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Conformal Vector Fields and Eigenvectors of Laplacian Operator

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Abstract In this paper, we consider an n-dimensional compact Riemannian manifold (M, g) of constant scalar curvature and show that the presence of a non-Killing conformal vector field ξ on M that is also an eigenvector of the Laplacian operator acting on smooth vector fields with eigenvalue λ together with a condition on Ricci curvature of M, that the Ricci curvature in the direction of a certain vector field is greater than or equal to $(n-1)\lambda$, forces M to be isometric to the n-sphere $S^n(\lambda)$.

Keywords Conformal vector field · Laplacian of vector fields · Ricci curvature · Scalar curvature

Mathematics Subject Classifications (2010) 53A30 · 53A50 · 53C99

1 Introduction

The interaction of analysis with geometry of Riemannian manifold has been subject of interest since the famous work of Lichnerowicz (cf. [4]) and Obata (cf. [13, 14]). Lichnerowicz initiated the study of the relation between the geometry of a compact Riemannian manifold and the bounds of eigenvalues of the Laplacian operator acting on smooth functions on the Riemannian manifold. He proved that if the Ricci curvature of the compact Riemannian

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manifold (M, g) satisfies $Ric \geqslant (n-1)k$ for a constant k, then the first nonzero eigenvalue λ_1 of the Laplacian Δ acting on smooth functions on M satisfies $\lambda_1 \geqslant nk$. Then question of equality $\lambda_1 = nk$ is addressed by the result of Obata [13], this equality holds if and only if M is isometric to the n-sphere $S^n(k)$ of constant curvature k. The work of Obata is the starting point of characterizing specific Riemannian manifolds by second order differential equations, his main result states "a necessary and sufficient condition for an n-dimensional complete and connected Riemannian manifold (M, g) to be isometric to the n-sphere $S^n(c)$ is that there exists a non constant smooth function f on M that satisfies the differential equation

$$H_f = -cfg$$

where H_f is the Hessian of the smooth function f". Then Tashiro [19] has shown that the Euclidean spaces R^n are characterized by the differential equation $H_f = cg$, and Tanno [17] obtained a similar characterization of spheres. Kanai [10] has shown that the differential equation $H_f = 0$ characterizes the Riemannian product $R \times M$, where M is a complete Riemannian manifold. Using the Riemannian submersion of the unit sphere $\pi : S^{2n+1} \rightarrow CP^n$ onto a complex projective space CP^n and Obata's differential equation, Blair [3] has obtained a differential equation that characterizes the complex projective space CP^n among complete connected complex manifolds. There are other generalizations of Obata's Theorem characterizing the complex projective space CP^n (cf. [12, 16]). Recently Garcia-Rio et al. [8, 9] have introduced the Laplacian operator Δ acting on smooth vector fields on a Riemannian manifold (M, g) and generalized the result of Obata using the differential equation satisfied by a vector field to characterize the n-sphere $S^n(c)$ (cf. Theorem 3.5 in [9]). These authors also have proved that the differential equation

$$\Delta Z = -cZ, c = \frac{S}{n(n-1)}$$

where Z is a smooth vector field on an n-dimensional compact Einstein manifold (M, g) of constant scalar curvature $S > 0$, (that is Z is the eigenvector of the Laplacian operator Δ), is a necessary and sufficient condition for M to be isometric to the n-sphere $S^n(c)$ (cf. Theorem 6 in [8]).

A smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if there exists a smooth function f on M that satisfies

$$\pounds_\xi g = 2fg$$

where $\pounds_\xi g$ is the Lie derivative of g with respect ξ. We say that ξ a nontrivial conformal vector field if the potential function f is a not a constant (note that on a compact M, if f is a constant it has to be zero and consequently ξ is Killing). If in addition ξ is a closed vector field, then ξ is said to be a closed conformal vector field. Riemannian manifolds admitting closed conformal vector fields or conformal gradient vector fields have been investigated in (cf. [1, 2, 6, 11, 15, 18, 19]) and it has been observed that there is a close relationship between the potential functions of conformal vector fields and Obata's differential equation.

In this paper we are interested in a nontrivial conformal vector field ξ on an n-dimensional compact Riemannian manifold (M, g) that is also an eigenvector of the Laplacian operator Δ acting on smooth vector fields on M that is $\Delta\xi = -\lambda\xi$. There are many vector fields of this type on the sphere $S^n(c)$. For instance if we treat $S^n(c)$ as hypersurface of the Euclidean space R^{n+1} with unit normal vector field N and take a constant vector field Z on R^{n+1}, which can be expressed as $Z = \xi + fN$, where ξ is the tangential component of Z to $S^n(c)$ and $f = \langle Z, N\rangle$ is the smooth function on $S^n(c)$, $\langle , \rangle$ being the Euclidean metric on R^{n+1}. Then it can be shown that

$$\pounds_\xi g = -2cfg, \quad \Delta\xi = -c\xi$$

where Δ is the Laplacian operator acting on the smooth vector fields on $S^n(c)$, that is ξ is a conformal vector field on S^n which is also an eigenvector of the Laplacian operator with nonzero eigenvalue c . It is not difficult to verify that the function $f = \langle Z, N\rangle$ is not a constant for at least one constant vector field Z on R^{n+1} and thus the corresponding vector field ξ is nontrivial conformal vector field. This example leads to a question: Is an n -dimensional compact Riemannian manifold (M, g) that admits a nontrivial conformal vector field ξ satisfying $\Delta\xi = -c\xi$ for a constant $c > 0$ necessarily isometric to $S^n(c)$? In this paper we show that the answer to this question is in affirmative for compact Riemannian manifolds of constant scalar curvature and Ricci curvature in certain direction is greater than or equal to $(n-1)c$. We prove the following:

Theorem *Let ξ be a nontrivial conformal vector field on an n-dimensional compact Riemannian manifold (M, g) of constant scalar curvature and potential function f. If ξ satisfies*

$$\Delta\xi = -\lambda\xi$$

for a constant $\lambda > 0$, where Δ is the Laplacian operator acting on smooth vector fields on M, and the Ricci curvature of M in the direction of the gradient of potential function ∇f is greater than or equal to $(n-1)\lambda$, then the Riemannian manifold (M, g) is isometric to $S^n(\lambda)$.

2 Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. Recently Garcia-Rio et al. [6] have initiated the study of the Laplacian operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by

$$\Delta X = \sum_{i=1}^{n} \left(\nabla_{e_i}\nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X \right)$$

where ∇ is the Riemannian connection and $\{e_1, .., e_n\}$ is a local orthonormal frame on M. This operator is self adjoint elliptic operator with respect to the inner product $\langle , \rangle$ on $\mathfrak{X}^C(M)$ the set of compactly supported vector fields in $\mathfrak{X}(M)$, defined by

$$\langle X, Y\rangle = \int_M g(X, Y), \quad X, Y \in \mathfrak{X}^C(M)$$

A vector field X is said to be an eigenvector of the Laplacian operator Δ if there is a constant μ such that $\Delta X = -\mu X$. On a compact Riemannian manifold (M, g), using the properties of Δ with respect to the inner product $\langle , \rangle$, it is easy to conclude that the eigenvalue $\mu \geqslant 0$. We shall denote by Δ both the Laplacian operators, the one acting on smooth functions on M as well as that acting on the smooth vector fields. For a smooth function $f \in C^\infty(M)$ on the Riemannian manifold (M, g), we denote by ∇f the gradient of f. For a smooth function f on M, we define the Hessian operator $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $A(X) = \nabla_X \nabla f$, ∇f being the gradient of f. The Ricci operator Q is a symmetric $(1, 1)$-tensor field that is defined by $g(QX, Y) = Ric(X, Y)$, $X, Y \in \mathfrak{X}(M)$, where Ric is the Ricci tensor of the Riemannian manifold. Then we have the following proved in ([5]).

Lemma 2.1 *Let (M, g) be a Riemannian manifold and f be a smooth function on M. Then the Hessian operator A of the function f satisfies*

$$\sum_i (\nabla A)(\mathrm{e}_i, \mathrm{e}_i) = \nabla(\Delta f) + Q(\nabla f)$$

where $\{\mathrm{e}_1, .., \mathrm{e}_n\}$ is a local orthonormal frame, Δ is the Laplacian operator acting on smooth functions on M and $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$.

On an Einstein manifold (M, g) of constant scalar curvature, each eigenfunction f of the Laplacian operator $\Delta : C^\infty(M) \to C^\infty(M)$ gives rise to the eigenvector of the Laplacian operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ as is seen in the following Lemma which in fact follows from Weitzenbock formula.

Lemma 2.2 *On an Einstein manifold (M, g) of constant scalar curvature S, $\Delta f = -\lambda f$ for a $f \in C^\infty(M)$ implies $\Delta \nabla f = -\mu \nabla f$, where ∇f is the gradient of f and the constants λ and μ are related by $n(\lambda - \mu) = S$.*

Proof Suppose $\Delta f = -\lambda f$ for a $f \in C^\infty(M)$ and constant λ. Then by Lemma 2.1, we have

$$\sum_{i=1}^{n} (\nabla A)(\mathrm{e}_i, \mathrm{e}_i) = \left(\frac{S}{n} - \lambda\right) \nabla f \tag{2.1}$$

where we used the fact $Ric(X, Y) = Sn^{-1} g(X, Y)$ for the Einstein manifold (M, g). Now we use (2.1) to compute $\Delta \nabla f$ as

$$\begin{aligned}\Delta \nabla f &= \sum_{i=1}^{n} \left(\nabla_{\mathrm{e}_i} \nabla_{\mathrm{e}_i} \nabla f - \nabla_{\nabla_{\mathrm{e}_i} \mathrm{e}_i} \nabla f\right) = \sum_{i=1}^{n} (\nabla A)(\mathrm{e}_i, \mathrm{e}_i) = \left(\frac{S}{n} - \lambda\right) \nabla f \\ &= -\mu \nabla f\end{aligned}$$

that finishes the proof. □

A vector field $\xi \in \mathfrak{X}(M)$ is said to be a conformal vector field if

$$\pounds_\xi g = 2fg \tag{2.2}$$

for a smooth function $f \in C^\infty(M)$, called the potential function, where $\pounds_\xi$ is the Lie derivative with respect to ξ. Using Kozul's formula (cf. [2]), we immediately obtain the following for a vector field ξ on M

$$2g(\nabla_X \xi, Y) = \left(\pounds_\xi g\right)(X, Y) + d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M) \tag{2.3}$$

where η is the 1-form dual to ξ that is $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$. Define a skew symmetric tensor field φ of type $(1, 1)$ on M by

$$d\eta(X, Y) = 2g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M) \tag{2.4}$$

Then using (2.2), (2.3) and (2.4) we immediately get the following (cf. [4]):

Lemma 2.3 [5] *Let ξ be a conformal vector field on an n-dimensional Riemannian manifold (M, g) with potential function f. Then,*

$$\nabla_X \xi = fX + \varphi X, \quad X \in \mathfrak{X}(M), \quad \operatorname{div} \xi = nf$$

Now we discuss some examples of Riemannian manifolds (M, g) with conformal vector fields which are eigenvectors of the Laplacian operator acting on smooth vector fields on M.

Example 1 Let (M, g) be an n-dimensional connected Einstein manifold of constant scalar curvature S. Let $\xi = \nabla\varphi$, for a $\varphi \in C^\infty(M)$ be the gradient conformal vector field on M with potential function f. Since ξ is closed vector field by Lemma 2.3 we have

$$\nabla_X \xi = fX, \quad X \in \mathfrak{X}(M) \tag{2.5}$$

which gives

$$\Delta\varphi = div(\nabla\varphi) = div\xi = nf \tag{2.6}$$

Also the Hessian operation A of the function φ satisfies $AX = fX$ and consequently for a local orthonormal frame $\{e_1, .., e_n\}$, we have $\sum_{i=1}^{n} (\nabla A)(e_i, e_i) = \nabla f$ and combining this with Lemma 2.1 for Einstein manifold and (2.6), we get

$$-n(n-1)\nabla f = S\nabla\varphi \tag{2.7}$$

and

$$\Delta f = -\frac{S}{(n-1)} f \tag{2.8}$$

Using (2.5) and (2.7), we get $\nabla_X \nabla f = hX$, $X \in \mathfrak{X}(M)$, where $h = -\frac{S}{n(n-1)} f$ is a smooth function, that is $u = \nabla f$ is a gradient conformal vector field and by Lemma 2.2 together with (2.8) we have $\Delta u = -\frac{S}{n(n-1)} u$. Thus on the Einstein manifold (M, g) the conformal vector field u is eigenvector of the Laplacian operator Δ acting on the smooth vector fields on M.

Example 2 Consider the Euclidean space (R^n, g) and define a metric $\overline{g}$ on R^n by

$$\overline{g}_u = \left(\frac{2}{1+\|u\|^2}\right)^2 g_u, \quad u \in R^n$$

Then the Riemannian connection $\overline{\nabla}$ on the Riemannian manifold $(R^n, \overline{g})$ and the Euclidean connection ∇ on (R^n, g) are related by

$$\overline{\nabla}_X Y = \nabla_X Y - X(f)Y - Y(f)X + g(X, Y)\nabla f \tag{2.9}$$

where $f = \log(1+\|u\|^2) - \log 2$. Let Ψ be the position vector field on R^n. Then using (2.9) we get

$$\overline{\nabla}_X \Psi = \left(\frac{1-\|u\|^2}{1+\|u\|^2}\right) X, \quad X \in \mathfrak{X}(R^n) \tag{2.10}$$

which proves that Ψ is a conformal vector field on the Riemannian manifold $(R^n, \overline{g})$. Using (2.10) we see that

$$\overline{\nabla}_X \overline{\nabla}_X \Psi - \overline{\nabla}_{\overline{\nabla}_X X} \Psi = X(h)X \tag{2.11}$$

where

$$h = \left(\frac{1-\|u\|^2}{1+\|u\|^2}\right)$$

For a local orthonormal frame $\{e_1, .., e_n\}$ on (R^n, g), we get the local orthonormal frame $\{e^f e_1, .., e^f e_n\}$ on $(R^n, \overline{g})$ and using the fact that the gradient ∇h of the function h on the Euclidean space (R^n, g) is given by

$$\nabla h = \frac{-4}{(1+\|u\|^2)^2}\Psi$$

and consequently by (2.11) we conclude $\overline{\Delta}\Psi = -\Psi$, that is the conformal vector field Ψ on the Riemannian manifold $(R^n, \overline{g})$ is an eigenvector of the Laplacian operator $\overline{\Delta}$ acting on the smooth vector fields on R^n.

Example 3 Consider a 3-dimensional Sasakian manifold $\overline{M}(\varphi, \xi, \eta, g)$, where φ is a (1,1) tensor field, ξ a unit vector field, η a smooth 1-form dual to ξ with respect to the Riemannian metric g. The structure (φ, ξ, η, g) satisfies

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.12}$$

$X, Y \in \mathfrak{X}(\overline{M})$ (cf. [2]). These manifolds are in abundance for example the unit sphere S^3, the Euclidean space R^3, the unit tangent bundle T_1S^2 of the unit sphere S^2, the special unitary group $SU(2)$, the Hiesenberg group nil^3 and the special linear group $SL(2, R)$ are 3-dimensional Sasakian manifolds. If $\overline{\nabla}$ is the Riemannian connection on the Sasakian manifold $\overline{M}(\varphi, \xi, \eta, g)$, we have (cf. [2])

$$\left(\overline{\nabla}\varphi\right)(X, Y) = g(X, Y)\xi - \eta(Y)X, \quad \overline{\nabla}_X\xi = -\varphi X, \quad X, Y \in \mathfrak{X}(\overline{M}). \tag{2.13}$$

Let M be an orientable totally umbilical and non-totally geodesic connected surface in the Sasakian manifold $\overline{M}(\varphi, \xi, \eta, g)$ which is neither tangential nor normal to the vector field ξ. We denote the induced metric on M by g and denote by N the unit normal vector field, by ∇ the Riemannian connection on (M, g). Then the Gauss and Weingarten formulas for the surface M are

$$\overline{\nabla}_X Y = \nabla_X Y + \lambda g(X, Y)N, \quad \overline{\nabla}_X N = -\lambda X, \quad X, Y \in \mathfrak{X}(M), \tag{2.14}$$

where λ is the constant mean curvature of M. As φ is skew symmetric φN is tangential to M and thus gives a vector field $v \in \mathfrak{X}(M)$ defined by $v = -\varphi N$. Also define a smooth function ρ on M by $\rho = g(\xi, N) \neq 0$ (by our assumption), and consequently, the restriction of the vector field ξ to M can be expressed as $\xi = u + \rho N$, where $u \in \mathfrak{X}(M)$ is the tangential component of ξ. Let α and β be the smooth 1-forms dual to the vector fields u and v on M. Then for $X \in \mathfrak{X}(M)$, we can express the restriction of φX to M as

$$\varphi X = \psi X + \beta(X)N, \tag{2.15}$$

where $\psi(X)$ is the tangential component of $\varphi(X)$ and it follows that $\psi : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a skew symmetric operator. Using $\xi = u + \rho N$, $v = -\varphi N$, and the (2.15) we get

$$\psi u = \rho v, \; g(u, v) = 0 \text{ and } \psi(v) = -\rho u. \tag{2.16}$$

Also, if $\nabla\rho$ denotes the gradient of the function ρ, using (2.13) and (2.14), we get

$$\nabla\rho = -\lambda u - v \tag{2.17}$$

Now, for $X, Y \in \mathfrak{X}(M)$, computing the covariant derivative $\left(\overline{\nabla}\varphi\right)(X, Y)$, using (2.13)–(2.15), and equating tangential and normal components we obtain

$$(\nabla_X\psi)(Y) = g(X, Y)u - \lambda g(X, Y)v + \lambda\beta(Y)X - \alpha(Y)X \tag{2.18}$$

and

$$\nabla_X v = \rho X + \lambda\psi X \tag{2.19}$$

Similarly, computing $\overline{\nabla}_X\xi$ with $\xi = u + \rho N$ and $X \in \mathfrak{X}(M)$, the tangential component gives

$$\nabla_X u = \lambda\rho X - \psi X \tag{2.20}$$

Using (2.17)–(2.20), it is easy to check that

$$(\pounds_u g) = 2\lambda\rho g, \quad (\pounds_v g) = 2\rho g, \quad \Delta u = -(1 + \lambda^2)u, \quad \Delta v = -(1 + \lambda^2)v$$

Since M is non-totally geodesic, we see that both vector fields u and v are conformal vector fields on (M, g), which are eigenvectors of the Laplacian operator Δ.

3 Proof of the Theorem

Let ξ be a nontrivial conformal vector field on an n-dimensional compact Riemannian manifold (M, g) of constant scalar curvature S with potential function f. Then by Lemma 2.3, we have

$$(\nabla\varphi)(X, Y) = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi - X(f)Y, \quad X, Y \in \mathfrak{X}(M), \tag{3.1}$$

where $(\nabla\varphi)(X, Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y)$. Summing above equation over a local orthonormal frame $\{e_1, .., e_n\}$ on M and using $\Delta\xi = -\lambda\xi$, we get

$$\sum_{i=1}^{n} (\nabla\varphi)(e_i, e_i) = \Delta\xi - \nabla f = -\lambda\xi - \nabla f \tag{3.2}$$

We use (3.1) to arrive at

$$(\nabla\varphi)(X, Y) - (\nabla\varphi)(Y, X) = R(X, Y)\xi + Y(f)X - X(f)Y$$

in which we choose $X = e_i$ and take inner product with e_i and add these n equations corresponding to a local orthonormal frame $\{e_1, .., e_n\}$ on M, to get

$$-g\left(\sum_{i=1}^{n} (\nabla\varphi)(e_i, e_i), Y\right) = Ric(Y, \xi) + (n-1)g(Y, \xi) \tag{3.3}$$

where we used the fact that φ is skew-symmetric and consequently $\sum g(\varphi e_i, e_i) = 0$ and that $g((\nabla\varphi)(X, Y), Z) = -g(Y, (\nabla\varphi)(X, Z))$. Combining (3.2) and (3.3) we arrive at

$$Q\xi = \lambda\xi - (n-2)\nabla f \tag{3.4}$$

Since, the scalar curvature S is a constant, using Lemma 2.3 and symmetry of Q, we immediately get that

$$\operatorname{div} Q\xi = fS \tag{3.5}$$

which together with the (3.4) gives

$$Sf = n\lambda f - (n-2)\Delta f \tag{3.6}$$

Using the Weitzenbock formula, we have

$$\delta d\xi + d\delta\xi = -\Delta\xi + Q\xi,$$

where $\delta = -\operatorname{div}$. Taking divergence on both sides in the above Weitzenbock formula together with Lemma 2.3 and (3.5) and the eigenvalue assumption on the conformal field shows that

$$n\Delta f = -\lambda n f - Sf. \tag{3.7}$$

The (3.6) and (3.7) immediately give

$$\Delta f = -n\lambda f \tag{3.8}$$

and consequently

$$\int_M \|\nabla f\|^2 = n\lambda \int_M f^2. \tag{3.9}$$

The Bochner formula gives

$$\int_M \left(\|A\|^2 + Ric(\nabla f, \nabla f) - n\lambda \|\nabla f\|^2\right) = 0,$$

which can be rearranged as

$$\int_M \left(\left(\|A\|^2 - \frac{1}{n}(\Delta f)^2\right) + Ric(\nabla f, \nabla f) - (n-1)\lambda \|\nabla f\|^2\right) = 0 \tag{3.10}$$

where we used (3.8) and (3.9). Note that as $TrA = \Delta f$, the Schwarz inequality implies $\|A\|^2 \geqslant \frac{1}{n}(\Delta f)^2$ with equality holding if and only if $A = \frac{\Delta f}{n} I$. Thus using the bound on the Ricci curvature in the statement of the Theorem and above inequality in the integral (3.10) together with (3.8), we conclude that the equality $A = -\lambda f I$ holds, that is for the non-constant function f (as ξ is nontrivial conformal vector field) the following hold

$$\nabla_X \nabla f = -\lambda f X, \quad X \in \mathfrak{X}(M)$$

for a positive constant λ and which by Obata's Theorem implies that M is isometric to $S^n(\lambda)$.

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