theorem 1.3 *Assume that (M^n, g) is an n-dimensional compact Riemannian manifold with $Ric(M) \geq -K$ for some constant $K > 0$. Given $a > 0$ some constant. Let V be a given smooth positive function on M such that $-\Delta V + \frac{|\nabla V|^2}{\tilde{K}} \leq 0$ on M for some $\tilde{K} > 0$. Let u be a positive solution to the nonlinear heat equation (1.1). Then we have, for all $t > 0$ and*

$$|\nabla \log u|^2 - \partial_t(\log u) + a \log u \leq \sqrt{\frac{(2K+\tilde{K})n}{2\delta - 1}} \sqrt{|\nabla \log u|^2 + d + \frac{e}{t}} + \frac{n}{2\delta t} - V,$$

where

$$d = \frac{na^2}{8(1-\delta)(2\delta-1)} \vee \frac{(2K+\tilde{K})n}{2\delta-1}, \qquad e = \frac{n}{2\delta(3\delta-2)}.$$

The result above can be proved true for the nonlinear heat equation (1.1) on the compact Riemannian manifold (M, g) with smooth boundary.

Theorem 1.4 *Assume that (M^n, g) is an n-dimensional compact Riemannian manifold with smooth convex boundary has $Ric(M) \geq -K$ for some constant $K > 0$. Given $a > 0$ some constant. Let V be a given smooth positive function on M such that $-\Delta V + \frac{|\nabla V|^2}{\tilde{K}} \leq 0$ on M for some $\tilde{K} > 0$. Let $u > 0$ be a positive smooth solution to (1.1) with Neumann boundary condition $u_\nu = 0$ where ν is the outward unit* *normal to the boundary. Assume that* $V_\nu \le 0$ *on the* ∂M. *Then we have, for all* $t > 0$ *and* $\delta \in \left(\frac{2}{3}, 1\right)$,

$$|\nabla \log u|^2 - \partial_t(\log u) + a \log u \le \sqrt{\frac{(2K+\tilde{K})n}{2\delta - 1}} \sqrt{|\nabla \log u|^2 + d + \frac{e}{t}} + \frac{n}{2\delta t} - V,$$

where

$$d = \frac{na}{8(1-\delta)(2\delta - 2)} \vee \frac{(2K+\tilde{K})n}{2\delta - 1}, \qquad e = \frac{n}{2\delta(3\delta - 2)}.$$

We remark that our results above are first new results for the (1.1). The arguments in the proofs are more tricky than previous works since the extra term Vu plays an important role.

Motivated by the works of S.-T. Yau [27], B. Qian [22], X. Cao et al. [5], and L. Ma [18], we study the gradient estimate for the positive solution to the following linear heat equation

$$\partial_t u = \Delta u + \sum W_i u_i + V u, \tag{1.4}$$

where W_i and V are smooth functions depending only on x. Recall that in [27], S.-T. Yau studied Harnack inequality for non-self-adjoint equation (1.4), where W_i and V may depend on t and Yau's result can be stated as below.

Theorem 1.5 *(Yau). Let* u *be a solution of* (1.4) *so that* $(d^2+1)^{-1}\sum_i W_i^2$ *and* $(d^2+1)^{-1}V$ *are bounded. Here* d *is the diameter of the manifold* M. *Fix* $\epsilon > 0$. *Suppose that we can choose constants* α *and* b *so that for any* $\xi = (\xi_1, ..., \xi_n) \in R^n$,

$$\frac{\alpha^2}{n}|\xi|^2 \ge 3\epsilon|\xi|^2 - 2\sum R_{ij}\xi_j\xi_j$$

and

$$\left(b - \sum W_{i,i}\right)^2 > \left(\frac{1}{4} + \frac{\alpha^2}{16}\right)\sum_{i,j}(W_{i,j} + W_{j,i})^2 + \Delta V - \alpha|\nabla V|$$
$$+\frac{1}{4}\sum(\Delta W_i)^2 + \frac{1}{\epsilon}\sum\left(\sum R_{ij}W_i\right)^2$$

hold. Then

$$-(\log u)_t + |\nabla \log u|^2 + \sum W_i(\log u)_i - V$$
$$\le \alpha\sqrt{|\nabla \log u|^2 + d + \frac{e}{t}} + \frac{n}{2t} + b,$$

where $d \ge \alpha^2$ *and* $e \ge \frac{n}{2}$, R_{ij} *is the Ricci curvature of the manifold* M.

It is clear that the assumptions in Yau's result above are not simple and Yau's argument is based on the deep study of gradient estimates for heat equation in the work of P. Li and S.-T. Yau [15]. Later, Li [16] generalized the estimates of P. Li and S.-T. Yau to the semi-linear parabolic equations on complete Riemannian manifolds. Yau can also generalize Harnack inequalities for heat equations to some non-self-adjoint evolution equations [26, 27]. Perelman [21] introduce new Harnack quantities and derived differential Harnack inequality for the fundamental solution to the backward heat equation coupled with the Ricci flow without any curvature assumption. Because Perelman's gradient estimate for heat kernel plays an important role in Ricci flow, people are motivated to find extensions of his result to various parabolic problems. In 2008, S. Kuang and Q. S. Zhang [13] established a point-wise gradient estimate for all positive solutions to the backward heat equation under the Ricci flow on closed manifolds with nonnegative scalar curvature. We point out that there have been many important contributions to heat equations on manifolds such as the works [2–4, 6, 7, 9].

We can prove the results below.

Theorem 1.6 *Assume that (M^n, g) is an n-dimensional compact Riemannian manifold with $Ric(M) \geq -K$ for some constant $K > 0$. Let u be a positive solution to the heat equation*

$$\partial_t u = \Delta u + \sum W_i u_i + V u, \tag{1.5}$$

where V is a smooth function on M and W_i are smooth functions on $M \times [0, \infty)$. Let $\frac{1}{2} < \delta < \frac{3}{5}$, $0 < \epsilon < \frac{1-\delta}{n}$, $\alpha \geq \sqrt{\frac{(3\epsilon+2K)n}{(1-\delta)}}$. Define,

$$\Lambda = \sup \left\{ \frac{1}{4\epsilon} \sum (\Delta W_i)^2 + \frac{1}{4\epsilon} \left(\sum R_{ij} W_i \right)^2 - \Delta V + \alpha |\nabla V| + \frac{4+\alpha^2}{16\epsilon} \sum (W_{i,j} + W_{j,i})^2 \right\}.$$

Then we have, for all $t > 0$,

$$|\nabla \log u|^2 - \partial_t (\log u) + \sum W_i (\log u)_i + V \leq \sqrt{\frac{(3\epsilon + 2K)n}{1-\delta}} \sqrt{|\nabla \log u|^2 + d + \frac{e}{t}} + \frac{n}{(\delta+1)t} + b,$$

where

$$d = \frac{2\sqrt{n\Lambda}}{(1-\delta)^{3/2}} \vee \alpha^2, \qquad e = \frac{2n}{(3-5\delta)(\delta+1)}, \qquad b = \sqrt{\frac{n\Lambda}{1-\delta}}.$$

We remark that our assumption in Theorem 1.6 is very different from that of Theorem 1.5 obtained by Yau. We think our assumption in the result above is almost the best hypotheses for the Harnack inequalities for the (1.5). The computation trick in our proof of Theorem 1.6 is different from that of Theorem 1.3. Similar result to Theorem 1.6 is also true for positive solutions to equation (1.5) on a complete non-compact Riemannian manifold, but we shall not formulation it here.

This paper is organized as follows. We prove Theorem 1.3 and Theorem 1.4 in Section 2. The proof of Theorem 1.6 will be given in Section 3.

2 The Proof of Theorem 1.3 and Theorem 1.4

Recall that a key tool in deriving the gradient estimates of positive solutions of heat equations is the following Weitzenbock-Bochner formula [1]: For a smooth function u on the Riemannian manifold (M, g), it holds that

$$\Delta|\nabla u|^2 = 2(\nabla u, \nabla\Delta u) + 2|\nabla^2 u|^2 + 2Rc(\nabla u, \nabla u),$$

where Rc is the Ricci tensor of the metric g.

We now prove Theorem 1.3.

Proof of Theorem 1.3 Let $\varphi = -\log u$. Compute,

$$\varphi_t = -\frac{u_t}{u}, \quad \varphi_j = -\frac{u_j}{u}, \quad \Delta\varphi = -\frac{\Delta u}{u} + |\nabla\varphi|^2.$$

Then we have

$$\varphi_t = \Delta\varphi - |\nabla\varphi|^2 + a\varphi - V,$$

and

$$a\varphi_t = \varphi_{tt} - \Delta\varphi_t + 2\varphi_i\varphi_{it}.$$

Let

$$\psi := \varphi_t + |\nabla\varphi|^2 - a\varphi - \alpha\sqrt{|\nabla\varphi|^2 + \beta} - \gamma(t) + V,$$

where α is a positive constant, β and γ are positive functions in t. All of them will be determined later. Then we have

$$\psi_t = \varphi_{tt} + 2\sum\varphi_i\varphi_{it} - a\varphi_t - \frac{\alpha\sum\varphi_i\varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t),$$

and

$$\begin{aligned}\Delta\psi =\ & (\Delta\varphi)_t + 2\sum\varphi_{ij}^2 + 2\sum\varphi_i(\Delta\varphi)_i + 2\sum R_{ij}\varphi_i\varphi_j - a\Delta\varphi + \Delta V \\ & - \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\sum\varphi_i(\Delta\varphi)_i + \alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum_j(\sum_i\varphi_i\varphi_{ij})^2}{(\sqrt{|\nabla\varphi|^2+\beta})^3} \\ \geq\ & (\Delta\varphi)_t + 2\sum\varphi_{ij}^2 + 2\sum\varphi_i(\Delta\varphi)_i + 2\sum R_{ij}\varphi_i\varphi_j - a\Delta\varphi + \Delta V \\ & - \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\sum\varphi_i(\Delta\varphi)_i}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}}.\end{aligned}$$

Thus,

$$\begin{aligned}
\psi_t - \Delta\psi \le\ & (\varphi_{tt} + 2\sum \varphi_i\varphi_{it} - a\varphi_t) - (\Delta\varphi)_t \\
& - \frac{\alpha\sum\varphi_i\varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2+\beta}} \\
& -\gamma'(t) - 2\sum\varphi_{ij}^2 - 2\sum\varphi_i(\Delta\varphi)_i - 2\sum R_{ij}\varphi_i\varphi_j + a\Delta\varphi - \Delta V \\
& + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i(\Delta\varphi)_i}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} \\
=\ & -2\sum\varphi_j\left(\psi + \alpha\sqrt{|\nabla\varphi|^2+\beta} + \gamma(t)\right)_j \\
& - \frac{\alpha\sum\varphi_i\varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2+\beta}} \\
& -\gamma'(t) - 2\sum\varphi_{ij}^2 - 2\sum R_{ij}\varphi_i\varphi_j + a\Delta\varphi - \Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& + \frac{\alpha\sum\varphi_i(\Delta\varphi)_i}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} \\
=\ & -2\sum\varphi_j\psi_j - \frac{\alpha\sum\varphi_j(|\nabla\varphi|^2)_j}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\sum\varphi_i\varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2+\beta}} \\
& -\gamma'(t) - 2\sum\varphi_{ij}^2 - 2\sum R_{ij}\varphi_i\varphi_j + a\Delta\varphi - \Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& + \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i\varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_j(|\nabla\varphi|^2)_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i V_i}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& - \frac{\alpha a\sum\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} \\
=\ & -2\sum\varphi_j\psi_j - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) - 2\sum\varphi_{ij}^2 - 2\sum R_{ij}\varphi_i\varphi_j \\
& + a\Delta\varphi - \Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha a\sum\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& + \frac{\alpha\sum\varphi_i V_i}{\sqrt{|\nabla\varphi|^2+\beta}} \\
\le\ & -2\nabla\varphi\nabla\psi - \left(\frac{\alpha(K+a)}{\sqrt{|\nabla\varphi|^2+\beta}} - 2K\right)|\nabla\varphi|^2 \\
& + \left(\frac{\alpha}{\sqrt{2|\nabla\varphi|^2+\beta(t)}} - \frac{2}{n}\right)|\Delta\varphi|^2 \\
& + a\Delta\varphi - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} + 2|\nabla\varphi||\nabla V| - \gamma'(t) - \Delta V, \qquad (2.1)
\end{aligned}$$

where we have used the assumption $\alpha \le 2\sqrt{|\nabla\varphi|^2+\beta(t)}$. Suppose that $\psi < 0$ when $t = 0$. Assume that at $t_0 > 0$, ψ becomes zero at some point x_0 in the interior of the manifold and $\psi < 0$ for all $t < t_0$. Then we have the first order condition $\psi_t \ge 0$, $\nabla\psi = 0$ at (t_0, x_0) and the second order conditon $\Delta\psi \le 0$ at (t_0, x_0). Note that at the point (t_0, x_0), we have $\Delta\varphi = \alpha\sqrt{|\nabla\varphi|^2+\beta(t)} + \gamma(t)$. Hence, at point (t_0, x_0), by (2.1), $\psi_t - \Delta\psi \ge 0$ becomes

$$
\begin{aligned}
&-\left(\frac{\alpha(K+a)}{\sqrt{|\nabla\varphi|^2+\beta}} - 2K\right)|\nabla\varphi|^2 + \left(\frac{\alpha}{\sqrt{2|\nabla\varphi|^2+\beta}} - \frac{2}{n}\right)|\Delta\varphi|^2 \\
&+ a\Delta\varphi - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) - \Delta V + 2|\nabla\varphi||\nabla V| \ge 0. \qquad (2.2)
\end{aligned}
$$

Let $\tilde{K} > 0$. Then we can write

$$
2K|\nabla\varphi|^2 = (2K+\tilde{K})|\nabla\varphi|^2 - \tilde{K}|\nabla\varphi|^2.
$$

Let $\delta \in (0, 1)$. By (2.2), at (t_0, x_0), we have

$$
\begin{aligned}
0 \le\ & (2K+\tilde{K})|\nabla\varphi|^2 - \tilde{K}|\nabla\varphi|^2 + 2|\nabla\varphi||\nabla V| \\
& + \left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}} - \frac{2\delta}{n}\right)|\Delta\varphi|^2 \\
& + \left(\frac{2\delta}{n} - \frac{2}{n}\right)|\Delta\varphi|^2 + a\Delta\varphi - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma' - \Delta V \\
\le\ & (2K+\tilde{K})|\nabla\varphi|^2 + \left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}} - \frac{2\delta}{n}\right)|\Delta\varphi|^2 + \frac{na^2}{8(1-\delta)} \\
& - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma' - \Delta V + \frac{|\nabla V|^2}{\tilde{K}}. \qquad (2.3)
\end{aligned}
$$

We only need to show that (2.3) is violated. We choose $\gamma(t) = \frac{f}{t}$, $\beta = d + \frac{e}{t}$, where $d > \alpha^2$, e, and f are positive constants, which will be determined later.

Note that for any $\frac{1}{2} < \delta < 1$ we have $\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}} < \frac{2\delta}{n}$ holds. Substituting $\Delta\varphi = \alpha\sqrt{|\nabla\varphi|^2 + \beta(t)} + \gamma(t)$ at (t_0, x_0) into (2.3), we have

$$\begin{aligned}
0 \le\ & (2K+\tilde{K})|\nabla\varphi|^2 + \left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}} - \frac{2\delta}{n}\right)(\alpha^2(|\nabla\varphi|^2+\beta)+2\alpha\gamma\sqrt{|\nabla\varphi|^2+\beta} \\
& +\gamma^2) + \frac{na^2}{8(1-\delta)} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma' - \Delta V + \frac{|\nabla V|^2}{\tilde{K}} \\
=\ & (2K+\tilde{K})|\nabla\varphi|^2 + \left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}} - \frac{2\delta}{n}\right)(\alpha^2(|\nabla\varphi|^2+\beta)+2\alpha\gamma\sqrt{|\nabla\varphi|^2+\beta} \\
& +\gamma^2) + \frac{na^2}{8(1-\delta)} + \frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2+\beta}} + \frac{f}{t^2} - \Delta V + \frac{|\nabla V|^2}{\tilde{K}} \\
=\ & \left((2K+\tilde{K}) + \frac{(1-2\delta)\alpha^2}{n}\right)|\nabla\varphi|^2 + \left(\frac{(1-2\delta)\alpha^2\beta}{n} + \frac{na^2}{8(1-\delta)}\right) + \frac{f}{t^2}\left(1 - \frac{2\delta f}{n}\right) \\
& + \frac{2\alpha f(1-2\delta)}{nt}\sqrt{|\nabla\varphi|^2+\beta} + \frac{\alpha f^2}{nt^2\sqrt{|\nabla\varphi|^2+\beta}} \\
& + \frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2+\beta}} - \Delta V + \frac{|\nabla V|^2}{\tilde{K}}.
\end{aligned}$$

To make the above be violate, we only need

$$\begin{aligned}
I_1 &:= 2K + \tilde{K} + \frac{(1-2\delta)\alpha^2}{n} \le 0, \\
I_2 &:= \frac{(1-2\delta)\alpha^2\beta}{n} + \frac{na^2}{8(1-\delta)} < 0, \\
I_3 &:= 1 - \frac{2\delta}{n} f \le 0, \\
I_4 &:= \frac{2\alpha(1-2\delta)f}{tn}\sqrt{|\nabla\varphi|^2+\beta} + \frac{\alpha f^2}{nt^2\sqrt{|\nabla\varphi|^2+\beta}} \\
&\quad + \frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2+\beta}} - \Delta V + \frac{|\nabla V|^2}{\tilde{K}} \le 0.
\end{aligned}$$

To this end, we choose

$$\alpha \ge \sqrt{(2K+\tilde{K})n/(2\delta-1)}, \qquad d \ge \frac{na}{8(1-\delta)(2\delta-1)},$$

$$f = \frac{n}{2\delta}, \qquad e \ge \frac{n}{2(3\delta-2)\delta},$$

where $\frac{2}{3} < \delta < 1$. We then complete the proof of Theorem 1.3. □

We remark that Theorem 1.3 can be extended to the complete noncompact manifolds (M^n, g) with the bounded function V. This can be done because the maximum principle hold; see [28] Theorem 1. We use the same function ψ as defined in Theorem 1.3. Provided that $\psi < 0$ near $t = 0$, we now applying the maximum principle to the function ψ on $[0, t] \times M$ for any fixed $t > 0$. Assume that we may find a point $t_0 \in [0, t]$ and a sequence of points $x_k \in M$ such that

$$\Delta\psi(t_0, x_k) \leq \frac{1}{k}, \qquad |\nabla\psi|(t_0, x_k) \leq \frac{1}{k},$$

and also

$$\partial_t\psi(t_0, x_k) \geq 0, \qquad \lim_{k\to\infty} \psi(t_0, x_k) = 0.$$

Hence, we have that $(\partial_t - \Delta)\psi \geq -\frac{1}{k}$ and $\Delta\varphi(t_0, x_k) = \psi(t_0, x_k) + \sqrt{|\nabla\varphi|^2 + \beta(t)} + \gamma(t) - V$ keep positive for k large enough when we choose $\beta = d + \frac{f}{t}, \alpha, d, f > 0$ as desired. The item $\nabla\varphi\nabla\psi$ appearing in (2.1) can be controlled by $\varepsilon|\nabla\varphi|^2$ and a constant $c(\varepsilon, k)$, where $c(\varepsilon, k) \to 0$ as $k \to \infty$. So all the steps following (2.1) will be valid with a bit of modification. Finally, letting $k \to \infty$ and then $\varepsilon \to 0$, we can get the following result.

Theorem 2.1 *Assume that (M^n, g) is an n-dimensional complete noncompact Riemannian manifold with $Ric(M) \geq$ for some $K > 0$. Given $a > 0$ some constant for some $K > 0$. Let V be a given smooth positive bounded function on M such that $-\Delta V + \frac{|\nabla V|^2}{\tilde{K}} \leq 0$ for some $\tilde{K} > 0$. Let u be a positive solution to the nonlinear heat equation (1.1) in $(M, g) \times [0, T)$. Then we have, for all $t > 0$ and $\delta \in \left(\frac{2}{3}, 1\right)$*

$$|\nabla \log u|^2 - \partial_t(\log u) + a \log u \leq \sqrt{\frac{2n(K + \tilde{K})}{2\delta - 1}} \sqrt{|\nabla \log u|^2 + d + \frac{e}{t}} + \frac{n}{2\delta t} + V,$$

where

$$d = \frac{na^2}{8(1-\delta)(2K + \tilde{K})} \vee \frac{2n(K + \tilde{K})}{2\delta - 1}, \qquad e = \frac{n}{2\delta(3\delta - 2)}.$$

We now consider the proof of Theorem 1.4. Note that Theorem 1.4 is a variant of Theorem 1.3 to any compact Riemannian manifold with smooth convex boundary and $V_\nu \leq 0$ on the boundary. The proof of Theorem 1.4 is similar to the proof of Theorem 1.3. We need only to exclude the possibility of the maximum point of ψ at boundary point. If the maximum occurs at the boundary point (t_0, x_0), then by the strong maximum principle we have $\psi_\nu > 0,\ \varphi_\nu = -\frac{u_\nu}{u} = 0,\ (\varphi_t)_\nu = 0$ at this point. Note that

$$\Delta\varphi = \varphi_t + |\nabla\varphi|^2 - a\varphi + V$$

and

$$|\nabla\varphi|^2_\nu = 2\varphi_j\varphi_{j\nu} = -2II(\nabla\varphi, \nabla\varphi) \leq 0.$$

So, at (t_0, x_0),

$$
\begin{aligned}
\psi_\nu &= |\nabla\varphi|^2_\nu + (\varphi_t)_\nu - a\varphi_\nu - \frac{\alpha(\nabla\varphi, \nabla_\nu\nabla\varphi)}{\sqrt{|\nabla\varphi|^2+\beta}} + V_\nu \\
&= -\left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}}\right) II(\nabla\varphi, \nabla\varphi) + V_\nu \\
&\leq 0.
\end{aligned}
$$

This is in contradiction with the $\psi_\nu > 0$. This proves Theorem 1.4.

3 The Proof of Theorem 1.6

As before, we let $\varphi = -\log u$. Then

$$
\varphi_t = -\frac{u_t}{u}, \quad \varphi_j = -\frac{u_j}{u}, \quad \Delta\varphi = -\frac{\Delta u}{u} + |\nabla\varphi|^2.
$$

Then we have

$$
\varphi_t = \Delta\varphi - |\nabla\varphi|^2 - V + \sum W_i\varphi_i.
$$

As in the work [26], we let

$$
\psi := \varphi_t + |\nabla\varphi|^2 + V - \sum W_i\varphi_i - \alpha\sqrt{|\nabla\varphi|^2+\beta} - \gamma(t) - b,
$$

where α and b are positive constants, β and γ are positive functions in t, which will be determined later. Then we have

$$
\begin{aligned}
\psi_t &= \varphi_{tt} + 2\sum\varphi_j\varphi_{it} - \sum(W_i)_t\varphi_i - \sum W_i\varphi_{it} \\
&\quad - \frac{\alpha\sum\varphi_i\varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta\psi &= (\Delta\varphi)_t + 2\sum\varphi_{ij}^2 + 2\sum\varphi_j(\Delta\varphi)_j + 2\sum R_{ij}\varphi_i\varphi_j - \sum R_{ij}W_i\varphi_j \\
&\quad -2\sum W_{i,j}\varphi_{ij} - \sum W_i(\Delta\varphi)_i - \sum(\Delta W_i)\varphi_i + \Delta V - \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} \\
&\quad - \frac{\alpha\sum\varphi_i(\Delta\varphi)_i}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum_j(\sum_i\varphi_i\varphi_{ij})^2}{\sqrt[3]{(|\nabla\varphi|^2+\beta)^2}}. \\
&\geq (\Delta\varphi)_t + 2\sum\varphi_{ij}^2 + 2\sum\varphi_j(\Delta\varphi)_j + 2\sum R_{ij}\varphi_i\varphi_j - \sum R_{ij}W_i\varphi_j \\
&\quad -2\sum W_{i,j}\varphi_{ij} - \sum W_i(\Delta\varphi)_i - \sum(\Delta W_i)\varphi_i + \Delta V \\
&\quad - \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\sum\varphi_i(\Delta\varphi)_i}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}}.
\end{aligned}
$$

By these relations, we get

$$
\begin{aligned}
\psi_t - \Delta\psi \ \le\ & (\Delta\varphi)_t - \frac{\alpha \sum \varphi_i \varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) - (\Delta\varphi)_t - 2\sum \varphi_{ij}^2 \\
& -2\sum \varphi_j(\Delta\varphi)_j - 2\sum R_{ij}\varphi_i\varphi_j + \sum R_{ij}W_i\varphi_j + 2\sum W_{i,j}\varphi_{ij} \\
& +\sum W_i(\Delta\varphi)_i + \sum (\Delta W_i)\varphi_i - \Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i(\Delta\varphi)_i}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& + \frac{\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} \\
=\ & -\frac{\alpha \sum \varphi_i \varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) - 2\sum \varphi_{ij}^2 - 2\sum \varphi_j(\Delta\varphi)_j \\
& -2\sum R_{ij}\varphi_i\varphi_j + \sum R_{ij}W_i\varphi_j + 2\sum W_{i,j}\varphi_{ij} + \sum W_i(\Delta\varphi)_i \\
& +\sum (\Delta W_i)\varphi_i - \Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i(\Delta\varphi)_i}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} \\
=\ & -2\varphi_j\left(\psi + \alpha\sqrt{|\nabla\varphi|^2+\beta} + \gamma(t) + b\right)_j \\
& +\sum W_i\left(\psi + \alpha\sqrt{|\nabla\varphi|^2+\beta} + \gamma(t) + b\right)_i \\
& -2\sum\varphi_{ij}^2 - 2\sum R_{ij}\varphi_i\varphi_j + \sum R_{ij}W_i\varphi_j + 2\sum W_{i,j}\varphi_{ij} + \sum W_i(\Delta\varphi)_i \\
& +\sum (\Delta W_i)\varphi_i - \Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i(\varphi_t + |\nabla\varphi|^2 + V - W_i\varphi_i)_i}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& + \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha \sum \varphi_i \varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) \\
=\ & -2\varphi_j\psi_j - \frac{2\alpha\varphi_j\varphi_i\varphi_{ij}}{\sqrt{|\nabla\varphi|^2+\beta}} + \sum W_i\psi_i + \frac{\alpha\sum W_i\varphi_j\varphi_{ij}}{\sqrt{|\nabla\varphi|^2+\beta}} - 2\sum\varphi_{ij}^2 \\
& -2\sum R_{ij}\varphi_i\varphi_j + \sum R_{ij}W_i\varphi_j + 2\sum W_{i,j}\varphi_{ij} + \sum(\Delta W_i)\varphi_i \\
& -\Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i(\varphi_{it} + \varphi_j\varphi_{ij} + V_i - W_{i,j}\varphi_i - W_i\varphi_{ij})}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& + \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha \sum \varphi_i \varphi_{it}}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) \\
=\ & -2\varphi_j\psi_j + \sum W_i\psi_i - 2\sum\varphi_{ij}^2 - 2\sum R_{ij}\varphi_i\varphi_j + \sum R_{ij}W_i\varphi_j \\
& +2\sum W_{i,j}\varphi_{ij} + \sum(\Delta W_i)\varphi_i - \Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i V_i}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& - \frac{\alpha\sum W_{i,j}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t). \qquad (3.1)
\end{aligned}
$$

Suppose that $\psi < 0$ when $t = 0$. Assume that at $t_0 > 0$, ψ becomes zero at some point x_0 in the interior of the manifold and $\psi < 0$ for $t < t_0$. Then $\psi_t \ge 0$,

$\nabla\psi = 0$ and $\Delta\psi \le 0$ at (t_0, x_0). Note that at the point (t_0, x_0), we have $\Delta\varphi = \alpha\sqrt{|\nabla\varphi|^2 + \beta(t)} + \gamma(t) + b$. Hence, at point (t_0, x_0),

$$
\begin{aligned}
0 \le\ & -2\sum \varphi_{ij}^2 - 2\sum R_{ij}\varphi_i\varphi_j + \sum R_{ij}W_i\varphi_j + 2\sum W_{i,j}\varphi_{ij} \\
& + \sum(\Delta W_i)\varphi_i - \Delta V + \frac{\alpha\sum\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i V_i}{\sqrt{|\nabla\varphi|^2+\beta}} \\
& - \frac{\alpha\sum W_{i,j}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum R_{ij}\varphi_i\varphi_j}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) \\
=\ & \left(\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}} - 2\right) R_{ij}\varphi_i\varphi_j + \left(\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}} - 2\right)\sum\varphi_{ij}^2 \\
& + \sum(W_{i,j} + W_{j,i})\varphi_{ij} + \sum(\Delta W_i)\varphi_i + \sum R_{ij}W_i\varphi_j \\
& - \frac{\alpha\sum(W_{i,j}+W_{j,i})\varphi_i\varphi_j}{2\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum\varphi_i V_i}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} \\
& - \gamma'(t) - \Delta V,
\end{aligned}
$$

where we have used the assumption $\alpha \le 2\sqrt{|\nabla\varphi|^2 + \beta}$. By the Ricci curvature lower bound assumption, we have at (t_0, x_0),

$$
\begin{aligned}
0 \le\ & -\left(\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}} - 2\right) K|\nabla\varphi|^2 + \left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}} - \frac{2}{n}\right)|\Delta\varphi|^2 + \epsilon|\Delta\varphi|^2 \\
& + \frac{1}{4\epsilon}\sum(W_{i,j}+W_{j,i})^2 + \epsilon|\nabla\varphi|^2 + \frac{1}{4\epsilon}\sum(\Delta W_i)^2 + \epsilon|\nabla\varphi|^2 \\
& + \frac{1}{4\epsilon}\left(\sum R_{ij}W_i\right)^2 + \epsilon|\nabla\varphi|^2 + \frac{1}{16\epsilon}\sum(W_{i,j}+W_{j,i})^2 + \alpha|\nabla V| \\
& - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) - \Delta V \\
=\ & \left(2K - \frac{\alpha K}{\sqrt{|\nabla\varphi|^2+\beta}} + 3\epsilon\right)|\nabla\varphi|^2 + \left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}} - \frac{2}{n} + \epsilon\right)|\Delta\varphi|^2 \\
& + \frac{4+\alpha^2}{16\epsilon}\sum(W_{i,j}+W_{j,i})^2 + \frac{1}{4\epsilon}\sum(\Delta W_i)^2 + \frac{1}{4\epsilon}\left(\sum R_{ij}W_i\right)^2 \\
& - \frac{\alpha\beta'(t)}{2\sqrt{|\nabla\varphi|^2+\beta}} - \gamma'(t) + \alpha|\nabla V| - \Delta V.
\end{aligned} \tag{3.2}
$$

We need only to show that (3.2) is violated. Choose $\gamma(t)=\frac{f}{t}$, $\beta=d+\frac{e}{t}$, where $d>\alpha^2$, e, f are positive constants, which will be determined later. For any $\frac{1}{2}<\delta<1$ and $0<\epsilon<\frac{1-\delta}{n}$, we can get $\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}}<\frac{2\delta}{n}$. Then

$$\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}}-\frac{2}{n}+\epsilon<\frac{\delta-1}{n}.$$

Substituting that, at (t_0,x_0), we have $\Delta\varphi=\alpha\sqrt{|\nabla\varphi|^2+\beta(t)}+\gamma(t)+b$ into (3.2)

$$\begin{aligned}
0 \le\ & (2K+3\epsilon)|\nabla\varphi|^2+\left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}}-\frac{2}{n}+\epsilon\right)\left(\alpha\sqrt{|\nabla\varphi|^2+\beta}+\gamma(t)+b\right)^2\\
&+\frac{4+\alpha^2}{16\epsilon}\sum(W_{i,j}+W_{j,i})^2+\frac{1}{4\epsilon}\sum(\Delta W_i)^2+\alpha|\nabla V|+\frac{1}{4\epsilon}\left(\sum R_{ij}W_i\right)^2\\
&+\frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2+\beta}}+\frac{f}{t^2}-\Delta V\\
=\ & (2K+3\epsilon)|\nabla\varphi|^2+\left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}}-\frac{2}{n}+\epsilon\right)\Big\{\alpha^2(|\nabla\varphi|^2+\beta(t))+2(\gamma(t)\\
&+b)\alpha\sqrt{|\nabla\varphi|^2+\beta(t)}+(\gamma(t)+b)^2\Big\}+\frac{4+\alpha^2}{16\epsilon}\sum(W_{i,j}+W_{j,i})^2\\
&+\frac{1}{4\epsilon}\sum(\Delta W_i)^2+\alpha|\nabla V|+\frac{1}{4\epsilon}\left(\sum R_{ij}W_i\right)^2+\frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2+\beta}}+\frac{f}{t^2}-\Delta V\\
\le\ & \left(\frac{(\delta-1)\alpha^2}{n}+3\epsilon+2K\right)|\nabla\varphi|^2+\frac{(\delta-1)\alpha^2}{n}\beta+\frac{2(\delta-1)\alpha f}{tn}\sqrt{|\nabla\varphi|^2+\beta}\\
&+\left(\frac{\alpha}{n\sqrt{|\nabla\varphi|^2+\beta}}-\frac{2}{n}+\frac{1-\delta}{n}\right)\frac{f^2}{t^2}+\frac{2b\alpha(\delta-1)}{n}\\
&\times\sqrt{|\nabla\varphi|^2+\beta}+\frac{2fb(\delta-1)}{nt}+\frac{(\delta-1)}{n}b^2+\frac{1}{4\epsilon}\sum(\Delta W_i)^2\\
&+\frac{1}{4\epsilon}\left(\sum R_{ij}W_i\right)^2+\frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2+\beta}}+\frac{f}{t^2}-\Delta V+\alpha|\nabla V|\\
&+\frac{4+\alpha^2}{16\epsilon}\sum(W_{i,j}+W_{j,i})^2\\
\le\ & \left(\frac{(\delta-1)\alpha^2}{n}+3\epsilon+2K\right)|\nabla\varphi|^2+\frac{f}{t^2}\left(1-\frac{(\delta+1)f}{n}\right)+\frac{(\delta-1)\alpha^2}{n}\beta+\frac{2b\alpha^2}{n}\\
&+\frac{2(\delta-1)\alpha f}{tn}\sqrt{|\nabla\varphi|^2+\beta}+\frac{\alpha f^2}{nt^2\sqrt{|\nabla\varphi|^2+\beta}}-\frac{2\alpha b(\delta+1)}{n}\sqrt{|\nabla\varphi|^2+\beta}\\
&+\frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2+\beta}}+\frac{\delta-1}{n}b^2+\frac{1}{4\epsilon}\sum(\Delta W_i)^2+\frac{1}{4\epsilon}\left(\sum R_{ij}W_i\right)^2-\Delta V\\
&+\alpha|\nabla V|+\frac{4+\alpha^2}{16\epsilon}\sum(W_{i,j}+W_{j,i})^2.
\end{aligned}$$

To make the above be violate, we only need

$$
\begin{aligned}
I_1 &:= \frac{(\delta-1)\alpha^2}{n} + 3\epsilon + 2K \le 0, \\
I_2 &:= \frac{(\delta-1)\alpha^2\beta}{n} + \frac{2b\alpha^2}{n} < 0, \\
I_3 &:= 1 - \frac{(\delta+1)f}{n} = 0, \\
I_4 &:= \frac{2(\delta-1)\alpha f}{tn}\sqrt{|\nabla\varphi|^2+\beta} + \frac{\alpha f^2}{nt^2\sqrt{|\nabla\varphi|^2+\beta}} \\
&\quad -\frac{2\alpha b(\delta+1)}{n}\sqrt{|\nabla\varphi|^2+\beta} + \frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2+\beta}} \le 0, \\
I_5 &:= \frac{\delta-1}{n}b^2 + \frac{1}{4\epsilon}\sum(\Delta W_i)^2 + \frac{1}{4\epsilon}\left(\sum R_{ij}W_i\right)^2 - \Delta V \\
&\quad +\alpha|\nabla V| + \frac{4+\alpha^2}{16\epsilon}\sum(W_{i,j}+W_{j,i})^2 \le 0.
\end{aligned}
$$

To this end, we need

$$
\alpha \ge \sqrt{(3\epsilon+2K)n/(1-\delta)}, \qquad d \ge \frac{2\sqrt{n\Lambda}}{(1-\delta)^{3/2}},
$$

$$
f = \frac{n}{\delta+1}, \qquad e \ge \frac{2n}{(3-5\delta)(\delta+1)}, \qquad b \ge \sqrt{\frac{n\Lambda}{1-\delta}},
$$

where

$$
\begin{aligned}
\Lambda = \sup\Bigg\{ &\frac{1}{4\epsilon}\sum(\Delta W_i)^2 + \frac{1}{4\epsilon}\left(\sum R_{ij}W_i\right)^2 - \Delta V + \alpha|\nabla V| \\
&+\frac{4+\alpha^2}{16\epsilon}\sum(W_{i,j}+W_{j,i})^2\Bigg\},
\end{aligned}
$$

and $\frac{1}{2} < \delta < \frac{3}{5}$. This completes the proof of Theorem 1.6.

Acknowledgements The authors would like to thank the unknown referees for a helpful suggestion.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Aubin, T.: Nonlinear Analysis on Manifolds. Springer, New York (1982)
2. Bailesteanu, M., Cao, X., Pulemotov, A.: Gradient estimates for the heat equation under the Ricci flow. J. Funct. Anal. **258**, 3517–3542 (2010)
3. Chow, B., Hamilton, R.S.: Constrained and linear Harnack inequalities for parabolic equations. Invent. Math. **129**(2), 213–238 (1997)
4. Cao, X., Hamilton, R.S.: Differential Harnack estimates for time dependent heat equations with potentials. Geom. Funct. Anal. **19**(4), 989–1000 (2009)

5. Cao, X., Fayyazuddin Ljungbergb, B., Liu, B.: Differential Harnack estimates for a nonlinear heat equation. J. Funct. Anal. **265**, 2312–2330 (2013)
6. Cao, X., Zhang, Q.S.: The conjugate heat equation and ancient solutions of the Ricci flow. Adv. Math. **228**(5), 2891–2919 (2011)
7. Cao, H.-D., Ni, L.: Matrix Li-Yau-Hamilton estimates for the heat equation on Kähler manifolds. Math. Ann. **331**(4), 795–807 (2005)
8. Chen, L., Chen, W.: Gradient estimates for a nonlinear parabolic equation on complete non-compact Riemannian manifolds. Ann. Global Anal. Geom. **35**, 397–404 (2009)
9. Hamilton, R.S.: A matrix Harnack estimate for the heat equation. Comm. Anal. Geom. **1**(1), 113–126 (1993)
10. Hamilton, R.S.: The Information of Singularities in the Ricci Flow. Survey in Differential Geometry, vol. 2, pp. 7–136. International Press, Boston (1995)
11. Huang, G., Ma, B.: Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds. Arch. Math. (Basel) **94**, 265–275 (2010)
12. Huang, G., Huang, Z., Li, H.: Gradient estimates and differential Harnack inequalities for a nonlinear parabolic equation on Riemannian manifolds. Ann. Global Anal. Geom. **43**(3), 209–232 (2013)
13. Kuang, S., Zhang, Q.S.: A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow. J. Funct. Anal. **255**(4), 1008–1023 (2008)
14. Ni, L.: The entropy formula for linear heat equation. J. Geom. Anal. **14**, 369–374 (2004)
15. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator. Acta Math. **156**(3–4), 153–201 (1986)
16. Li, J.: Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds. J. Funct. Anal. **100**, 233–256 (1991)
17. Ma, L.: Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds. J. Funct. Anal. **241**(1), 374–382 (2006)
18. Ma, L.: Gradient estimates for a simple nonlinear heat equation on manifolds. Appl. Anal. **96**, 225–230 (2017)
19. Ma, L.: Harnack's inequality and Green's functions on locally finite graphs. Nonlinear Anal. **170**, 226–237 (2018)
20. Ma, L., Zhao, L., Song, X.: Gradient estimate for the degenerate parabolic equation $u_t = \Delta F(u) + H(u)$ on manifolds. J. Differ. Equ. **244**, 1157–1177 (2008)
21. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:0211159v1 (2002)
22. Qian, B.: A uniform bound for the solutions to a simple nonlinear equation on Riemannian manifolds. Nonlinear Anal. **73**, 1538–1542 (2010)
23. Souplet, P., Zhang, Q.: Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds. Bull. Lond. Math. Soc. **38**(6), 1045–1053 (2006)
24. Wu, J.: Elliptic gradient estimates for a ninlinear harat equation and application. Nonlinear Anal. **151**, 1–17 (2017)
25. Xu, X.: Gradient estimates for on manifolds and some Liouville-type theorems. J. Differ. Equ. **252**(2), 1403–1420 (2012)
26. Yau, S.-T.: On the Harnack inequalities of partial differential equations. Comm. Anal. Geom. **2**(3), 431–450 (1994)
27. Yau, S.-T.: Harnack inequality for non-self-adjoint evolution equations. Math. Res. Lett. **2**(4), 387–399 (1995)
28. Yau, S.-T.: Harmonic function on complete Riemannian manifolds. Comm. Pure Appl. Math. **28**, 201–228 (1975)
29. Yang, Y.: Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds. Proc. Am. Math. Soc. **136**, 4095–4102 (2008)
30. Zhu, X., Li, Y.: Li-Yau estimates for a nonlinear parabolic equation on manifolds. Math. Phys. Anal. Geom. **17**, 273–288 (2014)

Affiliations

Li Ma[1] · Yang Liu[2]

Yang Liu
yangliu8223@163.com

1 School of Mathematics and Physics, University of Science and Technology Beijing, 30 Xueyuan Road, Haidian District, Beijing, 100083, People's Republic of China

2 College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan, 453007, People's Republic of China