



Note on the Intermediate Field Representation of ϕ^{2k} Theory in Zero Dimension

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Abstract

This note is a sequel to Rivasseau and Wang (J. Math. Phys. **51**, 092304, 2010). We correct the intermediate field representation for the stable ϕ^{2k} field theory in zero dimension introduced there and extend it to the case of complex conjugate fields. For $k = 3$ in the complex case we also provide an improved representation which relies on ordinary convergent Gaussian integrals rather than oscillatory integrals.

Keywords Constructive field theory · Loop vertex expansion · Borel summability

Mathematics Subject Classification (2010) 81T08

1 Introduction

The intermediate field (hereafter called IF) representation and the associated constructive loop vertex expansion (LVE) [2–5] have been increasingly used in recent years [6–13] for models with quartic interactions. It is important to extend such techniques to models with higher order stable interactions, as first attempted in [1]. The case of a ϕ^6 interaction is treated in sections 2 and 3 of [1], using imaginary Gaussian measures with a small contour deformation, and the case of a general ϕ^{2k} interaction is sketched in section 4 of [1]. Unfortunately Lemma 4.1 as stated there is not cor-

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rect and requires a modification.¹ Also the number of intermediate fields introduced in [1] is not optimal. Finally we found that the loop vertex expansion in [1] is not correct since the interpolation of imaginary Gaussian covariances through forest formulas is not fully justified. Hence the main purpose of this paper is to correct [1], to give a proof of Borel-Leroy summability (since the one in [1] is incorrect), and to give improved intermediate field representations for the partition functions of such toy models. It should lay the ground for future extensions to high order interactions of constructive techniques such as the LVE or its multiscale extension [14].

This multiscale loop vertex representation or MLVE adds a Fermionic expansion to the LVE. This Fermionic layer performs the usual removal of hard core constraints between momentum slices (as in a Mayer expansion). It is required to treat theories with renormalization. In particular it has been the key tool to construct superrenormalizable tensor models [13, 15, 16]. However it is not clear yet whether this MLVE is well-adapted to local quantum field theory, since the spatial decay of correlation functions is difficult to establish within the LVE formalism (see however [3]). This remains a key question for future constructive studies.

In the next section we gather some mathematical prerequisites on Gaussian imaginary integrals and Borel-Leroy summability. We also introduce the models discussed in this paper, namely the (stable) $\lambda\phi^{2k}$ model and its complex $\lambda(\bar{\phi}\phi)^k$ version. They are zero-dimensional, hence toy models useful to test constructive methods in quantum field theory [4]. Their partition functions $Z_k(\lambda)$ and $Z_k^c(\lambda)$ are the generating functions for counting ϕ^{2k} or $(\bar{\phi}\phi)^k$ vacuum Feynman graphs. Using their ordinary integral representation we check that these partition functions are the Borel-Leroy sum of their perturbative expansion in powers of λ .

In Section 3, starting from this ordinary representation, we guess the form of an intermediate field representation, using as in [1] imaginary Gaussian integrals with suitable integration contours ensuring convergence. This guess is based on commuting some integrals without caring about convergence.

Then in Section 4 we check that our guess is in fact an absolutely convergent integral, which is again the Borel sum of its perturbative expansion. Since this expansion is the same as the initial one, from unicity of the Borel-Leroy sum we conclude *a posteriori* that our guess is indeed a correct (non-perturbative) representation of the partition functions $Z_k(\lambda)$ and $Z_k^c(\lambda)$.

The “free energies” $\log Z_k(\lambda)$ and $\log Z_k^c(\lambda)$ are physically more interesting than the partition functions. They are the generating functions of *connected* ϕ^{2k} or $(\bar{\phi}\phi)^k$ vacuum Feynman graphs. The LVE combines an intermediate field representation with a forest formula and a replica trick to compute directly these functions through a convergent expansion. In spite of several attempts, we have not been able yet to define a convergent LVE for the imaginary Gaussian intermediate field representation of Section 3, hence correct the last problem in [1]. Therefore in Section 5 of this

¹In fact the corresponding correct Lemma happens to be also Lemma IV.1 of *this* paper. It combines bounds on complex arguments with a Hilbert-Schmidt bound on the norm of a matrix to reach (4.5). This rather subtle combination is necessary and had been missed in [1].

paper we introduce still another intermediate representation, better adapted to this task. Since it is also more complicated, we limit ourselves to give it in the $k = 3$ complex case.

After completion of this work our representation has been generalized for matrix and tensor models with positive interaction [17]. Moreover the improved representation of Section 5 has been extended to any $k \geq 3$ in [18]. It has been applied to the corresponding matrix models with stable $\lambda \text{Tr} M^{2k}$ interaction in [19], proving existence of a coupling constant analyticity domain for such models which does not shrink as the size N of the matrix tends to infinity.

2 Prerequisites

2.1 Imaginary Gaussian Measures

Consider a function $f(z)$ which is analytic in the strip $\Im z \leq \delta$ and exponentially bounded in that domain by $Ke^{\eta|z|}$ for some $0 \leq \eta < \delta$, where K is some constant.

The imaginary Gaussian integral of f with covariance $\pm iC$, where $C > 0$, is defined as

$$\int d\mu_{\pm iC}(x) f(x) := \int_{C_{\pm, \epsilon}} \frac{e^{-z^2/\pm 2iC} dz}{\sqrt{\pm 2\pi iC}} f(z) = \int_{C_{\pm, \epsilon}} \frac{e^{\pm iz^2/2C} dz}{\sqrt{\pm 2\pi iC}} f(z) \tag{2.1}$$

where the contour $C_{\pm, \epsilon}$ can be for instance chosen as $t \rightarrow z(t) = t \pm i\epsilon \tanh(t)$ for any $\epsilon \in]C\eta, \delta[$, where $t \in \mathbb{R}$. Remark indeed that from our hypotheses on f , the integral (2.1) is well defined and absolutely convergent for $C\eta < \epsilon < \delta$, and by Cauchy theorem, independent of $\epsilon \in]C\eta, \delta[$. The contour $C_{+, \epsilon}$ is shown in Fig. 1.

Although the result of integration does not depend on the contour, actual bounds on the result typically depend on choosing particular contours in which ϵ is not too small, see Section 4. Furthermore the Gaussian rules of integration still apply, e.g. defining $(2n - 1)!! := (2n - 1)(2n - 3) \dots 5.3.1$

$$\int d\mu_{\pm iC}(x) x^{2n} = (\pm iC)^n (2n - 1)!! . \tag{2.2}$$

This is easy to check since a polynomial is an entire function and we can deform the contour into $z = x + ix$, in which case we recover an ordinary Gaussian integration. Similarly

$$\int d\mu_{\pm iC}(x) e^{ax} = e^{\pm iCa^2/2}, \tag{2.3}$$

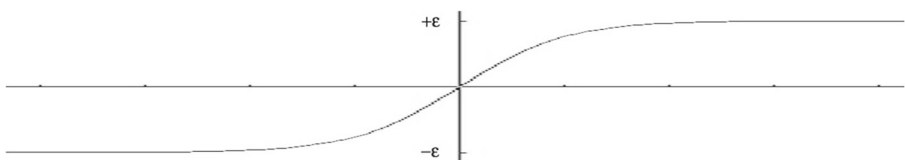


Fig. 1 The integration contour $C_{+, \epsilon}$

the integral being absolutely convergent for any contour such that $C|a| < \epsilon$.

The imaginary complex normalized Gaussian measures $d\mu_{\pm iC}^c(z)$ of covariance $\pm i$ for a complex variable $z = x + iy$ is similarly defined as a pair of independent real normalized Gaussian measures of covariance C , one for x and one for y . (2.2)-(2.3) generalize to

$$\int d\mu_{\pm iC}^c(x)z^n\bar{z}^{\bar{n}} = \delta_{n\bar{n}}(\pm iC)^n n! \tag{2.4}$$

and

$$\int d\mu_{\pm iC}^c(z)e^{az+b\bar{z}} = e^{\pm iabC}, \tag{2.5}$$

again this last integral being absolutely convergent if $C \sup\{|a|, |b|\} < \epsilon$. The integrals correspond to complexifications of the two dimensional integral $\int_{\mathbb{C}} dz d\bar{z} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy$ into the product of two contour integrals on $C_{+,\epsilon}$, one for x and one for y .

2.2 Borel-Leroy Summability

Usual Borel summability concerns functions analytic in a domain with opening angle π asymptotic to power series $\sum_{n=0}^{\infty} a_n \lambda^n$ with large n behavior $a_n \simeq c^n n!$. Borel-Leroy summability extends to the case of functions with larger analyticity domains, of opening angle $k\pi$, but also worse asymptotic series with large n behavior $a_n \simeq c^n [n!]^k$.

Consider the Riemann surface \mathcal{L} for the logarithm, namely the universal cover of \mathbb{C}^* with deck transformations $(z, \theta) \rightarrow (z, \theta + 2k\pi)$, $k \in \mathbb{Z}$. \mathcal{L} can be embedded in $\mathbb{C}^* \times \mathbb{R}$ since a point of \mathcal{L} can be defined as a pair (z, θ) in $\mathbb{C}^* \times \mathbb{R}$ such that θ is an argument of z . The logarithm is well-defined (single valued) on \mathcal{L} . For $\rho > 0$ we define the open domain $D_\rho^k \subset \mathcal{L}$ by the equation $\Re \lambda^{-\frac{1}{k}} > \rho^{-1}$, which means $\Re [e^{-\frac{1}{k} \log \lambda}] > \rho^{-1}$. D_ρ^1 is a disk tangent to the imaginary axis of diameter ρ . For larger values of k D_ρ^k spreads over more and more sheets of \mathcal{L} but remains close to the vertical axis (the origin $\lambda = 0$). More precisely a point λ is in D_ρ^k iff its unique representative $(z, \theta) \in \mathbb{C}^* \times \mathbb{R}$ with $\theta = \arg z (2\pi)$ satisfies to $|\theta| < \frac{k\pi}{2}$ and $|z|^{1/k} < \rho \cos \frac{\theta}{k}$ so it is at distance less than ρ^k of the vertical axis $\{0\} \times \mathbb{R}$ in $\mathbb{C} \times \mathbb{R}$.

We note R^N the N -th order Taylor remainder operator at the origin. It acts on a smooth function $f(\lambda)$ through

$$R^N f = \lambda^N \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} f^{(N)}(t\lambda) dt. \tag{2.6}$$

Theorem 2.1 *A power series $\sum_{n=0}^{\infty} a_n \lambda^n$ is Borel-Leroy summable of order k to the function $f(\lambda)$ if the following conditions are met:*

- For some $\rho > 0$, $f(\lambda)$ is analytic in a domain D_ρ^k .

- The function $f(\lambda)$ admits $\sum_{n=0}^{\infty} a_n \lambda^n$ as a strong asymptotic expansion to all orders as $\lambda \rightarrow 0$ with uniform estimate in D_ρ^k :

$$\left| R^N f \right| \leq AB^N [(kN)!] |\lambda|^N. \tag{2.7}$$

where A and B are some positive constants.

Then the Borel-Leroy transform of order k defined by

$$B_f^{(k)}(u) = \sum_{n=0}^{\infty} \frac{a_n}{(kn)!} u^n, \tag{2.8}$$

is holomorphic for $|u| < B^{-1}$, it admits (for some $R > 0$) an analytic continuation to the strip $\{u \in \mathbb{C} : |\Im u| < R, \Re u > 0\}$ which does not grow too fast at infinity, so that one recovers $f(\lambda)$ for $\lambda \in D_\rho^k$ through the absolutely convergent inverse integral

$$f(\lambda) = \frac{1}{k\lambda} \int_0^{\infty} B_f^{(k)}(u) e^{-\left(\frac{u}{\lambda}\right)^{\frac{1}{k}}} \left(\frac{u}{\lambda}\right)^{\frac{1}{k}-1} du. \tag{2.9}$$

Proof For $k = 1$ this is exactly Nevanlinna’s theorem as rediscovers by Sokal [20]. For larger values of k , defining $g = \lambda^{1/k}$, we see that $\tilde{f}(g) = f(\lambda) \simeq \sum_n a_n g^{kn}$ satisfies Nevanlinna’s hypothesis, and through the change of variables $u = \lambda v^k$ (2.9) is nothing but the ordinary inverse Borel formula from B to \tilde{f} . \square

2.3 ϕ^{2k} Theory in 0 Dimension

The partition function of the ϕ^{2k} scalar theory in zero dimension for $k \geq 2$ is given by the one-dimensional integral

$$Z_k(\lambda) = \int_{-\infty}^{+\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} e^{-\lambda\phi^{2k}/2} = \int d\mu(\phi) e^{-\lambda\phi^{2k}/2}, \tag{2.10}$$

where $d\mu$ is the normalized one-dimensional Gaussian measure of covariance 1. Its “free energy” is simply $\log Z_k(\lambda)$. The integral (2.10) will from now on be called the *standard representation* of the theory. The factor 1/2 in front of λ is a suitable normalization to simplify the intermediate field representation below.

Illegally commuting series and integration leads to

$$Z_k(\lambda) \simeq \sum_{n=0}^{\infty} a_{k,n} \lambda^n, \quad a_{k,n} = \frac{(-1)^n}{2^n n!} \int d\mu(\phi) \phi^{2kn} = (-1)^n \frac{(2kn)!!}{2^n n!} \tag{2.11}$$

where we define $2p!! = \prod_{k=1}^p (2k - 1)$. Of course the power series a_n has zero radius of convergence but it is Borel-Leroy summable of order $k - 1$ (see Theorem 2.2 below).

The model exists also in a complex version, with partition function

$$Z_k^c(\lambda) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\phi d\bar{\phi}}{\pi} e^{-\bar{\phi}\phi} e^{-\lambda(\bar{\phi}\phi)^k} = \int_{\mathbb{C}} d\mu^c(\phi) e^{-\lambda(\bar{\phi}\phi)^k}, \tag{2.12}$$

where $d\mu^c$ is the normalized one-dimensional complex Gaussian measure of covariance 1. Its perturbative expansion is

$$Z_k^c(\lambda) \simeq \sum_{n=0}^{\infty} a_{k,n}^c \lambda^n, \quad a_{k,n} = \frac{(-1)^n}{n!} \int d\mu^c(\phi) (\bar{\phi}\phi)^{kn} = (-1)^n \frac{(kn)!}{n!} \tag{2.13}$$

Theorem 2.2 *The partition functions $Z_k(\lambda)$ and $Z_{k,c}(\lambda)$ are Borel-Leroy summable of order $k - 1$.*

Proof Let us give the proof only for $Z_k(\lambda)$ as the complex case is similar. We shall prove analyticity in a domain $D_\rho^k = \{\lambda \in \mathcal{L} : \Re\lambda^{-1/k} > \rho^{-1}\}$ obviously bigger than the needed domain D_ρ^{k-1} . But we shall prove Borel-Leroy summability of order $k - 1$ (not k) since we shall prove Taylor remainder estimates in $[(k - 1)N]!$ (see (2.11)-(2.13)).

The integrand of $Z_k(\lambda)$ is an entire function of λ which is uniformly bounded by the integrable function $e^{-\frac{1}{2}\phi^2}$ in the right-half complex plane $\mathbb{C}_+ = \{\lambda \mid \Re(\lambda) > 0\}$. Hence Z_k is analytic in \mathbb{C}_+ . For $k = 2$, \mathbb{C}_+ contains a disk D_ρ^1 . For $k > 2$ we continue $Z_k(\lambda)$ analytically to a domain of \mathcal{L} of wider opening angle by performing the change of variable $\phi = \lambda^{-\frac{1}{2k}} \psi$. Rotating the integration contour we find

$$Z_k(\lambda) = \lambda^{-\frac{1}{2k}} \int_{-\infty}^{+\infty} e^{-\frac{\psi^{2k}}{2}} e^{-\lambda^{-\frac{1}{k}} \frac{\psi^2}{2}} \frac{d\psi}{\sqrt{2\pi}}. \tag{2.14}$$

In the domain D_ρ^k the integrand is analytic and its absolute value is uniformly bounded by the integrable function $e^{-\frac{\psi^{2k}}{2}}$, hence we can conclude to analyticity of the integral $Z_k(\lambda)$ in this domain.² However because of the prefactor $\lambda^{-\frac{1}{2k}}$ it is not yet obvious that $Z_k(\lambda)$ is uniformly bounded in D_ρ^k , as it should for (2.7) to hold at $N = 0$. This can be checked through a single expansion step on the $e^{-\frac{\psi^{2k}}{2}}$ factor, writing

$$e^{-\frac{\psi^{2k}}{2}} = 1 - \int_0^1 dt \frac{\psi^{2k}}{2} e^{-t\frac{\psi^{2k}}{2}}, \tag{2.15}$$

taking one ψ factor out of ψ^{2k} and joining it to the Gaussian factor $e^{-\lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}$ to create a full derivative (correcting for the missing $-\lambda^{-\frac{1}{k}}$ factor) and then performing integration by parts. It leads to:

$$\begin{aligned} Z_k(\lambda) &= 1 - \lambda^{-\frac{1}{2k}} \int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} \int_0^1 dt e^{-t\frac{\psi^{2k}}{2}} \psi^{2k-1} \left[-\lambda^{\frac{1}{k}} \frac{d}{d\psi} e^{-\lambda^{-\frac{1}{k}} \frac{\psi^2}{2}} \right] \tag{2.16} \\ &= 1 - \lambda^{\frac{1}{2k}} \int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} \int_0^1 dt [2k - 1 - tk\psi^{2k}] \psi^{2k-2} e^{-t\frac{\psi^{2k}}{2} - \lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}, \end{aligned}$$

an expression now easy to bound for $\lambda \in D_\rho^k$ by a $(k$ dependent) constant.

²We could without too much pain prove analyticity in a domain of larger opening angle, but shall not need it.

The uniform estimates in $A_k B_k^N [(kN)!] |\lambda|^N$ of (2.7) for $N \geq 1$ (the constants A_k and B_k being of course allowed to depend on k), are similar. First:

$$\begin{aligned}
 R^N Z_k(\lambda) &= \lambda^N \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} dt \int d\mu(\phi) \left(-\frac{\phi^{2k}}{2}\right)^N e^{-t\lambda \frac{\phi^{2k}}{2}} \\
 &= \frac{(-1)^N \lambda^{-\frac{1}{2k}}}{2^N (N-1)!} \int_0^1 (1-t)^{N-1} dt \int_{-\infty}^{+\infty} \frac{d\psi}{\sqrt{2\pi}} \psi^{2kN} e^{-t \frac{\psi^{2k}}{2} - \lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}.
 \end{aligned}
 \tag{2.17}$$

Let us evaluate

$$I_N = \frac{1}{(N-1)!} \int_{-\infty}^{+\infty} \frac{d\psi}{\sqrt{2\pi}} \psi^{2kN} e^{-t \frac{\psi^{2k}}{2} - \lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}.
 \tag{2.18}$$

Applying as before a single expansion step on the

$$e^{-t \frac{\psi^{2k}}{2}} = 1 - t \int_0^1 dt' \frac{\psi^{2k}}{2} e^{-t't' \frac{\psi^{2k}}{2}}
 \tag{2.19}$$

factor and computing exactly the first factor which is a Gaussian integral, we find

$I_N = \frac{(2kN)!}{(N-1)!} \lambda^{N+\frac{1}{2k}} - t J_N$, with

$$J_N := \frac{1}{(N-1)!} \int_0^1 dt' \int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} \psi^{2k(N+1)} e^{-t't' \frac{\psi^{2k}}{2} - \lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}.
 \tag{2.20}$$

To bound J_N , we consider the factor $\psi^{2k(N+1)}$ as an initial vertex, of coordination $2k(N+1)$.

We apply exactly $kN + 1$ Wick contraction steps to (less than half) the fields of this initial vertex with respect to the Gaussian measure $e^{-\lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}$. Each such step is similar to the one of (2.16). More precisely each step

- selects a remaining field in the initial vertex $\psi^{2k(N+1)}$, joins it to the Gaussian factor $e^{-\lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}$ to create a full derivative (correcting for the missing $-\lambda^{-\frac{1}{k}}$ factor)
- then integrates by parts: the derivative either acts on the interaction $e^{-t \frac{\psi^{2k}}{2}}$ or on the remaining fields of the initial vertex or on the field created by previous integration steps.

The process cannot run short of fields in the initial vertex $\psi^{2k(N+1)}$ because each step consumes at most two of these fields and the number of steps is $kN + 1$, less than *half* the total number $(2kN + 2k)$ of such fields. We call p the number of times the integration by parts hits the exponential $e^{-t't' \frac{\psi^{2k}}{2}}$; hence $0 \leq p \leq kN + 1$, and it hits $kN + 1 - p$ times the fields down from the exponential. The result is a complicated sum of non perturbative amplitudes A_G for processes G .³ They all have exactly $kN + 1$ edges, hence $kN + 1$ covariance factors $\lambda^{1/k}$, which in total give a

³Processes are not exactly Feynman graphs because of the remaining non-perturbative interaction factor $e^{-t't' \frac{\psi^{2k}}{2}}$.

factor $\lambda^{N+\frac{1}{k}}$. The number of fields down at the end must be $2k(N+1) + p(2k-2) - 2(kN+1-p) = 2k(p+1) - 2$ because p steps destroy a field and create $2k-1$ fields (together with a $tt'k$ factor) and $kN+1-p$ steps destroy two fields.

Therefore any process with a given value of p has the same amplitude

$$J_{N,p} := \lambda^{N+\frac{1}{k}} \int_0^1 dt' (-tt'k)^p \int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} \psi^{2k(p+1)-2} e^{-tt' \frac{\psi^{2k}}{2} - \lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}. \tag{2.21}$$

Lemma 2.1

$$|J_{N,p}| \leq |\lambda|^{N+\frac{1}{k}} A_k B_k^N p! \tag{2.22}$$

for some constants A_k and B_k (possibly depending on k).

Proof If $p = 0$, $J_{N,0}$ is bounded by a (k dependent) constant times $|\lambda|^{N+\frac{1}{k}}$ since in D_ρ^k we have $\Re\lambda^{-\frac{1}{k}} > \rho^{-1}$. If $p \geq 1$,

$$J_{N,p} = (-k)^p \lambda^{N+\frac{1}{k}} \int_0^1 dt' \int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} K_{N,p}(\psi, t, t') L(\psi), \tag{2.23}$$

where we define

$$K_{N,p}(\psi, t, t') := (tt')^p \psi^{2k(p-1)+k} e^{-tt' \frac{\psi^{2k}}{2}} \tag{2.24}$$

$$L(\psi) := \psi^{3k-2} e^{-\lambda^{-\frac{1}{k}} \frac{\psi^2}{2}}. \tag{2.25}$$

Applying a Cauchy-Schwarz bound we find

$$|J_{N,p}| \leq k^p |\lambda|^{N+\frac{1}{k}} \int_0^1 dt' \sqrt{\int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} K_{N,p}^2(\psi, t, t')} \int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} L^2(\psi). \tag{2.26}$$

Obviously the integral over L is easily bounded by a k dependent constant since in D_ρ^k we have $\Re\lambda^{-\frac{1}{k}} > \rho^{-1}$. The other integral is also easily bounded by the change of variables $u = tt' \psi^{2k}$. Indeed

$$\begin{aligned} \sqrt{\int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} K_{N,p}^2(\psi, t, t')} &= \sqrt{\int_{-\infty}^{+\infty} \frac{d\psi}{2\sqrt{2\pi}} (tt')^{2p} \psi^{4k(p-1)+2k} e^{-tt' \psi^{2k}}} \\ &\leq \sqrt{\frac{(tt')^{\frac{2k-1}{2k}}}{2k\sqrt{2\pi}} \int_0^\infty u^{2p-2+\frac{1}{2k}} e^{-u} du} \\ &\leq A_k B_k^N p! \end{aligned} \tag{2.27}$$

since $tt' \leq 1$. Remembering that $p \leq kN+1$ completes the proof. □

To complete the bound on J_N we need only to multiply this bound on $J_{N,p}$ by $\frac{1}{(N-1)!}$ times the number of processes at given p , then sum over p . But it is not necessary to compute the exact number of such processes with a fixed p . We can just give a crude bound on it (we do not try to find optimal constants). This number is bounded by 2^{kN+1} (to choose the p particular steps which derive the exponential) times the product over the $kN+1-p$ steps which derive fields down from the exponential of the number of fields down the exponential at that step. This last number is certainly at

most the total maximal number of fields ever produced down the exponential, raised to the power $kN + 1 - p$, hence certainly bounded by

$$[2k(N + 1) + (2k - 2)p]^{kN+1-p} \leq N^{kN+1-p}[2k^2 + 4k]^{kN+1}. \tag{2.28}$$

Taking into account that $N^{1-p} p! \leq N(k + 1)^{(k+1)N}$ for $p \leq kN + 1$, we get a bound of the form $A_k B_k^N N^{kN}$, which is independent of p . Summing over p adds a factor $kN + 1$ which can be absorbed by changing B_k . Finally multiplying by $\frac{1}{(N-1)!}$ (and using Stirling’s formula) gives a bound of the form $A_k B_k^N N^{(k-1)N}$. Returning to (2.17) and gathering all factors proves (2.7) hence completes the proof of Theorem 2.2. \square

From now on our goal is to define new intermediate field representations for the functions $Z_k(\lambda)$, $Z_{k,c}(\lambda)$ and to check in these new representations their Borel-Leroy summability.

3 Imaginary Gaussian IF Representation

3.1 Real Case

We first split the interaction in two using an intermediate field σ with normalized Gaussian measure $d\mu(\sigma)$ of covariance 1. The result is:

$$e^{-\lambda\phi^{2k}/2} = \int d\mu(\sigma)e^{i\sqrt{\lambda}\phi^k\sigma}. \tag{3.1}$$

We define $g_k = \lambda^{\frac{1}{2k}}$, and as next step we decompose

$$i\sqrt{\lambda}\phi^k\sigma = \frac{i}{4}[(g_k\phi\sigma + (g_k\phi)^{k-1})^2 - (g_k\phi\sigma - (g_k\phi)^{k-1})^2]. \tag{3.2}$$

We introduce a pair of intermediate fields a_1 and b_1 with imaginary covariances $-i$ and $+i$, hence the Gaussian measure $d\mu_{\pm i}(a_1, b_1) = d\mu_{-i}(a_1)d\mu_i(b_1)$ so that

$$e^{i\sqrt{\lambda}\phi^k\sigma} = \int d\mu_{\pm i}(a_1, b_1)e^{\frac{i}{\sqrt{2}}[(g_k\phi\sigma + (g_k\phi)^{k-1})a_1 + (g_k\phi\sigma - (g_k\phi)^{k-1})b_1]} \tag{3.3}$$

$$= \int d\mu_{\pm i}(a_1, b_1)e^{i[g_k\phi\sigma \frac{a_1+b_1}{\sqrt{2}} + (g_k\phi)^{k-1} \frac{a_1-b_1}{\sqrt{2}}]}. \tag{3.4}$$

We now change variables for

$$\alpha_1 = \frac{a_1 + b_1}{\sqrt{2}}, \quad \beta_1 = \frac{a_1 - b_1}{\sqrt{2}}, \tag{3.5}$$

so that

$$e^{i\sqrt{\lambda}\phi^k\sigma} = \int d\mu_X(\alpha_1, \beta_1)e^{ig_k\phi\sigma\alpha_1 + i(g_k\phi)^{k-1}\beta_1}, \tag{3.6}$$

where the Gaussian measure $d\mu_X$ is defined by its covariance

$$\langle \alpha_1\beta_1 \rangle_X = -i, \quad \langle \alpha_1^2 \rangle_X = 0, \quad \langle \beta_1^2 \rangle_X = 0. \tag{3.7}$$

Remember this section is heuristic so do not worry yet about convergence and integration contours, which will be addressed in the next section.

We keep the term $ig_k\phi\sigma\alpha_1$ and decompose the $ig_k^{k-1}\phi^{k-1}\beta_1$ term as

$$e^{ig_k^{k-1}\phi^{k-1}\beta_1} = \int d\mu_X(\alpha_2, \beta_2) e^{ig_k\phi\beta_1\alpha_2 + i(g_k\phi)^{k-2}\beta_2}. \tag{3.8}$$

Continuing in this way we prove inductively the following representation:

$$e^{-\lambda\phi^{2k}/2} = \int d\mu(\sigma) \prod_{j=1}^{k-1} d\mu_X(\alpha_j, \beta_j) e^{ig_k[\phi\sigma\alpha_1 + \sum_{j=1}^{k-2} \phi\beta_j\alpha_{j+1} + \phi\beta_{k-1}]}, \tag{3.9}$$

where the α_j, β_j and the measure $d\mu_X$ are respectively defined as in (3.5) and (3.7).

We now integrate

- for k odd, over ϕ, σ and all even α_{2j}, β_{2j} , for $j \in \{1, \dots, \frac{k-1}{2}\}$. In that case we denote $\Phi = (\phi, \sigma, \alpha_2, \beta_2, \dots, \alpha_{k-1}, \beta_{k-1})$ the $k + 1$ integrated variables and $\Psi = (\alpha_1, \beta_1, \dots, \alpha_{k-2}, \beta_{k-2})$ the $k - 1$ remaining ones. The Gaussian measure $d\mu(\sigma) \prod_{j=1}^{k-2} d\mu_X(\alpha_j, \beta_j)$ factorizes as $d\nu(\Phi)d\chi(\Psi)$.
- for k even, over ϕ and all odd $\alpha_{2j-1}, \beta_{2j-1}$, for $j \in \{1, \dots, \frac{k}{2}\}$. In that case we denote $\Phi = (\phi, \alpha_1, \beta_1, \dots, \alpha_{k-1}, \beta_{k-1})$ the $k + 1$ integrated variables and $\Psi = (\sigma, \alpha_2, \beta_2, \dots, \alpha_{k-2}, \beta_{k-2})$ the $k - 1$ remaining ones. The Gaussian measure $d\mu(\sigma) \prod_{j=1}^{k-2} d\mu_X(\alpha_j, \beta_j)$ factorizes again as $d\nu(\Phi)d\chi(\Psi)$.

The partition function writes

$$Z_k(\lambda) = \int d\chi(\Psi) \left[d\nu(\Phi) \exp\left[\frac{ig_k}{2} \langle \Phi, H_k(\Psi) \cdot \Phi \rangle \right] \right], \tag{3.10}$$

where H_k is a $(k + 1) \times (k + 1)$ real symmetric matrix. More precisely :

- if $k = 2p + 1$ is odd, H_k is

$$H_k = \left(\begin{array}{c|cccc} 0 & \alpha_1 & \beta_1 & \cdots & \beta_{k-2} & 1 \\ \alpha_1 & & & & & \\ \beta_1 & & & & & \\ \vdots & & & & & \\ \beta_{k-2} & & & & & \\ 1 & & & & & \end{array} \right), \tag{3.11}$$

- if $k = 2p$ is even, H_k is

$$H_k = \left(\begin{array}{c|cccc} 0 & \sigma & \alpha_2 & \cdots & \beta_{k-2} & 1 \\ \sigma & & & & & \\ \alpha_2 & & & & & \\ \vdots & & & & & \\ \beta_{k-2} & & & & & \\ 1 & & & & & \end{array} \right). \tag{3.12}$$

The Gaussian integration over Φ gives a determinant. Rewritten as usual in field theory as exponential of an action, it leads to the *IF representation*

$$Z_k(\lambda) = \int d\chi(\Psi) \exp\left[-\frac{1}{2} \text{Tr} \ln(1 - g_k M_k(\Psi))\right], \tag{3.13}$$

where $M_k(\Psi) = iC_k \cdot H_k(\Psi)$ and C_k , the covariance for the Φ variables, is

$$C_{\text{odd}} = \begin{pmatrix} \left. \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right| & & & \\ & \frac{\begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix}} & \mathbf{0} & \\ & \mathbf{0} & \frac{\begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix}} & \\ & & & \ddots \end{pmatrix}, \tag{3.14}$$

$$C_{\text{even}} = \begin{pmatrix} \left. \begin{array}{c} 1 \\ \hline \end{array} \right| & & & \\ & \frac{\begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix}} & \mathbf{0} & \\ & \mathbf{0} & \frac{\begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix}} & \\ & & & \ddots \end{pmatrix}. \tag{3.15}$$

The proof that this integral representation (3.13) converges and that the integral is indeed $Z_k(\lambda)$ is postponed to Section III. In the simplest cases $k = 3, 4$, hence for the $e^{-\lambda\phi^6/2}$ and $e^{-\lambda\phi^8/2}$ models, we obtain the representations:

$$Z_3(\lambda) = \int d\chi(\alpha_1, \beta_1) e^{-\frac{1}{2} \text{Tr} \ln[1 - \lambda^{1/6} M_3]}, \quad M_3 = \begin{pmatrix} 0 & i\alpha_1 & i\beta_1 & i \\ i\alpha_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \tag{3.16}$$

and

$$Z_4(\lambda) = \int d\chi(\sigma, \alpha_2, \beta_2) e^{-\frac{1}{2} \text{Tr} \ln[1 - \lambda^{1/8} M_4]}, \quad M_4 = \begin{pmatrix} 0 & i\sigma & i\alpha_2 & i\beta_2 & i \\ \alpha_2 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

3.2 Complex Case

As in the previous section, we first split the interaction in two using a complex intermediate field σ with normalized Gaussian measure $d\mu(\sigma)$ of covariance 1. We obtain

$$e^{-\lambda(\phi\bar{\phi})^{2p+1}} = \int d\mu^c(\sigma) e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\bar{\phi}\sigma + \phi\bar{\sigma})}, \tag{3.18}$$

$$e^{-\lambda(\phi\bar{\phi})^{2p}} = \int d\mu^c(\sigma) e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\sigma + \bar{\sigma})}. \tag{3.19}$$

For λ real positive and respectively $k = 2p + 1$ and $k = 2p$, defining $g_k = \lambda^{\frac{1}{2k}}$, we write

$$i\sqrt{\lambda}(\phi\bar{\phi})^p(\bar{\phi}\sigma + \phi\bar{\sigma}) = \frac{i}{2} [|g_k\bar{\phi}\sigma + (g_k^2\phi\bar{\phi})^p|^2 - |g_k\bar{\phi}\sigma - (g_k^2\phi\bar{\phi})^p|^2], \tag{3.20}$$

$$i\sqrt{\lambda}(\phi\bar{\phi})^p(\sigma + \bar{\sigma}) = \frac{i}{2} [|g_k\bar{\phi}\sigma + (g_k^2\phi\bar{\phi})^{p-1}g_k\bar{\phi}|^2 - |g_k\bar{\phi}\sigma - (g_k^2\phi\bar{\phi})^{p-1}g_k\bar{\phi}|^2].$$

We introduce a pair of complex intermediate fields a_1 and b_1 with imaginary covariances $-i$ and $+i$, hence the Gaussian measure

$$d\mu_{\pm i}^c(a_1, b_1) = d\mu_{-i}^c(a_1, \bar{a}_1) d\mu_i^c(b_1, \bar{b}_1). \tag{3.21}$$

More precisely it means that writing a_1^R, b_1^R and a_1^I, b_1^I for the real and imaginary parts of a_1 and b_1 , the measure $d\mu_{-i}^c(a_1, \bar{a}_1) d\mu_i^c(b_1, \bar{b}_1)$ is the product of four independent Gaussian measures on these four real variables. Each of the four can be independently complexified and we should use an appropriate contour of integration $C_{+, \epsilon}$ in each of the four corresponding complex planes, as defined in Section 2.1. However remember that in this heuristic section we do not consider convergence questions. Then we have

$$e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\bar{\phi}\sigma + \phi\bar{\sigma})} = \int d\mu_{\pm i}^c(a_1, b_1) e^{\frac{i}{\sqrt{2}} [(g_k\bar{\phi}\sigma + (g_k^2\phi\bar{\phi})^p)a_1 \tag{3.22}$$

$$+ (g_k\bar{\phi}\sigma - (g_k^2\phi\bar{\phi})^p)b_1 + c.c.]}$$

$$= \int d\mu_{\pm i}^c(a_1, b_1) e^{i[g_k\bar{\phi}\sigma \frac{a_1+b_1}{\sqrt{2}} + (g_k^2\phi\bar{\phi})^p \frac{a_1-b_1}{\sqrt{2}} + c.c.]},$$

$$e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\sigma + \bar{\sigma})} = \int d\mu_{\pm i}^c(a_1, b_1) e^{i[g_k\bar{\phi}\sigma \frac{a_1+b_1}{\sqrt{2}} + (g_k^2\phi\bar{\phi})^p g_k\bar{\phi} \frac{a_1-b_1}{\sqrt{2}} + c.c.]},$$

where $c.c.$ means complex conjugate. We can change variables as in the real case for

$$\alpha_1 = \frac{a_1 + b_1}{\sqrt{2}}, \quad \beta_1 = \frac{a_1 - b_1}{\sqrt{2}}, \tag{3.23}$$

and complex conjugates, so that

$$e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\bar{\phi}\sigma + \phi\bar{\sigma})} = \int d\mu_X^c(\alpha_1, \beta_1) e^{i[g_k\bar{\phi}\sigma\alpha_1 + (g_k^2\phi\bar{\phi})^p\beta_1 + c.c.]}, \tag{3.24}$$

$$e^{i\sqrt{\lambda}(\phi\bar{\phi})^p(\sigma + \bar{\sigma})} = \int d\mu_X^c(\alpha_1, \beta_1) e^{i[g_k\bar{\phi}\sigma\alpha_1 + (g_k^2\phi\bar{\phi})^p g_k\bar{\phi}\beta_1 + c.c.]}, \tag{3.25}$$

where the Gaussian measure $d\mu_X^c(\alpha_1, \beta_1)$ is defined by its covariance

$$\langle \alpha_1 \bar{\beta}_1 \rangle_X = \langle \bar{\alpha}_1 \beta_1 \rangle_X = -i, \tag{3.26}$$

$$\langle \alpha_1^2 \rangle_X = \langle \bar{\alpha}_1^2 \rangle_X = \langle \beta_1^2 \rangle_X = \langle \bar{\beta}_1^2 \rangle_X = 0 \tag{3.27}$$

$$\langle \alpha_1 \bar{\alpha}_1 \rangle_X = \langle \beta_1 \bar{\beta}_1 \rangle_X = \langle \alpha_1 \beta_1 \rangle_X = \langle \bar{\alpha}_1 \bar{\beta}_1 \rangle_X = 0. \tag{3.28}$$

We invite the reader to check in particular (3.26), which may be surprising at first sight but is perfectly consistent with the imaginary-Gaussian complex integration rule (2.4)).

An inductive reasoning strictly parallel to the previous subsection leads to:

$$e^{-\lambda(\phi\bar{\phi})^k} = \int d\mu^c(\sigma) \prod_{j=1}^{k-1} d\mu_X^c(\alpha_j, \beta_j) e^{igk[\bar{\phi}\sigma\alpha_1 + \sum_{j=1}^{k-2} \bar{\phi}\beta_j\alpha_{j+1} + \bar{\phi}\beta_{k-1} + c.c.]}, \tag{3.29}$$

where the α_j, β_j and the measure $d\mu_X^c$ are respectively defined as in (3.23) and (3.26). Integrating again

- for k odd, over ϕ, σ , all even α_{2j}, β_{2j} , for $j \in \{1, \dots, \frac{k-1}{2}\}$ and complex conjugates. In that case we denote $\Phi = (\phi, \sigma, \alpha_2, \beta_2, \dots, \alpha_{k-1}, \beta_{k-1})$ the $k + 1$ integrated variables and $\Psi = (\alpha_1, \beta_1, \dots, \alpha_{k-2}, \beta_{k-2})$ the $k - 1$ remaining ones. The Gaussian measure $d\mu^c(\sigma) \prod_{j=1}^{k-2} d\mu_X^c(\alpha_j, \beta_j)$ factorizes as $d\chi(\Psi, \bar{\Psi})d\nu(\Phi, \bar{\Phi})$.
- for k even, over ϕ , all odd $\alpha_{2j-1}, \beta_{2j-1}$, for $j \in \{1, \dots, \frac{k}{2}\}$ and complex conjugates. In that case we denote $\Phi = (\phi, \alpha_1, \beta_1, \dots, \alpha_{k-1}, \beta_{k-1})$ the $k + 1$ integrated variables and $\Psi = (\sigma, \alpha_2, \beta_2, \dots, \alpha_{k-2}, \beta_{k-2})$ the $k - 1$ remaining ones. The Gaussian measure $d\mu^c(\sigma) \prod_{j=1}^{k-2} d\mu_X^c(\alpha_j, \beta_j)$ factorizes again as $d\chi(\Psi, \bar{\Psi})d\nu(\Phi, \bar{\Phi})$.

the partition function writes

$$Z_{k,c}(\lambda) = \int d\chi(\Psi, \bar{\Psi}) \left[d\nu(\Phi, \bar{\Phi}) \exp[igk \langle \Phi, H_k(\Psi, \bar{\Psi}) \cdot \Phi \rangle] \right], \tag{3.30}$$

where H_k is a $(k + 1) \times (k + 1)$ Hermitian matrix. More precisely :

- if $k = 2p + 1$ is odd, H_k is

$$H_k = \left(\begin{array}{c|cccc} 0 & \alpha_1 & \beta_1 & \cdots & \beta_{k-2} & 1 \\ \hline \bar{\alpha}_1 & & & & & \\ \bar{\beta}_1 & & & & & \\ \vdots & & & & & \\ \bar{\beta}_{k-2} & & & & & \\ 1 & & & & & \end{array} \right), \tag{3.31}$$

- if $k = 2p$ is even, H_k is

$$H_k = \left(\begin{array}{c|cccc} 0 & \sigma & \alpha_2 & \cdots & \beta_{k-2} & 1 \\ \hline \bar{\sigma} & & & & & \\ \bar{\alpha}_2 & & & & & \\ \vdots & & & & & \\ \bar{\beta}_{k-2} & & & & & \\ 1 & & & & & \end{array} \right). \tag{3.32}$$

Gaussian integration over Φ leads to the *IF representation*

$$Z_{k,c}(\lambda) = \int d\chi(\Psi) \exp[-\text{Tr} \ln(1 - g_k M_k(\Psi))], \tag{3.33}$$

where $M_k(\Psi) = iC_k.H_k(\Psi)$ and C_k , the covariance for the Φ variables, is given as in the real case by (3.14) and (3.15).

In the simplest cases $k = 3, 4$ we obtain the representations :

$$Z_{3,c}(\lambda) = \int d\chi(\alpha_1, \beta_1) e^{-\text{Tr} \ln[1 - \lambda^{1/6} M_3]}, \quad M_3 = \begin{pmatrix} 0 & i\alpha_1 & i\beta_1 & 1 \\ i\bar{\alpha}_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \bar{\beta}_1 & 0 & 0 & 0 \end{pmatrix} \tag{3.34}$$

and

$$Z_{4,c}(\lambda) = \int d\chi(\sigma, \alpha_2, \beta_2) e^{-\text{Tr} \ln[1 - \lambda^{1/8} M_4]}, \quad M_4 = \begin{pmatrix} 0 & i\sigma & i\alpha_2 & i\beta_2 & i \\ \bar{\alpha}_2 & 0 & 0 & 0 & 0 \\ \bar{\sigma} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \bar{\beta}_2 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.35}$$

This integral represnettaion has been established for λ real positive and in the next section will be used to perform the analytic continuation of the functions Z_k to complex values of λ .

4 Analyticity Domains

In this section we prove the following theorem:

Theorem 4.1 *The integral $\int d\chi(\Psi) \exp[-\frac{1}{2} \text{Tr} \ln(1 - g_k M_k(\Psi))]$ is absolutely convergent in a domain $D_\rho^{k-1} = \{\lambda \in \mathbb{C} : \Re \lambda^{-\frac{1}{k-1}} > \rho^{-1}\}$ (for ρ sufficiently small). It defines a Borel-Leroy summable function of order $k - 1$ in this domain, whose Taylor series at the origin is the same as the Taylor series of $Z_k(\lambda)$. Hence (by unicity of the Borel sum) (3.13) and (3.33) hold.*

Proof We give the proof in the real ϕ^{2k} case, the argument for the complex case $(\bar{\phi}\phi)^k$ being essentially identical. The key step is an upper bound on the norm of the resolvent $[1 - g_k M(\Psi)]^{-1}$ in the Nevanlinna domain for Borel-Leroy summability of order $k - 1$. This bound must be uniform both in λ in that domain and uniform in the intermediate fields along the contours associated to $d\chi$.

Let us prove such a uniform bound in a slightly larger domain E_ρ^{k-1} consisting of all $\lambda = \rho e^{i\theta}$ with ρ small and $|\theta| < \frac{(k-1)\pi}{2}$ (hence in a half-disk for $\lambda^{\frac{1}{k-1}}$). Obviously it contains the disk $D_{\rho/2}^{k-1}$ necessary for Nevanlinna’s Theorem. We need to compute the eigenvalues of the matrix $1 - g_k M_k$, and to take into account the contours of integration.

Lemma 4.1 For $\lambda \in E_1^{k-1}$ and Ψ on the contours of integration $C_{\pm\epsilon}$ with $\epsilon = \frac{1}{4}k^{-1/2} \sin \frac{\pi}{4k}$ we have

$$\|(1 - g_k M_k)^{-1}\| \leq 2[\sin \frac{\pi}{4k}]^{-1}. \tag{4.1}$$

where, from now on the notation $\|M\|$ means the operator norm of M .

Proof Returning to the parametrization of our contour integrals we recall that $a_j = \Re a_j - i\epsilon \tanh(\Re a_j)$ and $b_j = \Re b_j - i\epsilon \tanh(\Re b_j)$, where \Re is the real part. Hence remembering (3.5), and putting $\Psi = X + iY$, where the vectors X and Y are real, each coefficient of Y is bounded in absolute value by $\epsilon\sqrt{2}$. The matrix $M(\Psi)$ being linear in Ψ , we have $M_k(\Psi) = M_k(X) + iM_k(Y)$, and since each of the $2k$ non-zero coefficients of $M_k(Y)$ is bounded in absolute value by $\epsilon\sqrt{2}$, we can bound its Hilbert-Schmidt norm $\|M_k(Y)\|_2$ by $2\epsilon\sqrt{k}$, hence

$$\|M_k(Y)\| \leq \|M_k(Y)\|_2 = 2\epsilon\sqrt{k} \tag{4.2}$$

Now let us compute the eigenvalues of the matrix $1 - g_k M_k(X)$. It has eigenvalue 1 with multiplicity $k - 1$ and two non trivial eigenvalues,

$$x_{\pm} = 1 \pm g_k \sqrt{R_k}, \tag{4.3}$$

where R_k is $-(\Re \alpha_1)^2 + i(\Re \beta_1 \Re \alpha_3 + \dots + \Re \beta_{k-4} \Re \alpha_{k-2} + \Re \beta_{k-2})$ if k is odd and is $i(\sigma \Re \alpha_2 + \Re \beta_2 \Re \alpha_4 + \dots + \Re \beta_{k-4} \Re \alpha_{k-2} + \Re \beta_{k-2})$ if k is even.

If R_k is not zero we can state something about the argument of $\pm\sqrt{R_k}$. In the odd case, if $\Re \alpha_1 \neq 0$, we have $R_k = -a^2(1 + ib)$ with a and b real, hence $\pm\sqrt{R_k} = ia\sqrt{1 + ib}$ and the argument of $\pm\sqrt{R_k}$ lies in $I = [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [-\frac{3\pi}{4}, -\frac{\pi}{4}]$. If $\Re \alpha_1 = 0$ or in the even case the argument of $\pm\sqrt{R_k}$ belongs to $\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}\}$, hence to the boundary of I .

But in the domain E_ρ^{k-1} the argument of g_k is bounded by $\frac{(k-1)\pi}{4k}$ hence the argument of $\pm g_k \sqrt{R_k}$ (when $g_k R_k \neq 0$) lies in

$$\begin{aligned} I_k &= [\frac{\pi}{4} - \frac{(k-1)\pi}{4k}, \frac{3\pi}{4} + \frac{(k-1)\pi}{4k}] \cup [-\frac{3\pi}{4} - \frac{(k-1)\pi}{4k}, -\frac{\pi}{4} + \frac{(k-1)\pi}{4k}] \\ &= [\frac{\pi}{4k}, \pi - \frac{\pi}{4k}] \cup [-\pi + \frac{\pi}{4k}, -\frac{\pi}{4k}], \end{aligned} \tag{4.4}$$

hence in that domain the spectrum of $1 - g_k M_k$ lies out of the disk of center 0 and radius $\sin \frac{\pi}{4k}$. Choosing $\epsilon = \frac{\sin \frac{\pi}{4k}}{4\sqrt{k}}$, and assuming $\rho \leq 1$, we have by (4.2) $\|g_k M_k(Y)\| \leq \frac{1}{2} \sin \frac{\pi}{4k}$, hence the spectrum of $(1 - g_k M_k(X) - i g_k M_k(Y))$ lies out of the disk of center 0 and radius $\frac{1}{2} \sin \frac{\pi}{4k}$, and

$$\|(1 - g_k M_k)^{-1}\| \leq 2[\sin \frac{\pi}{4k}]^{-1}. \tag{4.5}$$

□

Since $(1 - g_k M_k)^{-1}$ has only two non trivial eigenvalues not equal to 1, this bound implies the same bound on the inverse square root of its determinant $\det(1 - g_k M_k)$, hence

$$|\exp[-\frac{1}{2} \text{Tr} \ln(1 - g_k M(\Psi))]| = |\det^{-1/2}(1 - g_k M_k)| \leq 2[\sin \frac{\pi}{4k}]^{-1}. \tag{4.6}$$

which means a uniform upper bound on the integrand in (5.4). Analyticity of Z_k then follows by the standard argument based on Morera’s theorem that a uniformly convergent integral of an analytic integrand is analytic.

In fact there is clearly some margin still in the proof of Lemma 4.1 and with a little additional work we could check analyticity in a D_ρ^k domain as done in the proof of Theorem 2.2.

Finally the uniform estimates on the Taylor remainder at order n can be obtained simply by Taylor expanding Z_k by the Taylor formula with integral remainder. This is similar to the proof of Theorem 2.2 in Section 2.3 and left to the reader.

To complete the proof of Theorem 4.1, one needs also to check that the perturbative expansion in λ of this intermediate field representation is identical to the ordinary one, which is easy and left as an exercise to the reader. Then by unicity of the Borel sum, one concludes that the two integral representations (2.10) and (3.13) must be equal. \square

The LVE [2] as in [1] is a technique to compute explicitly the logarithm of such partition functions and check its Borel sumability. However we have not found yet how to adapt it to such intermediate field representations with Gaussian imaginary integrals. The problem comes from the many replicas introduced by the LVE (one per vertex). Each of them should have its own small contour deformation and these deformations add up in a way which we do not know how to control as the number n of loop vertices tends to infinity. Hence as a way out of this dilemma we give now, for the complex $k = 3$ case, hence for the $(\bar{\phi}\phi)^3$ model, another intermediate field representation, this time with *bona fide* real Gaussian integrals rather than imaginary ones.

5 Improved IF Representation

We return to (2.12) in the $k = 3$ case, hence consider

$$Z_3^c(\lambda) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\mu(\phi, \bar{\phi}) e^{-\lambda(\bar{\phi}\phi)^3}. \tag{5.1}$$

We split the interaction in two using a complex intermediate field σ with normalized Gaussian measure $d\mu(\sigma)$ of covariance 1. The result is:

$$e^{-\lambda(\bar{\phi}\phi)^3} = \int d\mu(\sigma) e^{i\lambda^{1/2}(\bar{\phi}\phi)[\bar{\phi}\sigma + \phi\bar{\sigma}]}. \tag{5.2}$$

We introduce complex conjugate fields intermediate fields a and \bar{a} so that

$$e^{i\sqrt{\lambda}(\bar{\phi}\phi)[\bar{\phi}\sigma + \phi\bar{\sigma}]} = \int d\mu(a) e^{\sqrt{i}\lambda^{1/4}[\bar{\phi}\phi a + (\bar{\phi}\sigma + \phi\bar{\sigma})\bar{a}]}. \tag{5.3}$$

The Gaussian integrals over ϕ and σ can be explicitly performed, giving

$$Z_3^c(\lambda) = \int d\mu(\phi, \sigma, a) e^{\sqrt{i}\lambda^{1/4}[\bar{\phi}\phi a + (\bar{\phi}\sigma + \phi\bar{\sigma})\bar{a}]} \tag{5.4}$$

$$= \int d\mu(a) \det[1 - \sqrt{i}\lambda^{1/4} \begin{pmatrix} a & \bar{a} \\ \bar{a} & 0 \end{pmatrix}]^{-1} \tag{5.5}$$

$$= \int d\mu(a) f(a, \bar{a}). \tag{5.6}$$

Since $\det[1 - \sqrt{i}\lambda^{1/4} \begin{pmatrix} a & \bar{a} \\ \bar{a} & 0 \end{pmatrix}] = 1 - \sqrt{i}\lambda^{1/4}a - i\lambda^{1/2}\bar{a}^2$, we have

$$f(a, \bar{a}) := (1 - \sqrt{i}\lambda^{1/4}a - i\lambda^{1/2}\bar{a}^2)^{-1} = \sum_{n=0}^{\infty} (\sqrt{i}\lambda^{1/4}a + i\lambda^{1/2}\bar{a}^2)^n. \tag{5.7}$$

Since any integral $\int d\mu(a)a^p\bar{a}^q$ is zero unless $p = q$ we can, in the functional integral (5.6), substitute another *perturbatively equivalent* function $f_{\#a=\#\bar{a}}$ which simply discards, in the power series defining f , any term not satisfying that constraint. This is a priori not justified non-perturbatively but will be justified *a posteriori* if we can obtain a Borel summable series by this process. In our case $f_{\#a=\#\bar{a}}$ can be computed explicitly. More precisely

$$f_{\#a=\#\bar{a}} = \sum_{n=3p} (-1)^p \lambda^p (a\bar{a})^{2p} C_p^{3p}. \tag{5.8}$$

The binomial coefficient $C_p^{3p} = \frac{3p!}{p!(2p)!}$ is not far from the generalized Catalan number $C_p^{(3)} := \frac{1}{3p+1} C_p^{3p+1} = \frac{1}{2p+1} C_p^{3p}$. We know that the *alternating* generating function

$$h(x) = \sum_{p=0}^{\infty} (-1)^p C_p^{(3)} x^p \tag{5.9}$$

for such generalized Catalan numbers obeys the algebraic equation [21]

$$-xh^3(x) - h(x) + 1 = 0. \tag{5.10}$$

which is soluble by radicals. More precisely defining $f_{\#a=\#\bar{a}} = g(\lambda^{1/2}a\bar{a})$ and noting $u = \lambda^{1/2}a\bar{a}$, we have

$$g(u) = \sum_{p=0}^{\infty} (-1)^p [u^2]^p C_p^{3p} = \sum_{p=0}^{\infty} (2p+1)(-1)^p (u^2)^p C_p^{(3)}. \tag{5.11}$$

Hence the generating functions g and h are related through the simple equation

$$g(u) = \frac{d}{du} [uh(u^2)] = h(u^2) + 2u^2h'(u^2). \tag{5.12}$$

Returning to the solution of (5.9), we define

$$\Delta_{\pm}(y) := \left(\sqrt{1+y} \pm \sqrt{y} \right)^{1/3} = 1 \pm \frac{1}{3}\sqrt{y} + \frac{7y}{18} \mp \frac{4y^{3/2}}{81} + O(y^2) \tag{5.13}$$

where we used that $(1 + v)^{1/3} = 1 + v/3 - v^2/9 + 5v^3/81 + O(v^4)$. The derivatives are easily computed as

$$\Delta'_\pm(y) = \frac{1}{6} \left((1 + y)^{-1/2} \pm y^{-1/2} \right) \left(\sqrt{1 + y} \pm \sqrt{y} \right)^{-2/3} \tag{5.14}$$

Cardano’s solution gives

$$\begin{aligned} h(x) &= \frac{\Delta_+(\frac{27}{4}x) - \Delta_-(\frac{27}{4}x)}{\sqrt{3x}} = 1 - x + 3x^2 + O(x^3) \\ h'(x) &= (3x)^{-3/2} \left[\frac{81}{4}x(\Delta'_+ - \Delta'_-) - \frac{3}{2}(\Delta_+ - \Delta_-) \right] = -1 + 6x + O(x^2) \end{aligned} \tag{5.15}$$

where each Δ or Δ' is taken at $\frac{27}{4}x$. A nice simplification occurs in

$$g(u) := h(u^2) + 2u^2h'(u^2) = \frac{3^{5/2}}{2}u(\Delta'_+ - \Delta'_-)|_{y=\frac{27}{4}u^2}, \tag{5.16}$$

since the terms in $(\Delta_+ - \Delta_-)$ disappear. Let us compute

$$(\Delta'_\pm)|_{y=\frac{27}{4}u^2} = \frac{1}{6u} \left(u(1 + \frac{27}{4}u^2)^{-1/2} \pm \sqrt{\frac{4}{27}} \right) \left(\sqrt{1 + \frac{27}{4}u^2} \pm \sqrt{\frac{27}{4}u} \right)^{-2/3}. \tag{5.17}$$

It leads to an explicit expression for g , analytic in a disk around $u = 0$, namely

$$\begin{aligned} g(u) &= \frac{1}{2} \left[\left(1 + \sqrt{\frac{27}{4}u(1 + \frac{27}{4}u^2)^{-1/2}} \right) \left(\sqrt{1 + \frac{27}{4}u^2} + \sqrt{\frac{27}{4}u} \right)^{-2/3} \right. \\ &\quad \left. + \left(1 - \sqrt{\frac{27}{4}u(1 + \frac{27}{4}u^2)^{-1/2}} \right) \left(\sqrt{1 + \frac{27}{4}u^2} - \sqrt{\frac{27}{4}u} \right)^{-2/3} \right] \\ &= 1 - 3u^2 + 15u^4 + \dots \end{aligned} \tag{5.18}$$

Substituting in (5.6) we end up with the *improved* IF representation with true (i.e. non imaginary) Gaussian measures

Improved IF Representation

$$Z(\lambda) = \int d\mu(a) e^{S(a, \bar{a})}, \quad S(a, \bar{a}) = S_1 + S_2 - \log 2 \tag{5.19}$$

$$S_1 = \log \left[(\sqrt{1 + v^2} + v)^{1/3} + (\sqrt{1 + v^2} - v)^{1/3} \right]_{v=\sqrt{\frac{27\lambda}{4}}a\bar{a}} \tag{5.20}$$

$$S_2 = -\frac{1}{2} \log \left(1 + \frac{27\lambda}{4} (a\bar{a})^2 \right) \tag{5.21}$$

This formula seems now well adapted to a loop vertex expansion because derivatives of S remain bounded:

Lemma *The derivatives of S with respect to any (strictly positive) number of a and \bar{a} variables are all uniformly bounded as $a \in \mathbb{C}$ and λ runs in a suitable Borel-Leroy domain of order 2.*

$$\left| \frac{\partial^p}{\partial a^p} \frac{\partial^q}{\partial \bar{a}^q} S(a, \bar{a}) \right| \leq (p+q)! K^{p+q} |\lambda|^{\frac{p+q}{4}} \quad (5.22)$$

Proof (Sketch) The lemma is obviously proved if we prove it separately for S_1 and S_2 .

For S_2 it is quite trivial as the Faa di Bruno formula gives a sum of terms of the form $(1 + \frac{27\lambda}{4}(a\bar{a})^2)^{-r} [\frac{27\lambda}{4}]^s a^t (\bar{a})^u$ with $1 \leq r \leq p+q$, $s \geq (p+q)/4$ and $t+u \leq 3r$.

For S_2 it is slightly more complicated but a derivative acting on $\log D$ gives the factor $D' D^{-1}$ with $D := (\sqrt{1+v^2+v})^{1/3} + (\sqrt{1+v^2-v})^{1/3}$. Deriving $(\sqrt{1+v^2-v})^{1/3}$ seems to create possibly an a or \bar{a} times $D^{-1}[v(1+v^2)^{-1/2} - 1](\sqrt{1+v^2-v})^{-2/3}$, which would naively look unbounded. But in fact at large v positive $v(1+v^2)^{-1/2}$ tends to 1 and there is a compensation $[v(1+v^2)^{-1/2} - 1](\sqrt{1+v^2-v})^{-2/3} = v^{-2/3}[(1+v^{-2})^{-1/2} - 1][(1+v^{-2})^{1/2} - 1]^{-2/3} \simeq cv^{-4}$. \square

The generalization of this improved representation to higher values of k and the precise definition and convergence of the corresponding loop vertex expansion has (after completion of this paper) been developed in [18].

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