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Existence, Non-existence, and Uniqueness for a Heat Equation with Exponential Nonlinearity in $\mathbb{R}^2$

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Abstract We consider a semilinear heat equation with exponential nonlinearity in $\mathbb{R}^2$. We prove that local solutions do not exist for certain data in the Orlicz space $\exp L^2(\mathbb{R}^2)$, even though a small data global existence result holds in the same space $\exp L^2(\mathbb{R}^2)$. Moreover, some suitable subclass of $\exp L^2(\mathbb{R}^2)$ for local existence and uniqueness is proposed.

Keywords Heat equation · Existence · Non-existence · Uniqueness · Critical nonlinearity

Mathematics Subject Classification (2010) Primary 35K05 · 35A01 · 35A02 · Secondary 46E30

1 Introduction and Main Results

In this paper we consider the Cauchy problem for a semilinear heat equation

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } (0,\infty)\times\mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}, \tag{1.1}$$

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where $u(t, x) : (0, \infty) \times \mathbb{R}^N \to \mathbb{R}$ is the unknown function, f is the nonlinearity and u_0 is the given initial data.

It is well known that if the initial data $u_0 \in L^\infty(\mathbb{R}^N)$ and the nonlinear term $f \in C^1(\mathbb{R})$ with $f(0) = 0$ then there exists $T(u_0) > 0$ and a unique solution $u \in L^\infty\left(0, T; L^\infty(\mathbb{R}^N)\right)$ of (1.1) (cf. [12]). On the other hand, the first result with a singular initial data is due to Weissler [24, 25]. He considered the Cauchy problem with power nonlinearities

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1}u \text{ in } (0, \infty) \times \mathbb{R}^N \\ u(0, x) \quad = u_0(x) \quad \text{ in } \mathbb{R}^N \end{cases}, \tag{1.2}$$

with initial data in the Lebesgue spaces $u_0 \in L^q(\mathbb{R}^N)$, where $p > 1$ and $1 < q < \infty$.

For such power type nonlinearities the scale invariance property plays an essential role. That is, if the function $u(t, x)$ satisfies (1.2), then for any $\lambda > 0$, the scaled function

$$u_\lambda = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$$

also satisfies (1.2). Moreover, the L^q norm is invariant under this scaling if and only if

$$q = q_c := \frac{N(p-1)}{2}.$$

With regard to this critical exponent, one can classify the existence and uniqueness results of the (1.2) into the following two cases:

Case 1. If $q \geq q_c$ and $q > 1$ or $q > q_c$ and $q \geq 1$, Weissler [24] and Brezis-Cazenave [4] proved that for any $u_0 \in L^q(\mathbb{R}^N)$ there exists a positive time $T = T(u_0)$ and a unique function $u \in C\left([0, T]; L^q(\mathbb{R}^N)\right) \cap L^\infty_{loc}(0, T; L^\infty(\mathbb{R}^N))$ which is a solution of (1.2).

Case 2. If $q < q_c$, Weissler [24] and Brezis-Cazenave [4] indicated that there exists no local solution in any suitable weak sense. Moreover, it is proved by Haraux-Weissler [7] that uniqueness does not hold for $u_0 = 0$ if $1 + \frac{2}{N} < p < \frac{N+2}{N-2}$. Tayachi [21] generalized this result to parabolic equations with a nonlinear gradient term.

When $q \geq p$ it is meaningful to consider weak solution $u \in C([0, T], L^q(\mathbb{R}^N))$ in the integral sense

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}|u(s)|^{p-1}u(s)ds, \tag{1.3}$$

where $e^{t\Delta}$ is the standard heat evolution operator (see [24, 25]) and to study the problem of uniqueness in the larger class of functions belonging to $C\left([0, T]; L^q(\mathbb{R}^N)\right)$ (unconditional uniqueness). Brezis-Cazenave [4] proved unconditional uniqueness for $q > q_c$, $q \geq p$ and for $q = q_c$, $q > p$. In the double critical case $q = q_c = p$, namely $p = q = \frac{N}{N-2}$, Ni-Sacks [14] proved that unconditional uniqueness does not hold on the unit ball in $\mathbb{R}^N$. In [22] Terraneo extended this non-uniqueness result to the whole space $\mathbb{R}^N$ for suitable initial data.

Finally, in the critical case $q = q_c$ and $N \geq 3$ Weissler [25] proved a global existence result under a smallness assumption on the initial data. Some generalization to a semilinear parabolic equation with a nonlinear gradient term was done by Snoussi-Tayachi-Weissler [20].

If we consider the doubly critical case ($p = q = q_c = \frac{N}{N-2}$) in 2 dimension, one can observe that the critical index q_c becomes infinite. This implies that any power nonlinearity $1 < p < \infty$ behaves like Case 1. Namely, for every power nonlinearity one can choose a Lebesgue space L^q where the well-posedness for the Cauchy problem (1.3) can be proved. However, if we consider some nonlinear term with higher growth than any power type nonlinearity, e.g. an exponential nonlinearity, Weissler's result only holds on L^∞ (at least in the category of Lebesgue spaces). Therefore one can wonder if there is any critical space and nonlinearity for 2-D problems.

From this standpoint, Ibrahim-Jrad-Majdoub-Saanouni [9] considered the following 2-D case, which is critical in regard to the energy method and the critical Trudinger embedding [23]:

$$\begin{cases} \partial_t u - \Delta u &= \pm u(e^{u^2} - 1) \ \text{ in } \ (0, \infty) \times \mathbb{R}^2, \\ u(0, x) &= u_0 \qquad\qquad\quad \text{ in } \ \mathbb{R}^2, \end{cases} \tag{1.4}$$

where $u_0 \in H^1(\mathbb{R}^2)$. They proved a local existence result and uniqueness. Moreover, a global existence for the defocusing case and a blow up result for the focusing case were discussed. It is worth to comment that exponential nonlinearities in the framework of Sobolev spaces for nonlinear dispersive partial differential equations were considered in several works in the literature, see for example [5, 8, 10, 13] and references therein.

On the other hand, it is expected that the problem (1.4) for the heat equation can be solved in spaces which are defined by an integrability of functions such as Lebesgue spaces. In this respect, the Orlicz space $\exp L^2$ was considered in [11, 19] as an extension of the class of Sobolev spaces. The Orlicz space $\exp L^2$ is defined as the set of all functions u which belong to $L^1_{loc}(\mathbb{R}^2)$ and which satisfy

$$\int_{\mathbb{R}^2} \left(\exp\left(\frac{|u(x)|}{\lambda} \right)^2 - 1 \right) dx < \infty$$

for some $\lambda > 0$, and the norm is given by the Luxemburg type

$$\|u\|_{\exp L^2(\mathbb{R}^2)} := \inf \left\{ \lambda > 0; \int_{\mathbb{R}^2} \left(\exp\left(\frac{|u(x)|}{\lambda} \right)^2 - 1 \right) dx \le 1 \right\}. \tag{1.5}$$

We recall that Trudinger's inequality shows that $H^1(\mathbb{R}^2) \subset \exp L^2(\mathbb{R}^2)$ (see [1, 15, 18, 23].) Therefore, one of the virtues of the use of Orlicz spaces is that it is possible to consider the critical problem in 2-D (1.4) in a larger space than $H^1(\mathbb{R}^2)$. Indeed, in [11, 19] global existence is proved for (1.4) under a smallness assumption on u_0 in $\exp L^2(\mathbb{R}^2)$.

Let $t > 0$ and $e^{t\Delta}$ be the heat evolution operator given by

$$e^{t\Delta} u_0(x) := \int_{\mathbb{R}^2} \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy.$$

We remark that the evolution operator $e^{t\Delta} : \exp L^2(\mathbb{R}^2) \to \exp L^2(\mathbb{R}^2)$ is bounded, however $e^{t\Delta}$ is not continuous at $t = 0$ in $\exp L^2(\mathbb{R}^2)$ (see [11]).

We stress that for a function u belonging to $\exp L^2(\mathbb{R}^2)$ the nonlinear term $f(u) = u(\mathrm{e}^{u^2} - 1)$ might not be defined in the distributional sense (for example for a function u as in (2.4) with large norm). Nevertheless if we consider any initial data with small norm in $\exp L^2(\mathbb{R}^2)$ and we look for solutions with small norm, namely $u \in L^\infty(0,T; \exp L^2(\mathbb{R}^2))$ with $\sup_{t\in(0,T)} \|u(t)\|_{\exp L^2} < \varepsilon$ for a well-chosen $\varepsilon > 0$, then

$$f(u) = u(\mathrm{e}^{u^2} - 1) \in L^\infty(0,T; L^1(\mathbb{R}^2)).$$

Therefore, in the framework of solutions of small norm in $\exp L^2(\mathbb{R}^2)$ it is meaningful to consider the solutions in the sense of distribution.

Definition 1.1 (Weak solution) Let $u_0 \in \exp L^2(\mathbb{R}^2)$ and $T \in (0,\infty)$ we say that the function $u : [0,T) \to \exp L^2(\mathbb{R}^2)$ such that $u \in C\big((0,T); \exp L^2(\mathbb{R}^2)\big)$ and $u(t) \to u_0$ in weak*topology is a weak solution of the differential (1.1) if u verifies the (1.1) in the sense of distribution with $u(0) = u_0$.

We recall that $u(t) \to u_0$ in weak* sense if and only if

$$\lim_{t\to 0} \int_{\mathbb{R}^2} \big(u(t,x)\phi(x) - u_0(x)\phi(x)\big)dx = 0 \quad \text{for every } \phi \in L^1(\log L)^{\frac{1}{2}}(\mathbb{R}^2), \tag{1.6}$$

where $L^1(\log L)^{\frac{1}{2}}(\mathbb{R}^2)$ is a predual space of $\exp L^2(\mathbb{R}^2)$.

The space $L^1(\log L)^{\frac{1}{2}}(\mathbb{R}^2)$ is defined by

$$L^1(\log L)^{\frac{1}{2}}(\mathbb{R}^2) = \left\{ f \in L^1_{loc}(\mathbb{R}^2) : \int_{\mathbb{R}^2} |f(x)| \log^{1/2}(2 + |f(x)|)dx < \infty \right\}.$$

(See [3, 17]).

Moreover in the framework of solutions of small norm in $\exp L^2(\mathbb{R}^2)$ one can consider the integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f\big(u(s)\big)ds \tag{1.7}$$

for any $t \in (0,T)$ and one can prove that any function $u \in L^\infty(0,T; \exp L^2(\mathbb{R}^2))$ which verifies the integral equation for any $t \in (0,T)$ with $u(t) \to u_0$ in weak*topology for $t \to 0$ belongs to $C((0,T); \exp L^2(\mathbb{R}^2))$. Therefore u is a weak solution of the differential equation (1.1) with $u(0) = u_0$ (see [11, 19]).

By using the integral formulation (1.7) one has

Proposition 1.1 *([11, 19]) There exists $\varepsilon > 0$ such that for every $u_0 \in \exp L^2(\mathbb{R}^2)$ with $\|u_0\|_{\exp L^2} \le \varepsilon$, there exists a solution $u \in L^\infty(0,\infty; \exp L^2(\mathbb{R}^2))$ of the Cauchy problem (1.7)-(1.6) with $f(u) = \pm u(e^{u^2} - 1)$.*

This small data global existence result suggests that the critical space for problem (1.4) is $\exp L^2(\mathbb{R}^2)$. Therefore, a local existence in $\exp L^2(\mathbb{R}^2)$ without any smallness assumption might be expected.

However, in this paper we obtain a negative answer for this question. Moreover, we will propose some suitable subclass of $\exp L^2(\mathbb{R}^2)$ for local existence and uniqueness.

Definition 1.2 ($\exp L^2$-classical solution (see [16])) Given $u_0 \in \exp L^2(\mathbb{R}^2)$ and $T \in (0, \infty)$ we say that the function $u : [0, T) \to \exp L^2(\mathbb{R}^2)$ such that $u \in C((0, T); \exp L^2(\mathbb{R}^2)) \cap L^\infty_{loc}(0, T; L^\infty(\mathbb{R}^2))$ and $u(t) \to u_0$ in weak*topology is a $\exp L^2-$classical solution of (1.1) in $[0, T)$ if u is C^1 in $t \in (0, T)$ and C^2 in $x \in \mathbb{R}^2$, and u is a classical solution of the (1.1) for $t \in (0, T)$.

We obtain the following non-existence result:

Theorem 1.1 *Assume that the nonlinear term f is continuous, $f(x) \geq 0$ if $x \geq 0$, and*

$$\liminf_{\eta\to\infty} \left(f(\eta)e^{-\lambda\eta^2} \right) > 0 \tag{1.8}$$

for some $\lambda > 0$.

Then, there exist some initial data $u_0 \in \exp L^2(\mathbb{R}^2)$, $u_0 \geq 0$ such that for every $T > 0$ the Cauchy problem (1.1) has no nonnegative $\exp L^2$-classical solution in $[0, T)$.

Theorem 1.1 says that local solutions do not exist for certain data in $\exp L^2(\mathbb{R}^2)$ even though a small data global existence result holds in the same space $\exp L^2(\mathbb{R}^2)$.

This type of non-existence result of nonnegative solution for power type nonlinearities was proved by Brezis-Cazenave [4]. Their method is based on an approximation of the Dirac delta function by L^1 functions. It seems difficult to apply their methods to our problem since such an approximation does not hold in $\exp L^2(\mathbb{R}^2)$. We give a different and direct proof in Section 3.

We remark that the set of smooth and compactly supported functions $C_0^\infty(\mathbb{R}^2)$ is *not dense* in $\exp L^2(\mathbb{R}^2)$ (see Section 2). In order to consider the existence of local solutions and uniqueness, we introduce the closure of $C_0^\infty(\mathbb{R}^2)$ in $\exp L^2(\mathbb{R}^2)$;

$$\exp L_0^2(\mathbb{R}^2) := \left\{ u \in \exp L^2(\mathbb{R}^2) : \begin{array}{l} \text{there exists } \{u_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^2) \\ \text{such that } \lim_{n\to\infty} \|u_n - u\|_{\exp L^2} = 0 \end{array} \right\}.$$

Moreover, it is known that

$$H^1(\mathbb{R}^2) \subsetneq \exp L_0^2(\mathbb{R}^2) \subsetneq \exp L^2(\mathbb{R}^2).$$

We summarize some properties of $\exp L^2(\mathbb{R}^2)$ and $\exp L_0^2(\mathbb{R}^2)$ in the next section (see also [2, 17]).

Taking the initial data in the class $\exp L_0^2(\mathbb{R}^2)$, we will prove local existence and uniqueness. We assume the nonlinear continuous term f satisfies the following: there are constants $C, \lambda > 0$ such that

$$|f(x) - f(y)| \leq C|x - y| \left(e^{\lambda x^2} + e^{\lambda y^2} \right) \tag{1.9}$$

for every $x, y \in \mathbb{R}$, and $f(0) = 0$. Remark that no assumption on a behavior of f near 0 is needed since we consider time local existence. Typical examples satisfying (1.9) are

$$f(u) = \pm u e^{u^2},\ \pm(e^u - 1),\ \pm|u|^{p-1}u \text{ for every } 1 \leq p < \infty, \text{ and their combination.}$$

Also in this framework we can introduce the notion of weak solution as in Definition 1.1. Since $C_0^\infty(\mathbb{R}^2)$ is dense in $\exp L_0^2(\mathbb{R}^2)$, $e^{t\Delta}$ becomes a strongly continuous semigroup in $\exp L_0^2(\mathbb{R}^2)$. Therefore in the definition of the weak solution the convergence of the solution $u(t)$ to the initial data u_0 is a strong convergence in $\exp L^2$. Moreover one can prove that the Cauchy problem (1.1) admits the equivalent integral formulation (1.7) in the standard sense (see Proposition 4.1).

Theorem 1.2 *Suppose that f satisfies* (1.9). *Given any $u_0 \in \exp L_0^2(\mathbb{R}^2)$, there exist a time $T = T(u_0) > 0$ and a unique weak solution $u \in C([0, T]; \exp L_0^2(\mathbb{R}^2))$ of* (1.7).

We comment in Remark 4.1 that the solution u in Theorem 1.2 belongs to $L^\infty_{\mathrm{loc}}(0, T; L^\infty(\mathbb{R}^2))$. This implies that u is a classical solution, i.e. of class C^1 in $t \in (0, T)$ and of class C^2 in $x \in \mathbb{R}^2$, provided that in addition $f \in C^1(\mathbb{R})$.

Remark 1.1 In view of Theorem 1.1 and Theorem 1.2, it might be possible to classify the existence and uniqueness results for a exponential nonlinearity, similar to power type nonlinearities. Theorem 1.2 suggests that $\exp L_0^2(\mathbb{R}^2)$ is a subcritical space corresponding to Case 1 in the classification of power type nonlinearities. It follows from Theorem 1.1 that the Orlicz space $\exp L^2(\mathbb{R}^2)$ with large data is supercritical as in Case 2, even though the global existence result in [11, 19] suggests that $\exp L^2(\mathbb{R}^2)$ with small data is a critical space.

The paper is organized as follows. In Section 2 we recall some properties of the Orlicz spaces $\exp L^2(\mathbb{R}^2)$ and $\exp L_0^2(\mathbb{R}^2)$, and some basic properties of the heat semigroup $e^{t\Delta}$ in Orlicz spaces. In Section 3 we prove the non-existence result in Theorem 1.1 by using some explicit initial data which are typical examples of functions in $\exp L^2(\mathbb{R}^2) \setminus \exp L_0^2(\mathbb{R}^2)$. In Section 4 we prove a local existence result for $u_0 \in \exp L_0^2(\mathbb{R}^2)$ by decomposing the equation into two equations, one for smooth initial data, and the other one for small initial data in $\exp L^2(\mathbb{R}^2)$. The method can be found in [10]. In Section 5 we prove a uniqueness result by direct method and we do not need any additional smoothness assumption on f.

2 Preliminaries

In this section we give the definition and collect some properties of the spaces $\exp L^2(\mathbb{R}^2)$ and $\exp L_0^2(\mathbb{R}^2)$. More detailed properties of Orlicz spaces can be found in [17] or Section 8 of [2].

We denote the Orlicz space $\exp L^2(\mathbb{R}^2)$ by the set

$$\exp L^2(\mathbb{R}^2) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \int_{\mathbb{R}^2} \left(e^{\alpha |u(x)|^2} - 1 \right) dx < \infty \text{ for some } \alpha > 0 \right\},$$

where the norm is given by the Luxemburg type (1.5). By definition, it holds

$$\int_{\mathbb{R}^2} \left(\exp \left(\frac{u(x)}{\lambda} \right)^2 - 1 \right) dx \le 1 \quad \text{if } \lambda \ge \|u\|_{\exp L^2}. \tag{2.1}$$

We recall that $\exp L^2_0(\mathbb{R}^2)$ is the closure of $C_0^\infty(\mathbb{R}^2)$ by $\|\cdot\|_{\exp L^2}$, i.e.,

$$\exp L^2_0(\mathbb{R}^2) := \left\{ u \in \exp L^2(\mathbb{R}^2) : \begin{array}{l} \text{there exists } \{u_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^2) \\ \text{such that } \lim_{n\to\infty} \|u_n - u\|_{\exp L^2} = 0 \end{array} \right\}.$$

It is known that

$$\exp L^2_0(\mathbb{R}^2) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \int_{\mathbb{R}^2} \left(e^{\alpha |u(x)|^2} - 1 \right) dx < \infty \ \text{ for every } \alpha > 0 \right\}. \tag{2.2}$$

Here we give a brief proof of (2.2). Let $u \in \exp L^2_0(\mathbb{R}^2)$ and $\alpha > 0$. Then one can take a sequence $\{u_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^2)$ such that

$$\|u - u_n\|_{\exp L^2} < \frac{1}{2\sqrt{\alpha}}$$

if $n \in \mathbb{N}$ is sufficiently large. Therefore, by convexity of $u \mapsto \exp \alpha u^2$,

$$\begin{aligned}
\int_{\mathbb{R}^2} \left[\exp\left(\alpha u^2\right) - 1\right] dx &= \int_{\mathbb{R}^2} \left[\exp\left(4\alpha \left(\tfrac{u-u_n}{2} + \tfrac{u_n}{2}\right)^2\right) - 1\right] dx \\
&\le \tfrac{1}{2} \int_{\mathbb{R}^2} \left[\exp\left(4\alpha\,(u-u_n)^2\right) - 1\right] dx + \tfrac{1}{2} \int_{\mathbb{R}^2} \left[\exp\left(4\alpha\,(u_n)^2\right) - 1\right] dx \\
&\le \tfrac{1}{2} \int_{\mathbb{R}^2} \left[\exp\left(\tfrac{u-u_n}{\|u-u_n\|_{\exp L^2}}\right)^2 - 1\right] dx + \tfrac{1}{2} \int_{\mathbb{R}^2} \left[\exp\left(4\alpha u_n^2\right) - 1\right] dx \\
&< \infty
\end{aligned}$$

if $n \in \mathbb{N}$ is sufficiently large.

We now prove the converse. Let $u \in L^1_{loc}(\mathbb{R}^2)$ and suppose that $\int_{\mathbb{R}^2} \left(e^{\alpha |u(x)|^2} - 1 \right) dx < \infty$, for every $\alpha > 0$. Define

$$u_n(x) := \begin{cases} u(x)\chi_{B_n(0)}(x) & -n < u(x) < n \\ n \operatorname{sign} u(x) \chi_{B_n(0)}(x) & \text{otherwise.} \end{cases}$$

Fix $\varepsilon > 0$. Lebesgue's convergence theorem shows that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left[\exp\left(\tfrac{u-u_n}{\varepsilon}\right)^2 - 1\right] dx \\
&\quad \le \int_{\{|u|>n\}\cap B_n(0)} \left[\exp\left(\tfrac{|u|-n}{\varepsilon}\right)^2 - 1\right] dx + \int_{B_n(0)^c} \left[\exp\left(\tfrac{|u|}{\varepsilon}\right)^2 - 1\right] dx \\
&\quad \to 0
\end{aligned}$$

as $n \to \infty$. Hence, there exists $N \in \mathbb{N}$ such that if $n \ge N$ then $\|u - u_n\|_{\exp L^2} < \varepsilon$. This means that $u \in E$, where E denotes the closure in $\exp L^2(\mathbb{R}^2)$ of the space of functions u which are bounded on $\mathbb{R}^2$ and have bounded support in $\mathbb{R}^2$. It is known that $C_0^\infty(\mathbb{R}^2)$ is dense in E (for the proof, see Theorem 8.21 in [2]), so $E = \exp L_0^2(\mathbb{R}^2)$. This completes the proof of (2.2).

Using (2.2), we easily find a typical example of a function

$$u \in \exp L^2(\mathbb{R}^2) \setminus \exp L_0^2(\mathbb{R}^2), \tag{2.3}$$

that is,

$$u(x) := \begin{cases} \left(\log \frac{1}{|x|}\right)^{\frac{1}{2}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{2.4}$$

since $\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)dx$ is finite for $0 < \alpha < 2$ but infinite for $\alpha \geq 2$.

Moreover, (2.2) shows that

$$L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \subset \exp L_0^2(\mathbb{R}^2). \tag{2.5}$$

Indeed, for every $u \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and every $\alpha > 0$, it holds

$$\begin{aligned} \int_{\mathbb{R}^2} (e^{\alpha |u(x)|^2} - 1)dx &= \sum_{k=1}^\infty \frac{\alpha^k \|u\|_{L^{2k}}^{2k}}{k!} \\ &\leq \sum_{k=1}^\infty \frac{\alpha^k (\|u\|_{L^2} + \|u\|_{L^\infty})^{2k}}{k!} \\ &= e^{\alpha(\|u\|_{L^2} + \|u\|_{L^\infty})^2} - 1 < \infty. \end{aligned}$$

This also shows that

$$\|u\|_{\exp L^2} \leq \frac{1}{\sqrt{\log 2}} \left(\|u\|_{L^2} + \|u\|_{L^\infty}\right). \tag{2.6}$$

Using these spaces, we prepare some basic estimates of the heat semigroup.

Proposition 2.1 *For every $2 \leq p < \infty$, the following inequality holds:*

$$\|u\|_{L^p} \leq \left\{\Gamma\left(\frac{p}{2} + 1\right)\right\}^{\frac{1}{p}} \|u\|_{\exp L^2},$$

where Γ is the gamma function

$$\Gamma(p) := \int_0^\infty x^{p-1} e^{-x} dx.$$

Proof of Proposition 2.1 If p is a natural number then the inequality is proved immediately by Taylor expansion. The general case can be proved by a minor modification (see [19]). □

Proposition 2.2 *Let $1 \leq p \leq q \leq \infty$, then the following estimates hold.*

$$\|e^{t\Delta} u_0\|_{L^q} \leq t^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \|u_0\|_{L^p} \quad \textit{for } u_0 \in L^p(\mathbb{R}^2),\ t > 0, \tag{2.7}$$

$$\|e^{t\Delta} u_0\|_{\exp L^2} \leq \|u_0\|_{\exp L^2} \quad \textit{for } u_0 \in \exp L^2(\mathbb{R}^2),\ t > 0, \tag{2.8}$$

$$e^{t\Delta} u_0 \in C([0, T]; \exp L_0^2(\mathbb{R}^2)) \quad \textit{for } u_0 \in \exp L_0^2(\mathbb{R}^2). \tag{2.9}$$

Proof of Proposition 2.2 The L^p-L^q estimate for the heat semigroup (2.7) is well known. We only prove (2.8) and (2.9). The first inequality is shown by the L^p-L^p estimate (2.7) and Taylor expansion. Indeed, for any $\lambda > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\left(\exp\left(\frac{e^{t\Delta}u_0}{\lambda}\right)^2-1\right)dx &= \sum_{k=1}^{\infty}\frac{\|e^{t\Delta}u_0\|_{L^{2k}(\mathbb{R}^2)}^{2k}}{k!\lambda^{2k}} \\ &\leq \sum_{k=1}^{\infty}\frac{\|u_0\|_{L^{2k}(\mathbb{R}^2)}^{2k}}{k!\lambda^{2k}} \\ &= \int_{\mathbb{R}^2}\left(\exp\left(\frac{u_0}{\lambda}\right)^2-1\right)dx.\end{aligned} \tag{2.10}$$

Therefore we obtain

$$\begin{aligned}\|e^{t\Delta}u_0\|_{\exp L^2} &= \inf\left\{\lambda>0;\ \int_{\mathbb{R}^2}\left(\exp\left(\frac{e^{t\Delta}u_0}{\lambda}\right)^2-1\right)dx\leq 1\right\} \\ &\leq \inf\left\{\lambda>0;\ \int_{\mathbb{R}^2}\left(\exp\left(\frac{u_0}{\lambda}\right)^2-1\right)dx\leq 1\right\} \\ &= \|u_0\|_{\exp L^2}.\end{aligned}$$

This proves (2.8).

We turn to prove (2.9). By (2.10) and (2.2), $e^{t\Delta}u_0 \in \exp L_0^2(\mathbb{R}^2)$ for every $t > 0$ if $u_0 \in \exp L_0^2(\mathbb{R}^2)$. Thus it remains to prove continuity,

$$\lim_{t\to 0}\|e^{t\Delta}u_0 - u_0\|_{\exp L^2} = 0.$$

Since $u_0 \in \exp L_0^2(\mathbb{R}^2)$, there exist $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^2)$ such that $\lim_{n\to\infty}\|u_n - u_0\|_{\exp L^2} = 0$. By (2.6) and (2.8), it holds

$$\begin{aligned}\|e^{t\Delta}u_0 - u_0\|_{\exp L^2} &\leq \|e^{t\Delta}(u_0-u_n)\|_{\exp L^2} + \|e^{t\Delta}u_n - u_n\|_{\exp L^2} + \|u_n - u_0\|_{\exp L^2} \\ &\leq \frac{1}{\sqrt{\log 2}}\left(\|e^{t\Delta}u_n - u_n\|_{L^2} + \|e^{t\Delta}u_n - u_n\|_{L^\infty}\right) + 2\|u_n - u_0\|_{\exp L^2}.\end{aligned}$$

Since $u_n \in C_0^{\infty}(\mathbb{R}^2)$, we see that

$$\lim_{t\to 0}(\|e^{t\Delta}u_n - u_n\|_{L^2} + \|e^{t\Delta}u_n - u_n\|_{L^\infty}) = 0.$$

Hence

$$\limsup_{t\to 0}\|e^{t\Delta}u_0 - u_0\|_{\exp L^2} \leq 2\|u_n - u_0\|_{\exp L^2}$$

for every $n \in \mathbb{N}$. This proves (2.9). □

Proposition 2.3 *Let $u \in C([0,T]; \exp L_0^2(\mathbb{R}^2))$. Then for every $\alpha > 0$ there holds*

$$\left(e^{\alpha u^2} - 1\right) \in C([0,T]; L^1(\mathbb{R}^2)).$$

Proof of Proposition 2.3 Let $u \in C([0,T]; \exp L_0^2(\mathbb{R}^2))$. It follows from (2.2) that

$$\left(e^{\alpha u^2} - 1\right) \in L^1(\mathbb{R}^2)$$

for every $\alpha > 0$ and $t \in [0,T]$.

It remains to prove the continuity in the time variable. We follow an idea from [9]. Fix $t \in [0, T]$ and a sequence such that $t_n \to t$ as $n \to \infty$. Define $\tilde{u}_n := u(t_n)$. We prove

$$\left(e^{\alpha \tilde{u}_n^2} - 1\right) \to \left(e^{\alpha u^2} - 1\right) \quad \text{in } L^1(\mathbb{R}^2)$$

for every $\alpha > 0$. Set $U_n := \tilde{u}_n - u$. A simple calculation shows

$$\begin{aligned} e^{\alpha \tilde{u}_n^2} - e^{\alpha u^2} = &\left(e^{\alpha u^2} - 1\right)\left\{\left(e^{\alpha U_n^2} - 1\right)\left(e^{2\alpha u U_n} - 1\right) + \left(e^{\alpha U_n^2} - 1\right) + \left(e^{2\alpha u U_n} - 1\right)\right\} \\ &+ \left(\left(e^{\alpha U_n^2} - 1\right)\left(e^{2\alpha u U_n} - 1\right) + \left(e^{\alpha U_n^2} - 1\right) + \left(e^{2\alpha u U_n} - 1\right)\right). \end{aligned}$$

Hence it suffices to prove

$$\begin{aligned} &e^{\alpha U_n^2} - 1 \to 0 \quad \text{in } L^p(\mathbb{R}^2), \\ &e^{2\alpha u U_n} - 1 \to 0 \quad \text{in } L^p(\mathbb{R}^2), \end{aligned}$$

for every $1 \leq p < \infty$. Recall that $\|U_n\|_{\exp L^2} \to 0$ as $n \to \infty$, since $u \in C([0, T]; \exp L_0^2(\mathbb{R}^2))$. Indeed, Taylor expansion and Proposition 2.1 show that

$$\begin{aligned} \left\| e^{\alpha U_n^2} - 1 \right\|_{L^p}^p &\leq C \int_{\mathbb{R}^2} \left(e^{\alpha p U_n^2} - 1\right) dx \\ &= C \sum_{k=1}^{\infty} \frac{(\alpha p)^k}{k!} \|U_n\|_{L^{2k}}^{2k} \\ &\leq C \sum_{k=1}^{\infty} (\alpha p)^k \|U_n\|_{\exp L^2}^{2k} \\ &\leq C \frac{\alpha p \|U_n\|_{\exp L^2}^2}{1 - \alpha p \|U_n\|_{\exp L^2}^2} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\| e^{2\alpha u U_n} - 1 \right\|_{L^p}^p &\leq C \int_{\mathbb{R}^2} \left(e^{2\alpha p |u||U_n|} - 1\right) dx \\ &\leq C \sum_{k=1}^{\infty} \frac{(2\alpha p)^k}{k!} \|u\|_{L^{2k}}^k \|U_n\|_{L^{2k}}^k \\ &\leq C \sum_{k=1}^{\infty} (2\alpha p)^k \|u\|_{\exp L^2}^k \|U_n\|_{\exp L^2}^k \\ &\leq C \frac{2\alpha p \|u\|_{\exp L^2} \|U_n\|_{\exp L^2}}{1 - 2\alpha p \|u\|_{\exp L^2} \|U_n\|_{\exp L^2}} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This completes the proof. □

3 Proof of Theorem 1.1

To prove Theorem 1.1, we first construct an initial data which has diverging integrability.

Lemma 3.1 *Let* $\alpha > 0$ *and*

$$u_0(x) := \begin{cases} \alpha \left(\log \frac{1}{|x|}\right)^{\frac{1}{2}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{3.1}$$

Then, for every $\lambda > 0$, there exists some $\tilde{\alpha} > 0$ such that

$$\int_0^{\varepsilon} \int_{B_r(0)} \exp\left(\lambda (e^{t\Delta} u_0)^2\right) dx dt = \infty,$$

for every $\alpha > \tilde{\alpha}$, $\varepsilon > 0$, and $r > 0$, where $B_r(0) \subset \mathbb{R}^2$ is the ball centered at the origin with radius $r > 0$.

Proof of Lemma 3.1 Fix $0 < \tilde{r} < r$ so that if $x \in B_{\tilde{r}}(0)$ then $B_{|x|}(3x) \subset B_1(0)$. Clearly

$$\int_0^{\varepsilon} \int_{B_r(0)} \exp\left(\lambda (e^{t\Delta} u_0)^2\right) dx dt \geq \int_0^{\varepsilon} \int_{B_{\tilde{r}}(0)} \exp\left(\lambda (e^{t\Delta} u_0)^2\right) dx dt$$

and for $x \in B_{\tilde{r}}(0)$ there holds

$$e^{t\Delta} u_0(x) = \int_{B_1(0)} \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy \geq \alpha \int_{B_{|x|}(3x)} \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}} \left(\log \frac{1}{|y|}\right)^{\frac{1}{2}} dy.$$

Since $y \in B_{|x|}(3x)$, it holds $2|x| \leq |y| \leq 4|x|$ and $|x| \leq |x-y| \leq 3|x|$ and this implies

$$e^{t\Delta} u_0(x) \geq \alpha \int_{B_{|x|}(3x)} \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}} \left(\log \frac{1}{|y|}\right)^{\frac{1}{2}} dy \geq C\alpha \frac{|x|^2}{t} e^{-\frac{9}{4}\frac{|x|^2}{t}} \left(\log \frac{1}{4|x|}\right)^{\frac{1}{2}} \tag{3.2}$$

for some $C > 0$. Now choose $\tilde{\varepsilon} < \varepsilon$ small so that every $0 < t < \tilde{\varepsilon}$ satisfies $B_{\sqrt{t}}(0) \subset B_{\tilde{r}}(0)$. This and (3.2) imply that

$$\begin{aligned} \int_0^{\varepsilon} \int_{B_{\tilde{r}}(0)} \exp\left(\lambda (e^{t\Delta} u_0)^2\right) dx dt &\geq \int_0^{\tilde{\varepsilon}} \int_{B_{\sqrt{t}}(0) \setminus B_{\sqrt{t}/2}(0)} \exp\left(\lambda (e^{t\Delta} u_0)^2\right) dx dt \\ &\geq \int_0^{\tilde{\varepsilon}} \int_{B_{\sqrt{t}}(0) \setminus B_{\sqrt{t}/2}(0)} \exp\left(\lambda C^2 \alpha^2 \log \tfrac{1}{4|x|}\right) dx dt \\ &= \tilde{C} \int_0^{\tilde{\varepsilon}} t^{1-\frac{\lambda C^2 \alpha^2}{2}} dt \end{aligned}$$

for some $\tilde{C} > 0$. If α is large enough, the integral in the last line goes to infinity and therefore

$$\int_0^{\varepsilon} \int_{B_{\tilde{r}}(0)} \exp\left(\lambda (e^{t\Delta} u_0)^2\right) dx dt = \int_0^{\tilde{\varepsilon}} \int_{B_{\sqrt{t}}(0) \setminus B_{\sqrt{t}/2}(0)} \exp\left(\lambda (e^{t\Delta} u_0)^2\right) dx dt = \infty. \tag{3.3}$$

This proves Lemma 3.1. □

Using Lemma 3.1, we prove the non-existence of any nonnegative $\exp L^2(\mathbb{R}^2)$-classical solution.

Proof of Theorem 1.1 Recall that u_0 defined in (3.1) belongs to $\exp L^2(\mathbb{R}^2)$ for every $\alpha > 0$. By contradiction we assume that there exists $T > 0$ and a nonnegative $\exp L^2(\mathbb{R}^2)$-classical solution u to (1.1). For any $t > 0$, $s > 0$, $t + s < T$ we have

$$u(t+s) \geq e^{t\Delta} u(s)$$

since u is a $\exp L^2(\mathbb{R}^2)$-classical solution. Considering $s \to 0$, we have $u(t+s) \to u(t)$. On the other hand, combining the facts that

$$e^{-\frac{|x-\cdot|^2}{4t}} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \subset L^1(\log L)^{\frac{1}{2}}(\mathbb{R}^2)$$

for every $x \in \mathbb{R}^2$ and $u(s)$ converges in weak*-topology to u_0 (see (1.6)), we obtain

$$e^{t\Delta}u(s,x) = \int_{\mathbb{R}^2} \frac{e^{-\frac{|x-y|^2}{4t}}}{4\pi t} u(s,y)dy \to \int_{\mathbb{R}^2} \frac{e^{-\frac{|x-y|^2}{4t}}}{4\pi t} u_0(y)dy = e^{t\Delta}u_0(x) \ \ (s \to 0).$$

Thus we have

$$u(t) \geq e^{t\Delta}u_0 \geq 0. \tag{3.4}$$

Here we use that the initial data u_0 defined in (3.1) is nonnegative. Since u is a classical solution of (1.1) for $t \in (0,T)$, we obtain by multiplying (1.1) by $\xi \in C_0^\infty(\mathbb{R}^2)$, $\xi \geq 0$ on $\mathbb{R}^2$ and $\xi \geq 1$ on $B_r(0)$ that

$$\frac{d}{dt}\int_{\mathbb{R}^2} u\xi dx + \int_{\mathbb{R}^2} u(-\Delta\xi) = \int_{\mathbb{R}^2} f(u)\xi dx \geq \int_{B_r(0)} f(u)dx.$$

Therefore for $0 < \sigma < T' < T$ we have

$$\int_{\mathbb{R}^2} u(T')\xi dx - \int_{\mathbb{R}^2} u(\sigma)\xi dx + \int_\sigma^{T'}\int_{\mathbb{R}^2} u(s)(-\Delta\xi)dxds \geq \int_\sigma^{T'}\int_{B_r(0)} f(u(s))dxds.$$

Letting $\sigma \to 0$, since u belongs to $L^\infty(0,T';\exp L^2(\mathbb{R}^2))$ and $u(t) \to u_0$ in weak* topology, we have

$$\int_{\mathbb{R}^2} u(T')\xi dx - \int_{\mathbb{R}^2} u_0\xi dx + \int_0^{T'}\int_{\mathbb{R}^2} u(-\Delta\xi)dxds \geq \int_0^{T'}\int_{B_r(0)} f(u(s))dxds,$$

and therefore

$$\int_0^{T'}\int_{B_r(0)} f(u)dxds < \infty. \tag{3.5}$$

On the other hand, assumption (1.8) shows that there are some positive constants $C > 0$ and $\eta_0 > 0$ such that

$$f(\eta) \geq e^{\lambda\eta^2} \tag{3.6}$$

for every $\eta > \eta_0$. Now take $\tilde{r} < r$ and $\tilde{\varepsilon} < T'$ as in the proof of Lemma 3.1. Then the inequality (3.2) shows that

$$e^{t\Delta}u_0(x) \geq C\left(\log\frac{1}{4\sqrt{t}}\right)^{\frac{1}{2}} \geq \eta_0 \tag{3.7}$$

if $t > 0$ is small enough and $x \in B_{\sqrt{t}}(0) \setminus B_{\sqrt{t}/2}(0) \subset B_{\tilde{r}}(0)$. It follows from (3.4), (3.6), and (3.7) that

$$\int_0^{T'}\int_{B_r(0)} f(u)dxds \geq \int_0^{\tilde{\varepsilon}}\int_{B_{\sqrt{t}}(0)\setminus B_{\sqrt{t}/2}(0)} \exp\left(\lambda(e^{t\Delta}u_0)^2\right)dxdt. \tag{3.8}$$

This contradicts (3.5) and (3.3) if $\alpha > 0$ is sufficiently large. □

4 Proof of Existence of Theorem 1.2

In this section we prove the existence of a solution of (1.1) in $C([0, T]; \exp L_0^2(\mathbb{R}^2))$ for some $T > 0$. We emphasize that also in this framework as in the case of solutions of small norm the Cauchy problem (1.1) admits the equivalent integral formulation (1.7).

Proposition 4.1 *Let* $T > 0$ *and* u_0 *be in* $\exp L_0^2(\mathbb{R}^2)$. *If* u *belongs to* $C([0, T]; \exp L_0^2(\mathbb{R}^2))$, *then* u *is a weak solution of* (1.1) *if and only if* $u(t)$ *satisfies the integral* (1.7) *for any* $t \in (0, T)$.

Proof of Proposition 4.1 The key estimate of the proof is the property that for any $u \in C([0, T]; \exp L_0^2(\mathbb{R}^2))$ then $f(u) \in C([0, T]; L^p(\mathbb{R}^2))$ for all $1 \le p < \infty$ (see Proposition 2.3). Therefore the integral on the right hand side of (1.7) is well defined (see [4, 6]). □

In order to find a solution we will apply a fixed point argument to (1.7). To this end, we apply a decomposition developed in [10].

Let $u_0 \in \exp L_0^2(\mathbb{R}^2)$. Then, for every $\varepsilon > 0$ there exists $u_1 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\|u_0 - u_1\|_{\exp L^2} < \varepsilon.$$

We define $u_2 := u_0 - u_1$. Now we divide the problem into the following two problems. One is a semilinear heat equation with smooth initial data

$$\begin{cases} \partial_t v - \Delta v = f(v) & \text{in } (0, \infty) \times \mathbb{R}^2, \\ v(0) = u_1 \in C_0^\infty(\mathbb{R}^2) & \text{in } \mathbb{R}^2, \end{cases} \tag{4.1}$$

and the other one is a semilinear heat equation with small data in $\exp L^2$

$$\begin{cases} \partial_t w - \Delta w = f(w + v) - f(v) & \text{in } (0, \infty) \times \mathbb{R}^2, \\ w(0) = u_2, \quad \|u_2\|_{\exp L^2} < \varepsilon, & \text{in } \mathbb{R}^2. \end{cases} \tag{4.2}$$

We now construct local solutions of (4.1) and (4.2) separately.

Lemma 4.1 *Let* $u_1 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *Then there exist a time* $T > 0$ *and a solution* $v \in C([0, T]; \exp L_0^2(\mathbb{R}^2)) \cap L^\infty(0, T; L^\infty(\mathbb{R}^2))$ *of* (4.1).

Lemma 4.2 *Let* $\varepsilon > 0$ *small enough. Then, for any* $v \in L^\infty(0, T; L^\infty(\mathbb{R}^2))$, *there exist a time* $\tilde{T} = \tilde{T}(u_2, \varepsilon, v) > 0$ *and a solution* $w \in C([0, \tilde{T}]; \exp L_0^2(\mathbb{R}^2))$ *of* (4.2).

Clearly if v and w are solutions of (4.1) and (4.2) respectively, then $u := v + w$ is a solution of (1.7).

4.1 Proof of Lemma 4.1

In this subsection, we prove Lemma 4.1.

Proof of Lemma 4.1 We apply a standard contraction mapping argument. Let $M > 0$ and define

$$Y_{M,T} := \left\{ v \in C([0,T]; \exp L_0^2(\mathbb{R}^2)) \cap L^\infty(0,T; L^\infty(\mathbb{R}^2)) : \|v\|_{Y_T} \le M + \|u_1\|_{L^2} + \|u_1\|_{L^\infty} \right\},$$

where $\|v\|_{Y_T} := \|v\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \|v\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))}$, and set

$$\Phi(v) := e^{t\Delta} u_1 + \int_0^t e^{(t-s)\Delta} \Big(f\big(v(s)\big)\Big) ds.$$

We prove that if $T > 0$ is small enough then Φ is a contraction map from $Y_{M,T}$ to itself. First we prove that there exists some C independent of T such that

$$\|\Phi(v_1) - \Phi(v_2)\|_{Y_T} \le CT \|v_1 - v_2\|_{Y_T} \tag{4.3}$$

for every $v_1, v_2 \in Y_{M,T}$. Clearly, for $q = 2$ or $q = \infty$ it holds

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^\infty(0,T;L^q)} \le \int_0^T \|f(v_1) - f(v_2)\|_{L^q} ds \le T \|f(v_1) - f(v_2)\|_{L^\infty(0,T;L^q)}. \tag{4.4}$$

Moreover,

$$\begin{aligned} \|f(v_1) - f(v_2)\|_{L^q} &\le C \left(\sum_{i=1,2} \int_{\mathbb{R}^2} |v_1 - v_2|^q e^{\lambda q v_i^2} dx \right)^{\frac{1}{q}} \\ &\le C \sum_{i=1,2} e^{\lambda \|v_i\|_{L^\infty}^2} \|v_1 - v_2\|_{L^q}. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we obtain the desired estimate (4.3).
Now take $v_2 = 0$ in (4.3). Then we see

$$\|\Phi(v)\|_{Y_T} \le CT \|v\|_{Y_T} + \|u_1\|_{L^2} + \|u_1\|_{L^\infty}. \tag{4.6}$$

The inequality (4.6) shows that if $v \in Y_{M,T}$ and $u_1 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ then $\Phi(v) \in L^\infty(0,T; L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$. This and (2.5) implies $\Phi(v) \in L^\infty(0,T; \exp L_0^2(\mathbb{R}^2))$. Moreover the density of $C_0^\infty(\mathbb{R}^2)$ in $\exp L_0^2(\mathbb{R}^2)$ implies $\Phi(v) \in C([0,T]; \exp L_0^2(\mathbb{R}^2))$. Now, choosing $T = T(M) > 0$ small enough again, (4.6) and (4.3) show that Φ is a contraction map from $Y_{T,M}$ to itself. Thus, contraction mapping arguments imply that there is a solution $v \in C([0,T]; \exp L_0^2(\mathbb{R}^2))$ of (4.1). □

4.2 Proof of Lemma 4.2

Let $\tilde{M}, \tilde{T} > 0$, and define

$$W_{\tilde{M},\tilde{T}} := \{ w \in C([0,\tilde{T}]; \exp L_0^2(\mathbb{R}^2)) : \|w\|_{L^\infty(0,\tilde{T};\exp L^2(\mathbb{R}^2))} \le \tilde{M} \}.$$

To prove Lemma 4.2, we begin with the following useful lemma.

Lemma 4.3 *Let $v \in L^\infty(0,\tilde{T}; L^\infty(\mathbb{R}^2))$ and $w_1, w_2 \in W_{\tilde{M},\tilde{T}}$. For any $2 \le q < \infty$, and for sufficiently small $\tilde{M}$ there exists a constant $C = C(\|v\|_{L^\infty(L^\infty)}, \tilde{M}, q) > 0$ such that*

$$\|f(w_1+v)(t) - f(w_2+v)(t)\|_{L^q} \le C(\|v\|_{L^\infty(L^\infty)}, \tilde{M}, q) \|w_1(t) - w_2(t)\|_{\exp L^2}, \quad \text{for any } t \in (0,T).$$

Proof of Lemma 4.3 Let $q \geq 2$. By the assumption (1.9) on f, we have

$$\begin{aligned}
&\|f(w_1+v)-f(w_2+v)\|_{L^q} \\
&\leq C\left(\sum_{i=1,2}\int_{\mathbb{R}^2}|w_1-w_2|^q e^{\lambda q(w_i+v)^2}dx\right)^{\frac{1}{q}} \\
&\leq C\sum_{i=1,2}\left(\int_{\mathbb{R}^2}|w_1-w_2|^q e^{2\lambda q w_i^2+2\lambda q v^2}dx\right)^{\frac{1}{q}} \\
&\leq Ce^{2\lambda q\|v\|_{L^\infty(L^\infty)}}\left(2\|w_1-w_2\|_{L^q}+\sum_{i=1,2}\left(\int_{\mathbb{R}^2}|w_1-w_2|^q\left(e^{2\lambda q w_i^2}-1\right)dx\right)^{\frac{1}{q}}\right).
\end{aligned} \tag{4.7}$$

Here we used $v \in L^\infty(0,\tilde{T};L^\infty(\mathbb{R}^2))$. Fix $1 < r < \infty$. Applying Hölder's inequality, we obtain

$$\left(\int_{\mathbb{R}^2}|w_1-w_2|^q\left(e^{2\lambda q w_i^2}-1\right)dx\right)^{\frac{1}{q}} \leq C\|w_1-w_2\|_{L^{qr'}}\left(\int_{\mathbb{R}^2}\left(e^{2\lambda q r w_i^2}-1\right)dx\right)^{\frac{1}{qr}} \tag{4.8}$$

for some constant $C > 0$. Recall that $q \geq 2$, so $2qr \geq 2$ and $qr' \geq 2$. It follows from Proposition 2.1 that

$$\begin{aligned}
\|w_1-w_2\|_{L^{qr'}} &\leq C\|w_1-w_2\|_{\exp L^2}, \\
\|w_1-w_2\|_{L^q} &\leq C\|w_1-w_2\|_{\exp L^2},
\end{aligned} \tag{4.9}$$

for some $C > 0$ depending only on q and r. Moreover, choose $\tilde{M} > 0$ small so that $2\lambda qr < 1/(\tilde{M})^2$. Then, by (2.1) and $\|w_i\|_{\exp L^2} \leq \tilde{M}$, it holds

$$\int_{\mathbb{R}^2}\left(e^{2\lambda q r w_i^2}-1\right)dx \leq \int_{\mathbb{R}^2}\left(\exp\left(\frac{w_i}{\tilde{M}}\right)^2-1\right)dx \leq 1. \tag{4.10}$$

Substituting (4.8), (4.9), and (4.10) into (4.7), we obtain Lemma 4.3. □

Next we prove Lemma 4.2.

proof of Lemma 4.2 Define

$$\Phi(w) := e^{t\Delta}u_2 + \int_0^t e^{(t-s)\Delta}\left(f\left(w(s)+v(s)\right)-f\left(v(s)\right)\right)ds.$$

We prove that if $\tilde{M} > 0$ and $\tilde{T} > 0$ are sufficiently small, then Φ is a contraction map from $W_{\tilde{M},\tilde{T}}$ to itself. To this end, we start by proving

$$\|\Phi(w_1)-\Phi(w_2)\|_{L^\infty(\exp L^2)} \leq C(\|v\|_{L^\infty(L^\infty)},\tilde{M})\left(\tilde{T}+\tilde{T}^{1-\frac{1}{p}}\right)\|w_1-w_2\|_{L^\infty(\exp L^2)}$$

for every $w_1, w_2 \in W_{\tilde{M},\tilde{T}}$. Recall that $L^2 \cap L^\infty \subset \exp L^2$, i.e.,

$$\|\Phi(w_1)-\Phi(w_2)\|_{L^\infty(\exp L^2)} \leq \|\Phi(w_1)-\Phi(w_2)\|_{L^\infty(L^2)}+\|\Phi(w_1)-\Phi(w_2)\|_{L^\infty(L^\infty)}. \tag{4.11}$$

Let $p > 2$. Then the $L^p - L^q$ estimates for the heat semigroup (2.7) show that

$$\|\Phi(w_1)-\Phi(w_2)\|_{L^\infty} \leq \int_0^t (t-s)^{-\frac{1}{p}}\|f(w_1+v)-f(w_2+v)\|_{L^p}ds.$$

Applying Lemma 4.3 as $q = p$, we have

$$\begin{aligned}\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty} &\le C(\|v\|_{L^\infty(L^\infty)}, \tilde{M}) \int_0^t (t-s)^{-\frac{1}{p}} ds \|w_1 - w_2\|_{\exp L^2} \\ &\le C(\|v\|_{L^\infty(L^\infty)}, \tilde{M}) \tilde{T}^{1-\frac{1}{p}} \|w_1 - w_2\|_{\exp L^2}.\end{aligned} \tag{4.12}$$

On the other hand, again by Lemma 4.3 as $q = 2$ and $\tilde{M}$ sufficiently small, it holds

$$\begin{aligned}\|\Phi(w_1) - \Phi(w_2)\|_{L^2} &\le C(\|v\|_{L^\infty(L^\infty)}, \tilde{M}) \int_0^t \|w_1 - w_2\|_{\exp L^2} ds \\ &\le C(\|v\|_{L^\infty(L^\infty)}, \tilde{M}) \tilde{T} \|w_1 - w_2\|_{L^\infty(\exp L^2)}.\end{aligned} \tag{4.13}$$

Therefore, (4.11), (4.12), and (4.13) show that if we choose $\tilde{M} > 0$ small enough, then there is some $C = C(\|v\|_{L^\infty}, \tilde{M}) > 0$ such that

$$\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty(\exp L^2)} \le C(\|v\|_{L^\infty}, \tilde{M}) \left(\tilde{T} + \tilde{T}^{1-\frac{1}{p}}\right) \|w_1 - w_2\|_{L^\infty(\exp L^2)} \tag{4.14}$$

for every $w_1, w_2 \in W_{\tilde{M},\tilde{T}}$.

We next prove that Φ is a map from $W_{\tilde{M},\tilde{T}}$ to itself, provided that $\|u_2\|_{\exp L^2}$ and $\tilde{T} > 0$ are sufficiently small. To this end, we first prove that $\Phi(w) \in C([0, T]; \exp L_0^2(\mathbb{R}^2))$ if $u_2 \in \exp L_0^2(\mathbb{R})$ and $w \in W_{\tilde{M},\tilde{T}}$. The estimates (4.12)-(4.13) with $w_2 = 0$ show that the nonlinear term satisfies

$$\Phi(w) - e^{t\Delta} u_2 \in L^\infty(0, T; L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)).$$

Therefore, the inclusion relation (2.5) and the standard smoothing effect of the heat semigroup imply

$$\Phi(w) - e^{t\Delta} u_2 \in C([0, T]; \exp L_0^2(\mathbb{R}^2)). \tag{4.15}$$

Moreover, one can see by (2.9) that

$$e^{t\Delta} u_2 \in C([0, T]; \exp L_0^2(\mathbb{R}^2)).$$

This and (4.15) proves $\Phi(w) \in C([0, T]; \exp L_0^2(\mathbb{R}^2))$. Now again by (4.14) with $w_2 = 0$, we have

$$\|\Phi(w)\|_{L^\infty(\exp L^2)} \le \|u_2\|_{\exp L^2} + C(\|v\|_{L^\infty}, \tilde{M}) \times (\tilde{T} + \tilde{T}^{1-\frac{1}{p}}) \|w\|_{L^\infty(\exp L^2)}. \tag{4.16}$$

Fix $\tilde{M} > 0$ small enough so that Lemma 4.3 holds. Then choose $0 < \varepsilon < \tilde{M}/2$ and $\tilde{T} = \tilde{T}(\tilde{M}, \varepsilon, \|v\|_{L^\infty})$ small enough such that Φ is a contraction map from $W_{\tilde{M},\tilde{T}}$ to itself. This proves Lemma 4.2. □

Proof of Theorem 1.2 We choose small $T, \varepsilon, \tilde{M}, \tilde{T}$ in the following order. First of all, fix $M > 0$. Choose $\tilde{M} > 0$ in (4.10) so that Lemma 4.3 holds. Recall that $\tilde{M}$ does not depend on $T, \varepsilon, M, \tilde{T}$. After that, fix $\varepsilon(\tilde{M}) > 0$ small such that (4.16) holds. Next one can decompose $u_0 = u_1 + u_2$ with $u_1 \in C_0^\infty(\mathbb{R}^2)$ and $\|u_2\|_{\exp L^2} < \varepsilon$. Then taking small $T = T(M, \|u_1\|_{L^2}, \|u_1\|_{L^\infty}) > 0$ in (4.3) and (4.6), we have a solution v of (4.1). Finally we obtain a solution w of (4.2) by choosing small $\tilde{T} = \tilde{T}(\tilde{M}, \varepsilon, \|v\|_{L^\infty})$ in (4.16) and (4.14). In conclusion, $u := v + w$ is a solution of (1.7) in $C([0, \min\{T, \tilde{T}\}]; \exp L_0^2(\mathbb{R}^2))$. □

Remark 4.1 The solution in Theorem 1.2 belongs to $L^\infty_{loc}(0,T;L^\infty(\mathbb{R}^2))$. Indeed, let $u\in C([0,T];\exp L^2_0(\mathbb{R}^2))$ be a solution of the integral (1.7). The smoothing effect of the heat kernel shows that $e^{t\Delta}u_0\in L^\infty(\mathbb{R}^2)$ for every $0<t<T$. Thus we only have to consider the nonlinear term. Fixing $p>2$, it follows from Proposition 2.3 that there exists some $\tilde{C}>\lambda p$, such that for any $0<t<T$

$$\begin{aligned}\int_0^t\|e^{(t-s)\Delta}f(u(s))\|_{L^\infty}ds &\le \int_0^t(t-s)^{-\frac{1}{p}}\|f(u(s))\|_{L^p}ds\\ &\le \int_0^t(t-s)^{-\frac{1}{p}}\left(\int_{\mathbb{R}^2}|u|^p\exp(\lambda pu^2)dx\right)^{\frac{1}{p}}ds\\ &\le \int_0^t(t-s)^{-\frac{1}{p}}\left(\int_{\mathbb{R}^2}(\exp(\tilde{C}u^2)-1)dx\right)^{\frac{1}{p}}ds\\ &\le t^{1-\frac{1}{p}}\sup_{0<s<t}\left(\int_{\mathbb{R}^2}(\exp(\tilde{C}u(s)^2)-1)dx\right)^{\frac{1}{p}}<\infty,\end{aligned}$$

since $u\in C([0,T];\exp L^2_0(\mathbb{R}^2))$. This shows that $u\in L^\infty_{loc}(0,T;L^\infty(\mathbb{R}^2))$. In particular, if $f\in C^1(\mathbb{R})$ the solution $u\in C([0,T];\exp L^2_0(\mathbb{R}^2))\cap L^\infty_{loc}(0,T;L^\infty(\mathbb{R}^2))$ satisfies the (1.1) in the classical sense, i.e. C^1 in $t\in(0,T)$ and C^2 in $x\in\mathbb{R}^2$.

5 Proof of Uniqueness in $C([0,T];\exp L^2_0(\mathbb{R}^2))$

In this section, we prove uniqueness of the solution in $C([0,T];\exp L^2_0(\mathbb{R}^2))$.

Proof of Uniqueness of Theorem 1.2 Let us suppose that u,v are two solutions of (1.7) which for some $T>0$ belong to $C([0,T];\exp L^2_0(\mathbb{R}^2))$, and with the same initial data $u(0)=v(0)=u_0$. Let

$$t_0=\sup\left\{t\in[0,T]\ \text{ such that } u(s)=v(s)\ \text{ for every } s\in[0,t]\right\}.$$

Let us suppose by contradiction that $0\le t_0<T$. Since $u(t)$ and $v(t)$ are continuous in time we have $u(t_0)=v(t_0)$. Let us denote $\tilde{u}(t)=u(t+t_0)$ and $\tilde{v}(t)=v(t+t_0)$ then $\tilde{u}$ and $\tilde{v}$ verify the (1.7) on $(0,T-t_0]$ and $\tilde{u}(0)=\tilde{v}(0)=u(t_0)$. We will prove that there exists a positive time $0<\tilde{t}\le T-t_0$ such that

$$\sup_{0<t<\tilde{t}}\|\tilde{u}(t)-\tilde{v}(t)\|_{\exp L^2}\le C(\tilde{t})\sup_{0<t<\tilde{t}}\|\tilde{u}(t)-\tilde{v}(t)\|_{\exp L^2}\tag{5.1}$$

for a constant $C(\tilde{t})<1$, and so $\tilde{u}(t)=\tilde{v}(t)$ for any $t\in[0,\tilde{t}]$. Therefore $u(t+t_0)=v(t+t_0)$ for any $t\in[0,\tilde{t}]$ in contradiction with the definition of t_0. In order to establish inequality (5.1) we control both the L^2–norm and the L^∞–norm of the difference of the two solutions. Thanks to Proposition 2.1 and Hölder's inequality for some p,q such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and $p>2$, we have

$$\begin{aligned}\|\tilde{u}(t)-\tilde{v}(t)\|_{L^2} &\le \int_0^t\left\||\tilde{u}(s)-\tilde{v}(s)|(e^{\lambda\tilde{u}^2(s)}+e^{\lambda\tilde{v}^2(s)})\right\|_{L^2}ds\\ &\le 2\int_0^t\|\tilde{u}(s)-\tilde{v}(s)\|_{L^2}ds\\ &\quad+\int_0^t\left(\|\tilde{u}(s)-\tilde{v}(s)\|_{L^q}\left\|(e^{\lambda\tilde{u}^2(s)}-1)+(e^{\lambda\tilde{v}^2(s)}-1)\right\|_{L^p}\right)ds\\ &\le Ct\sup_{0<s<t}\|\tilde{u}(s)-\tilde{v}(s)\|_{\exp L^2}\\ &\quad+C\sup_{0<s<t}\|\tilde{u}(s)-\tilde{v}(s)\|_{\exp L^2}\int_0^t\left\|(e^{\lambda\tilde{u}^2(s)}-1)+(e^{\lambda\tilde{v}^2(s)}-1)\right\|_{L^p}ds.\end{aligned}$$

Moreover thanks to Proposition 2.3 the term in the integral is uniformly bounded in time. Indeed,

$$\begin{aligned}\sup_{0<s<T-t_0} & \left\| (e^{\lambda \tilde{u}^2(s)} - 1) + (e^{\lambda \tilde{v}^2(s)} - 1) \right\|_{L^p} \\ & \leq \sup_{0<s<T-t_0} \left[\int_{\mathbb{R}^2} (e^{\lambda p \tilde{u}^2(s)} - 1) + (e^{\lambda p \tilde{v}^2(s)} - 1) dx \right]^{1/p} \\ & \leq C(T, t_0, \tilde{u}, \tilde{v}) < \infty. \end{aligned} \tag{5.2}$$

Therefore, we obtain

$$\sup_{0<s<t} \|\tilde{u}(s) - \tilde{v}(s)\|_{L^2} \leq C(T, t_0, \tilde{u}, \tilde{v})\, t \sup_{0<s<t} \|\tilde{u}(s) - \tilde{v}(s)\|_{\exp L^2}. \tag{5.3}$$

In a similar way we control

$$\begin{aligned}\|\tilde{u}(t) - \tilde{v}(t)\|_{L^\infty} & \leq \int_0^t (t-s)^{-\frac{1}{p}} \left\| |\tilde{u}(s) - \tilde{v}(s)| \left(e^{\lambda \tilde{u}^2(s)} + e^{\lambda \tilde{v}^2(s)} \right) \right\|_{L^p} ds \\ & \leq 2 \int_0^t (t-s)^{-\frac{1}{p}} \|\tilde{u}(s) - \tilde{v}(s)\|_{L^p} \\ & \quad + \int_0^t (t-s)^{-\frac{1}{p}} \|\tilde{u}(s) - \tilde{v}(s)\|_{L^{\bar{q}}} \left\| \left(e^{\lambda \tilde{u}^2(s)} - 1 + e^{\lambda \tilde{v}^2(s)} - 1 \right) \right\|_{L^{\bar{p}}} ds \end{aligned}$$

for some $p > 2$ and some $\bar{q}$, $\bar{p}$ such that $\frac{1}{\bar{q}} + \frac{1}{\bar{p}} = \frac{1}{p}$. Since $2\bar{p} > \bar{p} \geq p > 2$, one can apply an estimate similar to (5.2) via Proposition 2.1 and Proposition 2.3, and obtain that

$$\sup_{0<s<t} \|\tilde{u}(s) - \tilde{v}(s)\|_{\exp L^2} \leq C(T, t_0, \tilde{u}, \tilde{v})\, t^{1-\frac{1}{p}} \sup_{0<s<t} \|\tilde{u}(s) - \tilde{v}(s)\|_{\exp L^2}. \tag{5.4}$$

Therefore the two inequalities (5.3) and (5.4) imply

$$\sup_{0<s<t} \|\tilde{u}(s) - \tilde{v}(s)\|_{\exp L^2} \leq C(T, t_0, \tilde{u}, \tilde{v})(t^{1-\frac{1}{p}} + t) \sup_{0<s<t} \|\tilde{u}(s) - \tilde{v}(s)\|_{\exp L^2},$$

and for t small enough we obtain the desired estimate. □

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