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Darboux Transformations for Energy-Dependent Potentials and the Klein–Gordon Equation

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Abstract We construct explicit Darboux transformations for a generalized Schrödinger-type equation with energy-dependent potential, a special case of which is the stationary Klein–Gordon equation. Our results complement and generalize former findings (Lin et al., Phys Lett A 362:212–214, 2007).

Keywords Generalized Schrödinger equation · Energy-dependent potential · Darboux transformation · Klein–Gordon equation

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1 Introduction

Energy-dependent potentials appear in various contexts of Quantum Mechanics and affine areas. They can be found in the magneto-hydrodynamic model of the dynamo effect [15], in hydrodynamics [19, 32], in the Hamiltonian formulation of relativistic quantum mechanics [8, 24], in the area of quantum wells and semiconductors [6, 23, 25, 30], in models of heavy quark systems [11], and many more. In non-relativistic Quantum Mechanics, the presence of energy-dependent potentials forces theoretical adjustments, such as the way the norm is obtained and the form of the completeness relation [12]. Mathematically, handling a spectral problem associated with, say, the Schrödinger equation for an energy-dependent potential, becomes more much complicated than in the usual case of Sturm–Liouville type. Since the first step in solving spectral problems is finding the general solution of the underlying

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equation, it is a principal issue to construct methods that allow for the generation of such solutions. Already existing methods that have proved useful for handling the Schrödinger equation with an energy-independent potential, can usually not transferred to the energy-dependent context. An example for such a method is the Darboux transformation [9], which has become famous as the mathematical engine of the quantum-mechanical supersymmetry (SUSY) formalism [7]. Besides the latter field of application, the formalism of Darboux transformations has been generalized to work for a large variety of linear and nonlinear models, such as nonlinear Schrödinger equations, the sine-Gordon equation, the Korteweg-de-Vries equation and many more, see [1, 14, 22] for detailed discussions. While Darboux transformations have been mainly constructed for the single-variable case, a series of results have been obtained for two- and higher dimensional Schrödinger equations. The formalism arises as an extension of the conventional SUSY formalism and was first established in [2–4]. In contrast to its above mentioned counterparts, this higher-dimensional version of the Darboux transformation can generate full solutions of spectral problems. After its first introduction, the formalism was extended in many ways, a complete review of which is beyond the scope of this note. As examples let us mention the introduction of higher-order transformations [5] and an arbitrary metric [13], the combination of which led to the construction of non-separable Schrödinger equations that render exactly-solvable [16, 17]. A further way of generalizing the conventional Darboux transformation has been found by considering matrix equations [21] and multi-component equations of Dirac type. As examples, let us mention recent results that concern the $(1 + 1)$-dimensional Dirac equation [31], its stationary counterpart [10], and corresponding findings in two variables [26]. While the potentials considered in the previously mentioned systems are independent of the energy, the purpose of the present note is to develop a Darboux formalism for Schrödinger-type equations with energy-dependent potentials. In fact, the Darboux transformation in its standard form is not applicable to such potentials, except in very particular cases. One of these cases takes place if the potential depends linearly on the energy [29], another case is given by potentials that depend on the energy's square root [20]. While linear energy-dependence can be subsumed under the usual case, the Darboux transformation constructed in [20] is essentially different. The latter case is especially interesting, as by an energy reparametrization the corresponding Schrödinger equation can be converted into a Klein–Gordon equation, which, as is well-known, depends quadratically on the energy. It is interesting to note that the Darboux transformation constructed in [20] allows for a slight generalization to an effective mass-type Schrödinger equation [27]. Therefore, the purpose of the present note is to study whether the results obtained in [20] admit an extension to a fully generalized linear Schrödinger equation which contains the effective mass setting as a special case. We will give a positive answer to this question and state Darboux transformation as well as the transformed potentials in explicit form. As a byproduct, we obtain Darboux transformations for the stationary Klein–Gordon equation, which to the best of our knowledge have not been constructed earlier. The remainder of this paper is organized as follows: in Section 2 we review the main findings from [20] and summarize our results in Section 3, the proof of which is done in Section 4. The final Section 5 is devoted to the stationary Klein–Gordon equation and its Darboux transformation.

2 Preliminaries: The Conventional Case

For the sake of completeness, we summarize the results from [20]. Consider the following two stationary Schrödinger equations in atomic units:

$$\Psi'' - (E^2 + E\ V_0 + U_0)\ \Psi = 0, \tag{1}$$

$$\Phi'' - (E^2 + E\ V_1 + U_1)\ \Phi = 0, \tag{2}$$

where $\Psi = \Psi(x)$, $\Phi = \Phi(x)$ denote the respective solutions, the constant E stands for the energy, and $V_j = V_j(x)$, $U_j = U_j(x)$ for $j = 0, 1$ are the components of the energy-dependent potentials. Let $h = h(x)$ be an auxiliary solution of (1) at energy $\lambda \neq E$, such that h and Ψ are linearly independent. Then (1) and (2) are related to each other via the Darboux transformation

$$D(\Psi) = \sqrt{\frac{1}{2\,\frac{h'}{h} - V_0 - 2\,\lambda}} \left[\left(-\frac{h'}{h} + \lambda - E \right) \Psi + \Psi' \right], \tag{3}$$

that is, $\Phi = D(\Psi)$ solves (2), if the potential terms in (1) and (2) fulfill the following constraints:

$$V_1 = V_0 + \frac{d}{dx} \log\left(2\,\frac{h'}{h} - V_0 - 2\,\lambda \right), \tag{4}$$

$$U_1 = U_0 + \sqrt{2\,\frac{h'}{h} - V_0 - 2\,\lambda} \times \left[\frac{d^2}{dx^2} \left(\sqrt{\frac{1}{2\,\frac{h'}{h} - V_0 - 2\,\lambda}} \right) + 2\,\frac{d}{dx} \left(\frac{\lambda - \frac{h'}{h}}{\sqrt{2\,\frac{h'}{h} - V_0 - 2\,\lambda}} \right) \right]. \tag{5}$$

Thus, given an auxiliary solution h and a solution Ψ of (1), the Darboux transformation (3) generates a solution of (2), together with the corresponding transformed potential components (4) and (5).

3 Results and Discussion

We are concerned with extending the Darboux transformation (3), such that it becomes applicable to a generalization of the Schrödinger equation (1). We will first state this generalized Darboux transformation, give the transformed potentials associated with it and afterwards discuss its special cases.

3.1 The Generalized Darboux Transformation

Consider the following pair of generalized Schrödinger equations

$$f\ \Psi'' + g\ \Psi' - (E^2 + E\ V_0 + U_0)\ \Psi = 0, \tag{6}$$

$$f\ \Phi'' + g\ \Phi' - (E^2 + E\ V_1 + U_1)\ \Phi = 0, \tag{7}$$

where the same notation as in (1), (2) is used. In addition, the coefficients $f = f(x)$ and $g = g(x)$ are assumed to be arbitrary smooth functions. Let h is a solution of our initial equation (6) at energy $\lambda \neq E$, such that h and Ψ are linearly independent and define the Darboux transformation of a solution Ψ to (6) as follows:

$$\mathcal{D}(\Psi) = \sqrt{\frac{2\, f^{\frac{3}{2}}\, h}{4\, f\, h' - h\left[f' - 2\, g + 2\sqrt{f}\,(2\,\lambda + V_0)\right]}} \times \left[\left(-\frac{h'}{h} + \frac{\lambda - E}{\sqrt{f}}\right)\Psi + \Psi'\right]. \tag{8}$$

Then the function $\Phi = \mathcal{D}(\Psi)$ is a solution of (7), provided the following constraints between the potential components in (6) and (7) are satisfied:

$$V_1 = V_0 - 2\sqrt{f}\,\frac{w_1'}{w_1}, \tag{9}$$

$$U_1 = U_0 + \frac{w_1'\, f'}{w_1} + \frac{2\, w_1''\, f}{w_1} + 2\sqrt{f}\,\frac{w_2'}{w_1}, \tag{10}$$

where the functions w_1 and w_2 are defined by

$$w_1 = \sqrt{\frac{2\sqrt{f}\, h}{4\, f\, h' - h\left[f' - 2\, g + 2\sqrt{f}\,(2\,\lambda + V_0)\right]}}, \tag{11}$$

$$w_2 = \left(\lambda - \frac{g}{2\sqrt{f}} + \frac{f'}{4\sqrt{f}} - \frac{\sqrt{f}\, h'}{h}\right) w_1. \tag{12}$$

Thus, as in the conventional case, the Darboux transformation (8) maps solutions of the initial equation (6) onto solutions of its transformed counterpart (7).

3.2 Special Cases

Our generalized Schrödinger equation (6) encompasses several particular cases, three of which we will now mention.

The Klein–Gordon Equation The first particular case is the stationary Klein–Gordon equation

$$\Psi'' + \left[(e - v)^2 - (m + s)^2\right]\Psi = 0, \tag{13}$$

for vector and scalar potentials v and s, respectively. Furthermore, the positive constant m stands for the mass and e denotes the energy. If we expand the terms in (13), we obtain the form

$$\Psi'' + \left[e^2 - 2\, e\, v + v^2 - m^2 - 2\, m\, s - s^2\right]\Psi = 0, \tag{14}$$

which can be subsumed under the generalized Schrödinger equation (6), provided the following settings are made:

$$f = 1 \qquad g = 0 \qquad E = -i\,e \qquad V = 2\,i\,v \qquad U = m^2 + 2\,m\,s + s^2 + \frac{1}{2}\,v^2. \tag{15}$$

These settings render (6) in the form of our Klein–Gordon equation (14) or, equivalently, in its form (13). We will study this special case of our Schrödinger equation (6) in Section 5.

The Effective Mass Equation Next, let us mention the self-adjoint form attained for $g = f'$, which arises in the situation of an energy-independent potential for $f = 1$, or as the effective mass equation for $f = 1/(2m)$, where $m = m(x)$ is a positive, smooth function:

$$\frac{1}{2\,m}\,\Psi'' - \frac{m'}{2\,m^2}\,\Psi' - (E^2 + E\,V_0 + U_0)\,\Psi = 0.$$

Darboux transformations for this equation have already been constructed [27], but only work if the solutions and potentials involved meet a certain, nontrivial constraint, which comes on top of the constraints (9) and (10) that the potentials must fulfill. Our present Darboux transformation (8) does not require any additional constraint to be fulfilled and generalizes its counterpart constructed in [27].

$$\mathcal{D}(\Psi) = \sqrt{\frac{\sqrt{2}\,(4\,m\,h' - h\,m')}{h\,\sqrt{m}} - 4\,m\,V - 8\,\lambda\,m}$$
$$\times \left[\left(-\frac{h'}{h} + \sqrt{2\,m}\,(\lambda - E)\right)\,\Psi + \Psi'\right].$$

For the sake of brevity we omit to show the explicit form of the potential components (9) and (10) in the present case of an effective mass.

Weighted Energy Another special case of (6) is given by the equation

$$f\,\Psi'' + g\,\Psi' - (E^2\,W + E\,V_0 + U_0)\,\Psi = 0,$$

where $W = W(x)$ is an arbitrary weight function for the energy. This equation is obtained from (6) simply via multiplication by a suitable factor, such that the Darboux transformation (8) maintains its form.

3.3 Reduction to the Conventional Case

Let us briefly verify that the expressions we gave for the Darboux transformation and the transformed potential components simplify correctly to their well-known, conventional forms if $f = 1$ and $g = 0$ is taken. Starting with the Darboux transformation (8), we get

$$
\begin{aligned}
\mathcal{D}(\Psi) &= \sqrt{\frac{2\, f^{\frac{3}{2}}\, h}{4\, f\, h' - h\left[f' - 2\, g + 2\sqrt{f}\,(2\,\lambda + V_0)\right]}} \\
&\quad \times \left[\left(-\frac{h'}{h} + \frac{\lambda - E}{\sqrt{f}}\right)\Psi + \Psi'\right] \\
&\overset{\substack{f=1 \\ g=0 \\ \downarrow}}{=} \sqrt{\frac{h}{2\, h' - h\,(2\,\lambda + V_0)}}\left[\left(-\frac{h'}{h} + \lambda - E\right)\Psi + \Psi'\right] \\
&= \sqrt{\frac{1}{2\,\frac{h'}{h} - V_0 - 2\,\lambda}}\left[\left(-\frac{h'}{h} + \lambda - E\right)\Psi + \Psi'\right] \\
&= D(\Psi),
\end{aligned}
$$

so the Darboux transformation (8) reduces correctly to the conventional expression (3). In order to see how the transformed potentials (9) and (10) reduce, let us first apply $f = 1,\ g = 0$ to the functions w_1 and w_2, as given in (11) and (12):

$$
\begin{aligned}
w_1 &\overset{\substack{f=1 \\ g=0 \\ \downarrow}}{=} \sqrt{\frac{h}{2\, h' - h\,(2\,\lambda + V_0)}}, \\
w_2 &\overset{\substack{f=1 \\ g=0 \\ \downarrow}}{=} \left(\lambda - \frac{h'}{h}\right) w_1 .
\end{aligned}
$$

On substituting the first of these functions into (9) and integrating once on the right hand side, we arrive immediately at (4). Next, we plug the simplified functions w_1 and w_2 into the potential component (10), which then becomes

$$
\begin{aligned}
U_1 &= U_0 + \frac{1}{w_1}\,(w_1'' + 2\, w_2') \\
&= U_0 + \sqrt{2\,\frac{h'}{h} - V_0 - 2\,\lambda} \\
&\quad \times \left[\frac{d^2}{dx^2}\left(\sqrt{\frac{1}{2\,\frac{h'}{h} - V_0 - 2\,\lambda}}\right) + 2\,\frac{d}{dx}\left(\frac{\lambda - \frac{h'}{h}}{\sqrt{2\,\frac{h'}{h} - V_0 - 2\,\lambda}}\right)\right].
\end{aligned}
$$

This is the desired expression (5). In summary, our Darboux transformation (8) and the transformed potentials (9) and (10) reduce correctly to the known expressions (3), (4) and (5) that take place in the conventional case.

3.4 Comparison with Other Darboux Transformations

The formalism we have introduced in the previous section is a special case of a Darboux transformation for linear, second-order equations in a single variable. All Darboux transformations for such equations are related to each other, as we will now see by discussion of former results and comparison with the present findings. The simplest type of equation in the class we are focusing on here, is the conventional, stationary Schrödinger equation in one dimension.

$$\Psi'' + (E - V)\,\Psi = 0, \tag{16}$$

where the energy E is a real constant and Ψ, V are functions of a single scalar variable. Note that the potential V does not depend on E. The original Darboux formalism, developed in [9], applies by means of the Darboux operator

$$L = -\frac{h'}{h} + \frac{d}{dx}. \tag{17}$$

Here, $h = h(x)$ is a solution of an auxiliary equation that has the form (16), but for an arbitrary energy. The Darboux transformation for (16), governed by the operator (17), has been applied particularly in the context of the quantum-mechanical SUSY formalism, see [7] for an overview. The Schrödinger equation (16) is a special case of

$$f\,\Psi'' + g\,\Psi' + (E - V)\,\Psi = 0, \tag{18}$$

for parameter functions f and g in the real scalar variable x. As before, the potential V does not depend on the energy E. The generalized Schrödinger equation (18) that encompasses effective-mass models [18] and allows for linearly energy-dependent potentials [12], admits the Darboux operator

$$\mathcal{L} = \sqrt{f}\left(-\frac{h'}{h} + \frac{d}{dx}\right), \tag{19}$$

where $h = h(x)$ solves (18) for an arbitrary energy. It has been shown that the Darboux operators (17) and (19) are conjugate mappings, related to each other through a point transformation [29]. While in both Schrödinger equations (16) and (18) the potential is independent of the energy, in [20] a Darboux transformation applicable to certain forms of energy-dependent potentials was introduced. Details on the latter model are summarized in Section 2 of this work, in particular, the initial equation and the associated Darboux operator are displayed in (1) and (3), respectively. Recently, we were able to show [28] that the latter Darboux operator can be understood as its conventional counterpart (17) for a function h that solves an equation different from (1). Finally, in the present note we use the latter concept for constructing a Darboux transformation that applies to generalized equations with energy-dependent potentials of the form (6). Hence, we have seen that the Darboux transformation (8) is related to its counterparts, applying to Schrödinger equations and their linear generalizations. Finally, it should be pointed out that the Darboux formalisms for nonlinear, multi-component or matrix equations can be entirely different from the schemes that are associated with the above Schrödinger equations, as they involve e.g. matrix methods like Lax pairs [14, 21]. An exception is given by Dirac-type equations of the form

$$\left(M\ \partial_x + N\ \partial_y + W\right)\ \Psi = 0, \tag{20}$$

where M, N and W are 4×4 matrix functions that depend on two real scalar variables x and y. The function $\Psi = \Psi(x, y)$ is a four-component solution vector, which allows application of the following Darboux operators [26]:

$$L_x = \partial_x - u_x\, u^{-1} \qquad\qquad L_y = \partial_y - u_y\, u^{-1}. \tag{21}$$

Here, u is a 4×4 matrix function that solves the auxiliary equation

$$M\, u_x + N\, u_y + W\, u - u\, C = 0,$$

where the 4×4 matrix C depends only on x (first Darboux operator in (21) or only on y (second Darboux operator in (21). Similar results for the single-variable case have been reported in [10, 31]. Hence, for equations of the form (20) in one and two dimensions, the Darboux formalism is similar to its counterpart that is studied in the present note.

4 Construction of the Darboux Transformation

We will now prove the statements made in the previous section by constructing the generalized Darboux transformation (8) and the potentials (9) and (10) explicitly. To this end, let us explain our construction by means of the commutative diagram depicted in Fig. 1. The terms GSE_1 and GSE_2 stand for the generalized Schrödinger equations (6) and (7), respectively, while SE_1 and SE_2 denote the conventional equations (1) and (2), respectively. Starting with the initial generalized equation in the upper left corner of the diagram, we apply a point transformation P that takes our equation into its conventional counterpart SE_1. From there, we can apply the known Darboux transformation D, as given in (3), and arrive at the transformed equation SE_2, displayed in the lower right corner of the diagram. Finally, we use the inverse

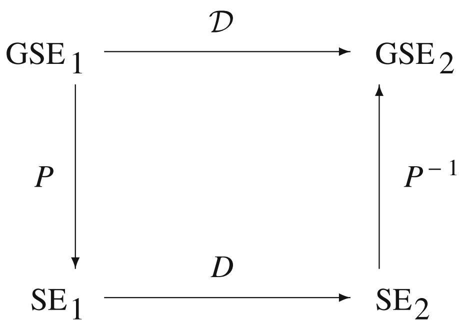

Fig. 1 Definition of the Darboux transformation

point transformation P^{-1} for restoring the generalized form of our Schrödinger equation GSE_2. In summary, the sought mapping $\mathcal{D}$ can be expressed as the following composition:

$$\mathcal{D} = P^{-1} \circ D \circ P.$$

In order to evaluate this explicitly, our first step is to construct the mapping P. Consider the generalized Schrödinger equation (6) and perform the following point transformation to its solution Ψ:

$$\begin{aligned} \Psi(x) &= \exp(F(x))\; \hat{\Psi}(u(x)), \\ F &= \int^{x} \left(\frac{f'}{4\,f} - \frac{g}{2\,f} \right) dx', \\ u &= \int^{x} \sqrt{\frac{1}{f}}\, dx', \end{aligned} \tag{22}$$

Note that in subsequent calculations we will state arguments of functions if there is a possibility of confusing the two coordinates x and u. Now, after applying the point transformation (22), the generalized Schrödinger equation (6) renders in the following form:

$$\hat{\Psi}''(u(x)) - \Big(E^2 + E\; V(x) + U(x) + W(x) \Big)\; \hat{\Psi}(u(x)) = 0, \tag{23}$$

where the function W is expressed through f, g and its derivatives as

$$W = \frac{g^2}{4\,f} - \frac{g\,f'}{2\,f} + \frac{3\,f'}{16\,f} + \frac{g'}{2} - \frac{f''}{4}. \tag{24}$$

Equation (23) is a conventional Schrödinger equation that allows for the application of the Darboux transformation (3). Let us take an auxiliary solution $\hat{h}$ of (23) at energy $\lambda \neq E$, such that $\hat{\Psi}$ and $\hat{h}$ are linearly independent. The Darboux transformation applied to $\hat{\Psi}$ then becomes

$$D(\hat{\Psi}) = \sqrt{\frac{1}{2\,\frac{\hat{h}'}{\hat{h}} - V_0 - 2\,\lambda}} \left[\left(-\frac{\hat{h}'}{\hat{h}} + \lambda - E \right) \hat{\Psi} + \hat{\Psi}' \right], \tag{25}$$

such that the transformed solution $\hat{\Phi} = D(\hat{\Psi})$ solves the Schrödinger equation

$$\hat{\Phi}''(u(x)) - (E^2 + E\; V_1(u(x)) + U_1(u(x)) + W(x))\; \hat{\Phi}(u(x)) = 0. \tag{26}$$

Here the transformed potential components V_1 and U_1 are given by expressions (4) and (5), note that the auxiliary function h must be replaced by $\hat{h}$. Furthermore, let us point out that V_1 and U_1 in (23) are expressed through the coordinate $u(x)$, while W is still given in terms of x. After having performed the Darboux transformation, we use the inverse of our point transformation (22) for reinstalling the generalized form of our Schrödinger equation (26). We set

$$\hat{\Phi}(u(x)) \;=\; \exp(-F(x))\; \Phi(x). \tag{27}$$

Equation (26) is then converted into the desired generalized form (7), that is,

$$f(x)\;\Phi''(x) + g(x)\;\Phi'(x) - (E^2 + E\;V_1(u(x)) + U_1(u(x)))\;\Phi(x) \;=\; 0, \tag{28}$$

where the transformed potentials V_1 and U_1 are still expressed through the auxiliary solution $\hat{h}$ and the coordinate $u(x)$. Let us now find our transformed solution Φ of (28) in explicit form. To this end, we combine (25) and (27):

$$\begin{aligned}\Phi(x) &= \exp(F(x))\;\hat{\Phi}(u(x)) \\ &= \exp(F(x))\sqrt{\frac{1}{2\,\frac{\hat{h}'(u(x))}{(\hat{h}(u(x))} - V_0(u(x)) - 2\,\lambda}} \\ &\quad\times\left[\left(-\frac{\hat{h}'(u(x))}{\hat{h}(u(x))} + \lambda - E\right)\hat{\Psi}(u(x)) + \hat{\Psi}'(u(x))\right].\end{aligned} \tag{29}$$

Before we continue by substituting $\hat{\Psi}$, let us express the auxiliary solution $\hat{h}$ of the conventional Schrödinger equation (24) by an auxiliary solution h of the generalized Schrödinger equation (6). The two auxiliary solutions are connected by means of our point transformation (22), that is,

$$\hat{h}(u(x)) \;=\; \exp(-F(x))\; h(x).$$

Hence, the derivative $\hat{h}'$ changes as follows:

$$\begin{aligned}\hat{h}'(u(x)) &= \frac{d}{dx}\left[\exp(-F(x))\;h(x)\right]\frac{1}{u'(x)} \\ &= \left[-\exp(-F(x))\;F'(x)\;h(x) + \exp(-F(x))\;h'(x)\right]\sqrt{f(x)}.\end{aligned}$$

This gives the following expression for the ratio $\hat{h}'/\hat{h}$, as it appears in (29):

$$\frac{\hat{h}'(u(x))}{\hat{h}(u(x))} \;=\; -\sqrt{f(x)}\;F'(x) + \frac{\sqrt{f(x)}\;h'(x)}{h(x)}.$$

Finally we replace F' by its explicit form, which is the integrand of F in (22):

$$\frac{\hat{h}'(u(x))}{\hat{h}(u(x))} \;=\; \frac{g(x)}{2\,\sqrt{f(x)}} - \frac{f'(x)}{4\,\sqrt{f(x)}} + \frac{\sqrt{f(x)}\;h'(x)}{h(x)}.$$

Note that the solutions $\hat{\Psi}$ and Ψ fulfill the same relation as the auxiliary functions $\hat{h}$ and h do, so we can replace the corresponding expressions in (29). The second factor on the right hand side then changes as follows:

$$\sqrt{\frac{1}{2\,\frac{\hat{h}'(u(x))}{(\hat{h}(u(x))}-V_0(u(x))-2\,\lambda}}$$
$$=\sqrt{\frac{1}{2\left(\frac{g(x)}{2\,\sqrt{f(x)}}-\frac{f'(x)}{4\,\sqrt{f(x)}}+\frac{\sqrt{f(x)}\,h'(x)}{h(x)}\right)-V_0(u(x))-2\,\lambda}}$$
$$=\sqrt{\frac{2\,\sqrt{f(x)}\,h(x)}{2\,g(x)\,h(x)-f'(x)\,h(x)+4\,f(x)\,h'(x)-\sqrt{f(x)}\,h(x)\,V_0(x)-2\,\sqrt{f(x)}\,h(x)\,\lambda}}$$
$$=\sqrt{\frac{2\,\sqrt{f(x)}\,h(x)}{4\,f(x)\,h'(x)-h(x)\left[f'(x)-2\,g(x)+2\,\sqrt{f(x)}\,(2\,\lambda+V_0(x))\right]}}. \tag{30}$$

In the same fashion we rewrite the first and the third factor on the right hand side of (29), note that the exponentials cancel out:

$$\left(-\frac{\hat{h}'(u(x))}{\hat{h}(u(x))}+\lambda-E\right)\hat{\Psi}(u(x))+\hat{\Psi}'(u(x))$$
$$=\left[-\left(\frac{g(x)}{2\,\sqrt{f(x)}}-\frac{f'(x)}{4\,\sqrt{f(x)}}+\frac{\sqrt{f(x)}\,h'(x)}{h(x)}\right)+\lambda-E\right]\Psi(x)$$
$$+\left[\left(\frac{g(x)}{2\,f(x)}-\frac{f'(x)}{4\,f(x)}\right)\,\Psi(x)+\,\Psi'(x)\right]\sqrt{f(x)}$$
$$=\,f(x)\left[\left(-\frac{h'(x)}{h(x)}+\frac{\lambda-E}{\sqrt{f(x)}}\right)\,\Psi(x)+\Psi'(x)\right]. \tag{31}$$

Now, on multiplying our results (30) and (31), we obtain the rewritten form of (29):

$$\Phi(x)\,=\sqrt{\frac{2\,f^{\frac{3}{2}}(x)\,h(x)}{4\,f(x)\,h'(x)-h(x)\left[f'(x)-2\,g(x)+2\,\sqrt{f}(x)\,(2\,\lambda+V_0(x))\right]}}$$
$$\times\left[\left(-\frac{h'(x)}{h(x)}+\frac{\lambda-E}{\sqrt{f(x)}}\right)\Psi(x)+\Psi'(x)\right],$$

which, as desired, coincides with expression (8). It remains to determine the explicit form of the transformed potential components (9) and (10), which is done similarly to the way we obtained the transformed solution (8). In the first step we take the conventional potential components (4) and (5), where the solutions h and Ψ are to be replaced by $\hat{h}$ and $\hat{\Psi}$, respectively, and all expressions are understood to be expressed through the coordinate $u(x)$. Next, the solutions $\hat{h}$ and $\hat{\Psi}$ are substituted via the inverse of our point transformation (22) and the derivatives with respect to $u(x)$ are rewritten in terms of x. Since this process is straightforward and due to the length of the expressions involved in the corresponding calculations, we omit to give details here.

5 Darboux Transformations for the Klein–Gordon Equation

In Section 3 we have seen that our generalized Schrödinger equation (6) matches the form of a stationary Klein–Gordon equation (13) if potential components and energy are redefined according to (15). This fact allows for the application of our Darboux transformation (8) to the Klein–Gordon equation. Consider the pair of Klein–Gordon equations

$$\Psi'' + \left[(e - v_0)^2 - (m + s_0)^2\right] \Psi = 0, \tag{32}$$

$$\Phi'' + \left[(e - v_1)^2 - (m + s_1)^2\right] \Phi = 0, \tag{33}$$

where $v_j = v_j(x),\ s_j = s_j(x),\ j = 0, 1$ stand for vector and scalar potentials, respectively. The constant e denotes the energy, and $m = m(x)$ represents the mass, which for the sake of generality we assume to be position-dependent. Although the Klein–Gordon equation is equivalent to our Schrödinger equation (6), the Darboux transformation (8) cannot be adapted immediately, because the transformed Schrödinger equation (7) does not necessarily take Klein–Gordon form. In particular, the coefficient of Φ in (7) must match the form in (33). In order to make this happen, let us consider the transformed potential components V_1 and U_1 separately. Since we have $f = 1,\ g = 0$ in (7), the potential component V_1 can be taken from (4). After applying the settings (15) to the present case and letting $\lambda = -i\mu$, we obtain

$$2\,i\,v_1 = 2\,i\,v_0 + \frac{d}{dx}\log\left(2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu\right),$$

$$v_1 = v_0 - \frac{1}{2}\,i\,\frac{d}{dx}\log\left(2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu\right). \tag{34}$$

This is the relation between the initial and the transformed vector potential of our Klein–Gordon equations (32) and (33), respectively. In the same manner we apply the settings (15) to the transformed potential component U_1, as displayed in (5):

$$\begin{aligned} &m^2 + 2\,m\,s_1 + s_1^2 + \frac{1}{2}\left[v_0 - \frac{1}{2}\,i\,\frac{d}{dx}\log\left(2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu\right)\right]^2 \\ &\quad = m^2 + 2\,m\,s_0 + s_0^2 + \frac{1}{2}\,v_0^2 + \sqrt{2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu} \\ &\qquad \times \left[\frac{d^2}{dx^2}\left(\sqrt{\frac{1}{2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu}}\right) + 2\,\frac{d}{dx}\left(\frac{-i\,\mu - \frac{h'}{h}}{\sqrt{2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu}}\right)\right]. \end{aligned} \tag{35}$$

Note that the function v_1 was replaced by (34). Condition (35) can be fulfilled by solving for the transformed scalar potential s_1:

$$s_1 = -m \pm \sqrt{G}, \tag{36}$$

where the function $G = G(x)$ stands for the abbreviation

$$G = \frac{1}{2}\left[v_0 - \frac{1}{2}\, i\, \frac{d}{dx}\, \log\left(2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu\right)\right]^2 + m^2 + 2\,m\,s_0 + s_0^2 + \frac{1}{2}\,v_0^2 + \sqrt{2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu} \times \left[\frac{d^2}{dx^2}\left(\sqrt{\frac{1}{2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu}}\right) + 2\,\frac{d}{dx}\left(\frac{-i\,\mu - \frac{h'}{h}}{\sqrt{2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu}}\right)\right].$$

Hence, if the transformed vector and scalar potentials are constrained as in (34) and (36), respectively, then our Darboux transformation (8) can be adapted to the Klein–Gordon equation (32). In particular, the function

$$\mathcal{D}(\Psi) = \sqrt{\frac{1}{2\,\frac{h'}{h} - 2\,i\,v_0 + 2\,i\,\mu}}\left[\left(-\frac{h'}{h} - i\,\mu + i\,e\right)\Psi + \Psi'\right], \tag{37}$$

solves the transformed Klein–Gordon equation (33), provided the conditions on the potentials (34) and (36) are fulfilled. It should be pointed out that the Darboux transformation (37) as well as the transformed potentials are in general complex functions, as can be seen from their imaginary terms. This is not a consequence of the Klein–Gordon equation's structure, but comes merely from the fact that our generalized Schrödinger equation (6) carries a negative sign in front of the term that contains energy and potentials, while usually this term has a positive sign. In order to guarantee reality of the transformed potentials or the transformed solution, a reality condition must be constructed. This, however, is beyond the scope of the present work.

6 Concluding remarks

In this note we have constructed a generalized Darboux transformation for the Schrödinger equation with a particular energy-dependence in its potential. While technically our transformation is fully functional and generalizes former results [19, 27], we are still to elaborate on its physical properties, such as incorporation of boundary conditions, normalizability of the solutions, and reality of the transformed potential. From a mathematical viewpoint, key issues concern the relation of our Darboux transformation to its well-known counterpart for energy-independent potentials and transformations of higher order. We will comment on these issues in forthcoming work.

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