Math Phys Anal Geom (2012) 15:1–16
DOI 10.1007/s11040-011-9100-8

On the Generalized Mittag-Leffler Function and its Application in a Fractional Telegraph Equation

Rubens Figueiredo Camargo · Edmundo Capelas de Oliveira · Jayme Vaz Jr.

Received: 13 April 2009 / Accepted: 14 November 2011 / Published online: 8 December 2011

Abstract The classical Mittag-Leffler functions, involving one- and two-parameter, play an important role in the study of fractional-order differential (and integral) equations. The so-called generalized Mittag-Leffler function, a function with three-parameter which generalizes the classical ones, appear in the fractional telegraph equation. Here we introduce some integral transforms associated with this generalized Mittag-Leffler function. As particular cases some recent results are recovered.

Keywords Fractional calculus · Mittag-Leffler functions · Integral transforms · H-Fox function

Mathematics Subject Classifications (2010) 33E12 · 35R11 · 26A33 · 34A08

1 Introduction

In a recent paper Langlands et al. [1] present a discussion involving the so-called cable equation models, particularly associated with the study of anom-

R. Figueiredo Camargo
Departamento de Matemática, Faculdade de Ciências – UNESP,
17033-360, Bauru, SP, Brazil
e-mail: rubens@fc.unesp.br

E. Capelas de Oliveira (✉) · J. Vaz Jr.
Departamento de Matemática Aplicada, IMECC – Unicamp,
13083-859, Campinas, SP, Brazil
e-mail: capelas@ime.unicamp.br

J. Vaz Jr.
e-mail: vaz@ime.unicamp.br

alous electrodiffusion in nerve cells. They present the solution of fractional Nernst–Planck equation in terms of the Fox's H-function. Another recent paper [2] presents the solution of a modified fractional diffusion equation also in terms of a Fox's H-function. In both papers, the authors use juxtaposition of integral transform, namely, the Laplace transform in time variable and Fourier (Mellin) in space variable. On the other hand, the fundamental solutions of the space and space-time Riesz fractional equations with periodic conditions were recently discussed by means of the Laplace transform technique [3] and the solutions of the problem associated with the diffusion-wave equations on finite interval was presented in terms of the Fox's H-function [4].

The machinery of the fractional calculus is still growing up and the functions associated with it must be developed, specifically; functions of Mittag-Leffler type that appear in several areas, among them, in Biology [1], in Physics [2], particularly, in fractional diffusion equation [5–7] and in Statistics [8], just to mention a few.

The classical Mittag-Leffler function, as introduced by himself, is a generalization of the exponential function [9]. In the fifthies it was introduced a first generalization of the classical Mittag-Leffler, the two-parameter Mittag-Leffler function [10]. In the seventies was introduced a more general three-parameter Mittag-Leffler function [11], a function that arises on problems involving the fractional telegraph equation [12], for example. Mathematically, there are several other generalizations, some of them, which can be seen in the book of Srivastava et al. [13].

As we have already said, functions of Mittag-Leffler type arise naturally in the solution of several fractional differential equations. In the paper [5] the authors introduce another function of Mittag-Leffler type which is important in the study of kinetic equation [14], random walks and anomalous diffusion [15]. On the other hand, the methodology of the juxtaposition of integral transforms is the most important tool to affront many fractional differential equations, particularly, the so-called wave-diffusion equation and the telegraph equation. Thus, the Laplace transform, the Mellin transform and the Fourier transform of the three parameter Mittag-Leffler function play an important role on this theme.

This paper proposes to discuss these integral transform in the context of the fractional differential equations, in the sense of to study a fractional differential (and integral) equation by means of the juxtaposition of integral transforms. The structure of the article is: In Section 2, we recover the definition and some particular cases of the three-parameter Mittag-Leffler function; in Section 3, we calculate the integral transform (Laplace, Mellin and Fourier) of the three-parameter Mittag-Leffler function, presenting all results in terms of the Fox's H-function. In Section 4, we discuss some few applications and finally we present our concluding remarks. An appendix involving the definition and some properties associated with the Fox's H-function and the particular case, the Meijer's G-function, close the paper.

2 Three-Parameter Mittag-Leffler Function

The three-parameter Mittag-Leffler function, also called generalized Mittag-Leffler function, was introduced by Prabhakar [11] by means of the following series representation

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \tag{1}$$

with $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\rho) > 0, z \in \mathbb{C}$ and $(a)_k$ is the Pochhammer symbol. Recently, this function was introduced in the calculation of the Green's function associated with the fractional telegraph equation [12]. Some addition theorems, involving this function, were presented at [16]. Other applications of it can be found in [17, 18].

We note that, for $\rho = 1$ we recover the two-parameter Mittag-Leffler function, i.e.,

$$E^{1}_{\alpha,\beta}(z) \equiv E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

whereas for $\rho = 1 = \beta$ we recover the classical Mittag-Leffler function

$$E^{1}_{\alpha,1}(z) \equiv E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

as introduced by Mittag-Leffler himself, which for $\alpha = 1$ reproduces the exponential function, $E_1(z) = \exp(z)$.

We mention also the formal relation [17]

$$E^{\rho}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\rho)}\, {}_1\Psi_1\left[z \,\middle|\, \begin{matrix} (\rho, 1) \\ (\beta, \alpha) \end{matrix} \right] = \frac{1}{\Gamma(\gamma)} H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix} (1-\gamma, 1) \\ (0, 1), (1-\beta, \alpha) \end{matrix} \right],$$

where ${}_1\Psi_1\left[z \,\middle|\, \begin{matrix} (a, b) \\ (p, q) \end{matrix} \right]$ is the Wright's function which was presented as solution of the fractional equation associated with the renewal processes of Mittag-Leffler and Wright type [19]. This function generalizes the classical confluent hypergeometric function, ${}_1F_1(a; c; z)$, since we can write

$$E^{\rho}_{1,\beta}(z) = \frac{1}{\Gamma(\rho)}\, {}_1F_1(\beta; \rho; z),$$

for $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\rho) > 0$.

3 Integrals of Mittag-Leffler Function

In this section we present our main results: the Laplace transform, the Mellin transform and the Fourier transform of the three-parameter Mittag-Leffler function. We begin with the Laplace transform, then we consider the Mellin transform and using a convenient relation involving the Mellin and Fourier transforms, we calculate the Fourier transform of the three-parameter Mittag-Leffler function. In what follow, we consider the modified generalized Mittag-Leffler function

$$\mathcal{E}^{\rho}_{\alpha,\beta}(t, y, \gamma) \equiv t^{\beta-1} E^{\rho}_{\alpha,\beta}(-\mathcal{K}|y|^{\gamma} t^{\alpha}), \tag{2}$$

where $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$, $\text{Re}(\rho) > 0$, $\mathcal{K}$ is a positive constant, γ is a constant with $0 < \gamma \leqslant 1$, and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. We associate t as a time variable and y as a space variable. We also note that in all three transforms, for $\rho = 1$, we will recover the results obtained by Yu and Zhang [5].

3.1 The Laplace Transform

Denoting by $F^{\rho}_{\alpha,\beta}(p)$ the Laplace transform of generalized Mittag-Leffler function, we have

$$F^{\rho}_{\alpha,\beta}(p) \equiv \mathfrak{L}\left[\mathcal{E}^{\rho}_{\alpha,\beta}(t, y, \gamma);\, p\right] = \int_0^{\infty} \mathrm{e}^{-pt} \mathcal{E}^{\rho}_{\alpha,\beta}(t, y, \gamma)\mathrm{d}t,$$

with $\text{Re}(p) > 0$. Substituting (1) in this last equation and evaluating formally the integral in the variable t, we get

$$F^{\rho}_{\alpha,\beta}(p) = \frac{1}{p^{\beta}} \sum_{k=0}^{\infty} \frac{(\rho)_k}{p^{\alpha k}} \frac{(-\mathcal{K}|y|^{\gamma})^k}{k!}.$$

Considering the geometric series we can write

$$F^{\rho}_{\alpha,\beta}(p) = \frac{p^{\alpha\rho-\beta}}{(p^{\alpha} + \mathcal{K}|y|^{\gamma})^{\rho}}.$$

Thus, we obtain the pair of the Laplace transforms associated with the generalized Mittag-Leffler function, i.e.,

$$\mathfrak{L}\left[\mathcal{E}^{\rho}_{\alpha,\beta}(t, y, \gamma);\, p\right] = \frac{p^{\alpha\rho-\beta}}{(p^{\alpha} + \mathcal{K}|y|^{\gamma})^{\rho}}. \tag{3}$$

3.2 The Mellin Transform

In order to evaluate the Mellin transform of the generalized Mittag-Leffler function we first present a relation involving the Laplace transform and the Mellin transform. Let p and s be the parameters associated with the Laplace and Mellin transform, respectively. The Mellin transform is defined formaly by the following improper integral

$$\mathfrak{M}[f(t); s] = \int_0^\infty f(t)\, t^{s-1} \mathrm{d}t.$$

Thus, we calculate the Mellin transform of a convenient Laplace transform, i.e.,

$$\mathfrak{M}\{\mathfrak{L}[f(x);\, p];\, 1-s\} = \int_0^\infty f(x)\,\mathrm{d}x \int_0^\infty \mathrm{e}^{-px} p^{-s} \mathrm{d}p.$$

Performing the integral in the variable p, we get the relation

$$\mathfrak{M}\{\mathfrak{L}[f(x);\, p];\, 1-s\} = \Gamma(1-s)\mathfrak{M}[f(x); s].$$

We then evaluate the Mellin transform of the modified generalized Mittag-Leffler function given by (2)

$$\begin{aligned} \mathfrak{M}[\mathcal{E}^{\rho}_{\alpha,\beta}(t, y, \gamma); s] &= \frac{1}{\Gamma(1-s)} \mathfrak{M}\left\{\mathfrak{L}[\mathcal{E}^{\rho}_{\alpha,\beta}(t, y, \gamma);\, p];\, 1-s\right\} \\ &= \frac{1}{\Gamma(1-s)} \int_0^\infty \frac{p^{\alpha\rho-\beta-s}}{(p^\alpha + \mathcal{K}|y|^\gamma)^\rho} \mathrm{d}p. \end{aligned} \tag{4}$$

Introducing a new variable ξ defined by means of the relation $p = (\mathcal{K}|y|^\gamma)^{\frac{1}{\alpha}}\xi$ we can write

$$\mathfrak{M}[\mathcal{E}^{\rho}_{\alpha,\beta}(t, y, \gamma); s] = \frac{(\mathcal{K}|y|^\gamma)^{\frac{1}{\alpha}(1-\beta-s)}}{\Gamma(1-s)} \int_0^\infty \frac{\xi^{\alpha\rho-\beta-s}}{(\xi^\alpha + 1)^\rho} \mathrm{d}\xi,$$

which, by means of the definition of beta function, furnishes

$$\mathfrak{M}[\mathcal{E}^{\rho}_{\alpha,\beta}(t, y, \gamma); s] = \frac{(\mathcal{K}|y|^\gamma)^{\frac{1}{\alpha}(1-\beta-s)}}{\alpha\Gamma(1-s)} B\left(\frac{\beta+s-1}{\alpha}, \rho + \frac{1-\beta-s}{\alpha}\right). \tag{5}$$

where $B(p, q)$ is the beta function.

3.3 The Fourier Transform

Finally, we discuss the Fourier transform of the modified generalized Mittag-Leffler function. Denoting by $\mathcal{E}(y)$ this Fourier transform we have

$$\mathcal{E}(y) \equiv \mathfrak{F}\left[\mathcal{E}^{\rho}_{\alpha,\beta}(t, x, \gamma);\, y\right] = \int_{\mathbb{R}^n} \mathrm{e}^{i\vec{x}\cdot\vec{y}} \mathcal{E}^{\rho}_{\alpha,\beta}(t, x, \gamma)\, \mathrm{d}^n x,$$

with $n \geqslant 1$, whose corresponding inverse is given by

$$\widehat{\mathcal{E}}^{\rho}_{\alpha,\beta}(t,x,\gamma) \equiv \mathfrak{F}^{-1}\left[\mathcal{E}^{\rho}_{\alpha,\beta}(t,y,\gamma);x\right] = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{-i\vec{x}\cdot\vec{y}}\mathcal{E}^{\rho}_{\alpha,\beta}(t,y,\gamma)\,\mathrm{d}^n y.$$

Then, taking the Mellin transform we get

$$\begin{aligned}\mathfrak{M}[\widehat{\mathcal{E}}^{\rho}_{\alpha,\beta}(t,x,\gamma);s] &= \mathfrak{F}^{-1}\left\{\mathfrak{M}[\mathcal{E}^{\rho}_{\alpha,\beta}(t,y,\gamma);s];x\right\}\\ &= \mathfrak{F}^{-1}\left\{\frac{(\mathcal{K}|y|^{\gamma})^{\frac{1-\beta-s}{\alpha}}}{\alpha\Gamma(1-s)}B\left(\frac{\beta+s-1}{\alpha},\rho+\frac{1-\beta-s}{\alpha}\right);x\right\}\\ &= \frac{(\mathcal{K}|y|^{\gamma})^{\frac{1-\beta-s}{\alpha}}}{\alpha\Gamma(1-s)}B\left(\frac{\beta+s-1}{\alpha},\rho+\frac{1-\beta-s}{\alpha}\right)\Lambda(x), \qquad (6)\end{aligned}$$

where we have introduced the function

$$\Lambda(x) \equiv \mathfrak{F}^{-1}\left[|y|^{\frac{\gamma}{\alpha}(1-\beta-s)};x\right],$$

which can be interpreted as the complex inverse Fourier transform of the function $|y|^{\frac{\gamma}{\alpha}(1-\beta-s)}$.

In what follow, we introduce spherical coordinates [20] and integrating in the angular variables we get

$$\mathfrak{F}^{-1}\left[|y|^{\frac{\gamma}{\alpha}(1-\beta-s)};x\right] = \frac{(1/2\pi)^{n/2}}{|x|^{-1+n/2}}\int_0^{\infty} r^{\frac{n}{2}+\frac{\gamma}{\alpha}(1-\beta-s)}J_{\frac{n}{2}-1}(r|x|)\,\mathrm{d}r,$$

where $J_{\mu}(z)$ is the first kind Bessel function and $x=(x_1,x_2,\cdots,x_n)$.

Performing the integral [21] in the last equation we obtain

$$\mathfrak{F}^{-1}\left[|y|^{\frac{\gamma}{\alpha}(1-\beta-s)};x\right] = \frac{(2/|x|)^{\frac{\gamma}{\alpha}(1-\beta-s)}}{(\sqrt{\pi}|x|)^n}\frac{\Gamma\left[\frac{n}{2}+\frac{\gamma}{2\alpha}(1-\beta-s)\right]}{\Gamma\left[\frac{\gamma}{2\alpha}(\beta+s-1)\right]}.$$

Using the relation involving the gamma function and the beta function, we obtain the Mellin transform of the inverse Fourier transform as follows

$$\begin{aligned}\mathfrak{M}\left[\widehat{\mathcal{E}}^{\rho}_{\alpha,\beta}(t,x,\gamma);s\right] &= \frac{(2\mathcal{K}^{1/\gamma}/|x|)^{\frac{\gamma}{\alpha}(1-\beta-s)}}{\alpha\,(\sqrt{\pi}|x|)^n\Gamma(1-s)}\\ &\quad\times\frac{\Gamma\left(\frac{\beta+s-1}{\alpha}\right)\Gamma\left(\rho+\frac{1-\beta-s}{\alpha}\right)}{\Gamma(\rho)}\frac{\Gamma\left[\frac{n}{2}+\frac{\gamma}{2\alpha}(1-\beta-s)\right]}{\Gamma\left[\frac{\gamma}{2\alpha}(\beta+s-1)\right]}.\end{aligned}$$

Thus, we must calculate the inverse Mellin transform to get the inverse Fourier transform of the generalized Mittag-Leffler function. To this end, we first introduce the notation

$$z = \left(\frac{|x|^{\gamma}}{2^{\gamma}\mathcal{K}t^{\alpha}}\right)^{\frac{1}{\alpha}},$$

and then we can write

$$\widehat{\mathcal{E}}^{\rho}_{\alpha,\beta}(t,x,\gamma) = \frac{(2\mathcal{K}^{1/\gamma}/|x|)^{\frac{\gamma}{\alpha}(1-\beta)}}{\alpha(\sqrt{\pi}|x|)^n\Gamma(\rho)} \times \frac{1}{2\pi i}\int_{\mathrm{Br}} \frac{\Gamma\left(\frac{\beta+s-1}{\alpha}\right)\Gamma\left(\rho+\frac{1-\beta-s}{\alpha}\right)\Gamma\left[\frac{n}{2}+\frac{\gamma}{2\alpha}(1-\beta-s)\right]}{\Gamma(1-s)\Gamma\left[\frac{\gamma}{2\alpha}(\beta+s-1)\right]} z^s\,\mathrm{d}s,$$

where Br is a modified Bromwich contour [22].

The above integral can be written in terms of the Fox's H-function (see Appendix). Identifying with the Fox's H-function we have, for the integral on the Bromwich contour, only

$$H^{2,1}_{2,3}\left(z \left| \begin{array}{l} \left(1+\frac{1-\beta}{\alpha},\frac{1}{\alpha}\right)\ (1,1) \\ \left(\rho+\frac{1-\beta}{\alpha},\frac{1}{\alpha}\right)\left[\frac{n}{2}+\frac{\gamma}{2\alpha}(1-\beta),\frac{\gamma}{2\alpha}\right]\left[1+\frac{\gamma}{2\alpha}(1-\beta),\frac{\gamma}{2\alpha}\right] \end{array} \right.\right).$$

Using (15) in Appendix we can write the last equation in the following form

$$\alpha\, H^{2,1}_{2,3}\left(\frac{|x|^\gamma}{2^\gamma\mathcal{K}t^\alpha} \left| \begin{array}{l} \left(1+\frac{1-\beta}{\alpha},1\right)\ (1,\alpha) \\ \left(\rho+\frac{1-\beta}{\alpha},1\right)\left[\frac{n}{2}+\frac{\gamma}{2\alpha}(1-\beta),\frac{\gamma}{2}\right]\left[1+\frac{\gamma}{2\alpha}(1-\beta),\frac{\gamma}{2}\right] \end{array} \right.\right).$$

Finally, taking into account (16) in Appendix and rearrange the terms we obtain

$$\widehat{\mathcal{E}}^{\rho}_{\alpha,\beta}(t,x,\gamma) = \frac{t^{\beta-1}}{(\sqrt{\pi}|x|)^n\Gamma(\rho)} H^{2,1}_{2,3}\left(\frac{|x|^\gamma}{2^\gamma\mathcal{K}t^\alpha} \left| \begin{array}{l} (1,1)\ (\beta,\alpha) \\ (\rho,1)\left(\frac{n}{2},\frac{\gamma}{2}\right)\left(1,\frac{\gamma}{2}\right) \end{array} \right.\right), \qquad (7)$$

which is the inverse Fourier transform of the generalized Mittag-Leffler function. The corresponding Fourier transform can be obtained by the following expression

$$\mathcal{E}^{\rho}_{\alpha,\beta}(t,y,\gamma) = \mathfrak{F}[\widehat{\mathcal{E}}^{\rho}_{\alpha,\beta}(t,x,\gamma)],$$

where $\widehat{\mathcal{E}}^{\rho}_{\alpha,\beta}(t,x,\gamma)$ is given by (7).

As we have already said, in all above equations, (3), (5) and (7), if we consider $\rho = 1$ we recover the results presented in [5].

4 Fractional Telegraph Equation

In this section we discuss, as an application, the so-called tridimensional fractional telegraph equation. The procedure used is a general one, but to specify the calculation we present the tridimensional case, $n = 3$, only, i.e., the spacial dimension is equal to three.

The general case is given by the fractional differential equation

$$a\mathsf{D}_t^{2\alpha} u + b\,\mathsf{D}_t^{\beta} u = -\mathcal{K}(-\Delta)^{\gamma} u, \qquad t > 0; \qquad x \in \mathbb{R}^n; \tag{8}$$

with $\mathsf{D} \equiv \partial/\partial t$, $1/2 < \alpha \leqslant 1$, $0 < \beta \leqslant 1$ and $0 < \gamma \leqslant 1$, with $u = u(t, x; \alpha, \beta, \gamma)$, $x = x(x_1, \ldots, x_n)$. Here $(-\Delta)^{\gamma}$ denotes the fractional Laplace operator [23], $a, b \in \mathbb{R}$, $\mathcal{K}$ a real physical constant, t is the time variable and x is the space variable. In the case with $\alpha = \beta = \gamma = 1$ we recover the classical telegraph equation as we will see below. Thus, (8) can be considered as a generalization of the classical differential equation associated with the telegraph problem. Here, we do not discuss the problem associated with the physical unities. We refer the reader to the Inizan's paper [24] where the problem is discussed.

The boundary and initial conditions are taken as follow

$$\lim_{|x|\to\infty} u(t, x; \alpha, \beta, \gamma) = 0 \qquad \text{and} \qquad \mathsf{D}_t^k u(t, x; \alpha, \beta, \gamma) = f_k(x),$$

respectively, for $k = 0, 1, \ldots, m-1$. Considering the time-fractional derivatives in the Caputo's sense,[1] the space-fractional derivative in the Riesz's sense and using the relation [23]

$$\mathfrak{F}\left[(-\Delta)^{\gamma} u(t, x; \alpha, \beta, \gamma); \omega\right] = |\omega|^{2\gamma} \mathfrak{F}\left[u(t, x; \alpha, \beta, \gamma); \omega\right],$$

where ω is the parameter associated with the Fourier transform, we get another fractional differential equation

$$a\mathsf{D}_t^{2\alpha} \widehat{u} + b\,\mathsf{D}_t^{\beta} \widehat{u} = -\mathcal{K}|\omega|^{2\gamma} \widehat{u}, \tag{9}$$

satisfying the following initial conditions

$$\mathsf{D}_t^k \widehat{u}(t, \omega, \alpha, \beta, \gamma) = F_k(\omega),$$

with $k = 0, 1, \ldots, m-1$, where $\widehat{u} = \widehat{u}(t, \omega; \alpha, \beta, \gamma)$ is the Fourier transform of $u = u(t, x; \alpha, \beta, \gamma)$ and $F_k(\omega)$ is the Fourier transform of $f_k(x)$.

As we have already said, in what follow, we consider the case $n = 3$, only. Then, taking the Laplace transform in (9) and using the initial conditions we have an algebraic equation whose solution is given by

$$\widehat{U}(p, \omega; \alpha, \beta, \gamma) = F_0(\omega) \frac{a\, p^{2\alpha-1} + b\ p^{\beta-1}}{a\, p^{2\alpha} + b\ p^{\beta} + \mathcal{K}|\omega|^{2\gamma}}, \tag{10}$$

[1] In the Caputo sense, the Laplace transform associated with the fractional derivative is $\mathfrak{L}[\mathsf{D}_t^{\mu} f(t); p] = p^{\mu} \mathfrak{L}[f(t); p] - \sum_{k=0}^{n-1} f^{(k)}(0)\, p^{\mu-k-1}$. For the Fourier transform we have $\mathfrak{F}[\mathsf{D}_t^{\mu} f(t); \omega] = (-i\omega)^{\mu-n} \mathfrak{F}[[\mathsf{D}_t^{n} f(t); \omega]$.

where p is the parameter of the Laplace transform and $\widehat{U}(p,\omega;\alpha,\beta,\gamma)$ is the juxtaposition of the inverse transform of $u(t,x;\alpha,\beta,\gamma)$. To obtain the last expression we have explicitly the conditions

$$u(0,x;\alpha,\beta,\gamma) = f_0(x) \qquad \Longleftrightarrow \qquad \widehat{u}(0,\omega;\alpha,\beta,\gamma) = F_0(\omega),$$

and

$$\left.\frac{\partial}{\partial t}u(t,x;\alpha,\beta,\gamma)\right|_{t=0} = f_1(x) = 0 \quad \Longleftrightarrow \quad \left.\frac{\partial}{\partial t}\widehat{u}(t,\omega;\alpha,\beta,\gamma)\right|_{t=0} = F_1(\omega) = 0.$$

Thus, we now proceed with the inversion. Taking the corresponding inverse Laplace transform, we can write [25]

$$\begin{aligned}\widehat{u}(t,\omega;\alpha,\beta,\gamma) = F_0(\omega)\sum_{r=0}^{\infty}\left(-\frac{b}{a}\right)^r t^{(2\alpha-\beta)r}E^{r+1}_{2\alpha,(2\alpha-\beta)r+1}\left(-\frac{\mathcal{K}}{a}|\omega|^{2\gamma}t^{2\alpha}\right)\\ +\frac{b}{a}F_0(\omega)\sum_{r=0}^{\infty}\left(-\frac{b}{a}\right)^r t^{(2\alpha-\beta)(r+1)}\\ \times E^{r+1}_{2\alpha,(2\alpha-\beta)(r+1)+1}\left(-\frac{\mathcal{K}}{a}|\omega|^{2\gamma}t^{2\alpha}\right),\end{aligned}$$

with $\mathrm{Re}(\alpha) > 1/2$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(s) > 0$, $\alpha > \beta$, $|b\, s^{\beta}/(a\, s^{\alpha} + \mathcal{K}|\omega|^{2\gamma})| < 1$ and $E^{\gamma}_{\alpha,\beta}(z)$ is the three-parameter Mittag-Leffler function presented in Section 2.

Using (2) we can rewrite (11) as follows

$$\widehat{u}(t,\omega;\alpha,\beta,\gamma) = F_0(\omega)\sum_{r=0}^{\infty}\left(-\frac{b}{a}\right)^r\left\{\mathcal{E}^{r+1}_{2\alpha,\mu r+1}(t,\omega;\gamma) + \frac{b}{a}\mathcal{E}^{r+1}_{2\alpha,\mu(r+1)+1}(t,\omega;\gamma)\right\},$$

where we have introduced the parameter $\mu = 2\alpha - \beta$.

To perform the corresponding inverse Fourier transform, we take into account the convolution theorem and (7) with $n = 3$,

$$\begin{aligned}u(t,x;\alpha,\beta,\gamma) &\equiv \mathfrak{F}^{-1}[\widehat{u}(t,\omega;\alpha,\beta,\gamma);\omega]\\ &= \sum_{r=0}^{\infty}\left(-\frac{b}{a}\right)^r\int_{\mathbb{R}^3}F_0(\xi)\mathcal{G}(t;x-\xi)\mathrm{d}\xi,\end{aligned} \tag{11}$$

where the function $\mathcal{G}(t,x)$, as in the integer case, is known as the fundamental solution, which is given by

$$\mathcal{G}(t;x) = \widehat{\mathcal{E}}^{r+1}_{2\alpha,\mu r+1}(t,x;\gamma) + \frac{b}{a}\widehat{\mathcal{E}}^{r+1}_{2\alpha,\mu(r+1)+1}(t,x;\gamma), \tag{12}$$

which can be written in terms of the Fox's H-function (see Appendix), as follows

$$u(t,x;\alpha,\beta,\gamma) = \frac{1}{(\sqrt{\pi}|x|)^3}\sum_{r=0}^{\infty}\left(-\frac{b}{a}\right)^r\frac{t^{\mu r}}{r!}\int_{\mathbb{R}^3}F_0(\xi)\mathfrak{H}^{2,1}_{2,3}(x-\xi)\,\mathrm{d}\xi,$$

where we have introduced

$$\mathfrak{H}_{2,3}^{2,1}(x) = H_{2,3}^{2,1}\left(\frac{a|x|^{2\gamma}}{2^{2\gamma}\mathcal{K}t^{2\alpha}}\left|\begin{matrix}(1,1)\,(\mu r+1,2\alpha)\\(r+1,1)\,\left(\frac{3}{2},\gamma\right)\,(1,\gamma)\end{matrix}\right.\right) + \frac{b}{a}t^{\mu}H_{2,3}^{2,1}\left(\frac{a|x|^{2\gamma}}{2^{2\gamma}\mathcal{K}t^{2\alpha}}\left|\begin{matrix}(1,1)\,(\mu(r+1)+1,2\alpha)\\(r+1,1)\,\left(\frac{3}{2},\gamma\right)\,(1,\gamma)\end{matrix}\right.\right), \tag{13}$$

with $\mu = 2\alpha - \beta$.

5 Particular Cases

In this section, as particular cases of our results, we present four cases, only, i.e., (i) $a = 0$ and $b = 1$, the fractional diffusion equation; (ii) $a = 1$ and $b = 0$, the fractional wave equation; (iii) $a = 0$, $b = 1$ and $\beta = 1 = \gamma$, the classical diffusion equation and (iv) $\alpha = 1 = \gamma$, the integer case, classical telegraph equation.

5.1 Fractional Diffusion Equation

Here, we discuss a particular case associated with the fractional diffusion equation. To this end, we consider in (8) $a = 0$ and $b = 1$ and in consequence the results are α-independent, and in our notation we put $\alpha = 0$. It is inappropriate to derive the standard diffusion equation from the telegraph equation.

With this consideration, after the juxtaposition of the Laplace and Fourier transforms, we obtain an algebraic equation whose solution can be written as follows

$$\widehat{U}(p,\omega;0,\beta,\gamma) = F_0(\omega)\frac{p^{\beta-1}}{p^{\beta}+\mathcal{K}|\omega|^{2\gamma}}.$$

Taking the inverse Laplace transform, using (3), we obtain

$$\widehat{u}(t,\omega;0,\beta,\gamma) = F_0(\omega)\,E_{\beta}(-\mathcal{K}|y|^{2\gamma},t^{\beta}),$$

where $E_{\beta}(\eta)$ is the classical Mittag-Leffler function, as showed in Section 2. Taking the inverse Fourier transform we can write the solution of the fractional diffusion equation in terms of the Fox's H-function, as we will see in next subsection.

5.2 Fractional Wave Equation

In this section we present a calculation involving the fractional wave equation. Putting $a = 1$ and $b = 0$ in (8) we get

$$\mathsf{D}_t^{2\alpha}u(t,x;\alpha,0,\gamma) = -\mathcal{K}(-\Delta)^{\gamma}u(t,x;\alpha,0,\gamma),$$

with $1/2 < \alpha \leqslant 1$ and $0 < \gamma \leqslant 1$. The solution of the corresponding algebraic equation is given by

$$\widehat{U}(p,\omega;\alpha,0,\gamma) = F_0(\omega)\frac{p^{2\alpha-1}}{p^{2\alpha}+\mathcal{K}|\omega|^{2\gamma}}.$$

To proceed with the inversion, firstly, we take the inverse Laplace transform, using (3), we have

$$\widehat{u}(t,\omega;\alpha,0,\gamma) = F_0(\omega)E_{2\alpha}(-\mathcal{K}|y|^{2\gamma}t^{2\alpha}),$$

where $E_\beta(\eta)$ is the classical Mittag-Leffler function. Secondly, performing the inverse Fourier transform, using (11) and (13), we obtain

$$u(t,x;\alpha,0,\gamma) = \frac{1}{(\sqrt{\pi}|x|)^3}\int_{\mathbb{R}^3} F_0(\xi)\mathfrak{H}^{2,1}_{2,3}(x-\xi)\,\mathrm{d}\xi,$$

where

$$\mathfrak{H}^{2,1}_{2,3} = H^{2,1}_{2,3}\left(\frac{|x|^{2\gamma}}{2^{2\gamma}\mathcal{K}t^{2\gamma}}\left|\begin{matrix}(1,1)\,(1,2\alpha)\\(1,1)\left(\frac{3}{2},\gamma\right)\,(1,\gamma)\end{matrix}\right.\right),$$

which is the solution given in terms of the Fox's H-function.

5.3 Classical Diffusion Equation

To recover the classical diffusion equation, we consider $a=0$, $b=1$ and $\beta = 1 = \gamma$. Substituting those values in (8) we obtain the following partial differential equation

$$\mathsf{D}_t u(t,x;0,1,1) = -\mathcal{K}(-\Delta)u(t,x;0,1,1).$$

With the same considerations, i.e., after the juxtaposition of the Laplace and Fourier transforms, we obtain an algebraic equation whose solution can be written as follows

$$\widehat{U}(p,\omega;0,1,1) = \frac{F_0(\omega)}{p+\mathcal{K}|\omega|^2}.$$

Taking the corresponding inverse Laplace transform, we obtain

$$\widehat{u}(t,\omega;0,1,1) = F_0(\omega)E_1(-\mathcal{K}|y|^2 t) \equiv F_0(\omega)\exp(-\mathcal{K}|y|^2 t),$$

where $E_1(\eta) = \exp(\eta)$ is the exponential function. Taking the inverse Fourier transform we can write the solution in terms of the Fox's H-function, as we will see in next subsection.

5.4 Classical Telegraph Equation

Here we consider all parameter as equal, $\alpha = \beta = \gamma = 1$, and we obtain the partial differential equation

$$a\mathsf{D}_t^2 u + b\,\mathsf{D}_t u = -\mathcal{K}(-\Delta)u, \quad t > 0; \quad x \in \mathbb{R}^n,$$

with $u = u(t,x;1,1,1)$.

Taking the Laplace and Fourier transforms in the last equation we obtain an algebraic equation whose solution can be written as follows

$$\widehat{U}(p,\omega;1,1,1) = F_0(\omega)\frac{ap+b}{ap^2+bp+\mathcal{K}|\omega|^2},$$

with a, b and $\mathcal{K}$ are real parameters.

Proceed with the inversion, we first consider the inverse Laplace transform, then,

$$\widehat{u}(t,\omega;1,1,1) = F_0(\omega)\sum_{r=0}^{\infty}\left(-\frac{b}{a}\right)^r t^r \Lambda(r),$$

with

$$\Lambda(r) = E_{2,r+1}^{r+1}\left(-\frac{\mathcal{K}}{a}|\omega|^2 t^2\right) + \frac{b}{a}t\, E_{2,r+1}^{r+2}\left(-\frac{\mathcal{K}}{a}|\omega|^2 t^2\right),$$

where $E_{\alpha,\beta}^{\gamma}(z)$ is the three-parameter Mittag-Leffler function, presented in Section 2.

Second, taking the corresponding inverse Fourier transform of the equation above we have

$$u(t,x;1,1,1) = \frac{1}{(\sqrt{\pi}|x|)^3}\sum_{r=0}^{\infty}\left(-\frac{b}{a}\right)^r \frac{t^r}{r!}\int_{\mathbb{R}^3} F_0(\xi)\mathfrak{H}_{2,3}^{2,1}(x-\xi)\,\mathrm{d}\xi,$$

with

$$\mathfrak{H}_{2,3}^{2,1}(x) = H_{2,3}^{2,1}\left(\eta\left|\begin{matrix}(1,1)\,(r+1,2)\\ (r+1,1)\left(\frac{3}{2},1\right)\,(1,1)\end{matrix}\right.\right) + \frac{b}{a}t\, H_{2,3}^{2,1}\left(\eta\left|\begin{matrix}(1,1)\,(r+2,2)\\ (r+1,1)\left(\frac{3}{2},1\right)\,(1,1)\end{matrix}\right.\right),$$

where we have defined $\eta = \frac{a|x|^2}{4\mathcal{K}t^2}$, and $H_{2,3}^{2,1}(z)$ are the Fox's H-functions.

The last equation can also be written as follows (see (17), Appendix)

$$\mathfrak{H}_{1,2}^{2,0}(x) = H_{1,2}^{2,0}\left(\eta\left|\begin{matrix}(r+1,2)\\ (r+1,1)\left(\frac{3}{2},1\right)\end{matrix}\right.\right) + \frac{b}{a}t\, H_{1,2}^{2,0}\left(\eta\left|\begin{matrix}(r+2,2)\\ (r+1,1)\left(\frac{3}{2},1\right)\end{matrix}\right.\right).$$

6 Concluding Remarks

Some consequences involving the generalized Mittag-Leffler function were presented. Using the juxtaposition of integral transforms, namely, the Laplace transform in time variable and Fourier transform in space variable, we discussed the fractional telegraph equation and the solution obtained in terms of the Fox's H-function. Several particular cases were recovered. As a natural continuation of this paper we'll discuss the Green's function associated with the fractional telegraph equation to obtain a possible relation between the Green's function associated with the diffusion equation and the wave equation, both obtained as a particular cases [26].

Acknowledgements RFC is grateful to Capes for financial support. ECO is grateful to Fapesp (06/52475-8) for a research grant. We are grateful to Dr. F. S. Costa for interesting discussions and also the referee for the suggestions, particularly for some important references, that improve the paper.

Appendix: Fox's *H*-Function

The Fox's H-function, also known as H-function or Fox's function, was introduced in the literature as an integral of Mellin–Barnes type [13]. We mention also the important books on the subject [27–30].

Let m, n, p and q be integer numbers. Consider the function

$$\Lambda(s) = \frac{\prod_{i=1}^{m} \Gamma(b_i - B_i s) \prod_{i=1}^{n} \Gamma(1 - a_i + A_i s)}{\prod_{i=m+1}^{q} \Gamma(1 - b_i + B_i s) \prod_{i=n+1}^{p} \Gamma(a_i - A_i s)}, \tag{14}$$

with $1 \leqslant m \leqslant q$ and $0 \leqslant n \leqslant p$. The coefficients A_i and B_i are positive real numbers; a_i and b_i are complex parameters.

The Fox's H-function, denoted by,

$$H^{m,n}_{p,q}(x) = H^{m,n}_{p,q}\left(x \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right.\right) = H^{m,n}_{p,q}\left(x \left| \begin{matrix} (a_1, A_1)\cdots(a_p, A_p) \\ (b_1, B_1)\cdots(b_q, B_q) \end{matrix} \right.\right),$$

is defined as the inverse Mellin transform, i.e.,

$$H^{m,n}_{p,q}(x) = \frac{1}{2\pi i}\int_L \Lambda(s)\, x^s\, \mathrm{d}s,$$

where $\Lambda(s)$ is given by (14), and the contour L runs from $L - i\infty$ to $L + i\infty$ separating the poles of $\Gamma(1 - a_i + A_i s)$, $(i = 1, \ldots, n)$ from those of $\Gamma(b_i - B_i s)$, $(i = 1, \ldots, m)$. The complex parameters a_i and b_i are taken with the imposition that no poles in the integrand coincide.

Here, in this appendix, we mention three properties associated with the Fox's H-function, and we mention also an important particular case, the so-called Meijer's G-function.

P.1. Change the Independent Variable

Let c be a positive constant. We have

$$H^{m,n}_{p,q}\left(x \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right.\right) = c\, H^{m,n}_{p,q}\left(x^c \left| \begin{matrix} (a_p, c\,A_p) \\ (b_q, c\,B_q) \end{matrix} \right.\right). \tag{15}$$

To show this expression one introduce a change of variable $s \to c\,s$ in the integral of inverse Mellin transform.

P.2. Change the First Argument

Set $\alpha \in \mathbb{R}$. Then we can write

$$x^{\alpha} H^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right) = H^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(a_p + \alpha A_p, A_p)\\(b_q + \alpha B_q, B_q)\end{matrix}\right). \tag{16}$$

To show this expression first we introduce the change $a_p \to a_p + \alpha A_p$ and take $s \to s - \alpha$ in the integral of inverse Mellin transform.

P.3. Lowering of Order

If the first factor (a_1, A_1) is equal to the last one, (b_q, B_q), we have

$$\begin{aligned} &H^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(a_1, A_1)\cdots(a_p, A_p)\\(b_1, B_1)\cdots(b_{q-1}, B_{q-1})(a_1, A_1)\end{matrix}\right)\\ &\quad = H^{m,n-1}_{p-1,q-1}\left(x\,\middle|\,\begin{matrix}(a_2, A_2)\cdots(a_p, A_p)\\(b_1, B_1)\cdots(b_{q-1}, B_{q-1})\end{matrix}\right). \end{aligned} \tag{17}$$

To show this identity is sufficient to simplify the common arguments in the Mellin-Barnes integral.

P.4. Reciprocity

To complete the list of the important properties, we mention also the reciprocity. If the parameters m and n interchanged, the p and q also change but the argument converts into the corresponding inverse, we have

$$H^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right) = H^{n,m}_{q,p}\left(\frac{1}{x}\,\middle|\,\begin{matrix}(1-b_q, B_q)\\(1-a_p, A_p)\end{matrix}\right). \tag{18}$$

To show this identity is sufficient to introduce a change of variable $s \to -s$ in the integral of inverse Mellin transform. This is an important property because it enables us to transform a Fox's H-function into another Fox's H-function whose argument is the corresponding inverse, which is important in the study of the asymptotic expansions, for example.

The Meijer's G-Function

To close this appendix, we present an important particular case obtained with $A_p = 1 = B_q$, i.e., we have the Meijer's G-function,

$$H^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(a_1, 1)\cdots(a_p, 1)\\(b_1, 1)\cdots(b_q, 1)\end{matrix}\right) \equiv G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(a_1)\cdots(a_p)\\(b_1)\cdots(b_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_L \Omega(s)\, x^s\, \mathrm{d}s,$$

where we have defined

$$\Omega(s) = \frac{\prod_{i=1}^{m} \Gamma(b_i - s) \prod_{i=1}^{n} \Gamma(1 - a_i + s)}{\prod_{i=m+1}^{q} \Gamma(1 - b_i + s) \prod_{i=n+1}^{p} \Gamma(a_i - s)},$$

with $0 \leqslant m \leqslant q$ and $0 \leqslant n \leqslant p$. The a_i and b_i are complex parameters.

A recent study involving the Meijer's G-function, particularly, a discussion on the contour L, can be seen in [31].

References

1. Langlands, T.A.M., Henry, B.I., Wearne, S.L.: Fractional cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions. J. Math. Biol. **59**, 761–808 (2008)
2. Langlands, T.A.M.: Solution of a modified fractional diffusion equation. Physica A **367**, 136–144 (2006)
3. Zhang, H., Liu, F.: The fundamental solutions of the space, space-time Riesz fractional partial differential equations with periodic conditions. Numer. Math. J. Chinese Univ. (English Ser.) **16**, 181–192 (2007)
4. Duan, J.-s., Xu, M.-y.: The problem for fractional diffusion-wave equations on finite interval and laplace transform (in Chinese). Appl. Math. J. Chin. Univ. Ser. A **19**, 165–171 (2004)
5. Yu, R., Zhang, H.: New function of Mittag-Leffler type and its application in the fractional diffusion-wave equation. Chaos, Solitons Fractals **30**, 946–955 (2006)
6. Mainardi, F., Luchko, Y., Pagnini, G.: The fundamental solution of the space-time fractional diffusion equation. Frac. Calc. Appl. Anal. **4**, 153–192 (2001)
7. Mainardi, F., Pagnini, G., Saxena, R.K.: Fox H-function in fractional diffusion. J. Comput. Appl. Math. **178**, 321–331 (2005)
8. Mathai, A.M., Saxena, R.K., Haubold, H.J.: A certain class of Laplace transforms with applications to reaction and reaction–diffusion equations. Astrophys. Space Sci. **305**, 283–288 (2006)
9. Mittag-Leffler, G.M.: Sur la Nouvelle Fonction E_α. C. R. Acad. Sci. Paris **137**, 554–558 (1903)
10. Humbert, P., Agarwal, R.P.: Sur la Fonction de Mittag-Leffler e Quelques-unes de Ses Généralisations. Bull. Sci. Math. **77**, 180–185 (1953)
11. Prabhakar, T.R.: a singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math. J. **19**, 7–15 (1971)
12. Figueiredo Camargo, R., Chiacchio, A.O., Capelas de Oliveira, E.: Differentiation to fractional orders and the fractional telegraph equation. J. Math. Phys. **49**, 033505 (2008)
13. Srivastava, H.M., Gupta, K.C., Goyal, S.P.: The H-Function of One and Two Variables with Applications. South Asian Publishers, New Delhi (1982)
14. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
15. Metzler, R., Klafter, J.: The random walk's guides to anomalous diffusion: a fractional kinetic equation. Phys. Rep. **339**, 1–77 (2000)
16. Soubhia, A.L., Figueiredo Camargo, R., Vaz Jr., J., Capelas de Oliveira, E.: Theorems for series in three-parameter Mittag-Leffler functions. Frac. Cal. Appl. Anal. **13**, 9–20 (2010)
17. Saxena, R.K., Pagnini, G.: Exact solutions of triple-order time-fractional differential equations for anomalous relaxation and diffusion I: the accelerating case. Physica A **390**, 602–613 (2011)
18. Capelas de Oliveira, E., Mainardi, F., Vaz Jr., J.: Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics. Eur. Phys. J. Special Topics **193**, 161–171 (2011)
19. Mainardi, F., Gorenflo, R., Vivoli, A.: Renewal processes of Mittag-Leffler and Wright type. Frac. Calc. Appl. Anal. **8**, 7–38 (2005)
20. Romão Martins, E., Capelas de Oliveira, E.: Differential Equations and the Stäckel Systems (in Portuguese), vol. 4. IMECC-UNICAMP, Campinas (2006)

21. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products. Academic Press, New York (1980)
22. Capelas de Oliveira, E., Rodrigues Jr., W.A.: Analytic Functions with Applications (in Portuguese). Livraria da Física, São Paulo (2005)
23. Chen, W., Holm, S.: Physical interpretation of fractional diffusion-wave equation via lossy media obeying frequency power law. arXiv.org/abs/math-ph/0303040 (2003)
24. Inizan, P.: Homogeneous fractional embeddings. J. Math. Phys. **49**, 082901 (2008)
25. Figueiredo Camargo, R., Charnet, R., Capelas de Oliveira, E.: On some fractional Green's functions. J. Math. Phys. **50**, 043514 (2009)
26. Capelas de Oliveira, E.: Fractional Green function and diffusion-wave equation (2011, submitted)
27. Mathai, A.M., Saxena, R.K.: The H-Function with Applications in Statistics and Other Disciplines. Wiley Eastern Ltd., New Delhi (1978)
28. Mathai, A.M., Haulbold, H.J.: Special Functions for Applied Scientists. Springer, New York (2008)
29. Mathai, A.M., Saxena, R.K., Haulbold, H.J.: The H-Function Theory and Applications. Springer, New York (2010)
30. Kilbas, A.A., Saigo, M.: H-Transforms. Theory and Applications. Chapman and Hall/CRC, Boca Raton (2004)
31. Silva Costa, F.: Fox's H-function and applications (in Portuguese). Doctoral thesis, Unicamp (2011)