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Li-Yau Estimates for a Nonlinear Parabolic Equation on Manifolds

Xiaorui Zhu · Yi Li

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Abstract In this paper, we derive Li-Yau gradient estimates for the positive solution of a nonlinear parabolic equation $u_t = \Delta u - qu - au(\ln u)^\alpha$, where q is a C^2 function and a, α are constants, on a complete manifold (M, g) with bounded below Ricci curvature. The results generalize classical Li-Yau gradient estimates and some recent works on this direction.

Keywords Nonlinear parabolic equation · Li-Yau estimates

Mathematics Subject Classification (2010) 53C44

1 Introduction

In this paper, we consider a parabolic equation of the type

$$\left(\Delta - q(x,t) - \frac{\partial}{\partial t}\right) u(x,t) = a u(x,t) \left(\ln(u(x,t))\right)^\alpha, \tag{1.1}$$

X. Zhu
China Maritime Police Academy, Ningbo, 315801, China
e-mail: zjuzxr@gmail.com

X. Zhu
Center of Mathematical Sciences, Zhejiang University, Hangzhou, 310027, China

Y. Li (✉)
Department of Mathematics, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, China
e-mail: yilicms@gmail.com

Y. Li
Shanghai Center for Mathematical Sciences, Fudan University, 220 Handan road, Shanghai, 200433, China

on $M \times (0, \infty)$, where a, α are constants, and q is a C^2 function defined on $M \times (0, \infty)$. We sometimes write $u(x, t)$ as u and $q(x, t)$ as q, etc, also write $\frac{\partial}{\partial t}$ as ∂_t.

Gradient estimates is one of the fundamental tools in studying nonlinear partial differential equations from geometry. Li and Yau [6] obtained a gradient estimate, called *Li-Yau estimate*, for heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x, t) = 0, \tag{1.2}$$

on $M \times (0, \infty)$; that is, the (1.1) with $q = a = 0$. Using gradient estimates, Li and Yau proved the optimal upper and lower bounds for heat kernel. Later, this estimate have been extended to Ricci flow by Hamilton [4], and furthermore, by Perelman [9].

After the fundamental work of Li-Yau, there are variant estimate for heat-type equations. One of them arises from gradient Ricci soliton (M, g, c, f); that is,

$$\mathrm{Rc}_g = cg + \nabla^2 f, \tag{1.3}$$

where (M, g) is an n-dimensional Riemannian manifold, c is a constant, and f is a smooth function. Letting $u = e^f$, the (1.3) can be written as (see [8])

$$\Delta u + 2cu \ln u = (A_0 - cn)u, \tag{1.4}$$

for some constant A_0. On the other hand, Yang [13] considered the similar equation

$$\left(\Delta - b - \frac{\partial}{\partial t}\right) u(x, t) = a u(x, t) \ln(u(x, t)), \tag{1.5}$$

where $a, b \in \mathbf{R}$; moreover Qian [10] and Wu [12] studied the same (1.5) where a, b are functions. Observe that (1.2), (1.4), and (1.5) are special cases of (1.1). For gradient estimates for (1.1) under the Ricci flow, we refer to [5]. Our estimates give more refinement than that in [5]. In a later paper [7], we will consider the gradient estimates for a more general nonlinear parabolic equation under a geometric flow.

Throughout this paper, M is assumed to be an n-dimensional complete Riemannian manifold with (possibly empty) boundary ∂M. We denoted by $\frac{\partial}{\partial \nu}$ the outward pointing unit normal vector to the boundary ∂M, and II the second fundamental form of ∂M with respect to $\frac{\partial}{\partial \nu}$.

We now state our main results in this paper.

Theorem 1.1 *Let (M, g) be a compact manifold with nonnegative Ricci curvature. Suppose that the boundary ∂M of M is convex, i.e., the second fundamental form*

II is nonnegative, whenever $\partial M \neq \emptyset$. Let $u(x,t)$ be a positive solution of the equation

$$(\Delta - \partial_t)\, u = a\, u \ln u,$$

on $M \times (0,\infty)$ for some constant a, with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0,$$

on $\partial M \times (0,\infty)$.

(1) If $a \leqslant 0$, then u satisfies

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a \ln u \leqslant \frac{n}{2t} - \frac{na}{2},$$

on $M \times (0,\infty)$.

(2) If $a \geqslant 0$, then u satisfies

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a \ln u \leqslant \frac{n}{2t}.$$

To the general (1.1), we obtain the following Li-Yau gradient estimate.

Theorem 1.2 *Let (M,g) be a complete manifold with boundary ∂M. Assume that $p \in M$ and the geodesic ball $B_p(2R)$ does not intersect ∂M. We denote by $-K(2R)$ with $K(2R) \geqslant 0$, a lower bound of the Ricci curvature on the ball $B_p(2R)$. Let $q(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in the x-variable and C^1 in the t-variable. Assume that*

$$\Delta q \leqslant \theta(2R), \quad |\nabla q| \leqslant \gamma(2R),$$

on $B_p(2R) \times [0,T]$ for some constants $\theta(2R)$ and $\gamma(2R)$. If $u(x,t)$ is a positive solution of the equation

$$\left(\Delta - q - \frac{\partial}{\partial t}\right) u = a u (\ln u)^{\alpha}, \quad \alpha > 0, \tag{1.6}$$

on $M \times (0,T]$ for some constant a, then for any $\beta > 1$ and $\epsilon \in (0,1)$, on $B_p(R)$, $u(x,t)$ satisfies the following estimates:

(1) for $a \geqslant 0$, we have

$$\begin{aligned} |\nabla f|^2 - \beta f_t - \beta q - \beta a f^{\alpha} \leqslant\ & \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4 C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} \\ & + \frac{n\beta^2[K + a(\beta-1)|f^{\alpha-1}|_{\infty}]}{(1-\epsilon)(\beta-1)} \\ & + \frac{n\beta^3 a\alpha|\alpha-1||f^{\alpha-2}|_{\infty}}{2(\beta-1)(1-\epsilon)} + \sqrt{\frac{[\beta\theta + (\beta-1)\gamma]n\beta^2}{2(1-\epsilon)}}. \end{aligned}$$

(2) for $a \leqslant 0$, we have

$$|\nabla f|^2 - \beta f_t - \beta q - \beta a f^{\alpha} \leqslant \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4 C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} + \frac{n\beta^2[K - \frac{a}{2}(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{(1-\epsilon)(\beta-1)} + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2(1-\epsilon)}}.$$

Here $f(x,t) := log(u(x,t))$, $|f|_\infty := \max_M |f|$, and $A = [2C_1^2 + (n-1)C_1^2(1 + R\sqrt{K}) + C_2]/R^2$ for some positive constants C_1, C_2.

When $\alpha = 1$, the above theorem recovers the main result in [10, 12]. As an application, we prove the gradient estimate for the elliptic equation

$$(\Delta - q)u = a\,u(\ln u)^{\alpha}, \quad \alpha > 0, \tag{1.7}$$

where u is a positive solution.

Corollary 1.3 *Let (M, g) be a complete non-compact n-dimensional Riemannian manifold. Suppose that $u(x,t)$ is a positive solution on M of the (1.7). Assume that*

(a) the Ricci curvature of (M, g) is bounded from below by $-K$, for some constant $K \geqslant 0$, and
(b) there exists a constant θ, and a function $\gamma(t)$ such that $|\nabla q| \leqslant \gamma$ and $\Delta q \leqslant \theta$ on M.

Then

(1) for $a \geqslant 0$, we have

$$\frac{|\nabla u|^2}{u^2} - \beta a(\ln u)^{\alpha} \leqslant \beta q + \left(\frac{\gamma}{2} + a|(\ln u)^{\alpha-1}|_\infty + \frac{a\beta\alpha|\alpha-1||(\ln u)^{\alpha-2}|_\infty}{2(\beta-1)}\right)n\beta^2 + \frac{n\beta^2 K}{\beta-1} + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2}},$$

on M for all $\beta > 1$.

(2) for $a \leqslant 0$, we have

$$\frac{|\nabla u|^2}{u^2} - \beta a(\ln u)^{\alpha} \leqslant \beta q + \left(\frac{\gamma}{2} - \frac{a}{2}\alpha|(\ln u)^{\alpha-1}|_\infty\right)n\beta^2 + \frac{n\beta^2 K}{\beta-1} + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2}},$$

on M for all $\beta > 1$.

In particular, if u is a positive solution of the equation $(\Delta - q)u = au \ln u$, then

(1') for $a > 0$, we have a lower bound

$$u \geqslant \exp\left[-\frac{q}{a} - \left(1 + \frac{\gamma}{2a}\right)n\beta - \frac{n\beta K}{(\beta-1)a} - \frac{1}{a}\left(\frac{[\beta\theta + (\beta-1)\gamma]n}{2}\right)^{1/2}\right],$$

on M for all $\beta > 1$.

(2') for $a < 0$, we have a upper bound

$$u \leqslant \exp\left[-\frac{q}{a} + \left(\frac{1}{2} - \frac{\gamma}{2a}\right)n\beta - \frac{n\beta K}{(\beta-1)a} - \frac{1}{a}\left(\frac{[\beta\theta + (\beta-1)\gamma]n}{2}\right)^{1/2}\right],$$

on M for all $\beta > 1$.

Remark 1.4 When q is a constant, Theorem 1.1 reduces to Theorem 1.1 in [13]. Corollary 1.3 give a much better bound for a positive solution of (1.7) on M if $q = 0, \alpha = 1$ and the Ricci curvature of M is nonnegative (compared with Corollary 1.6 in [10] and Corollary 1.2 in [13]). In fact, in this case, taking $q = \gamma = \theta = K = 0$, we have

$$u \geqslant \mathrm{e}^{-n} \ \ (a > 0), \quad \text{or} \quad u \leqslant \mathrm{e}^{n/2} \ \ (a < 0).$$

Note that our constant a is actually the constant $-a$ used in [10, 13].

2 Gradient Estimates

Suppose that $u(x,t)$ is a positive solution of (1.1). Let

$$f(x,t) := \ln(u(x,t)). \tag{2.1}$$

Then the (1.1) now can be written as $(\Delta - \partial_t) f = -|\nabla f|^2 + q + af$. We would like to consider a more general situation:

$$(\Delta - \partial_t)\, f = -|\nabla f|^2 + q + af^\alpha, \tag{2.2}$$

where $\alpha > 0$.

Lemma 2.1 *Let $f(x,t)$ be a smooth function on $M \times [0,\infty)$ satisfying (2.2), where a is a constant, α is a positive constant, and q is a C^2 function defined on $M \times (0,\infty)$. For any given $\beta \geqslant 1$, the function*

$$F := t\left(|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha\right), \tag{2.3}$$

satisfies the inequality

$$\begin{aligned}(\Delta - \partial_t)\, F \ \geqslant\ & -2\langle \nabla f, \nabla F\rangle - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 \\ & - \beta t \Delta q - 2(\beta-1)t\langle \nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\ & - \beta t a\alpha(\alpha-1) f^{\alpha-2}|\nabla f|^2 - \beta a\alpha t f^{\alpha-1}\left(-|\nabla f|^2 + f_t + q + af^\alpha\right),\end{aligned} \tag{2.4}$$

where $-K(x)$, with $K(x) \geqslant 0$, is a lower bound of the Ricci curvature tensor of M at the point $x \in M$, and $f_t := \partial_t f$.

Proof Differentiating (2.3) we have

$$\nabla_i F = t\left(2\nabla^j f \nabla_i \nabla_j f - \beta \nabla_i f_t - \beta \nabla_i q - \beta a \alpha f^{\alpha-1} \nabla_i f\right).$$

Then the Laplace of F equals

$$\begin{aligned}\Delta F &= \nabla^i \nabla_i F \\ &= t\Big[2\left|\nabla^2 f\right|^2 + 2\langle \nabla f, \Delta \nabla f\rangle - \beta(\Delta f)_t - \beta \Delta q \\ &\quad - \beta a \alpha((\alpha-1) f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f)\Big].\end{aligned}$$

Using the Ricci formula yields

$$\Delta \nabla_i f = \nabla_i \Delta f + R_{ij} \nabla^j f,$$

from which the Laplacian of F can be simplified as

$$\begin{aligned}\Delta F &= t\Big[2\left|\nabla^2 f\right|^2 + 2\langle \nabla f, \nabla \Delta f\rangle + 2\mathrm{Ric}(\nabla f, \nabla f) - \beta(\Delta f)_t - \beta \Delta q \\ &\quad - \beta a \alpha\left((\alpha-1) f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right)\Big] \\ &\geqslant t\Big[\frac{2}{n}(\Delta f)^2 + 2\langle \nabla f, \nabla \Delta f\rangle - 2K|\nabla f|^2 - \beta(\Delta f)_t - \beta \Delta q \\ &\quad - \beta a \alpha\left((\alpha-1) f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right)\Big],\end{aligned}$$

since $|\nabla^2 f|^2 \geqslant \frac{(\Delta f)^2}{n}$. Recall from (2.2) that

$$\begin{aligned}\Delta f &= -|\nabla f|^2 + q + f_t + a f^{\alpha} \\ &= -\frac{F}{t} - (\beta-1)\left(q + f_t + a f^{\alpha}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta F \geqslant\ & \frac{2t}{n}\left(|\nabla f|^2 - q - f_t - a f^{\alpha}\right)^2 \\ & - 2t\left\langle \nabla f, \nabla\left(\frac{F}{t} + (\beta-1)(q + f_t + a f^{\alpha})\right)\right\rangle \\ & - 2Kt|\nabla f|^2 - t\beta\left(-\frac{F}{t} - (\beta-1)(q + f_t + a f^{\alpha})\right)_t - \beta t \Delta q \\ & - \beta a \alpha t\left[(\alpha-1) f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right].\end{aligned}$$

Since

$$\left(\frac{F}{t} + (\beta-1)(q + f_t + a f^{\alpha})\right)_t = \frac{F_t}{t} - \frac{F}{t^2} + (\beta-1)(q_t + f_{tt} + a\alpha f^{\alpha-1} f_t),$$

and

$$\frac{F}{t} = |\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha,$$

we obtain

$$\begin{aligned}\Delta F \geqslant\ & \frac{2t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 - 2\langle\nabla f, \nabla F\rangle - 2(\beta-1)t\langle\nabla f, \nabla f_t\rangle \\ & - 2(\beta-1)t\langle\nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 - 2Kt|\nabla f|^2 + \beta F_t \\ & - \beta\left(|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha\right) + \beta(\beta-1)tq_t + \beta(\beta-1)tf_{tt} \\ & + t\beta(\beta-1)a\alpha f^{\alpha-1}f_t - \beta t\Delta q \\ & - \beta a\alpha t(\alpha-1)f^{\alpha-2}|\nabla f|^2 - \beta a\alpha t f^{\alpha-1}\Delta f.\end{aligned}$$

On the other hand,

$$\begin{aligned}F_t =\ & |\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \\ & + t\left(\partial_t|\nabla f|^2 - \beta f_{tt} - \beta q_t - \beta a\alpha f^{\alpha-1}f_t\right).\end{aligned}$$

Combining above two formulas we conclude that

$$\begin{aligned}(\Delta - \partial_t)F \geqslant\ & \frac{2t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 - 2\langle\nabla f, \nabla F\rangle - (\beta-1)t\cdot\partial_t|\nabla f|^2 \\ & - 2(\beta-1)t\langle\nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 - 2Kt|\nabla f|^2 \\ & + (\beta-1)\left(|\nabla f|^2 - \beta f_t - \beta q - \beta af^\alpha\right) + t\beta(\beta-1)a\alpha f^{\alpha-1}f_t \\ & + (\beta-1)t\left(\partial_t|\nabla f|^2 - \beta f_{tt} - \beta q_t - \beta a\alpha f^{\alpha-1}f_t\right) - \beta t\Delta q \\ & - \beta\left(|\nabla f|^2 - \beta f_t - \beta q - \beta af^\alpha\right) + \beta(\beta-1)tq_t + \beta(\beta-1)tf_{tt} \\ & - \beta a\alpha t(\alpha-1)f^{\alpha-2}|\nabla f|^2 - \beta a\alpha f^{\alpha-1}\Delta f \\ =\ & -2\langle\nabla f, \nabla F\rangle - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 \\ & - \beta t\Delta q - 2(\beta-1)t\langle\nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\ & - \beta a\alpha t(\alpha-1)f^{\alpha-2}|\nabla f|^2 - \beta a\alpha f^{\alpha-1}\Delta f.\end{aligned}$$

Now, (2.4) immediately follows from (2.2). □

Theorem 2.2 *Let (M, g) be a compact manifold with nonnegative Ricci curvature. Suppose that the boundary ∂M of M is convex, i.e., the second fundamental form II is nonnegative, whenever $\partial M \neq \emptyset$. Let $u(x,t)$ be a positive solution of the equation*

$$(\Delta - \partial_t)u = au\ln u,$$

on $M \times (0, \infty)$ for some constant a, with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0,$$

on $\partial M \times (0, \infty)$.

(1) If $a \leqslant 0$, then u satisfies

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a\ln u \leqslant \frac{n}{2t} - \frac{na}{2}, \tag{2.5}$$

on $M \times (0, \infty)$.

(2) If $a \geqslant 0$, then u satisfies

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a\ln u \leqslant \frac{n}{2t}. \tag{2.6}$$

Proof Setting $q = 0, \alpha = \beta = 1$, and $K = 0$ in Lemma 2.1 yields

$$\begin{aligned}(\Delta - \partial_t)\, F &\geqslant -2\langle\nabla f, \nabla F\rangle - \frac{F}{t} + \frac{2t}{n}\left(|\nabla f|^2 - f_t - af\right)^2 + at\left(|\nabla f|^2 - f_t - af\right) \\ &= -2\langle\nabla f, \nabla F\rangle - \frac{F}{t} + \frac{2F^2}{nt} + aF \\ &= -2\langle\nabla f, \nabla F\rangle + \frac{2F}{nt}\left(F - \frac{n}{2} + \frac{ant}{2}\right),\end{aligned}$$

where $F = t\left(|\nabla f|^2 - f_t - af\right)$.

(1) $a \leqslant 0$. In this case we claim that $F \leqslant \frac{n}{2} - \frac{ant}{2}$. If not at the maximum point (x_0, t_0) of F on $M \times [0, T]$ for some $T > 0$, we have

$$F(x_0, t_0) > \frac{n}{2} - \frac{ant}{2} \geqslant \frac{n}{2} > 0.$$

Consequently, $t_0 > 0$. If x_0 is an interior point of M, we conclude from (x_0, t_0) being a maximum point of F in $M \times [0, T]$ that

$$\Delta F(x_0, t_0) \leqslant 0, \quad \nabla F(x_0, t_0) = 0, \quad F_t(x_0, t_0) \geq 0.$$

Together with the proved inequality $(\Delta - \partial_t)F \geqslant -2\langle\nabla f, \nabla F\rangle + \frac{2F}{nt}(F - \frac{n}{2} + \frac{ant}{2})$, we arrive at

$$0 \geqslant \frac{2}{nt_0} F(x_0, t_0)\left[F(x_0, t_0) - \frac{n}{2} + \frac{ant_0}{2}\right].$$

By the assumption, it implies that $F(x_0, t_0) \leqslant \frac{n}{2} - \frac{ant_0}{2}$, a contradiction.

Therefore we proved that x_0 is on the boundary of M. Now the strong maximum principle tells us

$$\frac{\partial F}{\partial \nu}(x_0, t_0) > 0.$$

Let $e_1, \cdots, e_n$, where $e_n := \partial/\partial\nu$, be an orthonormal frame field on M, and f_j means the covariant differentiation in the e_i direction. Calculate

$$F_\nu = t\left[2\sum_{1\leqslant j\leqslant n} f_j f_{j\nu} - (f_t)_\nu - af_\nu\right] = 2t\sum_{1\leqslant j\leqslant n-1} f_j f_{j\nu} + 2tf_\nu f_{\nu\nu} - (f_t)_\nu - af_\nu.$$

Since $u_\nu = 0$ on ∂M, it follows that $f_\nu = 0$ on ∂M and hence

$$F_\nu = 2t\sum_{1\leqslant j\leqslant n-1} f_j f_{j\nu} = -2t\sum_{1\leqslant j,k\leqslant n-1} h_{jk} f_j f_k = -2t\mathrm{II}(\nabla f, \nabla f),$$

because of $f_{j\nu} = -\sum_{1\leqslant k\leqslant n-1} h_{jk} f_k$, where h_{jk} are components of the second fundamental form of ∂M. Evaluating at the point (x_0, t_0), we get

$$\mathrm{II}(\nabla f, \nabla f)(x_0, t_0) < 0,$$

which contradicts the convexity of ∂M. Hence, $F \leqslant \frac{n}{2} - \frac{ant}{2}$.

(2) $a \geqslant 0$. Since the right side of (2.6) is positive, we may assume without loss of generality that $F \geqslant 0$. In this case we obtain

$$(\Delta - \partial_t)\, F \geqslant -2\langle \nabla f, \nabla F\rangle + \frac{2F}{nt}\left(F - \frac{n}{2}\right),$$

which reduces to the case in [6] and by the same computation we conclude that $F \leqslant \frac{n}{2}$. □

Theorem 2.3 *Let (M, g) be a complete manifold with boundary ∂M. Assume that $p \in M$ and the geodesic ball $B_p(2R)$ does not intersect ∂M. We denote by $-K(2R)$ with $K(2R) \geqslant 0$, a lower bound of the Ricci curvature on the ball $B_p(2R)$. Let $q(x, t)$ be a function defined on $M \times [0, T]$ which is C^2 in the x-variable and C^1 in the t-variable. Assume that*

$$\Delta q \leqslant \theta(2R), \quad |\nabla q| \leqslant \gamma(2R),$$

on $B_p(2R) \times [0, T]$ for some constants $\theta(2R)$ and $\gamma(2R)$. If $u(x, t)$ is a positive solution of the equation

$$\left(\Delta - q - \frac{\partial}{\partial t}\right) u = a u (\ln u)^{\alpha}, \quad \alpha > 0, \tag{2.7}$$

on $M \times (0, T]$ for some constant a, then for any $\beta > 1$ and $\epsilon \in (0, 1)$, on $B_p(R)$, $u(x, t)$ satisfies the following estimates:

(1) for $a \geqslant 0$, we have

$$\begin{aligned} |\nabla f|^2 - \beta f_t - \beta q - \beta a f^{\alpha} \leqslant\ & \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4 C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} \\ & + \frac{n\beta^2[K + a(\beta-1)|f^{\alpha-1}|_\infty]}{(1-\epsilon)(\beta-1)} \\ & + \frac{n\beta^3 a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2(\beta-1)(1-\epsilon)} + \sqrt{\frac{[\beta\theta + (\beta-1)\gamma]n\beta^2}{2(1-\epsilon)}}. \end{aligned}$$

(2) for $a \leqslant 0$, we have

$$\begin{aligned} |\nabla f|^2 - \beta f_t - \beta q - \beta a f^{\alpha} \leqslant\ & \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4 C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} \\ & + \frac{n\beta^2[K - \frac{a}{2}(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{(1-\epsilon)(\beta-1)} \\ & + \sqrt{\frac{[\beta\theta + (\beta-1)\gamma]n\beta^2}{2(1-\epsilon)}}. \end{aligned}$$

Here $f(x,t) := log(u(x,t))$, $|f|_\infty := \max_M |f|$, *and* $A = [2C_1^2 + (n-1)C_1^2(1 + R\sqrt{K}) + C_2]/R^2$ *for some positive constants* C_1, C_2.

Proof As before, we set $f = \log u$ and $F = t(|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha)$. As in [2, 6, 8, 13], we let $\widetilde{\varphi}(r)$ be a C^2 function defined on $[0,\infty)$ such that

$$\widetilde{\varphi}(r) = \begin{cases} 1, & r \in [0,1], \\ 0, & r \in [2,\infty), \end{cases},$$

and

$$-C_1 \leqslant \widetilde{\varphi}'(r)\widetilde{\varphi}^{-1/2}(r) \leqslant 0, \quad \widetilde{\varphi}(r) \geqslant -C_2,$$

for some positive constants C_1, C_2. If $r(x) := \mathrm{dist}(p,x)$ denotes the distance between p and x, we set

$$\varphi(x) := \widetilde{\varphi}\left(\frac{r(x)}{R}\right).$$

Using Calabi's argument (see, e.g., [1, 3, 11]), we may assume without loss of generality that $\varphi(x)$ is smooth in the ball $B_p(2R)$. Then by the Laplacian comparison theorem (see [11]) we have

$$\frac{|\nabla\varphi|^2}{\varphi} \leqslant \frac{C_1^2}{R^2}, \quad \Delta\varphi \geqslant -\frac{(n-1)C_1^2(1+R\sqrt{K})+C_2}{R^2}.$$

Combining Lemma 2.1 with $\Delta(\varphi F) = \Delta\varphi\cdot F + 2\langle\nabla\varphi,\nabla F\rangle + \varphi\cdot\Delta F$ yields

$$\begin{aligned}
\Delta(\varphi F) \geqslant{}& F\left[-\frac{(n-1)C_1^2(1+R\sqrt{K})+C_2}{R^2}\right] + 2\left\langle\nabla\varphi, \nabla\left(\frac{\varphi F}{\varphi}\right)\right\rangle \\
&+ \varphi\Big[F_t - 2\langle\nabla f,\nabla F\rangle - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n}\left(|\nabla f|^2 - f_t - q - af^\alpha\right)^2 \\
&- \beta t\Delta q - 2(\beta-1)t\langle\nabla f,\nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\
&- \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 + \beta at\alpha f^{\alpha-1}\left(|\nabla f|^2 - f_t - q - af^\alpha\right)\Big] \\
={}& -F\left[\frac{(n-1)C_1^2(1+R\sqrt{K})+C_2}{R^2}\right] + \frac{2}{\varphi}\langle\nabla\varphi,\nabla(\varphi F)\rangle - \frac{2F|\nabla\varphi|^2}{\varphi} \\
&+ \varphi\Big[F_t - 2\langle\nabla f,\nabla F\rangle - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n}\left(|\nabla f|^2 - f_t - q - af^\alpha\right)^2 \\
&- \beta t\Delta q - 2(\beta-1)t\langle\nabla f,\nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\
&- \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 + \beta at\alpha f^{\alpha-1}\left(|\nabla f|^2 - f_t - q - af^\alpha\right)\Big].
\end{aligned}$$

Fix a $T' \leqslant T$. Let (x_0,t_0) be a point in $M\times[0,T']$ where φF achieves its maximum. We may assume that $(\varphi F)(x_0,t_0) > 0$ (so that $t_0 > 0$), otherwise it is clear. Ay (x_0,t_0), we have

$$\nabla(\varphi F)(x_0,t_0) = 0, \quad (\varphi F)_t(x_0,t_0) \geqslant 0, \quad \Delta(\varphi F)(x_0,t_0) \leqslant 0.$$

An obvious consequence is $\nabla\varphi\cdot F+\varphi\cdot\nabla F=0$ at the point (x_0,t_0). From the inequality $|\nabla\varphi|^2/\varphi\leqslant C_1^2/R^2$ and introducing a constant

$$A:=\frac{2C_1^2+(n-1)C_1^2(1+R\sqrt{K})+C_2}{R^2}, \tag{2.8}$$

we obtain the following inequality

$$\begin{aligned}0\geqslant&-AF+2F\langle\nabla f,\nabla\varphi\rangle+\frac{2t_0}{n}\varphi\left(|\nabla f|^2-f_t-q-af^\alpha\right)^2-\frac{\varphi F}{t_0}\\&-2Kt_0\varphi|\nabla f|^2-\beta t_0\varphi\Delta q-2(\beta-1)t_0\varphi\langle\nabla f,\nabla q\rangle\\&-2(\beta-1)t_0a\varphi\alpha f^{\alpha-1}|\nabla f|^2-\beta t_0a\varphi\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2\\&+\beta at_0\varphi\alpha f^{\alpha-1}\left(|\nabla f|^2-f_t-q-af^\alpha\right),\end{aligned} \tag{2.9}$$

at (x_0,t_0). Set (see [2, 13])

$$\mu:=\frac{|\nabla f|^2(x_0,t_0)}{F(x_0,t_0)}\geqslant 0.$$

We calculate

$$|\nabla f|^2-f_t-q-af^\alpha=F\left(\mu-\frac{\mu t_0-1}{\beta t_0}\right),$$

and

$$\langle\nabla f,\nabla\varphi\rangle\leqslant|\nabla f||\nabla\varphi|\leqslant\frac{C_1}{R}\varphi^{1/2}|\nabla f|,$$

at the point (x_0,t_0). Simplifying (2.9) at (x_0,t_0) yields

$$\begin{aligned}0\geqslant&-AF-\frac{2C_1}{R}\varphi^{1/2}\mu^{1/2}F^{3/2}+\frac{2t_0\varphi}{n}\cdot\frac{[1+(\beta-1)\mu t_0]^2}{\beta^2t_0^2}F^2-\frac{\varphi F}{t_0}-2Kt_0\varphi\mu F\\&-\beta t_0\varphi\theta-2(\beta-1)t_0\varphi\gamma F^{1/2}\mu^{1/2}-2(\beta-1)t_0a\varphi\alpha f^{\alpha-1}\mu F\\&-\beta t_0a\varphi\alpha(\alpha-1)f^{\alpha-2}\mu F+a\varphi\alpha f^{\alpha-1}[1+(\beta-1)\mu t_0]F.\end{aligned}$$

Multiplying by φt_0 on both sides, we have

$$\begin{aligned}AFt_0\varphi\geqslant&-\frac{2C_1t_0}{R}\varphi^{3/2}\mu^{1/2}F^{3/2}-\varphi^2F+\frac{2\varphi^2}{n\beta^2}[1+(\beta-1)\mu t_0]^2F^2\\&-2(t_0\varphi)^2[K+a(\beta-1)\alpha f^{\alpha-1}]\mu F+at_0\varphi^2\alpha f^{\alpha-1}[1+(\beta-1)t_0\mu]F\\&-\beta(t_0\varphi)^2\theta-2(\beta-1)(t_0\varphi)^2\gamma(\mu F)^{1/2}-\beta(t_0\varphi)^2a\alpha(\alpha-1)f^{\alpha-2}\mu F.\end{aligned} \tag{2.10}$$

If we set $G:=\varphi F$, then at the point (x_0,t_0) the inequality (2.10) becomes

$$\begin{aligned}At_0G\geqslant&-\frac{2C_1t_0}{R}\mu^{1/2}G^{3/2}-\varphi G+\frac{2}{n\beta^2}[1+(\beta-1)\mu t_0]^2G^2\\&-2\varphi t_0^2[K+a(\beta-1)\alpha f^{\alpha-1}]\mu G+a\varphi t_0\alpha f^{\alpha-1}[1+(\beta-1)\mu t_0]G\\&-\beta(\varphi t_0)^2\theta-2(\beta-1)t_0^2\varphi^{3/2}\gamma\mu^{1/2}G^{1/2}-\beta t_0^2\varphi a\alpha(\alpha-1)f^{\alpha-2}\mu G.\end{aligned} \tag{2.11}$$

Using the inequalities, where $0<\epsilon<1$,

$$\begin{aligned}\frac{2C_1t_0}{R}\mu^{1/2}G^{3/2}&\leqslant\frac{2\epsilon}{n\beta^2}[1+(\beta-1)\mu t_0]^2G^2+\frac{n\beta^2C_1^2t_0^2\mu G}{2\epsilon R^2[1+(\beta-1)\mu t_0]^2},\\2\mu^{1/2}G^{1/2}&\leqslant 1+\mu G,\end{aligned}$$

we simplify (2.11) as the following inequality

$$\begin{aligned} At_0G \geqslant\ & \frac{2(1-\epsilon)}{n\beta^2}[1+(\beta-1)\mu t_0]^2G^2 - \varphi G - \frac{n\beta^2C_1^2t_0^2\mu}{2\epsilon R^2[1+(\beta-1)\mu t_0]^2}G \\ & - 2\varphi t_0^2[K+a(\beta-1)\alpha f^{\alpha-1}]\mu G + a\varphi t_0\alpha f^{\alpha-1}[1+(\beta-1)\mu t_0]G \\ & - \beta\varphi^2t_0^2\theta - (\beta-1)t_0^2\varphi^{\frac{3}{2}}\gamma - (\beta-1)t_0^2\varphi^{\frac{3}{2}}\gamma\mu G - \beta t_0^2\varphi a\alpha(\alpha-1)f^{\alpha-2}\mu G, \end{aligned}$$

or equivalently,

$$\begin{aligned} \frac{2(1-\epsilon)[1+(\beta-1)\mu t_0]^2G^2}{n\beta^2} \leqslant\ & \Big[At_0 + \varphi + \frac{n\beta^2C_1^2t_0^2\mu}{2\epsilon R^2[1+(\beta-1)\mu t_0]^2} \\ & + 2\varphi t_0^2[K+a(\beta-1)\alpha f^{\alpha-1}]\mu \\ & - a\varphi t_0\alpha f^{\alpha-1}[1+(\beta-1)\mu t_0] + (\beta-1)t_0^2\varphi^{\frac{3}{2}}\gamma\mu \\ & + \beta t_0^2\varphi a\alpha(\alpha-1)f^{\alpha-2}\mu\Big]G \\ & + \Big[\beta\varphi^2\theta + (\beta-1)\varphi^{\frac{3}{2}}\gamma\Big]t_0^2. \end{aligned}$$

Note that $0 \leqslant \varphi \leqslant 1$ and $1+(\beta-1)\mu t_0 \geqslant 1$. Therefore

$$\begin{aligned} \frac{2(1-\epsilon)G^2}{n\beta^2} \leqslant\ & \Bigg[At_0 + 1 + \frac{n\beta^2C_1^2t_0^2\mu}{2\epsilon R^2[1+(\beta-1)\mu t_0]} + \frac{2\varphi t_0^2[K+a(\beta-1)\alpha f^{\alpha-1}]\mu}{[1+(\beta-1)\mu t_0]^2} \\ & - \frac{a\varphi t_0\alpha f^{\alpha-1}}{1+(\beta-1)\mu t_0} + \frac{(\beta-1)\gamma t_0^2\mu}{1+(\beta-1)\mu t_0} + \frac{\beta t_0^2\varphi a\alpha|\alpha-1|f^{\alpha-2}\mu}{1+(\beta-1)\mu t_0}\Bigg]G \\ & + [\beta\theta + (\beta-1)\gamma]t_0^2 \qquad (2.12) \\ \leqslant\ & \Bigg[At_0 + 1 + \frac{n\beta^2C_1^2t_0}{2\epsilon R^2(\beta-1)} + \frac{2\varphi t_0^2[K+a(\beta-1)\alpha|f|^{\alpha-1}]\mu}{[1+(\beta-1)\mu t_0]^2} + \gamma t_0 \\ & - \frac{a\varphi t_0\alpha f^{\alpha-1}}{1+(\beta-1)\mu t_0} + \frac{\beta t_0\varphi a\alpha|\alpha-1||f|^{\alpha-2}}{\beta-1}\Bigg]G \\ & + [\beta\theta + (\beta-1)\gamma]t_0^2. \end{aligned}$$

Before completing the proof, we recall a fact: if $x^2 \leqslant ax + b$ for some $b, x \geqslant 0$ and $a \in \mathbf{R}$, then

$$x \leqslant \frac{a}{2} + \sqrt{b + \left(\frac{a}{2}\right)^2} \leqslant \frac{a}{2} + \sqrt{b} + \frac{a}{2} = a + \sqrt{b}. \qquad (2.13)$$

If $a \geqslant 0$ in (2.12), then from (2.12) we deduce that

$$\begin{aligned} G^2 \leqslant\ & \Bigg[\frac{An\beta^2t_0}{2(1-\epsilon)} + \frac{n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4C_1^2t_0}{4\epsilon(1-\epsilon)R^2(\beta-1)} + \frac{n\beta^3a\alpha|\alpha-1||f|^{\alpha-2}t_0}{2(\beta-1)(1-\epsilon)} \\ & + \frac{n\beta^2\gamma t_0}{2(1-\epsilon)} + \frac{n\beta^2[K+a(\beta-1)\alpha|f|^{\alpha-1}]t_0}{(1-\epsilon)(\beta-1)}\Bigg]G + \frac{[\beta\theta+(\beta-1)\gamma]n\beta^2t_0^2}{2(1-\epsilon)}. \qquad (2.14) \end{aligned}$$

Applying (2.13) to the inequality (2.14), we get an upper bound for G:

$$G \leqslant \left[\frac{(A+\gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} + \frac{n\beta^2[K+a(\beta-1)\alpha|f|^{\alpha-1}]}{(1-\epsilon)(\beta-1)}\right]T' + \frac{n\beta^3 a\alpha|\alpha-1||f|^{\alpha-2}}{2(\beta-1)(1-\epsilon)}T' + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2(1-\epsilon)}}T' + \frac{n\beta^2}{2(1-\epsilon)},$$

since $t_0 \leqslant T'$. By the construction of φ, we have

$$\sup_{B_p(R)} F(x,t) \leqslant \sup_{B_p(R)} (\varphi(x)F(x,t)) \leqslant G(x_0,t_0),$$

for all $t \in [0, T']$. Because $T' \leqslant T$ is arbitrary, it follows that

$$|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \leqslant \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} + \frac{n\beta^2[K+a(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{(1-\epsilon)(\beta-1)} + \frac{n\beta^3 a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2(\beta-1)(1-\epsilon)} + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2(1-\epsilon)}},$$

where $|f|_\infty := \max_M |f|$. Similarly, when $a \leqslant 0$, we have

$$G^2 \leqslant \left[\frac{(A+\gamma)n\beta^2 t_0}{2(1-\epsilon)} + \frac{n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4C_1^2 t_0}{4\epsilon(1-\epsilon)R^2(\beta-1)} + \frac{n\beta^2 K t_0}{(1-\epsilon)(\beta-1)} - \frac{n\beta^2 a t_0 \alpha|f|^{\alpha-1}}{2(1-\epsilon)}\right]G + \frac{[\beta\theta+(\beta-1)\gamma]n\beta^2 t_0^2}{2(1-\epsilon)}. \tag{2.15}$$

From (2.13), (2.15), and above argument, an upper bound for desired quantity in this case is

$$|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \leqslant \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} + \frac{n\beta^2[K-\frac{a}{2}(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{(1-\epsilon)(\beta-1)} + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2(1-\epsilon)}}.$$

Hence, we complete the proof. □

When $\alpha = 1$, the above theorem reduces the main result in [10, 12]. Letting $R \to \infty$ and then $\epsilon \to 0$, we have the following

Corollary 2.4 *Let (M, g) be a complete non-compact n-dimensiobal Riemanian manifold. Suppose that $u(x,t)$ is a positive solution on $M \times (0, T]$ of the (2.7). Assume that* *(a)* *the Ricci curvature of* (M, g) *is bounded from below by* $-K$*, for some constant* $K \geqslant 0$*, and*

(b) *there exists a constant* θ*, and a function* $\gamma(t)$ *such that*

$$|\nabla q|(x,t) \leqslant \gamma(t), \quad \Delta q(x,t) \leqslant \theta,$$

for any $(x,t) \in M \times (0,T]$.

Then

(1) *for* $a \geqslant 0$*, we have*

$$\frac{|\nabla u|^2}{u^2} - \beta\frac{u_t}{u} - \beta a(\ln u)^\alpha \leqslant \beta q + \frac{n\beta^2}{2t} + \left(\frac{\gamma}{2} + a\alpha|f^{\alpha-1}|_\infty\right)n\beta^2 + \frac{n\beta^2 K}{\beta-1} + \frac{n\beta^3 a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2(\beta-1)} + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2}},$$

on $M \times (0,T]$ *for all* $\beta > 1$.

(2) *for* $a \leqslant 0$*, we have*

$$\frac{|\nabla u|^2}{u^2} - \beta\frac{u_t}{u} - \beta a(\ln u)^\alpha \leqslant \beta q + \frac{n\beta^2}{2t} + \left(\frac{\gamma}{2} - \frac{a}{2}\alpha|f^{\alpha-1}|_\infty\right)n\beta^2 + \frac{n\beta^2 K}{\beta-1} + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2}},$$

on $M \times (0,T]$ *for all* $\beta > 1$.

We now apply Corollary 2.4 to the elliptic equation

$$(\Delta - q)\,u = a\,u(\ln u)^\alpha, \tag{2.16}$$

where u is a C^2 function on M, by letting $T \to \infty$.

Corollary 2.5 *Let* (M, g) *be a complete non-compact n-dimensional Riemannian manifold. Suppose that* $u(x,t)$ *is a positive solution on* M *of the equation (2.16). Assume that*

(a) *the Ricci curvature of* (M, g) *is bounded from below by* $-K$*, for some constant* $K \geqslant 0$*, and*

(b) *there exists a constant* θ*, and a function* $\gamma(t)$ *such that* $|\nabla q| \leqslant \gamma$ *and* $\Delta q \leqslant \theta$ *on* M.

Then

(1) *for* $a \geqslant 0$*, we have*

$$\frac{|\nabla u|^2}{u^2} - \beta a(\ln u)^\alpha \leqslant \beta q + \left(\frac{\gamma}{2} + a|(\ln u)^{\alpha-1}|_\infty + \frac{a\beta\alpha|\alpha-1||(\ln u)^{\alpha-2}|_\infty}{2(\beta-1)}\right)n\beta^2 + \frac{n\beta^2 K}{\beta-1} + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma]n\beta^2}{2}},$$

on M *for all* $\beta > 1$.

(2) for $a \leqslant 0$*, we have*

$$\frac{|\nabla u|^2}{u^2} - \beta a(\ln u)^\alpha \leqslant \beta q + \left(\frac{\gamma}{2} - \frac{a}{2}\alpha|(\ln u)^{\alpha-1}|_\infty\right) n\beta^2 + \frac{n\beta^2 K}{\beta - 1} + \sqrt{\frac{[\beta\theta + (\beta-1)\gamma]n\beta^2}{2}},$$

on M *for all* $\beta > 1$.

In particular, if u *is a positive solution of the equation* $(\Delta - q)u = au\ln u$*, then*

(1') for $a > 0$*, we have a lower bound*

$$u \geqslant \exp\left[-\frac{q}{a} - \left(1 + \frac{\gamma}{2a}\right)n\beta - \frac{n\beta K}{(\beta-1)a} - \frac{1}{a}\left(\frac{[\beta\theta + (\beta-1)\gamma]n}{2}\right)^{1/2}\right],$$

on M *for all* $\beta > 1$.

(2') for $a < 0$*, we have a upper bound*

$$u \leqslant \exp\left[-\frac{q}{a} + \left(\frac{1}{2} - \frac{\gamma}{2a}\right)n\beta - \frac{n\beta K}{(\beta-1)a} - \frac{1}{a}\left(\frac{[\beta\theta + (\beta-1)\gamma]n}{2}\right)^{1/2}\right],$$

on M *for all* $\beta > 1$.

Remark 2.6 When q is a constant, Theorem 2.3 reduces to Theorem 1.1 in [13]. Corollary 2.5 give a much better bound for a positive solution of (2.16) on M if $q = 0$ and the Ricci curvature of M is nonnegative (compared with Corollary 1.6 in [10] and Corollary 1.2 in [13]). In fact, in this case, taking $q = \gamma = \theta = K = 0$, we have

$$u \geqslant e^{-n} \quad (a > 0), \quad \text{or} \quad u \leqslant e^{n/2} \quad (a < 0).$$

Note that our constant a is actually the constant $-a$ used in [10, 13].

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