Math Phys Anal Geom (2016) 19: 1
DOI 10.1007/s11040-016-9203-3

On the Equivalence of Euler-Lagrange and Noether Equations

A. C. Faliagas[1]

Received: 21 October 2015 / Accepted: 5 January 2016 / Published online: 21 January 2016

Abstract We prove that, under the condition of nontriviality, the Euler-Lagrange and Noether equations are equivalent for a general class of scalar variational problems. Examples are position independent Lagrangians, Lagrangians of p-Laplacian type, and Lagrangians leading to nonlinear Poisson equations. As applications we prove certain propositions concerning the nonlinear Poisson equation and its generalisations, and the equivalence of admissible and inner variations for the systems under consideration.

Keywords Calculus of variations · Noether · Euler-lagrange · Equivalence · Nonlinear poisson · Conservation laws · Energy-momentum tensor · Stress tensor

Mathematics Subject Classification (2010) 35A15 · 35B99 · 35J20 · 49S05 · 49S99

1 Introduction

Alikakos in [1] presented a method for the derivation of monotonicity formulas and Derrick-Pohozaev identities [5] related to nonlinear variational problems, using energy-momentum tensors. The method was applied to the nonlinear Poisson

✉ A. C. Faliagas
apostol.faliagas@gmail.com

1 Department of Mathematics, University of Athens, Panepistemiopolis, 15784 Athens, Greece

system and extended in a systematic fashion to a larger class of Lagrangians [2, 7]. This method is more general than the previously used, as the starting point is Noether's equations [10], whereas previously used methods start from the Euler-Lagrange equations. Of course, Noether's equations are a weaker hypothesis than the Euler-Lagrange equations for C^2 functions. A simple proof of this fact is given in Counterexample 1 below (see also [9], Chapter 3), which is based on trivial solutions of Noether's equations.

However, for a large class of Lagrangians, in the *nontrivial* classical (C^2) solutions, it turns out that the Euler-Lagrange and Noether's equations are equivalent. The proof of this statement is the target of this paper. In particular, Theorem 1 proves that, for the Lagrangian of the nonlinear Poisson equation, every nontrivial classical solution of Noether's system is necessarily a solution of the Euler-Lagrange equation. Theorem 2 provides an extension of this theorem to more general Lagrangians. In Section 2 there is a short account of inner variations, energy-momentum tensors and other prerequisite material, serving mainly to introduce the notation. More details on these topics are found in [7, 9].

Aside from their mathematical interest, these results are of importance to theoretical physics, for they provide a link between the Lagrangian and Noetherian formulations of physics ([8], 52-3; [3], 11.4, p. 17). This paper concerns the purely technical mathematical-analytical aspects of the subject, leaving the applications to physics for subsequent work.

2 Noether's Equations

In this section we summarize background material, which is necessary for the statement and proof of the main results, and serves as a means for the introduction of the notation used [9].

2.1 General Notation

Throughout this paper Ω is a domain (open, connected subset) of $\mathbb{R}^N$, unless otherwise stated. The following abbreviated notation

$$u_{,i} = \frac{\partial u}{\partial x_i}$$

is used for partial derivatives. Einstein's summation convention applies everywhere, unless otherwise stated.

$C^r(\Omega)$ is the set of r times continuously differentiable functions in Ω and $C^r(\overline{\Omega})$ the set of restrictions to Ω of r times continuously differentiable functions in $\mathbb{R}^N$. The set of r times continuously differentiable, $\mathbb{R}^M$ (or $\mathbb{C}^M$) valued functions in Ω is denoted by $C^r(\Omega)^M$ and the corresponding set of restrictions to Ω of r times continuously differentiable functions in $\mathbb{R}^N$, $C^r(\overline{\Omega})^M$. $\mathcal{D}(\Omega)$ denotes the set of real (or complex) C^∞ functions on Ω with compact support in Ω and $\mathcal{D}(\Omega)^M$ the set of $\mathbb{R}^M$ (or $\mathbb{C}^M$) valued C^∞ functions on Ω with compact support in Ω.

2.2 Variational Functionals

We will be considering (nonlinear) functionals $J : C^1(\overline{\Omega})^M \to \mathbb{R}$, $M \in \mathbb{N}$, of the form

$$J(u) := \int_\Omega L(x, u(x), Du(x))dx, \tag{VF}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $L(x, y, z)$ a Lagrangian

$$L : \Omega \times \mathbb{R}^M \times \mathbb{R}^{N\cdot M} \to \mathbb{R},$$

$L \in C^1(\overline{\Omega} \times \mathbb{R}^M \times \mathbb{R}^{N\cdot M})$ and $u \in C^1(\overline{\Omega})^M$.

A function $u \in \mathrm{C}^1(\Omega)^M$ is a *critical point* of J when

$$\delta J(u)v := \tfrac{d}{dt} J(u + tv)\Big|_{t=0} = 0 \quad \forall v \in \mathcal{D}(\Omega)^N.$$

The derivative $\delta J(u)v$ is the *variation of* J *at* u *in direction* v.

When $u \in C^2(\Omega)^M$, an easy calculation shows

$$\delta J(u)v = \int_\Omega \delta L(u) \cdot v dx$$

where $\delta L(u) = (\delta L(u)_i)_{i=1,\cdots,M}$ is the vector field with components

$$\delta L(u)_i = \left(L_{y_i} - \tfrac{\partial}{\partial x_j} L_{z_{ij}}\right)\Big|_{(x,u(x),Du(x))}. \tag{2.1}$$

We will refer to δL as the *Euler-Lagrange derivative*. Every critical point $u \in C^2(\Omega)^M$ of J satisfies the *Euler-Lagrange equations*

$$\delta L(u) = 0.$$

2.3 Inner Variations

Inner variations are a special kind of variations. Let $I =]-\delta, \delta[$, $\delta > 0$. Fix a $u \in C^1(\Omega)^M$ and a set of diffeomorphisms $(\xi^t)_{t\in \mathrm{I}}$, $\xi^t : \Omega \to \Omega$. The functions

$$\widetilde{u}(x, t) := u(\xi^t(x)), \quad t \in I$$

are an inner variation of u. More precisely, we have the following definition.

Definition 1 A) Let $h \in \mathcal{D}(\Omega)^N$, $\delta > 0$ and $I =]-\delta, \delta[$. A set of diffeomorphisms $(\xi^t)_{t\in I}$ of Ω having the properties (i) - (iii) below and such that the function $\xi : \Omega \times I \to \Omega$, $\xi(x, t) = \xi^t(x)$ is C^∞-differentiable, is called an *inner variation of* Ω *in direction* h or *which is defined by* h:

(i) $\xi^0 = id_\Omega$, i.e. $\xi^0(x) = x$ in Ω.
(ii) $D_t\xi(x, 0) = h(x),\ x \in \Omega$.
(iii) $\xi^t|\partial\Omega = id_{\partial\Omega}$, i.e. $\xi^t(x) = x,\ x \in \partial\Omega$.

B) Let J be a functional satisfying (VF), $u \in C^1(\overline{\Omega})^M$, $h \in \mathcal{D}(\Omega)^N$ and $(\xi_h^t)_{t\in I}$ the inner variation of Ω defined by h. The set of functions

$$u\circ\xi_h^t, \quad t \in I$$

is called the *inner variation of* u in direction h. The derivative

$$\partial J(u)h := \tfrac{d}{dt} J(u\circ\xi_h^t)\big|_{t=0}, \tag{2.2}$$

is called the *inner variation of the functional* J *at* u *in direction* h.

For the calculation of inner variations the following proposition is used.

Proposition 1 *Let J be a functional satisfying (VF) and $u \in C^1(\overline{\Omega})^M$. The inner variation of J at u is given by*

$$\partial J(u)h = \int_\Omega \big(u_{k,i}L_{z_{kj}}h_{i,j} - L\mathrm{div}h - L_{x_i}h_i\big)\,dx, \quad h \in \mathcal{D}(\Omega)^N, \tag{2.3}$$

where the L, $L_{x_i} := \frac{\partial L}{\partial x_i}$, $L_{z_{kj}} := \frac{\partial L}{\partial z_{kj}}$ are taken at the point $(x, u(x), Du(x))$, i.e. $L = L(x, u(x), Du(x))$, $L_{x_i} = L_{x_i}(x, u(x), Du(x))$ etc.

The proof consists in considering the function $\varphi(t) := J(u \circ \xi^t)$, applying the change of integration variable $x = \eta^t(y)$, where η^t is the inverse function of ξ^t, differentiating with respect to t and interchanging differentiation with integration. For the details see [7].

The definition of an inner variation of Ω (Definition 1(A)) is essentially a special case of the more general definition of a variation for manifolds, rectifiable sets and varifolds [11] (16, p. 80).

2.4 Energy-Momentum Tensor

On using the notation

$$\partial L(u)h := u_{k,i}L_{z_{kj}}(\cdot, u, Du)h_{i,j} - L(\cdot, u, Du)h_{i,i} - L_{x_i}(\cdot, u, Du)h_i \tag{2.4}$$

the expression for the inner variation of J can be written compactly as

$$\partial J(u)h = \int_\Omega \partial L(u)h dx, \quad h \in \mathcal{D}(\Omega)^N. \tag{2.5}$$

When $L_x = 0$, formula (2.4) simplifies further to

$$\partial L(u)h := u_{k,i}L_{z_{kj}}h_{i,j} - L\delta_{ij}h_{i,j} = (u_{k,i}L_{z_{kj}} - L\delta_{ij})h_{i,j}.$$

This motivates the following definition of the energy-momentum tensor.

Definition 2 Let J be a functional satisfying (VF). The *energy-momentum tensor* of the variational problem specified by J, is defined by

$$T_{ij}(x, y, z) = z_{ki}L_{z_{kj}}(x, y, z) - \delta_{ij}L(x, y, z) \tag{2.6}$$

where $x = (x_i)_{i=1,\cdots,N} \in \Omega$, $y = (y_k)_{k=1,\cdots,M} \in \mathbb{R}^M$, $z = (z_{ki})_{k=1,\cdots,M;i=1,\cdots,N} \in \mathbb{R}^{NM}$.

Remark 1 The above definition holds for general variables $(x, y, z) \in \Omega \times \mathbb{R}^M \times \mathbb{R}^{NM}$. Given any vector field $u \in C^1(\Omega)^M$ we have the tensor field

$$T_{ij}(x) = T_{ij}(x, u(x), Du(x)) = u_{k,i} L_{z_{kj}}(x, u(x), Du(x)) - \delta_{ij} L(x, u(x), Du(x)), \tag{2.7}$$

for which we use the same symbol. Again, in this formula it is not necessary that u be a solution of the Euler-Lagrange equations.

2.5 Noether's Equations

Let $u \in C^2(\Omega)$, which is not necessarily a solution of the Euler-Lagrange equations and $T_{ij}(x) = T_{ij}(x, u(x), Du(x))$. By the definition of energy-momentum tensor

$$\begin{aligned} T_{ij,j} &= \tfrac{\partial}{\partial x_j}\left(u_{k,i} L_{z_{kj}} - \delta_{ij} L\right) \\ &= u_{k,ij} L_{z_{kj}} + u_{k,i} \tfrac{\partial}{\partial x_j} L_{z_{kj}} - L_{x_i} - L_{y_k} u_{k,i} - L_{z_{kj}} u_{k,ij} \\ &= \left(\tfrac{\partial}{\partial x_j} L_{z_{kj}} - L_{y_k}\right) u_{k,i} - L_{x_i} \end{aligned}$$

where $L = L(x, u(x), Du(x))$, $L_{y_k} = L_{y_k}(x, u(x), Du(x))$ and $L_{z_{kj}} = L_{z_{kj}}(x, u(x), Du(x))$. From this we obtain

$$T_{ij,j} + L_{x_i} = \left(\tfrac{\partial}{\partial x_j} L_{z_{kj}} - L_{y_k}\right) u_{k,i} \tag{2.8}$$

which motivates the following definition.

Definition 3 The system of second order partial differential equations

$$T_{ij,j}(x, u(x), Du(x)) + L_{x_i}(x, u(x), Du(x)) = 0$$

or in index-free notation

$$\operatorname{div} T(x, u(x), Du(x)) + L_x(x, u(x), Du(x)) = 0 \tag{2.9}$$

is called *Noether's equations*.

The *inner critical points* of J, defined by $\partial J(u) = 0$, i.e. $\partial J(u)h = 0\ \forall h \in \mathcal{D}(\Omega)^N$, satisfy Noether's equations. By (2.8) every solution $u \in C^2(\Omega)$ of the Euler-Lagrange equations is a solution of Noether's equations. The converse of this statement is in general not true. Giaquinta and Hildebrandt [9] presented the following simple counterexample to demonstrate this.

Counterexample 1 Let $F \in C^1(\mathbb{R})$, $F \neq$ const. and

$$J(u) := \int_\Omega F(u(x))dx, \quad u \in C^2(\Omega) \cap C^1(\overline{\Omega}).$$

The energy-momentum tensor is calculated by (2.6)

$$T = -F(u)I,$$

and Noether's (2.9) reduce to

$$F'(u)Du = 0.$$

It is obvious that every constant function $u = c_0$ is a solution of this system, but not of the Euler-Lagrange equations, which for this functional assume the form

$$F'(u) = 0.$$

We will show in the next two sections that only trivial counterexamples are possible for a large class of Lagrangians.

3 Nonlinear Poisson Equation

The nonlinear Poisson equation, see (3.3) below, in a bounded domain Ω is the Euler-Lagrange equation of a variational functional J (see (VF) of Section 2.2) with the Lagrangian

$$L(u, z) = \frac{1}{2}|z|^2 + F(u), \tag{3.1}$$

where z corresponds to Du when u is a C^1 function and $F : \mathbb{R} \to \mathbb{R}$. When $F \in C^1(\mathbb{R})$, J clearly conforms to requirements (VF).

Theorem 1 *Let Ω be a bounded domain of $\mathbb{R}^N$, $F \in C^1(\mathbb{R})$, L a Lagrangian of the form (3.1) and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ a non-trivial classical solution of Noether's equations*

$$\operatorname{div} T(u, Du) = 0. \tag{3.2}$$

where T is the energy-momentum tensor corresponding to the Lagrangian L, with components $T_{ij} = u_{,i}u_{,j} - \delta_{ij}L$. Then u is a solution of the Euler-Lagrange equation

$$\Delta u = F'(u) \tag{3.3}$$

in Ω.

Remark 2 By (2.8), (3.2) is equivalently written in the form

$$(\Delta u - F'(u))Du = 0. \tag{3.4}$$

The non-triviality condition $Du \neq 0$ means Du is not identically 0, i.e. there is a $x_0 \in \Omega$ such that $Du(x_0) \neq 0$.

Proof Let u be a solution of (3.2) and set

$$A_0 := \{x \in \Omega : \ Du(x) = 0\}$$

and

$$A_1 := \{x \in \Omega : \ Du(x) \neq 0\}.$$

Obviously A_0 is closed in Ω, A_1 is open and $A_0 \cup A_1 = \Omega$. It is clear that (3.3) is satisfied in A_1. We have to show (3.3) is also satisfied in A_0.

Step 1. Let D be the subset of Ω in which the Euler-Lagrange (3.3) is satisfied, i.e.

$$D := \{x \in \Omega : \ \Delta u(x) = f(u(x))\},$$

where $f := F'$. D is obviously closed in Ω and we have already shown that

$$A_1 \subset D.$$

From this, keeping in mind that closures and boundaries are taken in Ω, it follows immediately that $\overline{A}_1 \subset D$, hence also

$$\partial A_1 \subset D.$$

We will show that

$$\partial A_0 \subset D. \tag{3.5}$$

The equality of the boundaries, $\partial A_0 = \partial A_1$, follows from the fact that each set is the complement of the other. From this (3.5) follows immediately.

If $\overset{\circ}{A}_0 = \emptyset$ we are finished, for $A_0 = \partial A_0 \subset D$. Let $\overset{\circ}{A}_0 \neq \emptyset$. By hypothesis, for all $x \in \overset{\circ}{A}_0$ we have $Du(x) = 0$, hence also $D^2u(x) = 0$ on $\overset{\circ}{A}_0$ and $u(x) = \text{const.}$ on connected components of A_0.

Step 2. Fix $x_0 \in \overset{\circ}{A}_0$. We will show that there is a $x_1 \in \partial A_0$ and a continuous curve $\gamma : \overline{I} \to A_0$, $I =]0, 1[$, such that $\gamma(0) = x_0$, $\gamma(1) = x_1$ and $\gamma([0, 1[) \subset \overset{\circ}{A}_0$, i.e. the curve lies in the interior of A_0, with the exception of x_1. Let $y \in A_1 \neq \emptyset$ by hypothesis and $\alpha : \overline{I} \to \Omega$ a continuous curve connecting $x_0 = \alpha(0)$ and $y = \alpha(1)$. The set

$$\Gamma := \{\alpha(t) : t \in \overline{I},\ \alpha(t) \in \partial A_0\}$$

is not empty ([6], (3.19.9) and following Remark, p. 70). Since $\{t \in \overline{I} : \alpha(t) \in \partial A_0\} = \alpha^{-1}(\partial A_0)$ is closed, $\tau := \inf\{t \in \overline{I} : \alpha(t) \in \partial A_0\} \in \alpha^{-1}(\partial A_0)$, hence $x_1 := \alpha(\tau) \in \partial A_0$ and it is clear that $\alpha([0, \tau[) \subset \overset{\circ}{A}_0$. For if there were a $\tau_1 < \tau$ such that $y' = \alpha(\tau_1) \notin \overset{\circ}{A}_0$, then $y' \notin A_0$ and application of the same procedure for $x_0, y' \in A_1$ would yield the existence of a $x_1' = \alpha(\tau') \in \partial A_0$ with $\tau' < \tau$, which contradicts the definition of τ. Reparametrisation of $\alpha|[0, \tau]$ yields γ.

Step 3. Now let $u(x_0) =: c_0$. Since x_0 and x_1 belong to the same connected component of A_0, we have $u(x_1) = c_0$ and $f(u(x_0)) = f(u(x_1)) = f(c_0) =: d_0$. Since by (3.5) $x_1 \in D$, we have

$$\Delta u(x_1) = f(u(x_1)) = d_0. \tag{3.6}$$

But

$$\Delta u(x_1) = \Delta u(\lim_{t \to 1-} \gamma(t)) = \lim_{t \to 1-} \Delta u(\gamma(t)) = 0 \tag{3.7}$$

since $\gamma(t) \in \overset{\circ}{A}_0$ for all $t \in [0, 1[$. Combination of (3.6) and (3.7) yields $d_0 = 0$, hence

$$f(u(x_0)) = 0. \tag{3.8}$$

This means in particular

$$\Delta u(x_0) - f(u(x_0)) = 0.$$

With this we have proved $\overset{\circ}{A}_0 \subset D$, and by (3.5) $A_0 \subset D$. □

From the proof of this theorem we conclude without difficulty the following corollary.

Corollary 1 *Under the assumptions of Theorem 1, if the set* $A_0 := \{x \in \Omega : Du(x) = 0\}$ *has an interior point, then* $f(u) = 0$ *on all of* $\overset{\circ}{A}_0$.

Proof By hypothesis $A_1 \neq \emptyset$. Let $x_0 \in \overset{\circ}{A}_0$. Now Steps 2 and 3 of the proof of Theorem 1 apply and from (3.8) it follows that $f(u(x_0)) = 0$. Since x_0 was arbitrary, the assertion is proved. □

As an application, we state the following result for the nonlinear Poisson equation.

Corollary 2 *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *and* $f \in C(\mathbb{R})$ *such that* $f(t) \neq 0$ *for all* $t \in \mathbb{R}$. *Then for every solution* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *of the nonlinear Poisson equation*

$$\Delta u = f(u) \tag{3.9}$$

the set $A_0 := \{x \in \Omega : \ Du(x) = 0\}$ *has no interior point.*

Proof Equation (3.9) is the Euler-Lagrange equation of the functional (3.1) with $F(t) = \int_0^t f(s)ds$, for which we have $F \in C^1(\mathbb{R})$. If $\overset{\circ}{A}_0 \neq \emptyset$, by Corollary 1 we would have $f(u) = 0$ on $\overset{\circ}{A}_0$, which is absurd. □

Corollary 3 *Let* Ω *be a bounded domain of* $\mathbb{R}^N$, $F \in C^1(\mathbb{R})$, $g \in C(\partial\Omega)$ *a non-constant function and* $\mathcal{H} := \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}) : u|\partial\Omega = g\}$. *Then the Euler-Lagrange and Noether equations for the Lagrangian (3.1) are equivalent in* $\mathcal{H}$.

Proof It follows immediately by Theorem 1, since every $u \in \mathcal{H}$ is non-trivial. □

Remark 3 We have restricted the above discussion to bounded domains, only for the sake of convenience. Indeed, this hypothesis serves to maintain integrability in (VF), and guarantees the validity of the interchange of differentiation and integration. The same purpose also serves the hypothesis $u \in C^1(\overline{\Omega})$, with the exception of Corollary 3. Thus, the above results, with proper modifications, are applicable to unbounded domains as well.

4 More General Lagrangians

We proceed to generalizing the results of the previous section by considering Lagrangians of the form $L(x, u, z)$, which, along with (VF), satisfy the condition (H) below. Again, u is a scalar function and argument z corresponds to ∇u.

$$\begin{aligned} L_{x_i z_i}(x, u, 0) &= 0 \text{ for all } x, u. \\ L_{x_i u}(x, u, 0) &= 0 \text{ forall } x, u \text{ andall } i = 1, \cdots, N. \end{aligned} \tag{H}$$

Example 1 (i) All Lagrangians which are independent of x satisfy (H). In particular the Lagrangians of classes I and II in [2] satisfy (H).
(ii) Lagrangians of the form

$$L(x, u, z) = \tfrac{1}{2}\varphi(x, u)|z|^2 + F(u),$$

where $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$, satisfy (H).

Recall the definition of the Euler-Lagrange derivative, formula (2.1).

Theorem 2 *Let Ω be a bounded domain of $\mathbb{R}^N$, $L \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ a Lagrangian satisfying (H) and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ a non-trivial classical solution of Noether's equations*

$$\operatorname{div} T(x, u, Du) + L_x(x, u, Du) = 0. \tag{4.1}$$

Then u is a solution of the Euler-Lagrange equation

$$\tfrac{\partial}{\partial x_j} L_{u,j} - L_u = 0 \tag{4.2}$$

in Ω.

Remark 4 By (2.8), (4.1) is equivalently written in the form

$$\left(\tfrac{\partial}{\partial x_j} L_{u,j} - L_u\right) u_{,i} = 0$$

or in index-free notation

$$\delta L(u) \cdot Du = 0. \tag{4.3}$$

Notice that (4.1) is a second order system of partial differential equations in u and (4.2) is a single second order partial differential equation in u.

Proof The proof begins exactly as the proof of Theorem 1 up to Step 3 where the proof of existence of the curve γ is complete, with the obvious modification

$$D := \{x \in \Omega : \ \delta L(u)(x) = 0\}.$$

For fixed $x_0 \in \overset{\circ}{A}_0$ let $u(x_0) =: c_0$. Further let $f := L_u$. Since x_0 and x_1 belong to the same connected component of A_0, we have $u(x_1) = c_0$ and by (H)

$$f(x_0, u(x_0), 0) = f(x_1, u(x_0), 0) = f(x_0, c_0, 0) =: d_0. \tag{4.4}$$

We have

$$\begin{aligned} \tfrac{\partial}{\partial x_i} L_{z_i}(x, u, Du)\Big|_{x_0} &= L_{x_i z_i}(x_0, c_0, 0) + L_{u z_i}(x_0, c_0, 0) u_{,i}(x_0) + \\ &\quad L_{z_i z_j}(x_0, c_0, 0) u_{,ij}(x_0) \\ &= L_{x_i z_i}(x_0, c_0, 0) = 0 \end{aligned} \tag{4.5}$$

by (H). Since by (3.5) $x_1 \in D$, we have in a similar fashion

$$\begin{aligned} \tfrac{\partial}{\partial x_i} L_{z_i}(x, u, Du)\Big|_{x_1} &= L_{x_i z_i}(x_1, c_0, 0) + L_{u z_i}(x_1, c_0, 0) u_{,i}(x_1) + \\ &\quad L_{z_i z_j}(x_1, c_0, 0) u_{,ij}(x_1) \\ &= L_{x_i z_i}(x_1, c_0, 0) = f(x_1, c_0, 0) = d_0 \end{aligned} \tag{4.6}$$

where the second equality from the end follows from the Euler-Lagrange equation:

$$\frac{\partial}{\partial x_i} L_{z_i}(x, u, Du)\Big|_{x_1} = L_u(x_1, u(x_1), Du(x_1)) = f(x_1, c_0, 0).$$

Note that we have omitted a step involving a continuity argument along γ as $t \to 1-$. From the first of (H) and $\gamma(t) \in \overset{\circ}{A}_0$ for all $t \in [0, 1[$ we obtain

$$\begin{aligned} L_{x_i z_i}(x_1, c_0, 0) &= L_{x_i z_i}(\lim_{t\to 1-} \gamma(t), u(\lim_{t\to 1-} \gamma(t)), Du(\lim_{t\to 1-} \gamma(t))) \\ &= \lim_{t\to 1-} L_{x_i z_i}(\gamma(t), u(\gamma(t)), Du(\gamma(t))) \\ &= \lim_{t\to 1-} L_{x_i z_i}(\gamma(t), c_0, 0) = 0. \end{aligned}$$

Combination of this equation with (4.6) yields $d_0 = 0$, hence by (4.4)

$$f(x_0, c_0, 0) = 0.$$

This means in particular

$$\frac{\partial}{\partial x_i} L_{z_i}(x, u, Du)\Big|_{x_0} - f(x_0, u(x_0), Du(x_0)) = L_{x_i z_i}(x_0, c_0, 0) - f(x_0, c_0, 0) = 0.$$

With this we have proved $\overset{\circ}{A}_0 \subset D$ and by (3.5) $A_0 \subset D$. □

Corollaries 1 and 2 transfer to the general Lagrangians conforming to (H), with the obvious modifications:

Corollary 4 *Under the assumptions of Theorem 1, if the set* $A_0 := \{x \in \Omega : Du(x) = 0\}$ *has an interior point, then* $L_u = 0$ *on all of* $\overset{\circ}{A}_0$.

Corollary 5 *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *and* $L_u(x, u, z) \neq 0$ *in* $\Omega \times \mathbb{R} \times \mathbb{R}^N$. *Then for every solution* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *of the partial differential equation in* u

$$\frac{\partial}{\partial x_j} L_{u,j} - L_u = 0$$

the set $A_0 := \{x \in \Omega : \ Du(x) = 0\}$ *has no interior point.*

Remark 3 and Corollary 3 hold as they are.

Example 2 We consider the Lagrangian of p-Laplacian type [4]

$$L(u, z) = \frac{1}{2}\varphi(|z|^2) + F(u)$$

where $F \in \mathrm{C}(\mathbb{R})$ and $\varphi \in C^2(\mathbb{R}^+)$ such that $\varphi(0) = 0$ and $\varphi'(s) \geqslant 0 \ \forall s \geqslant 0$, which satisfies condition (H). The energy-momentum tensor for this Lagrangian is given by

$$T_{ij} = \varphi'(|Du|^2) u_{,i} u_{,j} - \delta_{ij} \left(\frac{1}{2}\varphi(|Du|^2) + F(u) \right).$$

By Corollary 3 the PDE

$$\operatorname{div}(\varphi'(|Du|^2)Du \otimes Du) - D\left(\frac{1}{2}\varphi(|Du|^2) + F(u)\right) = 0$$

and

$$\frac{\partial}{\partial x_i}\left(\varphi'(|Du|^2)\frac{\partial u}{\partial x_i}\right) = F'(u)$$

with the boundary condition $u|\partial\Omega = g$, where $g \in C(\partial\Omega)$ is a non-constant function, are equivalent.

5 Application: Equivalence of Admissible and Inner Variations

The C^2 solutions of Euler-Lagrange (respectively Noether) equations are critical (respectively inner critical) points of the corresponding variational functional. Since each inner variation h gives rise to an admissible variation [7]

$$w = Du \cdot h,$$

the question arises if these two types of variation are equivalent. Notice that the set of inner variations is smaller than the set of admissible variations. The next theorem gives an answer to this question.

Theorem 3 *Let J be a variational functional with Lagrangian $L \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ satisfying (H) and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ a nontrivial function. Then the variational problem*

$$\delta J(u)v = 0 \quad \forall v \in \mathcal{D}(\Omega) \tag{5.1}$$

is equivalent to the problem

$$\delta J(u)w = 0 \quad \forall w \in \mathcal{I}(\Omega) \tag{5.2}$$

where $\mathcal{I}(\Omega) := \{Du \cdot h \,:\, h \text{ is an inner variation of } u\}$. In simple words one can consider only inner variations of u in Problem (5.1).

Proof Equation (5.2) follows immediately from (5.1) when u is C^∞. Otherwise $w \in C^1_c(\Omega)$ while (5.1) requires $v \in \mathcal{D}(\Omega)$. In this case the proof follows by a simple density argument. For the converse, if (5.2) is valid, then $\delta J(u)Du \cdot h = 0$ for all $h \in \mathcal{D}(\Omega)^N$. By the fundamental lemma of calculus of variations $\delta J(u)Du = 0$, hence u is a solution of Noether's equations and by Theorem 2 also a solution of the Euler-Lagrange equation. Since u is C^2, this means u is a critical point of J, i.e. (5.1) is satisfied. □

References

1. Alikakos, N.: Some basic facts on the system $\Delta u - W_u(u) = 0$. Proc. Amer. Math. Soc. **139**, 153–162 (2011)
2. Alikakos, N., Faliagas, A.: The stress-energy tensor and Pohozaev's identity for systems. Acta Math. Sci. **32**, 433–439 (2012)

3. Bjorken, J.D., Drell, S.D.: Relativistic quantum fields. McGraw-Hill (1964)
4. Caffarelli, L., Garofalo, N., Segala, F.: A gradient bound for entire solutions of quasi-linear equations and its consequences. Comm. Pure Appl. Math. **47**, 1457–1473 (1994)
5. Derrick, G.H.: Comments on nonlinear wave equations as models for elementary particles. J. Math. Phys. **5**, 1252 (1964)
6. Dieudonné, J.-A.: Treatise on analysis. Volume I. Foundations of modern analysis. Enlarged and corrected printing. Academic (1969)
7. Faliagas, A.: Construction of energy-momentum tensors in calculus of variations and applications to partial differential equations. Master's thesis. University of Athens (2012). http://dl.dropbox.com/u/56524894/masters-thesis.pdf
8. Feynman, R.P., Leighton, R.B., Sands, M.: The Feynman lectures on physics Vol. 1. Mainly mechanics, radiation, and heat. Addison-Wesley (1963)
9. Giaquinta, M., Hildebrandt, S.: Calculus of variations I. The lagrangian formalism. Grundlehren der mathematischen Wissenschaften. Springer (2010)
10. Noether, E.: Invariant variation problems. English translation (2005). ArXiv:0503066v1
11. Simon, L.: Lectures on geometric measure theory. Australian National University, Centre for Mathematical Analysis, Canberra (1983)