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Weakly Periodic Gibbs Measures of the Ising Model on the Cayley Tree of Order Five and Six

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Abstract For Ising model on the Cayley tree of order five and six we present new weakly periodic (non-periodic) Gibbs measures corresponding to normal subgroups of indices two in the group representation of the Cayley tree.

Keywords Cayley tree · Gibbs measure · Ising model · Weakly periodic measure

Mathematics Subject Classification (2010) 82B20

1 Introduction

A Gibbs measure is a mathematical idealization of an equilibrium state of a physical system which consists of a very large number of interacting components. In the language of Probability Theory, a Gibbs measure is simply the distribution of a stochastic process which, instead of being indexed by the time, is parametrized by the

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sites of a spatial lattice, and has the special feature of admitting prescribed versions of the conditional distributions with respect to the configurations outside finite regions. The physical phenomenon of phase transition is reflected by the non-uniqueness of the Gibbs measures for the considered model. The Ising model is realistic enough to exhibit this non-uniqueness of Gibbs measures in which a phase transition is predicted by physics. This fact is one of the main reasons for the physical interest in Gibbs measures. The problem of non-uniqueness, and also the converse problem of uniqueness, are central themes of the theory of Gibbs measures.

Let $M(H)$ be the set of all Gibbs measures defined by Hamiltonian H. Note that this set contains translation-invariant Gibbs measures, periodic Gibbs measures and non-periodic Gibbs measures and one can consider the problem of phase transition in the classes of translation-invariant Gibbs measures, periodic Gibbs measures and non-periodic Gibbs measures respectively. In [1–5] the translational invariant Gibbs measures of the Ising model and some of its generalization on the Cayley tree were studied. The papers [6–8] are devoted to periodic Gibbs measures with period 2 for models with finite radius of interaction. In [9–12] the authors introduced a new class of Gibbs measures, the so-called weakly periodic Gibbs measures and proved the existence of such measures for the Ising model on the Cayley tree. In [1, 12, 13] continuum sets of non periodic Gibbs measures for the Ising model on the Cayley tree are constructed . In [9, 10, 14] the authors considered non-periodic weakly periodic Gibbs measures for the Ising model on the Cayley tree of order $k < 5$. The present paper is a continuation of the investigations in [14] and in this paper we study weakly periodic Gibbs measures on the Cayley tree of order five and six.

2 Basic Definitions and Formulation of the Problem

Let $\Gamma^k = (V, L)$, be the Cayley tree of order $k \geqslant 1$, i.e. an infinity graph every vertex of which is incident to exactly $k + 1$ edges. Here V is the set of all vertices, L is the set of all edges of the tree Γ^k. It is known that Γ^k can be represented as a non-commutative group G_k, which is the free product of $k + 1$ cyclic groups of the second order [13]. Therefore we identify V and G_k.

For an arbitrary point $x^0 \in V$ we set $W_n = \{x \in V | d(x^0, x) = n\}$, $V_n = \bigcup\limits_{m=0}^{n} W_m$, $L_n = \{< x, y >\in L | x, y \in V_n\}$, where $d(x, y)$ is the distance between the vertices x and y in the Cayley tree, i.e. the number of edges in the shortest path joining the vertices x and y. We write $x \prec y$ if the path from x^0 to y goes through x. We call the vertex y a direct successor of x, if $y \succ x$ and x, y are nearest neighbors. The set of the direct successors of x is denoted by $S(x)$, i.e., if $x \in W_n$, then

$$S(x) = \{y_i \in W_{n+1} | d(x, y_i) = 1, i = 1, 2, \cdots , k\}.$$

Let $\Phi = \{-1, 1\}$ and let $\sigma \in \Omega = \Phi^V$ be a configuration, i.e. $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. For subset $A \subset V$ we denote by Ω_A the space of all configurations defined on the set A and taking values in Φ.

We consider the Hamiltonian of the Ising model:

$$H(\sigma) = -J \sum_{<x,y>\in L} \sigma(x)\sigma(y), \tag{2.1}$$

where $J \in R, \sigma(x) \in \Phi$ and $< x, y >$ are nearest neighbors.

For every n, we define a measure μ_n on Ω_{V_n} setting

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{ -\beta H(\sigma_n) + \sum_{x\in W_n} h_x \sigma(x) \right\}, \tag{2.2}$$

where $h_x \in R, x \in V, \beta = \frac{1}{T}$ (T is temperature, $T > 0$), $\sigma_n = \{\sigma(x), x \in V_n\} \in \Omega_{V_n}$, Z_n^{-1} is the normalizing factor, and

$$H(\sigma_n) = -J \sum_{<x,y>\in L_n} \sigma(x)\sigma(y).$$

The compatibility condition for the measures $\mu_n(\sigma_n), n \geqslant 1$, is

$$\sum_{\sigma^{(n)}} \mu_n(\sigma_{n-1}, \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \tag{2.3}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

Let $\mu_n, n \geqslant 1$ be a sequence of measures on the sets Ω_{V_n} that satisfy compatibility condition (2.3). By the Kolmogorov theorem, we then have a unique limit measure μ on $\Omega_V = \Omega$ (called a limit Gibbs measure) such that

$$\mu(\sigma_n) = \mu_n(\sigma_n)$$

for every $n = 1, 2, ...$ It is known that measures (2.2) satisfies the condition (2.3) if and only if the set $h = \{h_x, x \in G_k\}$ of real numbers satisfies the condition

$$h_x = \sum_{y\in S(x)} f(h_y, \theta), \tag{2.4}$$

where $S(x)$ is the set of direct successors of the vertex $x \in V$ (see [1–3]). Here, $f(x, \theta) = arctanh(\theta \tanh x), \theta = \tanh(J\beta), \beta = \frac{1}{T}$.

Let $G_k/\widehat{G}_k = \{H_1, ..., H_r\}$ be a factor group, where $\widehat{G}_k$ is a normal subgroup of index $r \geqslant 1$.

Definition 1 A set $h = \{h_x, x \in G_k\}$ of quantities is called $\widehat{G}_k$*-periodic* if $h_{xy} = h_x$, for all $x \in G_k$ and $y \in \widehat{G}_k$.

For $x \in G_k$ we denote by $x_\downarrow$ the unique point of the set $\{y \in G_k : \langle x, y\rangle\} \setminus S(x)$.

Definition 2 A set of quantities $h = \{h_x, x \in G_k\}$ is called $\widehat{G}_k$*-weakly periodic*, if $h_x = h_{ij}$, for any $x \in H_i, x_\downarrow \in H_j$.

We note that the weakly periodic h coincides with an ordinary periodic one (see Definition 1) if the quantity h_x is independent of $x_{\downarrow}$.

Definition 3 A Gibbs measure μ is said to be $\widehat{G}_k$-(weakly) periodic if it corresponds to a $\widehat{G}_k$-(weakly) periodic h. We call a G_k-periodic measure a translation-invariant measure.

In this paper, we study weakly periodic Gibbs measures and demonstrate that such measures exist for the Ising model on a Cayley tree of order five and six.

3 Weakly Periodic Measures

The level of difficulty in the description of weakly periodic Gibbs measures is related to the structure and index of the normal subgroup relative to which the periodicity condition is imposed. It is known (see Chapter 1 of [1]) that in the group G_k, there is no normal subgroup of odd index different from one. Therefore, we consider normal subgroups of even indices. Here, we restrict ourself to the case of index two.

We describe $\widehat{G}_k$-weakly periodic Gibbs measures for any normal subgroup $\widehat{G}_k$ of index two. We note (see Chapter 1 of [1]) that any normal subgroup of index two of the group G_k has the form

$$H_A = \left\{ x \in G_k : \sum_{i \in A} \omega_x(a_i)\text{-even} \right\},$$

where $\emptyset \neq A \subseteq N_k = \{1, 2, \ldots, k+1\}$, and $\omega_x(a_i)$ is the number of letters a_i in a word $x \in G_k$.

Let $A \subseteq N_k$ and H_A be the corresponding normal subgroup of index two. We note that in the case $|A| = k+1$, i.e., in the case $A = N_k$, weak periodicity coincides with ordinary periodicity. Therefore, we consider $A \subset N_k$ such that $A \neq N_k$. Then, in view of (2.4), the H_A-weakly periodic set of h has the form

$$h_x = \begin{cases} h_1, \ x \in H_A, \ x_{\downarrow} \in H_A, \\ h_2, \ x \in H_A, \ x_{\downarrow} \in G_k \backslash H_A, \\ h_3, \ x \in G_k \backslash H_A, \ x_{\downarrow} \in H_A, \\ h_4, \ x \in G_k \backslash H_A, x_{\downarrow} \in G_k \backslash H_A, \end{cases} \tag{3.1}$$

where $h_i, i = 1, 2, 3, 4$, satisfy the following equations:

$$\begin{cases} h_1 = |A| f(h_3, \theta) + (k - |A|) f(h_1, \theta), \\ h_2 = (|A| - 1) f(h_3, \theta) + (k + 1 - |A|) f(h_1, \theta), \\ h_3 = (|A| - 1) f(h_2, \theta) + (k + 1 - |A|) f(h_4, \theta), \\ h_4 = |A| f(h_2, \theta) + (k - |A|) f(h_4, \theta). \end{cases} \tag{3.2}$$

Consider operator $W : R^4 \to R^4$, defined by

$$\begin{cases} h_1' = |A|f(h_3,\theta) + (k-|A|)f(h_1,\theta) \\ h_2' = (|A|-1)f(h_3,\theta) + (k+1-|A|)f(h_1,\theta) \\ h_3' = (|A|-1)f(h_2,\theta) + (k+1-|A|)f(h_4,\theta) \\ h_4' = |A|f(h_2,\theta) + (k-|A|)f(h_4,\theta). \end{cases} \quad (3.3)$$

Note that the system of (3.2) describes fixed points of the operator W, i.e. $h = W(h)$.

It is obvious that the following sets are invariant with respect to operator W:

$$I_1 = \{h \in R^4 : h_1 = h_2 = h_3 = h_4\}, \quad I_2 = \{h \in R^4 : h_1 = h_4; h_2 = h_3\},$$

$$I_3 = \{h \in R^4 : h_1 = -h_4; h_2 = -h_3\}.$$

In [9] it was proved that the system of (3.2), on the invariant set I_2, has solutions which belong to I_1. The system of (3.2) on the invariant set I_1 reduces to the following equation

$$h = kf(h,\theta). \quad (3.4)$$

The solutions of (3.4) correspond to translation-invariant Gibbs measures. In this paper we will study weakly periodic (non-periodic, in particular non translation-invariant) Gibbs measures, i.e. we will investigate the fixed points of operator W in the invariant set I_3.

Let $\alpha = \frac{1-\theta}{1+\theta}$. In [14] the following statement is proved.

Theorem 1 *Let* $|A| = k, \alpha > 1$.

1) *For* $k \leqslant 3$ *all* H_A*-weakly periodic Gibbs measures on* I_3 *are translational invariant.*
2) *For* $k = 4$ *there exists a critical value* $\alpha_{cr} (\approx 6.3716)$ *such that for* $\alpha < \alpha_{cr}$ *on* I_3 *there exists one* H_A*-weakly periodic Gibbs measure; for* $\alpha = \alpha_{cr}$ *on* I_3 *there exist three* H_A*-weakly periodic Gibbs measures; for* $\alpha > \alpha_{cr}$ *on* I_3 *there exist five* H_A*-weakly periodic Gibbs measures.*

Remark 1 Note that one of the measures described in item 2) of Theorem 1, is translation-invariant, but the other measures are H_A-weakly periodic (non-periodic) and differ from measures considered in [9, 10].

In Theorem 1 have been considered the cases with $k \leqslant 4$ (see [14]). In this paper we consider the cases with $k \geqslant 5$.

Using the fact that

$$f(h,\theta) = \operatorname{arctanh}(\theta \tanh h) = \frac{1}{2} \ln \frac{(1+\theta)e^{2h} + (1-\theta)}{(1-\theta)e^{2h} + (1+\theta)},$$

and introducing the variables $z_i = e^{2h_i}$ $i = 1, 2, 3, 4$ one can transform the system of (3.2) to the following:

$$\begin{cases} z_1 = \left(\frac{z_3+\alpha}{\alpha z_3+1}\right)^{|A|} \cdot \left(\frac{z_1+\alpha}{\alpha z_1+1}\right)^{(k-|A|)} \\ z_2 = \left(\frac{z_3+\alpha}{\alpha z_3+1}\right)^{|A|-1} \cdot \left(\frac{z_1+\alpha}{\alpha z_1+1}\right)^{(k+1-|A|)} \\ z_3 = \left(\frac{z_2+\alpha}{\alpha z_2+1}\right)^{|A|-1} \cdot \left(\frac{z_4+\alpha}{\alpha z_4+1}\right)^{(k+1-|A|)} \\ z_4 = \left(\frac{z_2+\alpha}{\alpha z_2+1}\right)^{|A|} \cdot \left(\frac{z_4+\alpha}{\alpha z_4+1}\right)^{(k-|A|)} . \end{cases} \tag{3.5}$$

Theorem 2 *Let $|A| = k$. Then for arbitrary k the number of H_A-weakly periodic (non-periodic) Gibbs measures which correspond to fixed points of operator W on the invariant set I_3 does not exceed four.*

Proof Let $|A| = k$. Then the system of (3.5) has the form

$$\begin{cases} z_1 = (f(z_3))^k \\ z_2 = (f(z_3))^{k-1} \cdot (f(z_1)) \\ z_3 = (f(z_2))^{k-1} \cdot (f(z_4)) \\ z_4 = (f(z_2))^k , \end{cases} \tag{3.6}$$

where $f(x) = \frac{x+\alpha}{\alpha x+1}$. The system of (3.6) on the invariant set I_3 has the following form:

$$\begin{cases} z_1 = \left(f\left(\frac{1}{z_2}\right)\right)^k \\ z_2 = \left(f\left(\frac{1}{z_2}\right)\right)^{k-1} \cdot (f(z_1)) \end{cases} \tag{3.7}$$

and it can be transformed to the following equation

$$z_2 = \left(\frac{1+\alpha z_2}{\alpha + z_2}\right)^{k-1} \frac{\alpha(\alpha + z_2)^k + (1+\alpha z_2)^k}{(\alpha + z_2)^k + \alpha(1+\alpha z_2)^k}. \tag{3.8}$$

Assuming $u = f(z_2)$ we reduce the (3.8) to the equation

$$u^{2k} - \alpha u^{2k-1} + \alpha^2 u^{k+1} - \alpha^2 u^{k-1} + \alpha u - 1 = 0. \tag{3.9}$$

According to Descartes' rule of signs (see for example [15]) the number of positive roots of the polynomial (3.9) is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Therefore, the (3.9) has at most five positive solutions. It is easy to verify that this (3.9) is factorized as follows:

$$(u^2 - 1)P_{2k-2}(u) = 0, \tag{3.10}$$

where $P_{2k-2}(u)$ is a polynomial of degree $2k - 2$. Since one of the roots of (3.9) is $u = 1$ which corresponds to a translational-invariant Gibbs measure, the number of H_A-weakly periodic (non-periodic) Gibbs measures does not exceed of four. □

Remark 2 In general, the total number of H_A-weakly periodic (non-periodic) Gibbs measures (considered everywhere, not only on the invariant set I_3) may be greater than four.

Recall that a polynomial $P = \sum_{i=0}^{n} a_i x^i$ of degree n , is called palindromic (antipalindromic) if $a_i = a_{n-i}$ (respectively $a_i = -a_{n-i}$) for $i = 0, 1, \cdots, n$. Note that the polynomial (3.9) is antipalindromic. It is known that if antipalindromic polynomial of even degree is a multiple of $x^2 - 1$ (it has -1 and 1 as a roots) then its quotient by $x^2 - 1$ is palindromic (see for example [15]).

Theorem 3 *Let $|A| = k, k = 5$.*

For the weakly periodic Gibbs measures corresponding to the set of quantities from I_3 there exists a critical value $\alpha_{cr}(\approx 2.65)$ such that there is not any H_A- weakly periodic (nonperiodic) Gibbs measure for $0 < \alpha < \alpha_{cr}$, there are two H_A- weakly periodic (nonperiodic) Gibbs measures for $\alpha = \alpha_{cr}$, and there are four H_A- weakly periodic (nonperiodic) Gibbs measures for $\alpha_{cr} < \alpha$.

Proof Let $k = 5$.

In this case (3.9) has the form

$$u^{10} - \alpha u^9 + \alpha^2 u^6 - \alpha^2 u^4 + \alpha u - 1 = 0. \tag{3.11}$$

Now (3.10) has the form

$$(u^2 - 1)\left(u^8 - \alpha u^7 + u^6 - \alpha u^5 + (\alpha^2 + 1)u^4 - \alpha u^3 + u^2 - \alpha u + 1\right) = 0. \tag{3.12}$$

From (3.12) we have $u^2 - 1 = 0$ or

$$u^8 - \alpha u^7 + u^6 - \alpha u^5 + (\alpha^2 + 1)u^4 - \alpha u^3 + u^2 - \alpha u + 1 = 0. \tag{3.13}$$

Since $u > 0$, we have that $u = 1$ is the solution of (3.12). We assume that $u \neq 1$. Setting $\xi = u + \frac{1}{u} > 2$, from (3.13) we obtain the equation

$$\xi^4 - \alpha\xi^3 - 3\xi^2 + 2\alpha\xi + \alpha^2 + 1 = 0. \tag{3.14}$$

The (3.14) has at most two positive solutions. From (3.14) we find the parameter α:

$$\alpha_1 = \frac{\xi^3 - 2\xi + \sqrt{\xi^6 - 8\xi^4 + 16\xi^2 - 4}}{2} := \gamma_1(\xi), \tag{3.15}$$

$$\alpha_2 = \frac{\xi^3 - 2\xi - \sqrt{\xi^6 - 8\xi^4 + 16\xi^2 - 4}}{2} := \gamma_2(\xi). \tag{3.16}$$

Assume $v = \xi^2$ and $\varphi(v) = v^3 - 8v^2 + 16v - 4$. We consider $\varphi'(v) = 3v^2 - 16v + 16$. It is clear that $\varphi'(v) > 0$ for $v > 4$. On the other hand $\varphi(4) < 0$, $\varphi(+\infty) > 0$. It follows that $\varphi(v) = 0$ has a unique solution v_0 for $v > 4$. Therefore, the system of inequalities

$$\begin{cases} \xi^6 - 8\xi^4 + 16\xi^2 - 4 \geqslant 0 \\ \xi > 2, \end{cases}$$

valid for $\xi \in [\xi_0, +\infty)$, where $\xi_0 = \sqrt{v_0}(\approx 2.214)$.

Note that $\gamma_1(\xi_0) = \gamma_2(\xi_0)$.

One can check that

$$\lim_{\xi \to +\infty} \gamma_i(\xi) = +\infty, i = 1, 2. \tag{3.17}$$

It is clear that the function $\gamma_1(\xi)$ is increasing on the $[\xi_0, +\infty)$. Then we get following: for $\alpha \in (0, \gamma_1(\xi_0))$ there is not $\xi > 2$ satisfying the (3.14); if $\alpha \in [\gamma_1(\xi_0), +\infty)$ then there exists a unique $\xi > 2$ which satisfying the (3.14).

Note that if $\xi \in [\xi_0, +\infty)$ then the equation $\gamma_2'(\xi) = 0$ has a unique solution, which is $\xi_1 \approx 2.3841$, and also we get $\gamma_2(\xi_0) \approx 3.21$, $\gamma_2(\xi_1) \approx 2.65$. Denote $\alpha_{cr} = \gamma_2(\xi_1)$.

Hence it is evident that the function $\gamma_2(\xi)$ reaches its minimum in $[\xi_0, +\infty)$ at ξ_1. Consequently, for $\alpha \in (0, \gamma_2(\xi_1))$ there is not $\xi > 2$ satisfying the (3.14), for $\alpha \in \{\gamma_2(\xi_1)\} \cup (\gamma_2(\xi_0); +\infty)$ there exists a unique $\xi > 2$ satisfying the (3.14), if $\alpha \in (\gamma_2(\xi_1), \gamma_2(\xi_0))$ then there exist two $\xi > 2$ satisfying the (3.14).

Let n_α be the number of solutions of the (3.14). Then n_α has the following form

$$n_\alpha = \begin{cases} 0, & \text{if } \alpha \in (0, \alpha_{cr}) \\ 1, & \text{if } \alpha = \alpha_{cr} \\ 2, & \text{if } \alpha \in (\alpha_{cr}, +\infty). \end{cases}$$

For $\alpha > \alpha_{cr}$ from $u + \frac{1}{u} = \xi$ we get four solutions of the (3.13). In this case the (3.12) has five solutions. For $\alpha = \alpha_{cr}$ from $u + \frac{1}{u} = \xi$ we get that the (3.13) has two solutions. Consequently (3.12) has three solutions. In the case $\alpha \in (0, \alpha_{cr})$ the (3.12) has a unique solution $u = 1$. □

Theorem 4 *Let* $|A| = k, k = 6$.

For the weakly periodic Gibbs measures corresponding to the set of quantities from I_3 *there exists a critical value* $\alpha_c (\approx 1.89)$ *such that there is not any* H_A- *weakly periodic (nonperiodic) Gibbs measure for* $\alpha \in (0, \alpha_c)$*, there are two* H_A- *weakly periodic (nonperiodic) Gibbs measures for* $\alpha \in [2, 3] \cup \{\alpha_c\}$*, and there are four* H_A- *weakly periodic (nonperiodic) Gibbs measures for* $\alpha \in (\alpha_c, 2) \cup (3, +\infty)$.

Proof Let $k = 6$.

In this case (3.9) has the form

$$u^{12} - \alpha u^{11} + \alpha^2 u^7 - \alpha^2 u^5 + \alpha u - 1 = 0. \tag{3.18}$$

The function $y := y(u) = u^{12} - \alpha u^{11} + \alpha^2 u^7 - \alpha^2 u^5 + \alpha u - 1$ with $\alpha = 4.1$ is plotted in Fig. 1.

In this case, the (3.10) has the form

$$(u^2 - 1)\left(u^{10} - \alpha u^9 + u^8 - \alpha u^7 + u^6 + (\alpha^2 - \alpha)u^5 + u^4 - \alpha u^3 + u^2 - \alpha u + 1\right) = 0. \tag{3.19}$$

From (3.19) we have $u^2 - 1 = 0$ or

$$u^{10} - \alpha u^9 + u^8 - \alpha u^7 + u^6 + (\alpha^2 - \alpha)u^5 + u^4 - \alpha u^3 + u^2 - \alpha u + 1 = 0. \tag{3.20}$$

Since $u > 0$, we have that $u = 1$ is a solution of (3.19). We assume that $u \neq 1$. Setting $\xi = u + \frac{1}{u} > 2$, from (3.20) we obtain the equation

$$\xi^5 - \alpha \xi^4 - 4\xi^3 + 3\alpha \xi^2 + 3\xi + \alpha^2 - \alpha = 0. \tag{3.21}$$

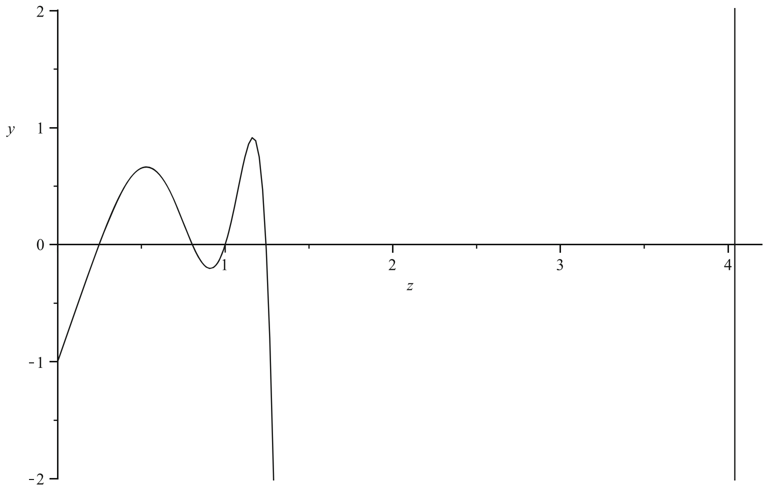

Fig. 1 The function $y = y(u)$ with $\alpha = 4.1$

From (3.21) we find the parameter α:

$$\alpha_1 = \frac{\xi^4 - 3\xi^2 + 1 - \sqrt{\xi(\xi^2-1)(\xi^2-3)(\xi-2)(\xi^2+2\xi+2)+1}}{2} := \alpha_1(\xi), \tag{3.22}$$

$$\alpha_2 = \frac{\xi^4 - 3\xi^2 + 1 + \sqrt{\xi(\xi^2-1)(\xi^2-3)(\xi-2)(\xi^2+2\xi+2)+1}}{2} := \alpha_2(\xi). \tag{3.23}$$

One can check that

$$\lim_{\xi \to +\infty} \alpha_i(\xi) = +\infty, i = 1, 2,$$

and $\xi(\xi^2-1)(\xi^2-3)(\xi-2)(\xi^2+2\xi+2)+1$ is positive for all $\xi \geqslant 2$.

Note that if $\xi \in [2, +\infty)$ the the equation $\alpha_1'(\xi) = 0$ has a unique solution which is $\xi_0 \approx 2.077$, and also we get $\alpha_1(\xi_0) \approx 1.89$. Hence it is clear that the function $\alpha_1(\xi)$ reaches its minimum in $[2, +\infty)$ at ξ_0. Consequently, for $\alpha \in (0, \alpha_1(\xi_0))$ there is not $\xi \geqslant 2$ satisfying the (3.21), for $\alpha \in \{\alpha_1(\xi_0)\} \cup (\alpha_1(2); +\infty)$ there exist unique $\xi \geqslant 2$ satisfying the (3.21), $\alpha \in (\alpha_1(\xi_0), \alpha_1(2)]$ there exists two $\xi \geqslant 2$ satisfying the (3.21).

It is clear that the function $\alpha_2(\xi)$ is increasing on the $[2, +\infty)$. Then we get following: for $\alpha \in (0, \alpha_2(2))$ there is not $\xi > 2$ satisfying the (3.21); $\alpha \in [\alpha_2(2), +\infty)$ there exist a unique $\xi > 2$ which satisfying the (3.21).

Denote $\alpha_1(\xi_0) = \alpha_c$.

Let n_α be the number of solutions of the (3.20). Then n_α has the following form

$$n_\alpha = \begin{cases} 0, & \text{if } \alpha \in (0, \alpha_c) \\ 1, & \text{if } \alpha \in (\alpha_1(2), \alpha_2(2)) \cup \{\alpha_c\} \\ 2, & \text{if } \alpha \in (\alpha_c, \alpha_1(2)] \cup [\alpha_2(2), +\infty). \end{cases}$$

For $\alpha \in (0, \alpha_c)$ (3.19) has a unique solution $u = 1$. For $\alpha \in (2, 3) \cup \{\alpha_c\}$ from $u + \frac{1}{u} = \xi$ we get two solutions of the (3.20). In this case the (3.19) has five solutions.

Note that $\alpha_1(2) = 2$ and $\alpha_2(2) = 3$ and in the case $\xi = 2$ from $u + \frac{1}{u} = \xi$ we get $u = 1$. Consequently, for $\alpha = 2$ and $\alpha = 3$ we get two solutions of (3.20) which differ from $u = 1$. In this case (3.19) has three solutions. For $\alpha \in (\alpha_c, \alpha_1(2)) \cup (\alpha_2(2), +\infty)$ from $u + \frac{1}{u} = \xi$ we get that the (3.20) has two solutions. Consequently, the (3.19) has five solutions.

Let N_α be the number of solutions of the (3.19). Then N_α has the following form

$$N_\alpha = \begin{cases} 1, & \text{if } \alpha \in (0, \alpha_c) \\ 3, & \text{if } \alpha \in [2, 3] \cup \{\alpha_c\} \\ 5, & \text{if } \alpha \in (\alpha_c, 2) \cup (3, +\infty). \end{cases}$$

□

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