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Amoebas of Complex Hypersurfaces in Statistical Thermodynamics

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Abstract The amoeba of a complex hypersurface is its image under the logarithmic projection. A number of properties of algebraic hypersurface amoebas are carried over to the case of transcendental hypersurfaces. We demonstrate the potential that amoebas can bring into statistical physics by considering the problem of energy distribution in a quantum thermodynamic ensemble. The spectrum $\{\varepsilon_k\} \subset \mathbb{Z}^n$ of the ensemble is assumed to be multidimensional; this leads us to the notions of multidimensional temperature and a vector of differential thermodynamic forms. Strictly speaking, in the paper we develop the multidimensional Darwin–Fowler method and give the description of the domain of admissible average values of energy for which the thermodynamic limit exists.

Keywords Amoeba of complex hypersurface · Asymptotics of Laurent coefficients · Logarithmic Gauss mapping · Darwin–Fowler method

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1 Introduction

The amoeba of a complex hypersurface V defined in a Reinhardt domain is the image of V in the logarithmic scale. The notion of the amoeba of an algebraic hypersurface, introduced in [1], plays a fundamental role in the study of zero distributions of polynomials in $\mathbb{C}^n$. Over the last decade, the amoeba proved to be a useful tool and a convenient language in the diverse questions such as the classification of topological types of simple Harnack curves [2], the description of phase diagrams of dimer models [3, 4], the study of the asymptotic behavior of solutions to multidimensional difference equations [5]. Adelic (non-archimedean) amoebas turned out to be helpful in the computation of nonexpansive sets for dynamical systems [6].

The main purpose of the present paper is to demonstrate the advantages of using amoebas in statistical physics. As an example of such usage we consider the statistical problem of finding the preferred states of a thermodynamic ensemble when its spectrum is discrete.

In the classical formulation of this problem, which was studied by Maxwell et al., it is assumed that the energy levels occupied by the ensemble systems form a *one-dimensional* spectrum $\{\varepsilon_k\} \subset \mathbb{R}$ (see, for example, [7, 8]). By contrast, we consider the case of a *multidimensional* spectrum $\{\varepsilon_k\} \subset \mathbb{R}^n$ for $n > 1$. Then such important notions as temperature and the differential thermodynamic form become vector quantities. Our point of view is closely related to interpretations of the moment map in the context of thermodynamical formalism that were explored in a recent paper [9] by M. Kapranov.

In fact, the major part of the paper is devoted to the generalization of the asymptotic Darwin–Fowler method [10, 11], that gives a way to describe a state of a quantum thermodynamic ensemble with a multidimensional spectrum. For this purpose, we introduce the notion of an amoeba of a general (not only algebraic) complex hypersurface and describe the structure of the amoeba complement (Theorem 1). Next, we prove an asymptotic formula (Theorem 2) for diagonal Laurent coefficients of a meromorphic function; the polar hypersurface, its amoeba and the logarithmic Gauss mapping are significantly used in the proof.

There are two main reasons motivating to apply amoebas to the asymptotic investigations of Laurent coefficients of meromorphic functions in several complex variables. First, the connected components of the amoeba complement are in one-to-one correspondence with the Laurent expansions of a meromorphic function centered at the origin and define the domain of convergence of the corresponding series. Second, by the multidimensional residues, the asymptotics of the Laurent coefficients is given by the oscillating integral over a chain on a polar hypersurface V. In the logarithmic scale the critical points of the phase function of such an integral comprise the contour of the amoeba of V.

Thus, our generalization of the Darwin–Fowler method (Section 7) is grounded on Theorems 1 and 2. Theorem 3 provides the asymptotics of the average values for occupation numbers of energy ε_k from a given spectrum.

These average values are expressed by the Laurent coefficients of the meromorphic function constructed by means of the partition function of an ensemble. Although Theorem 3 requires tricky integration techniques, its statement is a quite expected generalization of the Darwin–Fowler results. This is not the case with Theorem 4, which is totally inspired by the geometry brought in our investigation by the theory of amoebas. Theorem 4 gives the answer to the question whether an average energy of an ensemble permits the thermodynamical limit. Namely, the domain of admissible average energies coincides with the interior of the convex hull of the spectrum.

Working on this paper, we became certain that further development of the thermodynamical formalism will stimulate research in the theory of amoebas and tropical geometry. Other convincing examples are recent papers [9, 12].

In conclusion, we want to express our admiration for our friend and coauthor Mikael Passare, who died tragically in September 2011. Mikael was one of the pioneers of studying the amoebas and coamoebas of complex algebraic sets and a keen researcher of their properties. Since the late 1990s, on numerous occasions he emphasized the significance of these notions and vigorously encouraged and provided support of any kind to us and many mathematicians interested in algebraic tropical geometry.

2 Amoebas of Complex Hypersurfaces

For convenience we shall denote by $(\mathbb{C}^\times)^n$ the set $(\mathbb{C} \setminus \{0\})^n$.

Definition 1 [1] The *amoeba* $\mathcal{A}_V$ of a complex algebraic hypersurface

$$V = \{z \in (\mathbb{C}^\times)^n : Q(z) = 0\}$$

(or of the polynomial Q) is the image of V under the mapping $\mathrm{Log}\colon (\mathbb{C}^\times)^n \to \mathbb{R}^n$ that is determined by the formula

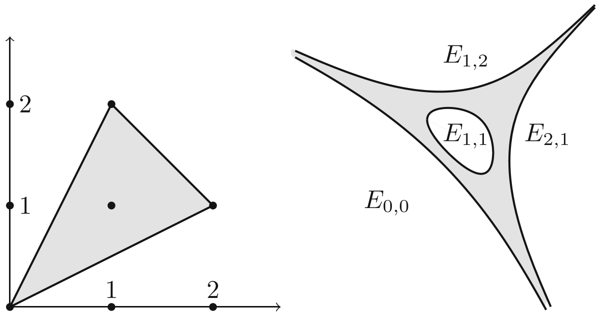
$$\mathrm{Log}\colon (z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|).$$

The term amoeba is motivated by the specific appearance of $\mathcal{A}_V$ in the case $n = 2$. It has a shape with thin tentacles going off to infinity (see Fig. 1). The complement $\mathbb{R}^n \setminus \mathcal{A}_V$ consists of a finite number of connected components, which are open and convex [1]. The basic results on amoebas of algebraic hypersurfaces can be found in [2, 13–15].

We denote by $\mathcal{N}_Q$ the *Newton polytope* of the polynomial Q, that is, the convex hull in $\mathbb{R}^n$ of all the exponents of the monomials occurring in the polynomial Q. For each integer point $\nu \in \mathcal{N}_Q$ we define the dual cone C_ν to the polytope $\mathcal{N}_Q$ at the point ν to be the set

$$C_\nu = \left\{ s \in \mathbb{R}^n : \langle s, \nu \rangle = \max_{\alpha \in N_Q} \langle s, \alpha \rangle \right\}.$$

Fig. 1 The Newton polytope (*left*) and the amoeba with its complement components E_ν (*right*) for the polynomial $Q(z)=z_1^2z_2-4z_1z_2+z_1z_2^2+1$



We recall that the *recession cone* of a convex set $E \subset \mathbb{R}^n$ is the largest cone that after a suitable translation is contained in E. The connection between the combinatorics of the Newton polytope $\mathcal{N}_Q$ of the polynomial Q and the structure of the complement of the amoeba $\mathcal{A}_V$ is described by the following result.

Theorem [13] *On the set $\{E\}$ of connected components of the complement $\mathbb{R}^n \setminus \mathcal{A}_V$ there exists an injective order function*

$$\nu\colon \{E\} \to \mathbb{Z}^n \cap \mathcal{N}_Q$$

such that the dual cone $C_{\nu(E)}$ to the Newton polytope at the point $\nu(E)$ is equal to the recession cone of the component E.

This means that the connected components of the complement $\mathbb{R}^n \setminus \mathcal{A}_V$ can be labelled as E_ν by means of the integer vectors $\nu = \nu(E) \in \mathcal{N}_Q$ (see Fig. 1).

The value $\nu(E)$ of the order function allows two interpretations. On the one hand, $\nu(E)$ is the gradient of the restriction to E of the Ronkin function for the polynomial Q (see [14]). The Ronkin function is a multidimensional analogue of Jensen's function and finds numerous applications in the theory of value distribution of meromorphic functions. On the other hand, the components of the vector $\nu(E)$ are the linking numbers of the basis loops in the torus $\mathrm{Log}^{-1}(x)$, for any $x \in E$, and the hypersurface V (see [13] or [2]).

Remark The set $\mathrm{vert}\,\mathcal{N}_Q$ of vertices of the polytope $\mathcal{N}_Q$ belongs to the image of the order function ν. In other words, for each vertex $\beta \in \mathcal{N}_Q$ there is a component E_β with recession cone C_β ([1, 16]). The existence of components E_ν corresponding to other integer points $\nu \in \mathcal{N}_Q \setminus \mathrm{vert}\,\mathcal{N}_Q$ depends on the coefficients of the polynomial Q.

There is a bijective correspondence between the connected components $\{E_\nu\}$ of the complement $\mathbb{R}^n \setminus \mathcal{A}_V$ and the Laurent expansions (centered at the origin) of an irreducible rational fraction $F(z) = P(z)/Q(z)$ (see [1, Section 6.1]). The sets $\mathrm{Log}^{-1}(E_\nu)$ are the domains of convergence for the corresponding Laurent expansions. One may therefore label such an expansion using the components of the amoeba complement, or using the integer points in the Newton polytope. For instance, the Taylor expansion of a function that is holomorphic at the origin will always correspond to the vertex of the Newton polytope $\mathcal{N}_Q$ with coordinates $(0, \ldots, 0)$.

In Sections 5–7 we shall see that, when working with partition functions, one needs to consider amoebas also of non-algebraic complex hypersurfaces. Let Q be a Laurent series in the variables $z = (z_1, \ldots, z_n)$:

$$Q(z) = \sum_{\alpha \in A \subset \mathbb{Z}^n} a_\alpha z^\alpha \,.$$

We assume that its domain of (absolute) convergence G is non-empty, and that $Q(z) \not\equiv 0$. We shall also make the assumption that Q does have zeros in $G \cap (\mathbb{C}^\times)^n$. Let

$$V = \{z \in G \cap (\mathbb{C}^\times)^n : Q(z) = 0\}$$

be the hypersurface given by the zeros of the analytic function $Q(z)$. The amoeba for V is defined as in the algebraic case: $\mathcal{A}_V = \mathrm{Log}(V)$.

We introduce the notation $\mathcal{G} = \mathrm{Log}(G)$ for the image of the convergence domain G of the series Q. It is well known that $\mathcal{G}$ is a convex domain. In the algebraic case, when Q is a polynomial, the set $\mathcal{G}$ is all of $\mathbb{R}^n$, and the amoeba $\mathcal{A}_V$ is a proper subset of $\mathcal{G}$. In the general case it may well happen that there is an equality $\mathcal{A}_V = \mathcal{G}$. To avoid this situation, we require that the summation support A of the series Q lies in some acute cone, that is, the closure $\mathcal{N}$ of the convex hull $\mathrm{ch}(A)$ does not contain any lines.

Theorem 1 *If for the series Q the set $\mathcal{N} = \overline{\mathrm{ch}(A)}$ does not contain any lines, then the complement $\mathcal{G} \setminus \mathcal{A}_V$ is non-empty. To the set $\{\nu\}$ of vertices of the polyhedron $\mathcal{N}$ there corresponds a family $\{E_\nu\}$ of pairwise distinct connected components of the complement $\mathcal{G} \setminus \mathcal{A}_V$. The dual cone C_ν to $\mathcal{N}$ at the vertex ν coincides with the recession cone for E_ν.*

Proof The assumption of the theorem implies that the set of the vertices $\mathrm{vert}(\mathcal{N})$ is non-empty. The argument is similar to the one for the algebraic case (when Q is a polynomial and $\mathcal{G} = \mathbb{R}^n$) that is given in [13, 16]. First, one shows that for each vertex $\nu \in \mathcal{N}$ a suitable translation of the cone C_ν is disjoint from $\mathcal{A}_V$, so that one can associate with the vertex ν the component E_ν of the complement $\mathcal{G} \setminus \mathcal{A}_V$ that contains this translated cone. Here the only difference is that, when $\mathcal{G} \neq \mathbb{R}^n$, one must show that the translated cones are contained in $\mathcal{G}$. This follows from the fact that the dual cones C_ν at the vertices of $\mathcal{N}$ all lie in the cone $-C^\vee(\mathcal{N})$, where $C^\vee(\mathcal{N})$ is the dual cone of the recession cone $C(\mathcal{N})$ of $\mathcal{N}$, together with the multidimensional Abel lemma [17], which says that the cone $-C^\vee(\mathcal{N})$ lies in the recession cone of the domain $\mathcal{G}$.

Next, just as in [16], one associates to the collection of n-cycles $\Gamma_\nu = \mathrm{Log}^{-1}(x_\nu)$, with the point x_ν taken in the translation of C_ν, a collection of de

Rham dual n-forms ω^μ which are meromorphic in $G \cap (\mathbb{C}^\times)^n$ with poles on V. Namely, we choose

$$\omega^\mu = \frac{1}{(2\pi i)^n} \cdot \frac{a_\mu z^\mu}{Q(z)} \cdot \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}, \qquad \mu \in \operatorname{vert}(\mathcal{N})$$

(recall that a_μ is the Laurent coefficient of Q). For points $z \in \Gamma_\nu$ we have $|a_\nu z^\nu| > |g_\nu(z)|$, where $g_\nu(z) = Q(z) - a_\nu z^\nu$. Indeed, choosing a complex curve $z = t^b = (t^{b_1}, \ldots, t^{b_n})$ with integer b from the interior of C_ν, we see that in the series

$$Q(t^b) = \sum_{\alpha \in A} a_\alpha t^{\langle \alpha, b \rangle}$$

the maximum of exponents $\langle \alpha, b \rangle$ is attained only for $\alpha = \nu$. For $|t| \gg 1$ we have $\operatorname{Log} t^b = b \cdot \log |t| \in \mathcal{G}$, and thus the series $Q(t^b)$ converges and

$$|a_\nu t^{\langle \nu, b \rangle}| > |g_\nu(t^b)|.$$

The same inequality holds for any shifted such curve $z = c \cdot t^b$ with $|c_j| = 1$, $j = 1, \ldots, n$, and therefore the required inequality holds on Γ_ν .

Hence, the meromorphic function $1/Q(z)$ can be developed into a geometric progression

$$\frac{1}{Q(z)} = \sum_{k=0}^{\infty} (-1)^k \frac{g_\nu^k(z)}{(a_\nu z^\nu)^{k+1}}$$

uniformly converging on Γ_ν, and one has

$$\int_{\Gamma_\nu} \omega^\mu = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\pi i)^n} \int_{\Gamma_\nu} \frac{a_\mu z^\mu}{a_\nu z^\nu} \cdot \left(\frac{g_\nu(z)}{a_\nu z^\nu} \right)^k \cdot \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}.$$

The leading term of $Q(z)$ with respect to the orders, defined by weight vectors from C_ν, is equal to $a_\nu z^\nu$. This yields that all the integrals in the sum vanish for $\nu \neq \mu$, and if $\nu = \mu$, the only one nonzero summand occurs for $k = 0$ and equals 1. Therefore,

$$\int_{\Gamma_\nu} \omega^\mu = \delta_{\nu\mu},$$

and by the de Rham duality [18] the cycles Γ_ν, $\nu \in \operatorname{vert} \mathcal{N}$ are linearly independent in the homology group $H_n((G \cap (\mathbb{C}^\times)^n) \setminus V)$. The cycles $\operatorname{Log}^{-1}(x)$ for x from the same connected component of $\mathcal{G} \setminus \mathcal{A}_V$ are homologically equivalent; this implies that the connected components $\{E_\nu\}_{\nu \in \operatorname{vert}(\mathcal{N})}$ are pairwise distinct. Since the n-dimensional cones of a fan dual to $C(\mathcal{N})$ coincide with the cones C_ν and $C_\nu \subset E_\nu$, one has that C_ν coincides with the recession cone for E_ν. □

3 The Amoeba Contour and the Logarithmic Gauss Mapping

In Section 2 we saw that certain information about the position of the amoeba of a complex hypersurface is given by the combinatorics of the integer points of the Newton polytope (or polyhedron) of the polynomial (or series) that defines this hypersurface. Here we shall describe an object associated with the amoeba that reflects the differential geometry of the hypersurface. The study of this object can be carried out with more analytic methods.

The *contour* $\mathcal{C}_V$ of the amoeba $\mathcal{A}_V$ is defined (see [15]) as the set of critical values of the mapping $\mathrm{Log}\colon V \to \mathbb{R}^n$, that is, the mapping Log restricted to the hypersurface V. We observe that the boundary $\partial\mathcal{A}_V$ is included in the contour $\mathcal{C}_V$, but the inverse inclusion does not hold in general. Note that the contour of an amoeba for a simple Harnack curve coincides with the boundary of the amoeba [2, 5] (the amoeba of a simple Harnack curve is depicted on Fig. 1). Herewith, a real section $V \cap \mathbb{R}^2$ of a Harnack curve consists of fold critical points of the projection $\mathrm{Log} : V \mapsto \mathcal{A}_V$. Figure 2 depicts the amoebas of two complex curves whose contours do not correspond to their boundaries. In these, the points a, b, c and d are the images of Whitney pleats.

We recall (see [2, 19]) that the *logarithmic Gauss mapping* of a complex hypersurface $V \subset (\mathbb{C}^\times)^n$ is defined to be the mapping

$$\gamma = \gamma_V \colon \mathrm{reg}\, V \to \mathbb{CP}_{n-1},$$

which to each regular point $z \in \mathrm{reg}\, V$ associates the complex normal direction to the image $\log(V)$ at the point $\log(z)$. (Here log, in contrast to Log, denotes the full complex coordinatewise logarithm.) The image $\gamma(z)$ does not depend

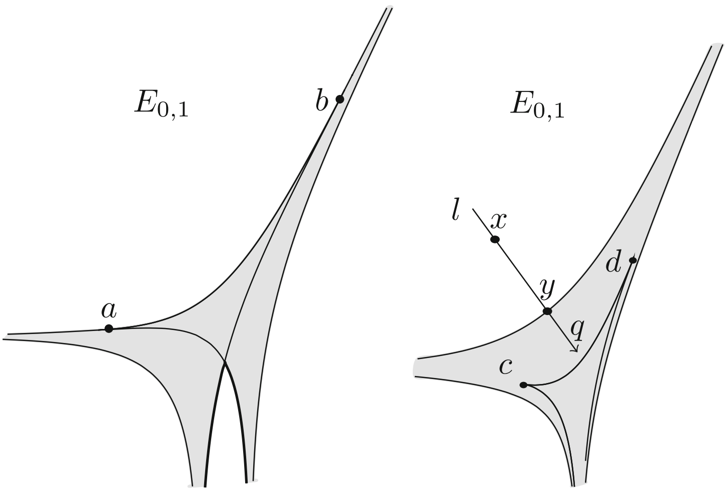


Fig. 2 The amoebas and their contours for the graphs of polynomials $1 - 2z - 3z^2$ (*left*) and $1 + z + z^2 + z^3$ (*right*, the *normal line* l to $\partial E_{0,1}$ with a directional vector q and points x, y illustrate the proof of Theorem 2)

on the choice of a branch of log and it is given in coordinates by the explicit formula [2]:

$$\gamma(z) = \left(z_1 Q'_{z_1}(z) : \ldots : z_n Q'_{z_n}(z)\right) .$$

The connection between the contour $\mathcal{C}_V$ and the logarithmic Gauss mapping is given as follows.

Proposition 1 [2] *The contour $\mathcal{C}_V$ is expressed by the identity*

$$\mathcal{C}_V = \operatorname{Log}\left(\gamma^{-1}(\mathbb{RP}_{n-1})\right) .$$

In other words, the mapping γ sends the critical points z of $\operatorname{Log}\big|_V$ to real direction $\gamma(z)$ which is orthogonal to the contour $\mathcal{C}_V$ at $\operatorname{Log} z$.

The inverse $z = \gamma^{-1}(q)$ of the logarithmic Gauss mapping is given by the solutions to the system of equations

$$\begin{cases} Q(z) = 0 \,, \\ q_n z_j Q'_{z_j} - q_j z_n Q'_{z_n} = 0 \,, \qquad j = 1, \ldots, n-1 \,. \end{cases} \tag{1}$$

For a fixed vector $q \in \mathbb{Z}^n_* = \mathbb{Z}^n \setminus \{0\}$ the solutions to system (1) consist of the points $z \in V$ at which the Jacobian of the mapping $(Q(z), z^q)$ has rank $\leqslant 1$, which means that the following statement holds.

Proposition 2 *A point $w \in \operatorname{reg} V$ is a critical point for the monomial function $z^q|_V$ if and only if the logarithmic Gauss mapping takes the value q at w, that is, $\gamma(w) = q$.*

Notice that if V is the graph of a function of n variables $z = (z_1, \ldots, z_n)$, so that it is the zero set of the function $Q(z, w) = w - f(z)$, then the logarithmic Gauss mapping is given in the affine coordinates $s_j = q_j/q_{n+1}, \;\; j = 1, \ldots, n$ of $\mathbb{CP}_n$ by the formula

$$z_j \frac{f'_{z_j}}{f} = -s_j \,, \qquad j = 1, \ldots, n \,. \tag{2}$$

4 Asymptotics of Laurent Coefficients

Let E be a connected component of the amoeba complement with smooth boundary ∂E. The cone generated by the outward normals to ∂E will be called the *component cone* of E and denoted by K_E. It is clear that K_E is a cone over the image of ∂E under the ordinary Gauss mapping $\sigma : \partial E \to S^{n-1}$.

Definition 2 The smooth boundary ∂E of a connected component E is said to be *simple* if for each $x \in \partial E$ the real torus $\operatorname{Log}^{-1}(x)$ intersects V in a unique point, and if moreover the logarithmic Gauss mapping γ of the hypersurface V is locally invertible at this intersection point.

The following Proposition is a consequence of the triangle inequality (see for the details [20]) and exhibits a class of simple boundaries in the case where $V = \Gamma_f$ is the graph over the convergence domain of a power series $f(z) = \sum_{\alpha \in A \subset \mathbb{N}^n} \omega_\alpha z^\alpha$.

Proposition 3 *If $\bar{0} \in A$, the coefficients ω_α are positive, and the set A generates the lattice $\mathbb{Z}^n$ as a group, then the boundary of the component $E_{\bar{0},1}$ of the complement of the amoeba $\mathcal{A}_{\Gamma_f}$ is simple.*

In Proposition 3 it is essential that coefficients ω_α are positive, as follows from the example of the polynomial $f = 1 - 2z_1 - 3z_1^2$ (see Fig. 2). Namely, the preimage of the inner point of the arc $(a, b) \subset \partial E_{0,1}$ consists of two points on the graph Γ_f, and the boundary point a or b have one preimage on Γ_f, but the logarithmic Gauss mapping has no inverse at a and b.

Convexity and smoothness of ∂E implies that each point $x \in \partial E$ is the preimage $x = \sigma^{-1}(q)$ of a point $q \in K_E$.

We consider the expansion of the meromorphic function $F = P(z)/Q(z)$ in a Laurent series

$$F(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \tag{3}$$

that converges in the preimage $\mathrm{Log}^{-1}(E)$ of a complement component E of the amoeba of the polar hypersurface $V = \{z : Q(z) = 0\}$ of F. For a fixed $q \in \mathbb{Z}^n_*$ we define a diagonal sequence $c_{q\cdot k} = c_{(q_1,\ldots,q_n)\cdot k}$ of the Laurent coefficients c_α from (3).

Theorem 2 *Let the boundary ∂E be simple. Then for each $q \in \mathbb{Z}^n_* \cap K_E$ the diagonal sequence $\{c_{q\cdot k}\}$ has the asymptotics*

$$c_{q\cdot k} = k^{\frac{1-n}{2}} \cdot z^{-q\cdot k}(q) \cdot \left\{C(q) + O(k^{-1})\right\} \tag{4}$$

as $k \to +\infty$. Here $z(q) = V \cap \mathrm{Log}^{-1}(\sigma^{-1}(q))$, and the constant $C(q)$ vanishes only when $P(z(q)) = 0$.

Proof The idea of the proof is to choose the cycle of integration $\mathrm{Log}^{-1}(x)$ in the Cauchy formula

$$c_{q\cdot k} = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{F(z)}{z^{q\cdot k}} \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}, \qquad x \in E, \tag{5}$$

for those x that lie near the point $y = \mathrm{Log}\, z(q) \in \partial E$ on the line $l = \{y + qt : t \in \mathbb{R}\}$, which is transversal to ∂E (see Fig. 2). In view of the assumed simplicity of ∂E, the torus $\mathrm{Log}^{-1}(y) \subset \mathrm{Log}^{-1}(l)$ intersects V in a unique point, and $\mathrm{Log}^{-1}(l)$ intersects V in a neighborhood of $z(q)$ along an $(n-1)$-dimensional chain $h \subset V$. By means of residue theory one shows (see [21] for the case $n = 2$) that, as a function of the parameter k, the integral (5) is asymptotically equivalent, as $k \to +\infty$, to the oscillatory integral

$$2\pi i \int_h \operatorname{res}\omega \cdot e^{-k\langle q, \log z\rangle},$$

where

$$\omega = \frac{1}{(2\pi i)^n} \frac{P(z)}{Q(z)} \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n},$$

and $\operatorname{res}\omega = Q\omega/dQ$ denotes the residue form for ω (see [18]). The phase $\varphi = \langle q, \log z\rangle = \log z^q$ has a unique critical point $z(q)$ on $h \subset V \cap \operatorname{Log}^{-1}(l)$ (see Proposition 2), at which $\operatorname{Re}\varphi$ attains its minimal value. A direct computation shows that the Hessian $\operatorname{Hess}\varphi$ vanishes on V simultaneously with the Jacobian of the logarithmic Gauss mapping. Since ∂E is simple, this Jacobian is not equal to zero at $z(q)$, and hence $z(q)$ is a Morse critical point for the phase φ. Using the principle of stationary phase (see [22, Proposition 1.1 in Chapter V]) we obtain formula (4) with the constant $C(q)$ being the value at the point $z(q)$ of the function $P/z_1 \cdot \ldots \cdot z_n \cdot Q'_{z_n} \cdot (\operatorname{Hess}\varphi)^{1/2}$. □

5 The Thermodynamic Ensemble and its Most Probable Distribution

We consider a *thermodynamic ensemble* $\mathfrak{U}$, consisting of N copies of some physical system. Usually (see for instance [7, 10, 11, 22] or [8]) the system is characterized by energy values from a spectrum

$$0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \ldots, \qquad \varepsilon_j \in \mathbb{Z}.$$

Each choice of energies in the systems of the ensemble defines a state of the ensemble. A basic question in the study of the behavior of an ensemble concerns the preferred states of the ensemble as $N \to \infty$.

We will consider a more general situation where the system is characterized by a multidimensional quantity $\varepsilon_k = (\varepsilon_k^1, \ldots, \varepsilon_k^n)$ from a given spectrum

$$\mathfrak{S} = \{\varepsilon_k\}_{k=\overline{0,\infty}} \subset \mathbb{N}^n,$$

in which we for convenience shall assume that $\varepsilon_0 = \bar{0}$. Furthermore, we shall consider spectra from the lattice $\mathbb{Z}^n$ that lie in acute cones in $\mathbb{R}^n \supset \mathbb{Z}^n$.

We introduce the quantity

$$W(a) = W(a_0, a_1, \ldots) = \frac{N!}{a_0! a_1! a_2! \ldots}, \tag{6}$$

expressing the number of different states of the ensemble, for which exactly a_k of the systems is in the state with parameter value ε_k. We also say that a_k is the ε_k energy *occupation number* in the ensemble. It is clear that in (6) one should have

$$\sum_k a_k = N, \tag{7}$$

$$\sum_k a_k \varepsilon_k = \mathcal{E}, \tag{8}$$

where $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$ is the energy of the ensemble and the summation is over the index k that enumerates the elements ε_k of the spectrum. The collection of numbers $a = (a_k)$ is said to be *admissible* if it satisfies conditions (7) and (8).

By definition, the *most probable distributions* of energies among the systems of the ensemble (for $N \gg 1$) correspond to those a that occur most frequently, that is, those that realize the maximum

$$\max_a W(a)$$

among all admissible collections a.

When considering the problem of describing the most probable energy distributions one makes the assumption that the vector $\mathcal{E}/N = u$ is kept constant, that is, the average energy $u = (u_1, \ldots, u_n)$ of the ensemble systems is fixed. Under this condition, vector relation (8) written out coordinatewise gives n relations among the independent variables a_k. Just as in the case of a scalar spectrum ($n = 1$, see for instance [7]), following the approach of Boltzmann, one uses the Lagrange multiplier method to find the distributions that maximize $W(a)$, which we write now $W_u(a)$ (see [20] for details). The Lagrange multipliers μ_j, that correspond to the coordinate-wise connections of vector relation (8), provide an important language for the solution of the assigned problem. More precisely, by introducing the *partition function* as the series

$$Z(\mu) = Z(\mu_1, \ldots, \mu_n) = \sum_k \mathrm{e}^{-\langle \mu, \varepsilon_k \rangle},$$

we obtain the *fundamental thermodynamic relations*:

$$-\nabla_\mu \log Z = u\,, \qquad a_k = N\frac{\mathrm{e}^{-\langle \mu, \varepsilon_k \rangle}}{Z},$$

where ∇_μ is the gradient with respect to the variables μ.

In order to apply methods from analytic function theory and the method of stationary phase, it is more convenient for us to consider other (complex) coordinates $z_j = \mathrm{e}^{-\mu_j}$, $j = 1, \ldots, n$. In these coordinates the partition function has the form

$$Z(z) = \sum_k z^{\varepsilon_k} = \sum_{\alpha \in \mathfrak{S}} z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}. \tag{9}$$

Analogously, the fundamental thermodynamic relations assume the form

$$z_j \frac{Z'_{z_j}(z)}{Z(z)} = u_j\,, \qquad j = 1, \ldots, n, \tag{10}$$

$$a_k = N\frac{z^{\varepsilon_k}}{Z(z)}. \tag{11}$$

Let us give an interpretation of these relations by the following

Statement 1 *For $N \gg 1$ the occupation numbers* (11) *evaluated in the solutions $z(u)$ of* (10) *are the coordinates of the critical points for the function $W_u(a)$. In particular, the most probable distributions $a = (a_k)$ may be computed by means of the indicated formula for suitable solutions $z(u)$.*

The comparison of formulas (10) and (2) shows that the solutions $z(u)$ to system (10) is the inverse image $\gamma^{-1}(-u)$ of the logarithmic Gauss mapping $\gamma : \Gamma_Z \to \mathbb{CP}_n$ of the graph Γ_f of the partition function $Z(z)$. However, the list of links between the mathematical notions introduced in the first and the second sections and the fundamental thermodynamic relations goes beyond this shallow observation.

Let us point out another important link computing the critical values of the function $W_u(a)$. Since the logarithm is a smooth function, the critical points of $W(a)$ and $\log W(a)$ coincide. The latter function can be written for large N with the help of Stirling's asymptotic formula in the form

$$\log W(a) = N(\log N - 1) - \sum_k a_k (\log a_k - 1)\,.$$

The critical values of this function (under restriction $\mathcal{E}/N = u$) are

$$\log W_u = \log\left[z(u)^{-\mathcal{E}} Z(z(u))^N\right] = N(\log Z(z(u)) - \langle u, \log z(u)\rangle)\,. \qquad (12)$$

It is easy to check this equality substituting a_k (as in Statement 1) in the previous expression for $\log W(a)$ evaluated in the solutions $z(u)$ of (10) and taking into account relations (7) and (8).

We are interested in the critical values $\log W_u$ only for real u, i.e. $u \in \mathbb{R}^n$. The portion of a critical value attributed to one system of an ensemble, i.e. the value

$$S_u =: \frac{1}{N}\log W_u = \log Z(z(u)) - \langle u, \log z(u)\rangle$$

plays a role of *entropy*. Since in the logarithmic scale $\log z = -\mu$ one has

$$u = -\nabla_\mu \log Z = \nabla_{\log z} \log Z\,,$$

the entropy S_u, as a function of u, is the Legendre transform of the logarithm of a partition function in the logarithmic scale.

Thus, based on Proposition 1 we get the following

Statement 2 *The liftings of the solutions $z(u)$ of system* (10)*, for $u \in \mathbb{R}^n \subset \mathbb{RP}_n$, to the graph Γ_Z of the partition function coincide with the inverse image $\gamma^{-1}(-u)$ of the Gauss logarithmic map $\gamma : \Gamma_Z \to \mathbb{CP}_n$. On the amoeba $\mathcal{A}_{\Gamma_Z}$ of the graph these solutions parametrize its contour. The values S_u of the entropy coincide with the critical values of the linear function*

$$l_u(x) = x_{n+1} - u_1 x_1 - \cdots - u_n x_n,$$

restricted to the boundary $\partial E_{\bar{0},1}$ of the connected component $E_{\bar{0},1}$ of the complement $\mathbb{R}^{n+1} \setminus \mathcal{A}_{\Gamma_Z}$.

For certain spectra $\mathfrak{S}$ the partition function $Z(z)$ admits an analytic continuation outside the domain of convergence of its series representation (9) with the new "twin spectra" $\mathfrak{S}' \subset \mathbb{Z}^n$ appearing. Let us consider two examples.

Example 1 The partition function Z for the spectrum $\mathfrak{S} = \{0, 2, 3, 4, \ldots\}$, $n = 1$, is equal to the rational function $1 + z^2/(1-z)$, which outside the unit disk $|z| < 1$ has the development

$$Z = -\left(z + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) = -\sum_{\alpha \in \mathfrak{S}'} z^{\alpha},$$

where $\mathfrak{S}' = \{1, -1, -2, \ldots\}$. We can also consider the thermodynamic relations (10) and (11) in the complement $\{|z| > 1\}$ of the unit disk. The corresponding pieces of the amoeba of the graph of this rational function are depicted on Fig. 3 in the middle.

On Fig. 3, the points a and c' depict the points at infinity, where the normal vector $[-u : 1]$ to the contour of the amoeba at a and c' equals $[0 : 1]$ and $[-1 : 1]$ respectively. The boundaries $\partial E_{0,1}$ and $\partial E_{1,1}$ have a common tangent at the points b and b' (a simple computation shows that the normal vector $[-u_0 : 1]$ corresponds to the value $u_0 = 1/2$). The set of the normal vectors to the arc $(a, b) \subset \partial E_{0,1}$ coincides with the set of the normal vectors to $(a', b') \in \partial E_{1,1}$, the same holds for the pair of arcs (b, c) and (b', c'). The tangents at the points of the arc (b, c) lie higher than parallel to them tangents at the points of (b', c'), the tangents at the points of (a, b) and (a', b'). It follows from Statement 2 that the maximal value of the entropy S_u for $0 < u < u_0$ corresponds to a solution $z(u)$ projected on the arc (a', b'), and it is from the domain $\{|z| > 1\}$.

However, the combinatorial interpretation of $W(a)$ forbids us to consider the domain $\{|z| > 1\}$ because all the occupation numbers in (11) for $z > 1$ and some of them for $z < -1$ are negative. Moreover, the partition function is negative at the points that project on the boundary $\partial E_{1,1}$.

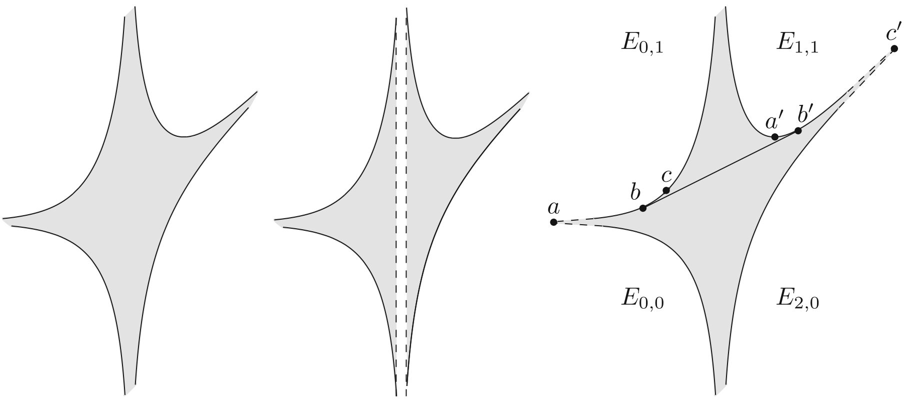


Fig. 3 Amoebas for the graph of the partition function $1 + z^2/(1-z)$: the full amoeba (on the *left*), its pieces over $|z| < 1$ and over $|z| > 1$ (in the *middle*) and with a common tangent segment $[b, b']$ to $\partial E_{0,1}$ and $\partial E_{1,1}$ (on the *right*)

The next example shows that in several dimensions we can overcome such limitations.

Example 2 Consider the spectrum

$$\mathfrak{S} = \{(0,0)\} \cup \{(2,2) + \mathbb{S}\} + \{(4,4) + \mathbb{S}\},$$

where $\mathbb{S}$ is the semigroup $(2,1)\cdot\mathbb{N} + (1,2)\cdot\mathbb{N}$ (see Fig. 4 on the left).

The partition function $\sum_{\alpha\in\mathfrak{S}} z^\alpha$ converges in the domain $D = \{|z_1^2 z_2| < 1, |z_1 z_2^2| < 1\}$ and equals

$$Z(z) = 1 + \frac{(1 + z_1^2 z_2^2) z_1^2 z_2^2}{(1 - z_1^2 z_2)(1 - z_1 z_2^2)}.$$

The development of $Z(z)$ in the domain $D' = \{|z_1^2 z_2| > 1, |z_1 z_2^2| > 1\}$ is again a partition function, i.e. it is a power series

$$Z(z) = \sum_{\alpha\in\mathfrak{S}'} z^\alpha$$

with summation over the spectrum

$$\mathfrak{S}' = \{(0,0)\} \cup \{(-1,-1) - \mathbb{S}\} \cup \{(1,1) - \mathbb{S}\}$$

(see Fig. 4 in the middle).

The full amoeba of the graph Γ_Z corresponds to the polynomial

$$(w-1)(1 - z_1^2 z_2)(1 - z_1 z_2^2) - z_1^2 z_2^2 - z_1^4 z_2^4$$

in three variables z_1, z_2, w. Points $(0,0,1)$ and $(3,3,1)$ are vertices of the Newton polytope for this polynomial, therefore the complement to the full amoeba of the graph Γ_Z contains connected components $E_{0,0,1}$ and $E_{3,3,1}$. Since the Laurent coefficients of the developments of Z in the domains D and D' are positive, the boundaries $\partial E_{0,0,1}$ and $\partial E_{3,3,1}$ are the Log-images of the graph Γ_Z over the real domains $D \cap \mathbb{R}_+^2$ and $D' \cap \mathbb{R}_+^2$. Consider the "diagonal" function

$$Z(t,t) = 1 + \frac{(1+t^4)t^4}{(1-t^3)^2}.$$

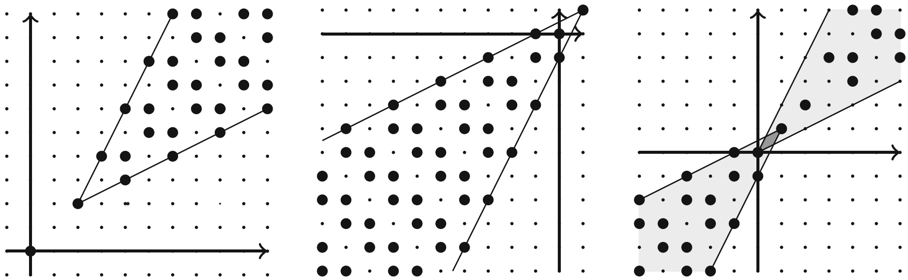


Fig. 4 "Twin-spectra" $\mathfrak{S}$ (on the *left*) and $\mathfrak{S}'$ (in the *middle*) and their convex hulls (on the *right*)

The amoeba of its graph can be embedded in the amoeba $\mathcal{A}_{\Gamma_Z}$ by the mapping

$$i : (\log|t|, \log|Z(t,t)|) \mapsto (\log(t), \log(t), \log|Z(t,t)|).$$

The boundaries of the components $E_{0,1}$ and $E_{6,1}$ of the complement to the amoeba of the graph of $Z(t,t)$ are the Log-images of pieces of the graph over the intervals $0 < t < 1$ and $1 < t < \infty$, respectively. The amoeba $\mathcal{A}_{\Gamma_Z}$ lives in the space $\mathbb{R}^3$ of variables x_1, x_2, x_3; the plane $x_1 = x_2$ cuts out in the surfaces $\partial E_{0,0,1}$ and $\partial E_{3,3,1}$ two pieces, the images $i(\partial E_{0,1})$ and $i(\partial E_{6,1})$, respectively. As in Example 1, the curves $\partial E_{0,1}$ and $\partial E_{6,1}$ have a common tangent line, lying below these curves, since they are convex.

In view of the symmetry of $\mathcal{A}_{\Gamma_Z}$ with respect to the plane $x_1 = x_2$, there exists a common tangent plane τ to surfaces $\partial E_{0,0,1}$ and $\partial E_{3,3,1}$ with the property that τ crosses the common tangent line to the embeddings $i(\partial E_{0,1})$ and $i(\partial E_{6,1})$ symmetrically with respect to the plane $x_1 = x_2$. As follows from the results of Section 7, the vector $[u_1 : u_2 : 1]$ is normal to the tangent plane τ if $u = (u_1, u_2)$ belongs to the intersection of interiors of convex hulls of the spectra $\mathfrak{S}$ and $\mathfrak{S}'$, i.e. to the double-shaded rhombus on the right of Fig. 4. In general, the rhombus is divided by some curve γ into two domains such that the value of the entropy S_u (corresponding to the ensemble with the spectrum $\mathfrak{S}$) is greater than that of the entropy S'_u (corresponding to the ensemble with the spectrum $\mathfrak{S}'$) in the first domain and is less in the second one. Perhaps, this phenomenon may be considered as a tunnelling transition from one ensemble to another in a way to increase the entropy, when we choose the value of the energy u on γ.

At the end of this section, we show that the notion of multidimensional spectrum, our starting point, leads to the notions of multidimensional temperature and the vector of thermodynamic forms. For this purpose, we compute the differential of logarithm of a partition function assuming that the variables $z_1, \ldots, z_n$ are positive and entries ε_k of the spectrum $\{\varepsilon_k\}$ vary in some neighbourhood of lattice points in $\mathbb{R}^n$, i.e. we consider the spectrum $\{\varepsilon_k\}$ to be variable.

In accordance with (10) and (11)

$$\begin{aligned} d\log Z = (d_z + d_\varepsilon)\log Z &= \sum_j z_j \frac{Z'_{z_j}}{Z}\frac{dz_j}{z_j} + \sum_k \sum_j \frac{Z'_{\varepsilon_k^j}}{Z} d\varepsilon_k^j \\ &= \langle u, d\log z\rangle + \sum_j \sum_k \frac{z^{\varepsilon_k^j}}{Z} \log z_j d\varepsilon_k^j \\ &= \langle u, d\log z\rangle + \left\langle \log z, \frac{1}{N}\sum_k a_k d\varepsilon_k \right\rangle. \end{aligned}$$

Hence, we get the following expression for the differential of the entropy

$$dS = d\,(\log Z - \langle u, \log z\rangle) = \langle -\log z, du\rangle + \left\langle \log z, \frac{1}{N}\sum_k a_k d\varepsilon_k \right\rangle = \left\langle \frac{1}{T}, \omega \right\rangle,$$

where

$$\omega = (\omega_1, \ldots, \omega_n), \qquad T = (T_1, \ldots, T_n)$$

denote the vector of the thermodynamic forms and the vector of temperature with components

$$\omega_j = du_j - 1/N \sum_k a_k d\varepsilon_k, \qquad T_j = -1/\log z_j.$$

6 The Average Value $\overline{a}$ of the Admissible Collections $\{a\}$

In the preceding section we gave a description, following Botzmann, of the most probable distributions of the ensemble. However, the method that was used is somewhat limited, since the extremal points (11) for (6) are obtained by applying the Stirling formula to $a_k!$, and this is only justified for large values of a_k. In the case of a scalar spectrum, the Darwin–Fowler method offers a possibility to avoid this drawback. It consists in a description of the asymptotics of the averages of the occupation numbers. We shall analogously describe the asymptotics of the averages of the occupation numbers, when the energy spectrum is composed of vector quantities. In this section we show that this problem is equivalent to the problem of describing the asymptotics of the diagonal coefficients of a Laurent expansion of the meromorphic function $w/(w - Z(z))$.

Definition 3 [7, 10] The *average value of the admissible collections* $\{a\}$ is the collection $\overline{a} = (\overline{a}_k)$ of numbers

$$\overline{a}_k = \frac{\sum_a a_k W(a)}{\sum_a W(a)},$$

where the summation is over all admissible collections $a = (a_k)$.

For the study of the averages $\overline{a}_k$ we introduce the sum

$$\sum_a W(a, \omega) = \sum_a \frac{N!}{a_0!a_1!\ldots a_k!\ldots}\omega_0^{a_0}\omega_1^{a_1}\ldots\omega_k^{a_k}\ldots \tag{13}$$

over all admissible collections $a = (a_k)$. Here the ω_j are real parameters, all varying in a small neighborhood of 1. We remark that $W(a, I) = W(a)$, where $I = (1, 1, \ldots)$ is the all ones vector. Hence, for $\omega = I$ the quantity (13) expresses the *total number of states* of the ensemble. It is not difficult to see that

$$\overline{a}_k = \frac{\partial}{\partial \omega_k} \log \sum_a W(a, \omega) \bigg|_{\omega = I}. \tag{14}$$

As in [7, 10] one proves the integral representation

$$\sum_a W(a, \omega) = \frac{1}{(2\pi i)^n} \int_{T_r} f^N(z) z^{-\mathcal{E}} \bigwedge_1^n \frac{dz_j}{z_j}, \tag{15}$$

where $T_r = \{|z_1| = r_1, \ldots, |z_n| = r_n\}$, and the r_j are chosen so small that on T_r one has convergence of the series

$$f(z) = f(z, \omega) = \sum_k \omega_k z^{\varepsilon_k} = \sum_k \omega_k z_1^{\varepsilon_k^1} \cdot \ldots \cdot z_n^{\varepsilon_k^n}.$$

Since $f(z, I) = Z(z)$ we refer this series to be a variation of partition function. Since the condition $0 < \omega_k < 1 + \delta$ is fulfilled, the domain of convergence G' of this series is non-empty and contains the origin $z = 0$.

We now introduce the function of $n + 1$ variables

$$F(z, w) = \frac{w}{w - f(z)},$$

which is meromorphic in the domain $G = G' \times \mathbb{C}_w$. The polar hypersurface of F is the graph

$$\Gamma_f = \{(z, w) \in G : w = f(z)\}.$$

Due to the fact that $\varepsilon_0 = \bar{0}$, the closure $\mathcal{N}$ of the convex hull of the summation support of the series $w - f(z)$ contains the vertex $\nu = (\bar{0}, 1)$. According to Theorem 1 this vertex corresponds to a connected component $E_{\bar{0},1}$ of the complement of the amoeba $\mathcal{A}_V$. Using a geometric progression we expand F in a Laurent series, convergent in $\text{Log}^{-1}(E_{\bar{0},1}) \subset \{(z, w) \in G : |w| > |f(z)|\}$:

$$F(z, w) = \sum_{N=0}^{\infty} \frac{f^N}{w^N} = \sum_N \sum_{\mathcal{E}} C_{\mathcal{E}, -N} z^{\mathcal{E}} w^{-N}. \tag{16}$$

For the Laurent coefficients $C_{\mathcal{E},-N}$ of this series one has the integral representation

$$C_{\mathcal{E},-N} = \frac{1}{(2\pi i)^{n+1}} \int_{\text{Log}^{-1}(x)} \frac{w}{w - f(z)} z^{-\mathcal{E}} w^N \bigwedge_1^n \frac{dz_j}{z_j} \wedge \frac{dw}{w},$$

where $x \in E_{\bar{0},1}$. Performing the integration with respect to w in this last integral, we immediately obtain (15).

We thus find that the problem of describing the asymptotics of the sum (13) is equivalent to the same problem for the coefficients $C_{\mathcal{E},-N}$ of the series (16), for $\mathcal{E} = u \cdot N$, with u being the vector of average energies. That is, it amounts to finding the asymptotics of the diagonal coefficients $C_{(u,-1)\cdot N}$ with direction vector $q = (u, -1)$.

7 The Asymptotics of the Average Values $\overline{a}_k$

Let the point (z_*, w_*) on the graph Γ_f of the variation f of partition function be such that $\mathrm{Log}(z_*, w_*) \in \partial E_{\bar{0},1}$.

Since $\partial E_{0,1}$ is a part of the amoeba contour, the first coordinates z_* of the given point on the graph satisfy (2) for some $u \in \mathbb{R}^n_+$, and the coordinate w_* is uniquely determined by z_*. As we let ω tend to the vector $I = (1, 1, \ldots)$, we get $f \to Z$, and the point (z_*, w_*) moves to the point $(z, w) = (z(u), w(u))$, whose logarithmic image lies on the boundary $\partial E_{\bar{0},1}$ of the component $E_{\bar{0},1}$ of the complement to the amoeba of the graph $\Gamma_Z = \{w = Z(z)\}$ of the partition function of the ensemble. Besides that, $z(u)$ satisfies system (10).

Theorem 3 *Suppose that the spectrum $\mathfrak{S} = \{\varepsilon_k\}$ generates the lattice $\mathbb{Z}^n$, and that the point $z = z(u) \in \mathbb{R}^n_+$ satisfies system* (10). *Then, as $N \to \infty$, the average values $\overline{a}_k$ for the occupation numbers of energy ε_k has the form*

$$\overline{a}_k \sim N \left. \frac{z^{\varepsilon_k}}{Z(z)} \right|_{z=z(u)} \tag{17}$$

and they coincide with most probable values of a_k.

Proof By assumption the spectrum $\mathfrak{S}$ generates the lattice $\mathbb{Z}^n$ and hence, according to Proposition 3 the boundary $\partial E_{\bar{0},1}$ is simple. Therefore we can apply Theorem 2 for the asymptotics of the diagonal sequence of Laurent coefficients of the series (16):

$$C_{(u,-1)\cdot N} \sim C(q) \cdot N^{-\frac{n}{2}} \cdot (z_*^{-u}(u) w_*(u))^N, \qquad N \to +\infty\,.$$

Hence, taking into account the summary in Section 6, we find that the asymptotics of the total number of states, as $N \to +\infty$, has the form

$$\sum_a W(a, \omega) \sim C(q) \cdot N^{-\frac{n}{2}} \cdot (z_*^{-u}(u) \cdot f(z_*(u)))^N.$$

Now, direct calculation leads us to the asymptotic equality

$$\frac{\partial}{\partial \omega_k} \log \sum_a W(a, \omega) \;\sim\; N \cdot \left\langle \nabla_z \varphi(z_*(u)), \frac{\partial}{\partial \omega_k} z_*(u) \right\rangle + N \cdot \frac{z_*^{\varepsilon_k}(u)}{f(z_*(u))},$$

where $\varphi = \log(z^{-u} f(z))$ denotes the phase (see the proof of Theorem 2). On the right hand side of the last formula the first term is equal to zero, because z_* is a critical point for the phase φ. Therefore, setting $\omega = I$, we get from formula (14) the desired asymptotics (17). □

Let us now raise the question about what the admissible values for the vector u of average energies are that guarantee the existence of a solution $z(u) \in \mathbb{R}^n_+$ to system (10), and hence provide asymptotics (17).

In the work of Darwin and Fowler [10, 11] this question was not considered. Apparently, it was first paid attention to in [22, Section 4.5.1], where it was observed that if the partition function is a polynomial of degree d, then the admissible average energies must be taken within the interval $0 < u < d$, that is, in the interior of the convex hull of the numbers $0 = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_k = d$.

The raised question is answered by the following theorem, where we use the notation $\mathcal{N}^\circ$ for the interior of the convex hull in $\mathbb{R}^n$ of the spectrum $\mathfrak{S} = \{\varepsilon_k\}$.

Theorem 4 *Suppose that the spectrum $\mathfrak{S} = \{\varepsilon_k\}$ generates the lattice $\mathbb{Z}^n$. Then for every value of the average energy $u \in \mathcal{N}^\circ$ system* (10) *has a unique solution $z = z(u)$ in $\mathbb{R}^n_+$, and hence for $u \in \mathcal{N}^\circ$ the average values $\overline{a_k}$ coincide with the most probable ones.*

Proof Lifting the solutions $z(u)$ of the system of equations (10) for $u \in \mathbb{R}^n$ to the graph Γ_Z of the partition function of the ensemble, we obtain the critical values for the mapping $\mathrm{Log}\big|_V$. On the amoeba $\mathcal{A}_{\Gamma_Z}$ of the graph these solutions parametrize its contour. In particular, the solutions $z(u) \in \mathbb{R}^n_+$ that are of interest to us parametrize the boundary of the complement component $E_{\bar{0},1}$. Thanks to the fact that the spectrum generates the lattice $\mathbb{Z}^n$, we know from Proposition 3 that to each point on $\partial E_{\bar{0},1}$ there corresponds a unique vector $q \in K_{E_{\bar{0},1}}$. Therefore, in order to obtain all solutions $z(u)$ from $\mathbb{R}^n_+$ one must go through all vectors q from the component cone $E_{\bar{0},1}$.

By Theorem 1 the recession cone of the component $E_{\bar{0},1}$ is the dual cone to $\hat{\mathcal{N}}$ at the vertex $\nu = (\bar{0}, 1)$, where $\hat{\mathcal{N}}$ denotes the closure of the convex hull of the summation support of the series $Q = w - Z(z)$. (See Fig. 5 where the recession cone is bounded by dashed lines.) The outward normals of those facets of the polyhedron $\hat{\mathcal{N}}$ that come together at the vertex ν span this dual cone. Therefore, the sought cone $K_{E_{\bar{0},1}}$ is spanned by the edges of $\hat{\mathcal{N}}$ that emanate from the vertex ν, and thus $K_{E_{\bar{0},1}}$ consists of all vectors of the form $q = (u, -1)$, with $u \in \mathcal{N}^\circ$. □

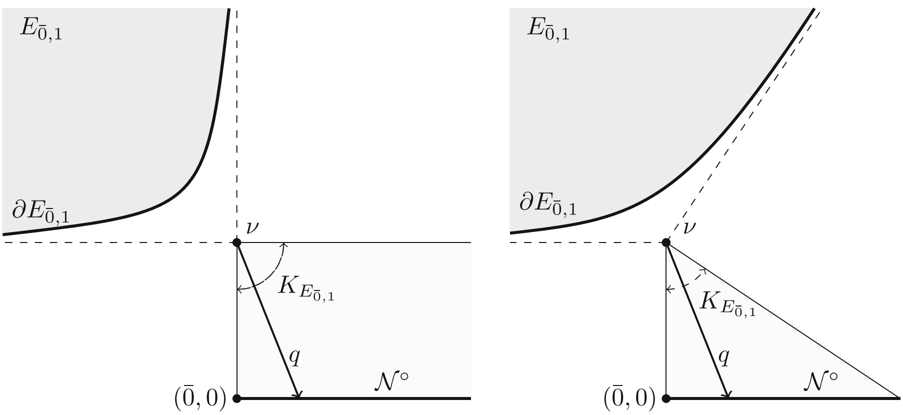


Fig. 5 The relations between $\mathcal{N}^\circ$, $K_{E_{\bar{0},1}}$ and $E_{\bar{0},1}$ for a finite (*right*) and an infinite (*left*) spectrum

We conclude with some remarks and illustrations to Theorem 4. First, the statement of the theorem still holds if one shifts the spectrum $\mathfrak{S}$ by a noninteger vector. For example, the domain of admissible average values of energy in the case of the Plank oscillator with the spectrum $\{1/2 + \mathbb{N}\}$ equals $\{u > 1/2\}$. Such domain for the Fermi oscillator with the spectrum $\mathfrak{S} = \{0, 1\}$ is the interval $\{0 < u < 1\}$ (see [7, Chapter 4]). The latter case is depicted on the right of Fig. 5. Example 2 of Section 5 deals with the "twin-spectra", and the sectors on Fig. 4 are the domains of admissible average values of energy in the corresponding cases. These sectors have a non-empty intersection: the double-shaded rhombus (Fig. 4, on the right).

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