

Euler's Theorem for Homogeneous White Noise Operators

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Abstract In this paper we introduce a new notion of λ -order homogeneous operators on the nuclear algebra of white noise operators. Then, we give their Fock expansion in terms of quantum white noise (QWN) fields $\{a_t, a_t^*; t \in \mathbb{R}\}$. The quantum extension of the scaling transform enables us to prove Euler's theorem in quantum white noise setting.

Keywords QWN-Euler operator · Euler's Theorem · QWN-scaling operator · Homogeneous operator · QWN-derivatives

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1 Introduction and Preliminaries

Let H be the real Hilbert space of square integrable functions on \mathbb{R} with norm $|\cdot|_0$ and $E \equiv \mathcal{S}(\mathbb{R})$ be the Schwartz space consisting of rapidly decreasing C^∞ -functions. Then, the nuclear Gel'fand triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, dx) \subset \mathcal{S}'(\mathbb{R}) \quad (1)$$

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can be reconstructed in a standard way (see Ref. [18]) by the harmonic oscillator $A = 1 + t^2 - d^2/dt^2$ and H . The eigenvalues of A are $2n$, $n = 1, 2, \dots$, the corresponding eigenfunctions $\{e_n; n \geq 1\}$ form an orthonormal basis for $L^2(\mathbb{R})$. In fact (e_n) are the Hermite functions and therefore each e_n is an element of E . The space E is a nuclear space equipped with the Hilbertian norms

$$|\xi|_p = |A^p \xi|_0, \quad \xi \in E, \quad p \in \mathbb{R}$$

and we have

$$E = \text{proj} \lim_{p \rightarrow \infty} E_p, \quad E' = \text{ind} \lim_{p \rightarrow \infty} E_{-p},$$

where, for $p \geq 0$, E_p is the completion of E with respect to the norm $|\cdot|_p$ and E_{-p} is the topological dual space of E_p . We denote by $N = E + iE$ and $N_p = E_p + iE_p$, $p \in \mathbb{Z}$, the complexifications of E and E_p , respectively. Throughout, we fix a Young function θ satisfying the condition

$$\limsup_{x \rightarrow \infty} \frac{\theta(x)}{x^2} < +\infty. \tag{2}$$

Its polar function θ^* is the Young function defined by

$$\theta^*(x) = \sup_{t \geq 0} (tx - \theta(t)), \quad x \geq 0.$$

For more details, see Refs. [8].

For a complex Banach space $(B, \|\cdot\|)$, $\mathcal{H}(B)$ denotes the space of all entire functions on B and for $m > 0$, $\text{Exp}(B, \theta, m)$ is the Banach space

$$\text{Exp}(B, \theta, m) = \left\{ f \in \mathcal{H}(B); \|f\|_{\theta, m} := \sup_{z \in B} |f(z)| e^{-\theta(m\|z\|)} < \infty \right\}.$$

The projective system $\{\text{Exp}(N_{-p}, \theta, m); p \in \mathbb{N}, m > 0\}$ and the inductive system $\{\text{Exp}(N_p, \theta, m); p \in \mathbb{N}, m > 0\}$ give the two nuclear spaces

$$\mathcal{F}_\theta(N') = \text{proj} \lim_{p \rightarrow \infty; m \downarrow 0} \text{Exp}(N_{-p}, \theta, m), \quad \mathcal{G}_\theta(N) = \text{ind} \lim_{p \rightarrow \infty; m \rightarrow 0} \text{Exp}(N_p, \theta, m). \tag{3}$$

It is noteworthy that, for each $\xi \in N$, the exponential function

$$e_\xi(z) := e^{(z, \xi)}, \quad z \in N',$$

belongs to $\mathcal{F}_\theta(N')$ and the set of such test functions spans a dense subspace of $\mathcal{F}_\theta(N')$. In the remainder of this paper we use simply \mathcal{F}_θ to denote the space $\mathcal{F}_\theta(N')$. The space of continuous linear operators from \mathcal{F}_θ into its topological dual space \mathcal{F}_θ^* is denoted by $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and assumed to carry the bounded convergence topology. For $z \in N'$ and $\varphi \in \mathcal{F}_\theta$ with Taylor expansions $\sum_{n=0}^\infty (x^{\otimes n}, f_n)$, the holomorphic derivative of φ at $x \in N'$ in the direction z is defined by

$$(a(z)\varphi)(x) := \lim_{\lambda \rightarrow 0} \frac{\varphi(x + \lambda z) - \varphi(x)}{\lambda}. \tag{4}$$

We can check that the limit always exists and $a(z) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$. Let $a^*(z) \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta^*)$ be the dual adjoint of $a(z)$, i.e., for $\Phi \in \mathcal{F}_\theta^*$ and $\phi \in \mathcal{F}_\theta$, $\langle\langle a^*(z)\Phi, \phi \rangle\rangle = \langle\langle \Phi, a(z)\phi \rangle\rangle$, where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the standard bilinear form on $\mathcal{F}_\theta^* \times \mathcal{F}_\theta$. Similarly,

for $\psi \in \mathcal{G}_{\theta^*}(N)$ with Taylor expansion $\psi(\xi) = \sum_{n=0}^{\infty} \langle \psi_n, \xi^{\otimes n} \rangle$ we use the common notation $a(z)\psi$ for the derivative (4) with $z \in N$.

The Wick symbol of $\Xi \in \mathcal{L}(\mathcal{F}_{\theta}, \mathcal{F}_{\theta}^*)$ is by definition [18] a \mathbb{C} -valued function on $N \times N$ defined by

$$\sigma(\Xi)(\xi, \eta) = \langle \Xi e_{\xi}, e_{\eta} \rangle e^{-\langle \xi, \eta \rangle}, \quad \xi, \eta \in N. \tag{5}$$

By a density argument, every operator in $\mathcal{L}(\mathcal{F}_{\theta}, \mathcal{F}_{\theta}^*)$ is uniquely determined by its Wick symbol. In fact, if $\mathcal{G}_{\theta^*}(N \oplus N)$ denotes the nuclear space obtained as in (3) by replacing N_p by $N_p \times N_p$, we have the following characterization theorem for operator Wick symbols.

Theorem 1 (See Ref. [13]) *The Wick symbol map σ yields a topological isomorphism between $\mathcal{L}(\mathcal{F}_{\theta}, \mathcal{F}_{\theta}^*)$ and $\mathcal{G}_{\theta^*}(N \oplus N)$.*

It is a fundamental fact in quantum white noise theory [18] (see, also Ref. [13]) that every white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_{\theta}, \mathcal{F}_{\theta}^*)$ admits a unique Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{6}$$

where, for each pairing $l, m \geq 0$, $\kappa_{l,m} \in (N^{\otimes(l+m)})'_{sym(l,m)}$ and $\Xi_{l,m}(\kappa_{l,m})$ is the integral kernel operator uniquely specified via the Wick symbol transform by

$$\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle, \quad \xi, \eta \in N. \tag{7}$$

For any $S_1, S_2 \in \mathcal{L}(\mathcal{F}_{\theta}, \mathcal{F}_{\theta}^*)$, there exists a unique $\Xi \in \mathcal{L}(\mathcal{F}_{\theta}, \mathcal{F}_{\theta}^*)$, denoted $S_1 \diamond S_2$, such that

$$\sigma(S_1 \diamond S_2) = \sigma(S_1)\sigma(S_2). \tag{8}$$

The operator $S_1 \diamond S_2$ will be referred to as *the Wick product* of S_1 and S_2 .

Let θ_n be given by $\theta_n = \inf_{r>0} e^{\theta(r)}/r^n$, $n \in \mathbb{N}$. Then, for $p \in \mathbb{N}$ and $\gamma_1, \gamma_2 > 0$, we define the Hilbert space

$$F_{\theta, \gamma_1, \gamma_2}(N_p \oplus N_p) = \left\{ \vec{\varphi} = (\varphi_{l,m})_{l,m=0}^{\infty}; \varphi_{l,m} \in (N_p^{\otimes l} \otimes N_p^{\otimes m})_{sym(l,m)}, \sum_{l,m=0}^{\infty} (\theta_l \theta_m)^{-2} \gamma_1^{-l} \gamma_2^{-m} |\varphi_{l,m}|_p^2 < \infty \right\}$$

Put

$$F_{\theta}(N \oplus N) = \bigcap_{p \in \mathbb{N}, \gamma_1 > 0, \gamma_2 > 0} F_{\theta, \gamma_1, \gamma_2}(N_p \oplus N_p).$$

Theorem 2 ([4]) *An operator $\Xi \in \mathcal{L}(\mathcal{F}_{\theta}^*, \mathcal{F}_{\theta})$ if and only if there exists a unique $(\kappa_{l,m})_{l,m} \in F_{\theta}(N \oplus N)$ such that*

$$\Xi = \Xi_{-\tau} \diamond \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{9}$$

where τ is the usual trace on $N \otimes N$, i.e., $\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle$ and

$$\Xi_{\pm\tau} = \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \Xi_{k,k}(\tau^{\otimes k}).$$

Let \mathcal{U}_θ be the space of white noise operators given by

$$\mathcal{U}_\theta = \left\{ \Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}); (\kappa_{l,m})_{l,m} \in F_\theta(N \oplus N) \right\}.$$

For $x, y \in N$, we put $\kappa_{l,m}(x, y) = \frac{x^{\otimes l}}{l!} \otimes \frac{y^{\otimes m}}{m!}$ and $\Xi^{x,y} := \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}(x, y))$. Then, the set $\{\Xi^{x,y}; x, y \in N\}$ spans a dense subspace of \mathcal{U}_θ .

Theorem 3 ([4]) *The map f_τ defined by*

$$f_\tau : \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta) \longrightarrow \mathcal{U}_\theta, \quad \Xi \longmapsto \Xi_\tau \diamond \Xi,$$

is a topological isomorphism.

We recall from Ref. [4] the dual pairing: for $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{U}_\theta$ and $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, we define

$$\langle \Xi, T \rangle := \sum_{l,m=0}^{\infty} l!m! \langle \kappa_{l,m}, \Phi_{l,m} \rangle.$$

For more details see [4–6, 22, 23] and [24].

In mathematics, a homogeneous function is a function with multiplicative scaling behavior: if the argument is multiplied by a factor, then the result is multiplied by some power of this factor. More precisely, for $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$, put $S_t f(x) = f(tx)$, $x \in \mathbb{R}^d$. For a given $\lambda \in \mathbb{R}$, an element $f \in L^2(\mathbb{R}^d)$ is said to be λ -order homogeneous if $S_t f(x) = t^\lambda f(x)$ for each $t \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^d$. It is well known that f is λ -order homogeneous if and only if it satisfies the so-called Euler equation

$$\sum_{i=1}^d x_i \frac{\partial}{\partial x_i} f = \lambda f. \tag{10}$$

In infinite dimension analysis, an analogue of the Euler operator $\sum_{i=1}^d x_i \frac{\partial}{\partial x_i}$ was introduced in [17] as follows

$$\Delta_E = \sum_{i=1}^{\infty} (a^*(e_i) + a(e_i))a(e_i) = \sum_{i=1}^{\infty} \langle \cdot, e_i \rangle a(e_i).$$

Moreover, the scaling transformation S_t is defined at $\varphi(x) = \sum_{n=0}^\infty \langle x^{\otimes n}, \varphi_n \rangle \in \mathcal{F}_\theta$ by

$$S_t \varphi(x) = \sum_{n=0}^\infty \left\langle x^{\otimes n}, \frac{t^n}{n!} \sum_{l=0}^\infty (t^2 - 1)^l \frac{(n + 2l)!}{l!2^l} \tau^{\otimes l} \widehat{\otimes}_l \varphi_{n+2l} \right\rangle, \quad x \in N'.$$

For $\lambda \in \mathbb{R}$, φ is said to be λ -order homogeneous if $S_t \varphi = t^\lambda \varphi$ for any $t \in \mathbb{R} \setminus \{0\}$. It is proved in [20] that φ is λ -order homogeneous if and only if it satisfies the Euler equation

$$\Delta_E \varphi = \lambda \varphi. \tag{11}$$

The main purpose of this paper is the study of the QWN-analogue of (11). We start by introducing a QWN-Scaling transformation and a QWN-second quantization. These transformations will be used to introduce the notion of λ -order homogeneous operators. Then, as a first main result we give their Fock expansions (see Theorem 5). Our second main result is stated in Theorem 7, where we show that a white noise operator Ξ is λ -order homogeneous if and only if it satisfies the following QWN-Euler equation

$$\Delta_E^Q \Xi = \lambda \Xi.$$

Here Δ_E^Q is the QWN-Euler operator defined in [6].

2 Fundamental QWN-Operators

2.1 QWN-Laplacians

$$\begin{aligned} \Delta_G^Q &= \sum_{j=1}^\infty D_{e_j}^+ D_{e_j}^+ + \sum_{j=1}^\infty D_{e_j}^- D_{e_j}^-, \\ N^Q &= \sum_{j=1}^\infty (D_{e_j}^+)^* (D_{e_j}^+)^+ + \sum_{j=1}^\infty (D_{e_j}^-)^* D_{e_j}^-, \end{aligned}$$

where, for $\zeta \in N$,

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \quad D_\zeta^- \Xi = -[a^*(\zeta), \Xi] \tag{12}$$

are the *creation derivative* and *annihilation derivative* of Ξ , (see [12]).

Lemma 1 For any $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$, we have

$$N^Q \Xi = \sum_{l,m=0}^\infty (l + m) \Xi_{l,m}(\kappa_{l,m}). \tag{13}$$

Proof From [6], we have, for $x, y \in N$

$$\sigma(N^Q \Xi^{x,y})(\xi, \eta) = (\langle x, \eta \rangle + \langle y, \xi \rangle) \sigma(\Xi^{x,y})(\xi, \eta).$$

On the other hand, denoting the right hand side of (13) by A^Q , we get

$$\begin{aligned} & \sigma(A^Q \Xi^{x,y})(\xi, \eta) \\ &= \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \langle x, \eta \rangle \frac{\langle x, \eta \rangle^{l-1}}{(l-1)!} \frac{\langle y, \xi \rangle^m}{m!} + \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \frac{\langle x, \eta \rangle^l}{l!} \langle y, \xi \rangle \frac{\langle y, \xi \rangle^{m-1}}{(m-1)!} \\ &= (\langle x, \eta \rangle + \langle y, \xi \rangle) \sigma(\Xi^{x,y})(\xi, \eta). \end{aligned}$$

Then, by a density argument we complete the proof. □

It is noteworthy that the identity (13) holds true for $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$.

Proposition 1 *Let $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. Then, we have*

$$(\Delta_G^Q)^* T = \{\Xi_{2,0}(\tau) + \Xi_{0,2}(\tau)\} \diamond T \tag{14}$$

$$(N^Q)^* T = N^Q T. \tag{15}$$

Proof From [1], for $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$, we have

$$\Delta_G^Q \Xi = \sum_{l,m=0}^{\infty} (l+2)(l+1) \Xi_{l,m}(\tau \otimes^2 \kappa_{l+2,m}) + \sum_{l,m=0}^{\infty} (m+2)(m+1) \Xi_{l,m}(\kappa_{l,m+2} \otimes_2 \tau), \tag{16}$$

where, for $z_p \in (N^{\otimes p})'$, and $\xi_{l+m-p} \in N^{\otimes(l+m-p)}$, $p \leq l + m$, the contractions $z_p \otimes_p \kappa_{l,m}$ and $\kappa_{l,m} \otimes^p z_p$ are defined by

$$\begin{aligned} \langle z_p \otimes^p \kappa_{l,m}, \xi_{l-p+m} \rangle &= \langle \kappa_{l,m}, z_p \otimes \xi_{l-p+m} \rangle, \\ \langle \kappa_{l,m} \otimes_p z_p, \xi_{l+m-p} \rangle &= \langle \kappa_{l,m}, \xi_{l+m-p} \otimes z_p \rangle. \end{aligned}$$

Then, for $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, we obtain

$$\begin{aligned} & (T, \Delta_G^Q \Xi) \\ &= \sum_{l,m=0}^{\infty} l!m!(l+2)(l+1) \langle \Phi_{l,m}, \tau \otimes^2 \kappa_{l+2,m} \rangle \\ & \quad + \sum_{l,m=0}^{\infty} l!m!(m+2)(m+1) \langle \Phi_{l,m}, \kappa_{l,m+2} \otimes_2 \tau \rangle \\ &= \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} l!m! \langle \tau \otimes \Phi_{l-2,m}, \kappa_{l,m} \rangle + \sum_{l=0}^{\infty} \sum_{m=2}^{\infty} l!m! \langle \Phi_{l,m-2} \otimes \tau, \kappa_{l,m} \rangle. \end{aligned}$$

Therefore, we get

$$(\Delta_G^Q)^* T = \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \Xi_{l,m}(\tau \otimes \Phi_{l-2,m}) + \sum_{l=0}^{\infty} \sum_{m=2}^{\infty} \Xi_{l,m}(\Phi_{l,m-2} \otimes \tau),$$

which yields

$$\begin{aligned} \sigma((\Delta_G^Q)^*T)(\xi, \eta) &= \sum_{l=2}^\infty \sum_{m=0}^\infty \langle \tau \otimes \Phi_{l-2,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \\ &\quad + \sum_{l=0}^\infty \sum_{m=2}^\infty \langle \Phi_{l,m-2} \otimes \tau, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \\ &= \{ \langle \eta, \eta \rangle + \langle \xi, \xi \rangle \} \sigma(T)(\xi, \eta) \\ &= \sigma(\Xi_{2,0}(\tau) + \Xi_{0,2}(\tau))(\xi, \eta) \sigma(T)(\xi, \eta). \end{aligned}$$

This gives

$$(\Delta_G^Q)^*T = \{ \Xi_{2,0}(\tau) + \Xi_{0,2}(\tau) \} \diamond T$$

as desired. (15) follows from (13). □

2.2 QWN-Second Quantization

We start by clarifying the topology of the nuclear algebra $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. From Theorem 1, we have the topological isomorphism:

$$\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*) \simeq \mathcal{G}_{\theta^*}(N \oplus N) = \bigcup_{p \geq 0, \gamma > 0} \text{Exp}(N_p \oplus N_p, \theta^*, \gamma).$$

For $p \geq 0$ and $\gamma > 0$, let $\mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ denotes the subspace of all $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ which correspond to elements in $\text{Exp}(N_p \oplus N_p, \theta^*, \gamma)$. The topology of $\mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is naturally induced from the norm of the Banach space $\text{Exp}(N_p \oplus N_p, \theta^*, \gamma)$ which will be denoted by $\| \cdot \|_{\theta,-p,\gamma}$, i.e., for $\Xi \in \mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$,

$$\| \Xi \|_{\theta,-p,\gamma} = \| \sigma \Xi \|_{\theta^*,-p,\gamma} = \sup_{\xi, \eta \in N_p} | \sigma(\Xi)(\xi, \eta) | e^{-\theta^*(\gamma|\xi|_p) - \theta^*(\gamma|\eta|_p)}.$$

For $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and $t \in \mathbb{R}$, we define the operator $\Gamma^Q(t)$ by

$$\Gamma^Q(t)\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(t^{l+m}\Phi_{l,m}). \tag{17}$$

We denote by $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ the group of all linear homeomorphisms from $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ onto itself.

Proposition 2 $\{ \Gamma^Q(e^t) \}_{t \in \mathbb{R}}$ is a regular one-parameter subgroup of $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ with infinitesimal generator N^Q .

Proof The proof of the fact that $\{ \Gamma^Q(e^t) \}_{t \in \mathbb{R}}$ is a one-parameter subgroup of $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ is straightforward. Since we have

$$e^{t(l+m)} = 1 + t(l+m) + t^2 \sum_{k=0}^\infty \frac{t^k}{(k+2)!} (l+m)^{(k+2)}$$

then, for $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, one can write

$$\Gamma^Q(e^t)\Xi = \Xi + tN^Q(\Xi) + t^2\Lambda(t)(\Xi) \tag{18}$$

where

$$\Lambda(t)(\Xi) = \sum_{l,m=0}^\infty \Xi_{l,m}(\Lambda_{l,m}(t)\Phi_{l,m})$$

with

$$\Lambda_{l,m}(t) = \sum_{k=0}^\infty \frac{t^k}{(k+2)!} (l+m)^{(k+2)}.$$

Now, for $|t| \leq 1$, using a similar computation as in [7], one can show that, there exist $c, r, r' > 0$ and $p, q \geq 0$ such that

$$\left\| \frac{\sigma(\Gamma^Q(e^t)\Xi) - \sigma(\Xi)}{t} - \sigma(N^Q\Xi) \right\|_{\theta^*, -p, r'} \leq c|t| \|\sigma(\Xi)\|_{\theta^*, -q, r}.$$

It then follows

$$\lim_{t \rightarrow 0} \sup_{\|\sigma(\Xi)\|_{\theta^*, -q, r} \leq 1} \left\| \frac{\sigma(\Gamma^Q(e^t)\Xi) - \sigma(\Xi)}{t} - \sigma(N^Q\Xi) \right\|_{\theta^*, -p, \gamma} = 0.$$

This proves the desired statement. □

2.3 QWN-Scaling Transformation

$$S_t^Q(\Xi) :=$$

$$\sum_{j,k,l,m=0}^\infty t^{l+m} (t^2 - 1)^{j+k} \frac{(l+2j)!(m+2k)!}{2^{j+k} j! k! l! m!} \Xi_{l,m}(\tau^{\otimes j} \otimes^{2j} \kappa_{l+2j, m+2k} \otimes_{2k} \tau^{\otimes k}) \tag{19}$$

We recall from Ref. [6] and Theorem 3 that the QWN-Fourier-Gauss transform $G_{K_1, K_2; B_1, B_2}^Q$ is a continuous linear operator from \mathcal{U}_θ into itself defined by

$$G_{K_1, K_2; B_1, B_2}^Q \Xi = \sum_{l,m}^\infty \Xi_{l,m}(g_{l,m}) \tag{20}$$

where $K_i, B_i \in \mathcal{L}(N', N') \cap \mathcal{L}(N, N)$, $i = 1, 2$ and $g_{l,m}$ is given by

$$g_{l,m} = \sum_{j,k=0}^\infty \frac{(l+2j)!(m+2k)!}{l! m! j! k!} \left(B_1^{\otimes l} \otimes B_2^{\otimes m} \right) \left(\tau_{\kappa_1}^{\otimes j} \otimes^{2j} \kappa_{l+2j, m+2k} \otimes_{2k} \tau_{\kappa_2}^{\otimes k} \right).$$

In our setting, we observe that $S_t^Q = G_{\frac{1}{2}(t^2-1)I, \frac{1}{2}(t^2-1)I; tI, tI}^Q$.

Theorem 4 Let $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. Then, $(S_t^Q)^* \Xi$ is given by

$$(S_t^Q)^* \Xi = F(t) \diamond \Gamma^Q(t)(\Xi),$$

where $F(t)$ is given by

$$F(t) = \sum_{j,k=0}^{\infty} \frac{(t^2 - 1)^{j+k}}{2^{j+k} j! k!} \Xi_{2j,2k}(\tau^{\otimes j} \otimes \tau^{\otimes k}). \tag{21}$$

Proof For $\Xi = \sum_{l,m=0} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and $T = \sum_{l,m=0} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{U}_\theta$, we have

$$\begin{aligned} & \langle \Xi, S_t^Q T \rangle \\ &= \sum_{j,k,l,m=0}^{\infty} l! m! \left\langle \kappa_{l,m}, (t^2 - 1)^{j+k} t^{l+m} \frac{(l+2j)!(m+2k)!}{2^{j+k} j! k!} \tau^{\otimes j} \otimes \tau^{2j} \Phi_{l+2j,m+2k} \otimes \tau^{2k} \tau^{\otimes k} \right\rangle \\ &= \sum_{p,q=0}^{\infty} p! q! \left\langle \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{(t^2 - 1)^{j+k}}{2^{j+k} j! k!} t^{p+q-2j-2k} \tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}, \Phi_{p,q} \right\rangle. \end{aligned}$$

This yields

$$(S_t^Q)^* \Xi = \sum_{p,q=0}^{\infty} \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{t^{p+q-2j-2k} (t^2 - 1)^{j+k}}{2^{j+k} j! k!} \Xi_{p,q}(\tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}). \tag{22}$$

On the other hand, we have

$$\begin{aligned} & \sigma(F(t) \diamond \Gamma^Q(t)(\Xi))(\xi, \eta) \\ &= \sigma(F(t))(\xi, \eta) \sigma(\Gamma^Q(t)(\Xi))(\xi, \eta) \\ &= \sum_{j,k,l,m=0}^{\infty} \frac{t^{l+m} (t^2 - 1)^{j+k}}{2^{j+k} j! k!} \left\langle \tau^{\otimes j} \otimes \kappa_{l,m} \otimes \tau^{\otimes k}, \eta^{\otimes l+2j} \otimes \xi^{\otimes m+2k} \right\rangle \\ &= \sum_{p,q=0}^{\infty} \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{t^{p+q-2j-2k} (t^2 - 1)^{j+k}}{2^{j+k} j! k!} \left\langle \tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}, \eta^{\otimes p} \otimes \xi^{\otimes q} \right\rangle \end{aligned}$$

or equivalently

$$\begin{aligned} & F(t) \diamond \Gamma^Q(t)(\Xi) \\ &= \sum_{p,q=0}^{\infty} \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{t^{p+q-2j-2k} (t^2 - 1)^{j+k}}{2^{j+k} j! k!} \Xi_{p,q}(\tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}). \end{aligned}$$

Comparing with (22), the statement follows. □

Remark 1 Using (19), for $x, y \in N$, we have

$$S_t^Q(\Xi^{x,y}) = \exp \left\{ \frac{1}{2} (t^2 - 1) \langle x, x \rangle + \frac{1}{2} (t^2 - 1) \langle y, y \rangle \right\} \Xi^{tx,ty}. \tag{23}$$

Then, for all $s, t \in \mathbb{R} \setminus \{0\}$, by a density argument, one can verify that

$$S_s^Q S_t^Q = S_{st}^Q. \tag{24}$$

In particular, for all $s \in \mathbb{R}$ and $t \in \mathbb{R} \setminus \{0\}$, we get

$$S_{1+\frac{s}{t}}^Q S_t^Q = S_{s+t}^Q. \tag{25}$$

3 Euler’s Theorem For Homogeneous Operator

3.1 Homogeneous Operator

Definition 1 Let $\Xi \in \mathcal{U}_\theta$ and $\lambda \in \mathbb{R}$. We say that Ξ is λ -order homogeneous if for each $t \in \mathbb{R} \setminus \{0\}$ we have

$$S_t^Q(\Xi) = t^\lambda \Xi. \tag{26}$$

This definition is motivated by the classical case studied in [17].

Lemma 2 Let $l, m \geq 0$ and $\kappa_{l,m} \in (N^{\otimes l} \otimes N^{\otimes m})_{\text{sym}(l,m)}$. Then, for $K_i, B_i \in \mathcal{L}(N', N') \cap \mathcal{L}(N, N)$, $i = 1, 2$, we have

$$G_{K_1, K_2; B_1, B_2}^Q \Xi_{l,m}(\kappa_{l,m}) = \sum_{p=0}^{\lfloor l/2 \rfloor} \sum_{q=0}^{\lfloor m/2 \rfloor} \Xi_{l-2p, m-2q}(g_{l-2p, m-2q}), \tag{27}$$

where $g_{l-2p, m-2q}$ is given by

$$g_{l-2p, m-2q} = \frac{l!m!}{(l-2p)!(m-2q)!p!q!} \left(B_1^{\otimes(l-2p)} \otimes B_2^{\otimes(m-2q)} \right) \left(\tau_{K_1}^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau_{K_2}^{\otimes q} \right) \tag{28}$$

and τ_{K_i} is the K_i -trace defined by $\langle \tau_{K_i}, z \otimes w \rangle = \langle K_i z, w \rangle$.

Proof The operator $\Xi_{l,m}(\kappa_{l,m})$ can be rewritten as

$$\Xi_{l,m}(\kappa_{l,m}) = \sum_{\alpha, \beta=0}^{\infty} \Xi_{\alpha, \beta}(f_{\alpha, \beta}),$$

where $f_{\alpha, \beta}$ is defined by

$$f_{\alpha, \beta} = \begin{cases} \kappa_{l,m} & \text{if } (l, m) = (\alpha, \beta) \\ 0 & \text{if } (l, m) \neq (\alpha, \beta). \end{cases} \tag{29}$$

Then, by using (20), we get

$$G_{K_1, K_2; B_1, B_2}^Q \Xi_{l,m}(\kappa_{l,m}) = \sum_{\alpha, \beta=0}^{\infty} \Xi_{\alpha, \beta}(g_{\alpha, \beta}),$$

with

$$g_{\alpha, \beta} = \sum_{j, k=0}^{\infty} \frac{(\alpha + 2j)!(\beta + 2k)!}{\alpha! \beta! j! k!} \left(B_1^{\otimes \alpha} \otimes B_2^{\otimes \beta} \right) \left(\tau_{K_1}^{\otimes j} \otimes^{2j} f_{\alpha+2j, \beta+2k} \otimes_{2k} \tau_{K_2}^{\otimes k} \right).$$

From (29), we observe that $g_{\alpha,\beta} = 0$ for $\alpha > l$ or $\beta > m$. Thus, we obtain

$$G_{K_1, K_2; B_1, B_2}^Q \Xi = \sum_{0 \leq \alpha \leq l} \sum_{0 \leq \beta \leq m} \Xi_{\alpha,\beta}(g_{\alpha,\beta}),$$

with

$$g_{\alpha,\beta} = \sum_{j,k=0}^{\infty} \sum_{2j=l-\alpha} \sum_{2k=m-\beta} \frac{l!m!}{\alpha!\beta!j!k!} \left(B_1^{\otimes \alpha} \otimes B_2^{\otimes \beta} \right) \left(\tau_{K_1}^{\otimes j} \otimes^{2j} \kappa_{l,m} \otimes_{2k} \tau_{K_2}^{\otimes k} \right).$$

Moreover, when $l - \alpha = 2p + 1$ or $m - \beta = 2q + 1$, we have $g_{\alpha,\beta} = 0$. The case $l - \alpha = 2p$ and $m - \beta = 2q$ gives

$$g_{\alpha,\beta} = \frac{l!m!}{\alpha!\beta!p!q!} \left(B_1^{\otimes \alpha} \otimes B_2^{\otimes \beta} \right) \left(\tau_{K_1}^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau_{K_2}^{\otimes q} \right).$$

Replacing α by $l - 2p$ and β by $m - 2q$, we get the desired statement. □

The following theorem gives the Fock expansion of the λ -order homogeneous operator in \mathcal{U}_θ .

Theorem 5 *Let $\lambda \in \mathbb{N}$ and $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$. Then, Ξ is λ -order homogeneous if and only if*

$$\begin{aligned} \Xi &= \sum_{l=0}^{\lambda} \sum_{p=0}^{[l/2]} \sum_{q=0}^{[\frac{\lambda-l}{2}]} \int_{\mathbb{R}^{\lambda-2p-2q}} \Upsilon_{l-2p,\lambda-l-2q}(s_1, \dots, s_{l-2p}, t_1, \dots, t_{\lambda-l-2q}) \\ &\quad a_{s_1}^* \cdots a_{s_{l-2p}}^* a_{t_1} \cdots a_{t_{\lambda-l-2q}} ds_1 \cdots ds_{l-2p} dt_1 \cdots dt_{\lambda-l-2q}, \end{aligned}$$

where $\Upsilon_{l-2p,\lambda-l-2q}$ is given by

$$\begin{aligned} \Upsilon_{l-2p,\lambda-l-2q} &= \\ \sum_{j,k=0}^{\infty} \frac{(l+2j)!(\lambda-l+2k)!(-1)^{j+k}}{j!k!p!q!(l-2p)!(\lambda-l-2q)!2^{j+k}2^{p+q}} &\left(\tau^{\otimes(j+p)} \otimes^{2(j+p)} \kappa_{l+2j,\lambda-l+2k} \otimes_{2(k+q)} \tau^{\otimes(k+q)} \right). \end{aligned}$$

Proof In the following we set

$$\mathcal{G}^Q := G_{-\frac{1}{2}I, -\frac{1}{2}I; -iI, -iI}^Q.$$

Motivated by the classical case (see [16]), we can show that \mathcal{G}^Q is a topological isomorphism from \mathcal{U}_θ into itself. Moreover,

$$(\mathcal{G}^Q)^{-1} \Xi = G_{-\frac{1}{2}I, -\frac{1}{2}I; iI, iI}^Q \Xi. \tag{30}$$

For any $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$ and $t \in \mathbb{R}$, the technical identity

$$S_t^Q(\Xi) = (\mathcal{G}^Q)^{-1} \Gamma^Q(t) \mathcal{G}^Q(\Xi) \tag{31}$$

holds true. Indeed, by direct computation, we have

$$\mathcal{G}^Q \Xi^{x,y} = \exp \left\{ -\frac{1}{2} \langle x, x \rangle - \frac{1}{2} \langle y, y \rangle \right\} \Xi^{-ix, -iy}, \quad x, y \in N.$$

Therefore,

$$\Gamma^Q(t) \mathcal{G}^Q(\Xi^{x,y}) = \exp \left\{ -\frac{1}{2} \langle x, x \rangle - \frac{1}{2} \langle y, y \rangle \right\} \Xi^{-itx, -ity}.$$

Then, we obtain

$$(\mathcal{G}^Q)^{-1} \Gamma^Q(t) \mathcal{G}^Q(\Xi^{x,y}) = \exp \left\{ \frac{1}{2} (t^2 - 1) (\langle x, x \rangle + \langle y, y \rangle) \right\} \Xi^{tx, ty}.$$

Hence, by (23) we deduce that

$$S_t^Q(\Xi^{x,y}) = (\mathcal{G}^Q)^{-1} \Gamma^Q(t) \mathcal{G}^Q(\Xi^{x,y}),$$

which proves (31) by density argument.

In view of (31), Equation (26) can be rewritten as follows

$$\Gamma^Q(t) \mathcal{G}^Q(\Xi) = t^\lambda \mathcal{G}^Q(\Xi). \tag{32}$$

Let $T = \sum_{l,m=0}^\infty \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{U}_\theta$ be the unique Fock expansion of the operator $T = \mathcal{G}^Q(\Xi)$, where $\Phi_{l,m}$ is given by

$$\Phi_{l,m} = \sum_{j,k=0}^\infty \frac{(l+2j)!(m+2k)!(-i)^{l+m}(-1)^{j+k}}{l!m!j!k!2^{j+k}} \left(\tau^{\otimes j} \otimes^{2j} \Phi_{l+2j,m+2k} \otimes_{2k} \tau^{\otimes k} \right).$$

Then (32) can be rewritten as

$$\Gamma^Q(t) T = t^\lambda T, \tag{33}$$

or equivalently

$$\sum_{l,m=0}^\infty t^{l+m} \Xi_{l,m}(\Phi_{l,m}) = \sum_{l,m=0}^\infty t^\lambda \Xi_{l,m}(\Phi_{l,m}).$$

From the uniqueness of the Fock expansion, this last equation is satisfied if and only if $\lambda = l + m$. Then, T satisfies (33) if and only if

$$T = \sum_{l=0}^\lambda \Xi_{l,\lambda-l}(\Phi_{l,\lambda-l}).$$

Therefore, by (27), we obtain

$$\begin{aligned} \Xi &= (\mathcal{G}^Q)^{-1} T \\ &= \sum_{l=0}^\lambda G^Q_{-\frac{1}{2}l, -\frac{1}{2}l; i, i, l}(\Xi_{l,\lambda-l}(\Phi_{l,\lambda-l})) \\ &= \sum_{l=0}^\lambda \sum_{p=0}^{[l/2]} \sum_{q=0}^{[\frac{\lambda-l}{2}]} \Xi_{l-2p, \lambda-l-2q}(\Upsilon_{l-2p, \lambda-l-2q}) \end{aligned}$$

where $\Upsilon_{l-2p,\lambda-l-2q}$ is given by

$$\Upsilon_{l-2p,\lambda-l-2q} = \frac{l!(\lambda-l)!i^\lambda}{(l-2p)!(\lambda-l-2p)!p!q!2^{p+q}} \left(\tau^{\otimes p} \otimes^{2p} \Phi_{l,\lambda-l} \otimes_{2q} \tau^{\otimes q} \right)$$

and

$$\Phi_{l,\lambda-l} = \sum_{j,k=0}^{\infty} \frac{(l+2j)!(\lambda-l+2k)!(-1)^\lambda(-1)^{j+k}}{l!(\lambda-l)!j!k!2^{j+k}} \left(\tau^{\otimes j} \otimes^{2j} \kappa_{l+2j,\lambda-l+2k} \otimes_{2k} \tau^{\otimes k} \right).$$

Then, we get

$$\begin{aligned} \Upsilon_{l-2p,\lambda-l-2q} &= \sum_{j,k=0}^{\infty} \frac{(l+2j)!(\lambda-l+2k)!(-1)^{j+k}}{j!k!p!q!(l-2p)!(\lambda-l-2q)!2^{j+k}2^{p+q}} \\ &\quad \times \left(\tau^{\otimes(j+p)} \otimes^{2(j+p)} \kappa_{l+2j,\lambda-l+2k} \otimes_{2(k+q)} \tau^{\otimes(k+p)} \right) \end{aligned}$$

as desired. □

Example 1 (The 1-order homogeneous operators). For $z, w \in N$, the operator

$$\Xi = a^*(z) + a(w)$$

is a 1-order homogeneous operator. In particular, for $z = w$, the multiplication operator $\Xi = M_{(\cdot,z)}$ is 1-order homogeneous.

Example 2 (The 2-order homogeneous operators). For $\kappa_{0,2}, \kappa_{2,0}, \kappa_{1,1} \in N \otimes N$, the operators

$$\begin{aligned} &\Xi_{2,0}(\kappa_{2,0}) + \Xi_{0,0}(\langle \tau, \kappa_{2,0} \rangle), \\ &\Xi_{0,2}(\kappa_{0,2}) + \Xi_{0,0}(\langle \tau, \kappa_{0,2} \rangle), \\ &\Xi_{1,1}(\kappa_{1,1}) \end{aligned}$$

are 2-order homogeneous. Note that if we take $\kappa_{0,2} = \kappa_{2,0} = \tau_K$ for $K \in \mathcal{L}(N', N)$ such that $\langle \tau, \tau_K \rangle = 0$, then the K -Gross Laplacian $\Delta_G(K) = \Xi_{2,0}(\tau_K)$ and its dual $\Delta_G^*(K) = \Xi_{0,2}(\tau_K)$ are 2-order homogeneous operators. Moreover, for $B \in \mathcal{L}(N', N)$, the conservation operator $N(B) = \Xi_{1,1}(\tau_B)$ is 2-order homogeneous operator.

Remark 2 Let $\lambda \in \mathbb{N}$. Then, using Theorem 3, $\Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ is λ -order homogeneous if and only if

$$\Xi = \Xi_{-\tau} \diamond \Xi_h$$

where Ξ_h is λ -order homogeneous in \mathcal{U}_θ .

3.2 Euler’s Theorem For Homogeneous Operator

$$\Delta_E^Q := \Delta_G^Q + N^Q = \sum_{j=1}^{\infty} M_{(\cdot,e_j)}^{Q+} D_{e_j}^+ + \sum_{j=1}^{\infty} M_{(\cdot,e_j)}^{Q-} D_{e_j}^-$$

where for $z \in N'$,

$$M_{\langle \cdot, z \rangle}^{Q-} = \sigma^{-1}(M_{\langle \cdot, z \rangle} \otimes I)\sigma, \quad M_{\langle \cdot, z \rangle}^{Q+} = \sigma^{-1}(I \otimes M_{\langle \cdot, z \rangle})\sigma,$$

and $M_{\langle \cdot, z \rangle}$ is the multiplication operator by $\langle \cdot, z \rangle$, see [18].

Theorem 6 *Let $\Xi \in \mathcal{U}_\theta$ and $t \in \mathbb{R} \setminus \{0\}$. Then for each $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$*

$$\lim_{s \rightarrow 0} \langle T, \frac{S_{t+s}^Q \Xi - S_t^Q \Xi}{s} \rangle = \frac{1}{t} \langle T, \Delta_E^Q S_t^Q \Xi \rangle$$

Proof By Theorem 4 and (25) we have

$$\begin{aligned} \lim_{s \rightarrow 0} \langle T, \frac{S_{t+s}^Q \Xi - S_t^Q \Xi}{s} \rangle &= \lim_{s \rightarrow 0} \langle \frac{1}{s} \left\{ F(1 + \frac{s}{t}) \diamond \Gamma^Q(1 + \frac{s}{t})(T) - T \right\}, S_t^Q \Xi \rangle \\ &= \lim_{s \rightarrow 0} \langle F(1 + \frac{s}{t}) \diamond \frac{1}{s} (\Gamma^Q(1 + \frac{s}{t})(T) - T) \\ &\quad + \frac{1}{s} (F(1 + \frac{s}{t}) - I) \diamond T, S_t^Q \Xi \rangle \end{aligned}$$

Since, we have

$$\lim_{s \rightarrow 0} \left\{ \frac{(1 + \frac{s}{t})^{l+m} - 1}{s} \right\} = \frac{d}{ds} (1 + \frac{s}{t})^{l+m} \Big|_{s=0} = \frac{1}{t} (l+m),$$

for $T = \sum_{l,m=0}^\infty \Xi_{l,m}(\Phi_{l,m})$, we get

$$\lim_{s \rightarrow 0} \frac{\Gamma^Q(1 + \frac{s}{t})(T) - T}{s} = \sum_{l,m=0}^\infty \frac{1}{t} (l+m) \Xi_{l,m}(\Phi_{l,m}) = \frac{1}{t} N^Q T. \tag{34}$$

On the other hand using (21), we have

$$\begin{aligned} &\lim_{s \rightarrow 0} \sigma \left(\frac{1}{s} \{ F(1 + \frac{s}{t}) - I \} \right) (\xi, \eta) \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \left\{ \exp \left[\left(\frac{(1 + \frac{s}{t})^2 - 1}{2} \right) (\langle \xi, \xi \rangle + \langle \eta, \eta \rangle) \right] - 1 \right\} \\ &= \frac{1}{t} (\langle \xi, \xi \rangle + \langle \eta, \eta \rangle) \\ &= \frac{1}{t} \sigma (\Xi_{0,2}(\tau) + \Xi_{2,0}(\tau)) (\xi, \eta). \end{aligned}$$

Then, from (14) and (15) we get

$$\begin{aligned} \lim_{s \rightarrow 0} \langle T, \frac{S_{t+s}^Q \Xi - S_t^Q \Xi}{s} \rangle &= \langle \frac{1}{t} N^Q T + \frac{1}{t} (\Delta_G^Q)^* T, S_t^Q \Xi \rangle \\ &= \frac{1}{t} \langle T, \Delta_E^Q S_t^Q \Xi \rangle \end{aligned}$$

Which gives the desired statement. □

Remark 3 Using Theorem 6, for $\Xi \in \mathcal{U}_\theta$, we have

$$\lim_{s \rightarrow 0} \frac{S_{e^s}^Q \Xi - \Xi}{s} = \lim_{s \rightarrow 0} \left(\frac{e^s - 1}{s} \right) \frac{S_{(e^s-1+1)}^Q \Xi - \Xi}{e^s - 1} = \Delta_E^Q \Xi.$$

This shows that $\{S_{e^t}^Q\}$ is a semigroup on \mathcal{U}_θ with infinitesimal generator Δ_E^Q . Hence, we deduce that $S_{e^t}^Q U_0$ is the unique solution of the Cauchy problem

$$\frac{\partial}{\partial t} U_t = \Delta_E^Q U_t, \quad U_0 \in \mathcal{U}_\theta.$$

Theorem 7 (Euler’s theorem). *Let $\Xi \in \mathcal{U}_\theta$. Then Ξ is λ -order homogeneous if and only if it satisfies the following QWN-Euler equation*

$$\Delta_E^Q \Xi = \lambda \Xi. \tag{35}$$

Proof If Ξ is λ -order homogeneous, then by (26) we have

$$\Delta_E^Q \Xi = \lim_{t \rightarrow 1} \frac{S_t^Q(\Xi) - S_1^Q(\Xi)}{t - 1} = \lim_{t \rightarrow 1} \frac{t^\lambda - 1}{t - 1} \Xi = \lambda \Xi.$$

Conversely, suppose that (35) is satisfied. Let $t \in \mathbb{R} \setminus \{0\}$. Put $G(t) = t^{-\lambda} S_t^Q(\Xi)$. Then, by (35) and Theorem 6 we get

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{G(t+s) - G(t)}{s} &= \lim_{s \rightarrow 0} \frac{1}{s} \left\{ (t+s)^{-\lambda} S_{t+s}^Q(\Xi) - t^{-\lambda} S_t^Q(\Xi) \right\} \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \left\{ (t+s)^{-\lambda} (S_{t+s}^Q(\Xi) - S_t^Q(\Xi)) \right\} \\ &\quad + \lim_{s \rightarrow 0} \frac{1}{s} \left\{ (t+s)^{-\lambda} - t^{-\lambda} \right\} S_t^Q(\Xi) \\ &= t^{-(\lambda+1)} \Delta_E^Q(S_t^Q(\Xi)) - \lambda t^{-(\lambda+1)} S_t^Q(\Xi). \end{aligned} \tag{36}$$

Now, let $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. Then, by Theorem 4 and Theorem 6, we have

$$\begin{aligned} \langle T, S_t^Q(\Delta_E^Q \Xi) \rangle &= \langle F(t) \diamond \Gamma^Q(t)(T), \Delta_E^Q \Xi \rangle \\ &= \lim_{s \rightarrow 0} \langle F(t) \diamond \Gamma^Q(t)(T), \frac{S_{1+s}^Q \Xi - \Xi}{s} \rangle \\ &= \lim_{s \rightarrow 0} \langle T, \frac{S_t^Q S_{1+s}^Q \Xi - S_t^Q \Xi}{s} \rangle \end{aligned}$$

But, by applying the QWN-Scaling transformation on $\Xi^{x,y}$, for $x, y \in N$, we can show that $S_u^Q S_v^Q = S_v^Q S_u^Q$ for all $u, v \in \mathbb{R}$. Then we get

$$\langle T, S_t^Q(\Delta_E^Q \Xi) \rangle = \lim_{s \rightarrow 0} \langle T, \frac{S_{1+s}^Q S_t^Q \Xi - S_t^Q \Xi}{s} \rangle$$

Hence, using Theorem 6, we obtain

$$\langle T, S_t^Q(\Delta_E^Q \Xi) \rangle = \langle T, \Delta_E^Q(S_t^Q \Xi) \rangle$$

from which we deduce that

$$S_t^Q(\Delta_E^Q \Xi) = \Delta_E^Q(S_t^Q \Xi).$$

Therefore, using (35) we get

$$\Delta_E^Q(S_t^Q(\Xi)) = \lambda S_t^Q(\Xi).$$

Thus, from (36) we deduce that $G'(t) = 0$ for all $t \in \mathbb{R} \setminus \{0\}$. In particular, $G(t) = G(1)$, i.e.,

$$t^{-\lambda} S_t^Q(\Xi) = S_1^Q(\Xi) = \Xi.$$

From which we deduce the desired statement. □

Remark 4 Euler’s theorem remains valid in $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ where Δ_E^Q is replaced by $\tilde{\Delta}_E^Q$ acting on $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ as follows

$$\tilde{\Delta}_E^Q(\Xi) = \Xi_{-\tau} \diamond \Delta_E^Q(\Xi_\tau \diamond \Xi).$$

Corollary 1 *Let $\lambda \in \mathbb{N}$ and $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$ such that $\Delta_G^Q(\Xi) = 0$. Then Ξ is λ -order homogeneous if and only if $\Xi = \sum_{l=0}^\lambda \Xi_{l,\lambda-l}(\kappa_{l,\lambda-l})$.*

Proposition 3 *Let $\Xi \in \mathcal{U}_\theta$ be a λ -order homogeneous operator such that $\Delta_G^Q(\Xi) = 0$. Then for each $\xi \in N$, $D_\xi^\pm(\Xi)$ are $(\lambda - 1)$ -order homogeneous and $(D_\xi^\pm)^*(\Xi)$ are $(\lambda + 1)$ -order homogeneous.*

Proof We recall from [6] that, for any $\xi \in N$, the following identities hold true

$$\begin{aligned} D_\xi^+ \Xi_{l,m}(\kappa_{l,m}) &= l \Xi_{l-1,m}(\xi \otimes^1 \kappa_{l,m}) \\ D_\xi^- \Xi_{l,m}(\kappa_{l,m}) &= m \Xi_{l,m-1}(\kappa_{l,m} \otimes_1 \xi) \\ (D_\xi^+)^* \Xi_{l,m}(\kappa_{l,m}) &= \Xi_{l+1,m}(\xi \otimes \kappa_{l,m}) \\ (D_\xi^-)^* \Xi_{l,m}(\kappa_{l,m}) &= \Xi_{l,m+1}(\kappa_{l,m} \otimes \xi). \end{aligned}$$

Then, if $\Xi = \sum_{l=0}^\lambda \Xi_{l,\lambda-l}(\kappa_{l,\lambda-l})$, we have

$$D_\xi^+ \Xi = \sum_{l=0}^{\lambda-1} (l+1) \Xi_{l,\lambda-l-1}(\xi \otimes^1 \kappa_{l+1,\lambda-1-l}). \tag{37}$$

Thus, the fact $\Delta_G^Q(D_\xi^+ \Xi) = D_\xi^+(\Delta_G^Q \Xi) = 0$ and identity (37) proves the statement for $D_\xi^+(\Xi)$ via Corollary 1. The others statements can be verified by slight modification. □

Theorem 8 *Let $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$. Then $\Xi_{l,m}(\kappa_{l,m})$ is $(l + m)$ -order homogeneous if and only if $\Delta_G^Q(\Xi_{l,m}(\kappa_{l,m})) = 0$.*

Proof From (16) we have

$$\Delta_G^Q \Xi_{l,m}(\kappa_{l,m}) = l(l - 1)\Xi_{l-2,m}(\tau \otimes^2 \kappa_{l,m}) + m(m - 1)\Xi_{l,m-2}(\kappa_{l,m} \otimes_2 \tau). \tag{38}$$

Then by iterating (38) we get

$$\begin{aligned} & (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) = \\ & \sum_{p=0}^{[l/2]} \sum_{p+q=k}^{[m/2]} \frac{l!m!k!}{(l - 2p)!(m - 2q)!p!q!} \Xi_{l-2p,m-2q}(\tau^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau^{\otimes q}). \end{aligned} \tag{39}$$

On the other hand from (27), we obtain

$$\begin{aligned} S_t^Q \Xi_{l,m}(\kappa_{l,m}) &= \sum_{p=0}^{[l/2]} \sum_{q=0}^{[m/2]} \frac{l!m!}{(l - 2p)!(m - 2q)!p!q!} t^{l+m-2p-2q} \left(\frac{1}{2}(t^2 - 1)\right)^{p+q} \\ &\times \Xi_{l-2p,m-2q}(\tau^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau^{\otimes q}). \end{aligned} \tag{40}$$

Then in view of (39), (40) becomes

$$\begin{aligned} & S_t^Q \Xi_{l,m}(\kappa_{l,m}) \\ &= \sum_{k=0}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \left(\frac{1}{2}(t^2 - 1)\right)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) \\ &= t^{l+m} \Xi_{l,m}(\kappa_{l,m}) + \sum_{k=1}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \left(\frac{1}{2}(t^2 - 1)\right)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}). \end{aligned}$$

It then follows that $\Xi_{l,m}(\kappa_{l,m})$ is $(l + m)$ -homogeneous if and only if

$$\sum_{k=1}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \left(\frac{1}{2}(t^2 - 1)\right)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) = 0. \tag{41}$$

Hence, using the fact that $\{P_k(X) = X^{l+m-2k}(X^2 - 1)^k; k = 1, 2, \dots, [(l+m)/2]\}$ is a linearly independent family of polynomials, one can show that (41) holds if and only if

$$(\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) = 0, \quad \forall k = 1, 2, \dots, [(l+m)/2].$$

This implies in particular that $\Delta_G^Q \Xi_{l,m}(\kappa_{l,m}) = 0$. The converse is straightforward by Euler’s theorem. □

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