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On Ising Model with Four Competing Interactions on Cayley Tree

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Abstract In the paper we consider an Ising model with four competing interactions (external field, nearest neighbor, second neighbors and triples of neighbors) on the Cayley tree of order two. We show that for some parameter values of the model there is a phase transition. Our second result gives complete description of the periodic Gibbs measures for the model. We also construct uncountably many non-periodic extreme Gibbs measures.

Keywords Cayley tree · Ising model · Competing interactions · Phase transition · Gibbs measure

Mathematics Subject Classifications (2000) 60K35 · 82B20 · 82B05

1 Introduction

Lattice spin systems are a large class of systems considered in statistical mechanics. Some of them have a real physical meaning, others are studied as suitably simplified models of more complicated systems. The structure of the

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lattice plays an important role in investigations of spin systems. For example, in order to study the phase transition problem for a system on Z^d and on Cayley tree there are two different methods: Pirogov-Sinai theory on Z^d, Markov random field theory and recurrent equations of this theory on Cayley tree. In [2–6] for several models on Cayley tree, using the Markov random field theory Gibbs measures are described.

In the paper we investigate a model with four competing interactions on the Cayley tree.

The paper is organized as follows.

In Section 2 we give definitions of the model, Cayley tree and Gibbs measures.

In Section 3 we reduce the problem of describing limit Gibbs measures to the problem of solving a nonlinear functional equations.

Section 4 devoted to describe translation-invariant Gibbs measures. We show that two (minimal and maximal) of translation-invariant Gibbs measures are extreme in the set of all Gibbs measures.

In Section 5 we study periodic Gibbs measures and show that our model admits only translation-invariant and periodic with period two (chess-board) Gibbs measures.

In the last section we construct uncountably many non-periodic extreme Gibbs measures.

2 Definitions

Cayley tree The Cayley tree Γ^k (see [1]) of order $k \geqslant 1$ is an infinite tree, i.e. a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$ where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called nearest neighboring vertices and we write $l =< x, y >$. The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d | x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \text{ such that the pairs } < x_0, x_1 >, ..., < x_{d-1}, x_d > \text{ are neighboring vertices}\}.$$

For the fixed $x^0 \in V$ we set

$$W_n = \left\{x \in V | d\left(x, x^0\right) = n\right\}, \qquad V_n = \left\{x \in V | d\left(x, x^0\right) \leqslant n\right\},$$
$$L_n = \{l =< x, y > \in L | x, y \in V_n\}.$$

A collection of the pairs $< x, x_1 >, ..., < x_{d-1}, y >$ is called a path from x to y. We write $x < y$ if the path from x^0 to y goes through x. We call the vertex y a direct successor of x, if $y > x$ and x, y are nearest neighbors. The set of the direct successors of x is denoted by $S(x)$, i.e.

$$S(x) = \{y \in W_{n+1} | d(x, y) = 1\}, \quad x \in W_n.$$

We observe that for any vertex $x \neq x^0$, x has k direct successors and x^0 has $k+1$.

The vertices x and y are called second neighbor which is denoted by $>x, y<$, if there exists a vertex $z \in V$ such that x, z and y, z are nearest neighbors. We will consider only second neighbors $>x, y<$, for which there exist n such that $x, y \in W_n$. Three vertices x, y and z are called a triple of neighbors and they are denoted by $<x, y, z>$, if $<x, y>, <y, z>$ are nearest neighbors and $x, z \in W_n$, $y \in W_{n-1}$, for some $n \in N$. The fixed vertex x^0 is called the 0-th level and the vertices in W_n are called the n-th level.

It is known [5] that there exists a one-to-one correspondence between the set V of the vertices of the Cayley tree of order $k \geqslant 1$ and the group G_k of the free products of $k+1$ cyclic groups of the second order with generators $a_1, a_2, \ldots, a_{k+1}$.

Let us define a group structure on the Γ^k as follows. Vertices which corresponds to the 'words' $g, h \in G_k$ are called nearest neighbors and are connected by an edge if either $g = ha_i$ or $h = ga_j$ for some i or j. The graph thus defined is a Cayley tree of order k. Consider a left (resp. right) transformation shift on G_k defined as : for $g_o \in G_k$ we put

$$T_{g_0}h = g_0h (\text{resp.} T_{g_0}h = g_0h) \ \ \forall h \in G_k.$$

Then the set of all left (resp. right) shifts on G_k is isomorphic to the group G_k.

The model The Ising model, which was originally regarded as a ferromagnetic model, has found some applications in many other physical, biological and chemical systems, and even in sociology. The model that considered in [8] is a natural generalization of the Ising model, and a model of the similar form has recently been investigated by Monroe [15, 16] to understand the physical aspects associated with the Husimi tree or the Kagome lattice. On a similar note, the topic of statistical mechanics on non amenable graphs is a modern growing field [2, 14]. In the same paper [8], we have presented the exact solution of an Ising model with competing restricted interactions and zero external magnetic field on the Cayley tree Γ^2 for order 2.

In this paper we consider the Ising Model with four competing interactions on the Cayley tree which is defined by the following Hamiltonian

$$H(\sigma) = -J_3 \sum_{<x,y,z>} \sigma(x)\sigma(y)\sigma(z) - J \sum_{>x,y<} \sigma(x)\sigma(y) -$$
$$-J_1 \sum_{<x,y>} \sigma(x)\sigma(y) - \alpha \sum_{x \in V} \sigma(x) \tag{1}$$

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors and the spin variables $\sigma(x)$ assume the values ± 1. (See [1] for models with competing interactions, and see [2, 14–16] for the physical motivation underlying the study of these models.)

Remark If one consider the Hamiltonian with all possible triple (without condition $x, z \in W_n$) and second neighbors (without condition $x, y \in W_n$) then the problem of describing of limit Gibbs measures becomes a difficult problem. Note that interactions defined in (1) depend on the chosen origin x^0, another choice of x^0 would give different interactions. Thus the tree-spin interactions are not translation invariant.

The various partial cases of this model have been investigated in numerous works, for example, the case $J_3 = \alpha = 0$ was considered in [15, 16] and [8]. In [8], the exact solution of an Ising model with competing restricted interactions with zero external field was presented. The case $J = \alpha = 0$ was considered in [7, 16], and [17]. In [7], the exact solution was found for the problem of phase transitions. In [17] it is proven that there are two translation—invariant and uncountable number of distinct non-translation—invariant extreme Gibbs measures. In [9] the phase transition problem was solved for $\alpha = 0,\ J \cdot J_1 \cdot J_3 \neq 0$ and for $J_3 = 0,\ \alpha \cdot J \cdot J_1 \neq 0$ as well.

In the paper we will consider the case $J \cdot J_1 \cdot J_3 \cdot \alpha \neq 0$.

Gibbs measures Let Λ be a finite subset of V. Assume $\Omega(\Lambda)$ is the set of all configuration on Λ, that is the functions $\{\sigma(x), x \in \Lambda\}$. Let $\overline{\sigma}(V \setminus \Lambda)$ be a fixed boundary configuration. The total energy of configuration $\sigma(\Lambda) \in \Omega(\Lambda)$ under condition $\overline{\sigma}(V \setminus \Lambda)$ is defined as

$$
\begin{aligned}
H(\sigma(\Lambda)|\overline{\sigma}(V \setminus \Lambda)) = -J_3 &\sum_{\substack{<x,y,z> \\ x,y,z \in \Lambda}} \sigma(x)\sigma(y)\sigma(z) - J \sum_{\substack{>x,y< \\ x,y \in \Lambda}} \sigma(x)\sigma(y) - \\
&- J_1 \sum_{\substack{<x,y> \\ x,y \in \Lambda}} \sigma(x)\sigma(y) - \alpha \sum_{x \in \Lambda} \sigma(x) - \\
&- J_3 \sum_{\substack{<x,y,z> \\ x \in \Lambda,\, y \notin \Lambda,\, z \notin \Lambda \text{ or} \\ x \in \Lambda,\, y \in \Lambda,\, z \notin \Lambda}} \sigma(x)\sigma(y)\sigma(z) - \\
&- J \sum_{\substack{>x,y< \\ x \in \Lambda,\, y \notin \Lambda}} \sigma(x)\overline{\sigma}(y) - J_1 \sum_{\substack{<x,y> \\ x \in \Lambda,\, y \notin \Lambda}} \sigma(x)\overline{\sigma}(y)\,. \qquad (2)
\end{aligned}
$$

When all boundary points $\{\overline{\sigma}(y), y \in V \setminus \Lambda\}$ are fixed as $+1$, we have the positive boundary condition and when they are fixed as -1, we have negative boundary condition. The free boundary condition corresponds to the case when the last three sums in the above are absent, that is formally all boundary points are fixed as 0.

The partition function $Z_\Lambda(\overline{\sigma}(V \setminus \Lambda))$ in volume Λ under boundary condition $\overline{\sigma}(V \setminus \Lambda))$ is defined as

$$
Z_\Lambda = \sum_{\sigma(\Lambda) \in \Omega(\Lambda)} \exp(-\beta H(\sigma(\Lambda))|\overline{\sigma}(V \setminus \Lambda)),
$$

where $\beta = 1/kT$ is the inverse temperature. Then the conditional Gibbs measure μ_Λ in volume Λ under boundary condition $\overline{\sigma}(V \setminus \Lambda)$ is defined as

$$\mu_\Lambda(\sigma(\Lambda)) = \frac{\exp(-\beta H(\sigma(\Lambda))|\overline{\sigma}(V \setminus \Lambda))}{Z_\Lambda}. \tag{3}$$

3 The Functional Equation

There are several approaches to derive the equation solutions of which describes the limit Gibbs measures for lattice models on the Cayley tree. One approach is based on properties of Markov random fields on Cayley tree [23] and [18]. Another approach is based on recurrence equations for partition functions [7, 12].

Here we shall use the Markov random field method.

Let $h : x \to R$ be a real valued function of $x \in V$. Given $n = 1, 2, \ldots$, consider the probability measure $\mu^{(n)}$ on $\{-1, +1\}^{V_n}$ which defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left\{ -\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x) \right\}.$$

Here, as before, $\beta = 1/kT$ and $\sigma_n : x \in V_n \to \sigma_n(x)$ and Z_n is the corresponding partition function

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega(V_n)} \exp\left\{ -\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_x \tilde{\sigma}(x) \right\}.$$

The consistency condition for $\mu^{(n)}(\sigma_n)$, $n \geqslant 1$ is

$$\sum_{\sigma^{(n)}} \mu^{(n)}\left(\sigma_{n-1}, \sigma^{(n)}\right) = \mu^{(n-1)}(\sigma_{n-1}), \tag{4}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

Let $V_1 \subset V_2 \subset \ldots, \cup_{n=1}^{\infty} V_n = V$ and $\mu_1, \mu_2, \ldots$ be a sequence of the probability measures on $\Phi^{V_1}, \Phi^{V_2}, \ldots$ satisfying the consistency condition, where $\Phi = \{-1, +1\}$. Then, according to the Kolmogorov theorem, (see, e.g. [21]), there is a unique limit Gibbs measure μ_h on Ω such that for every $n = 1, 2, \ldots$ and $\sigma_n \in \Phi^{V_n}$ the following equality holds

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n).$$

The following statement describes the conditions on h_x which guarantee the consistency condition of measures $\mu^{(n)}(\sigma_n)$.

Proposition 1 *The measure $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$ satisfies the consistency condition (4) if and only if for any $x \in V$ the following equation holds:*

$$h_x = \frac{1}{2} \log\left(\theta_4 \frac{\theta_1^2 \theta_2 \theta_3 e^{2(h_y + h_z)} + \theta_1\left(e^{2h_y} + e^{2h_z}\right) + \theta_2 \theta_3}{\theta_1^2 \theta_2 + \theta_1 \theta_3\left(e^{2h_y} + e^{2h_z}\right) + \theta_2 e^{2(h_y + h_z)}} \right), \tag{5}$$

here $S(x)=\{y,z\}$, $<y,x,z>$ *is a ternary neighbor and* $\theta_1=e^{2\beta J_1}, \theta_2=e^{2\beta J}$, $\theta_3=e^{2\beta J_3}, \theta_4=e^{2\beta\alpha}$.

Proof Necessity. According to the consistency condition (4) we have

$$\frac{Z_{n-1}}{Z_n}\sum_{\sigma^{(n)}} exp\Big\{-\beta H_{n-1}(\sigma_{n-1})+\beta J_1\sum_{\substack{x\in W_{n-1},\\ y,z\in S(x)}}\sigma(x)(\sigma(y)+\sigma(z))+$$
$$+\beta J\sum_{\substack{x\in W_{n-1},\\ y,z\in S(x)}}\sigma(y)\sigma(z)+\beta J_3\sum_{\substack{x\in W_{n-1},\\ y,z\in S(x)}}\sigma(x)\sigma(y)\sigma(z)+$$
$$+\sum_{x\in W_{n-1}}\beta\alpha\sigma(x)+\sum_{x\in W_{n-1}}h_x\sigma(x)\Big\}$$
$$=\exp\Big\{-\beta H_{n-1}(\sigma_{n-1})+\sum_{x\in W_{n-1}}h_x\sigma(x)\Big\}$$

Consequently we have

$$\frac{Z_{n-1}}{Z_n}\sum_{\sigma^{(n)}}\prod_{x\in W_{n-1}}\exp\{\beta J_1\sigma(x)(\sigma(y)+\sigma(z))+\beta J\sigma(y)\sigma(z)+$$
$$+\beta J_3\sigma(x)\sigma(y)\sigma(z)+\beta\alpha\sigma(x)+h_y\sigma(y)+h_z\sigma(z)\}$$
$$=\prod_{x\in W_{n-1}}\exp\{h_x\sigma(x)\}.$$

Assume $x\in W_{n-1}$ and $S(x)=\{y,z\}$, $\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}$. Since $\sigma^{(n)}=\cup_{x\in W_{n-1}}\sigma_x^{(n)}$, we get

$$\frac{Z_{n-1}}{Z_n}\prod_{x\in W_{n-1}}\sum_{\sigma_x^{(n)}}\exp\{\beta J_1\sigma(x)(\sigma(y)+\sigma(z))+\beta J\sigma(y)\sigma(z)+$$
$$+\beta J_3\sigma(x)\sigma(y)\sigma(z)+\beta\alpha\sigma(x)+h_y\sigma(y)+h_z\sigma(z)\}$$
$$=\prod_{x\in W_n}\exp\{h_x\sigma(x)\}. \tag{6}$$

□

Now fix $x\in W_{n-1}$ and rewrite (6) for the cases $\sigma(x)=1$ and $\sigma(x)=-1$. If $\sigma(x)=1$, we have

$$N=\sum_{\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}}\exp\{\beta J_1(\sigma(y)+\sigma(z))+\beta J\sigma(y)\sigma(z)+$$
$$+\beta J_3\sigma(y)\sigma(z)+\beta\alpha+h_y\sigma(y)+h_z\sigma(z)\}$$
$$=\exp\{h_x\};$$

and if $\sigma(x) = -1$, then

$$D = \sum_{\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}} \exp\{-\beta J_1(\sigma(y)+\sigma(z)) + \beta J\sigma(y)\sigma(z)\} + \\ + \beta J_3\sigma(y)\sigma(z) - \beta\alpha + h_y\sigma(y) + h_z\sigma(z)\} \\ = \exp\{-h_x\}.$$

So that

$$\frac{N}{D} = \exp\{2h_x\}. \tag{7}$$

The numerator N of the left-hand side is equal to

$$N = \exp(2\beta J_1 + \beta J + \beta J_3 + \beta\alpha + h_y + h_z) + \\ + \exp(-\beta J - \beta J_3 + \beta\alpha - h_y + h_z) + \exp(-\beta J - \beta J_3 + \beta\alpha + h_y - h_z) + \\ + \exp(-2\beta J_1 + \beta J + \beta J_3 + \beta\alpha - h_y - h_z),$$

while the denumerator D is equal to

$$D = \exp(-2\beta J_1 + \beta J + \beta J_3 - \beta\alpha + h_y + h_z) + \\ + \exp(-\beta J - \beta J_3 - \beta\alpha - h_y + h_z) + \exp(-\beta J - \beta J_3 - \beta\alpha + h_y - h_z) + \\ + \exp(2\beta J_1 + \beta J + \beta J_3 - \beta\alpha - h_y - h_z).$$

Then the equality $\frac{N}{D} = \exp\{2h_x\}$ implies (5).
Sufficiency. Assume that (5) is valid. From (7) we get

$$\sum_{\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}} \exp\{\beta J_1\sigma(x)(\sigma(y)+\sigma(z)) + \beta J\sigma(y)\sigma(z) + \\ + \beta J_3\sigma(x)\sigma(y)\sigma(z) + \beta\alpha\sigma(x) + h_y\sigma(y) + h_z\sigma(z)\} \\ = a(x)\exp\{\sigma(x)h_x\},$$

where $\sigma(x) = \pm 1$. This equality implies

$$\prod_{x\in W_{n-1}} \sum_{\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}} \exp\{\beta J_1\sigma(x)(\sigma(y)+\sigma(z)) + \beta J\sigma(y)\sigma(z) + \\ + \beta J_3\sigma(x)\sigma(y)\sigma(z) + \beta\alpha\sigma(x) + h_y\sigma(y) + h_z\sigma(z)\} \\ = \prod_{x\in W_{n-1}} a(x)\exp\{\sigma(x)h_x\}. \tag{8}$$

Denoting $A_n(x) = \prod_{x\in W_{n-1}} a(x)$, we have from (8)

$$Z_{n-1}A_{n-1}\mu^{(n-1)}(\sigma_{n-1}) = Z_n \sum_{\sigma^{(n)}} \mu^{(n)}\big(\sigma_{n-1}, \sigma^{(n)}\big).$$

As $\mu^{(n)}, n \geqslant 1$ is a probability, we have

$$\sum_{\sigma_{n-1}}\sum_{\sigma^{(n)}}\mu^{(n)}\big(\sigma_{n-1},\sigma^{(n)})\big)=\sum_{\sigma_{n-1}}\mu^{(n-1)}(\sigma_{n-1})=1.$$

From these equalities we get $Z_{n-1}A_{n-1}=Z_n$, which means that (4) holds.

According to Proposition 1 the problem of describing the Gibbs measures is reduced to the description of the solutions of the functional Eq. 5.

Denote $\Omega=\{-1,+1\}^V$. Note that any transformation S of the group G_k induces a shift automorphism $\tilde{S}:\Omega\to\Omega$ by

$$(\tilde{S}\sigma)(g)=\sigma(Sg),\qquad g\in G_k,\ \sigma\in\Omega.$$

By G_k we denote the set of all shifts on Ω. We say that a Gibbs measure μ on Ω is translation - invariant if for any $T\in G_k$ the equality $\mu(T(A))=\mu(A)$ is valid for all $A\in\mathcal{F}$, where $\mathcal{F}$ is a standard σ-algebra of subsets of Ω generated by cylinder subsets.

4 Translation-invariant Gibbs Measures: Phase Transition

The analysis of the solution of (5) is rather tricky. It is natural to begin with the translation-invariant solutions where $h_x=h$ is constant for all $x\in V$. In this case from (5), we have

$$u=\theta_4\frac{\theta_1^2\theta_2\theta_3u^2+2\theta_1u+\theta_2\theta_3}{\theta_1^2\theta_2+2\theta_1\theta_3u+\theta_2u^2}\tag{9}$$

where $u=e^{2h}$.

Note that if there is more than one positive solution for Eq. 9, then there is more than one translation-invariant Gibbs measure corresponding to these solutions. We say that a phase transition occurs for model (1), if Eq. 9 has more than one positive solution. The number of the solutions of Eq. 9 depends on the parameter $\beta=\frac{1}{kT}$. The phase transition usually occurs for low temperature. If it is possible to find an exact value of temperature T^* such that a phase transition occurs for all $T<T^*$ then T^* is called a critical value of temperature.

Finding the exact value of the critical temperature for some models means to exactly solve the models.

Proposition 2 *If* $\theta_1^2>3$, $\theta_2>\frac{2\theta_1}{\theta_1^2-3}$, *and*

$$\frac{\sqrt{\theta_2^2(\theta_1^4+2\theta_1^2-3)-4\theta_1^2-8\theta_1\theta_2}-\sqrt{\theta_2^2(\theta_1^4+2\theta_1^2-3)-4\theta_1^2-8\theta_1\theta_2-4\theta_1^2\theta_2^2}}{2\theta_1\theta_2}<\theta_3<$$

$$<\frac{\sqrt{\theta_2^2(\theta_1^4+2\theta_1^2-3)-4\theta_1^2-8\theta_1\theta_2}+\sqrt{\theta_2^2(\theta_1^4+2\theta_1^2-3)-4\theta_1^2-8\theta_1\theta_2-4\theta_1^2\theta_2^2}}{2\theta_1\theta_2},$$

$$\eta_1(\theta_1,\theta_2,\theta_3)<\theta_4^2<\eta_2(\theta_1,\theta_2,\theta_3)$$

then Eq. 9 has three positive roots $u_1^* < u_2^* < u_3^*$. Here

$$\eta_i(\theta_1, \theta_2, \theta_3) = \frac{1}{u_i} \frac{\theta_1^2\theta_2\theta_3 u_i^2 + 2\theta_1 u_i + \theta_2\theta_3}{\theta_1^2\theta_2 + 2\theta_1\theta_3 u_i + \theta_2 u_i^2}$$

where $u_i, i = 1, 2$ are the solutions of

$$\theta_1^2\theta_2^2\theta_3 u^4 + 4\theta_1\theta_2 u^3 + \theta_3\big(3\theta_2^2 - \theta_1^4\theta_2^2 + 4\theta_1^2\big)u^2 + 4\theta_1\theta_2\theta_3^2 u + \theta_1^2\theta_2^2\theta_3 = 0. \quad (10)$$

Proof Denote

$$f(u) = \frac{\theta_1^2\theta_2\theta_3 u^2 + 2\theta_1 u + \theta_2\theta_3}{\theta_1^2\theta_2 + 2\theta_1\theta_3 u + \theta_2 u^2}.$$

We have

$$f'(u) = 2\theta_2 \frac{\theta_1\big(\theta_1^2\theta_3^2 - 1\big)u^2 + \theta_2\theta_3\big(\theta_1^4 - 1\big)u + \theta_1\big(\theta_1^2 - \theta_3^2\big)}{\big(\theta_1^2\theta_2 + 2\theta_1\theta_3 u + \theta_2 u^2\big)^2},$$

$$f''(u) = 2\theta_2\big(\theta_2 u^2 + 2\theta_1\theta_3 u + \theta_1^2\theta_2\big)^{-3} \times$$

$$\times\big(-2\theta_1\theta_2\big(\theta_1^2\theta_3^2 - 1\big)u^3 - 3\theta_2^2\theta_3\big(\theta_1^4 - 1\big)u^2 + 6\theta_1\theta_2\big(\theta_3^2 - \theta_1^2\big)u +$$

$$+ \theta_1^2\theta_3\big(\theta_2^2\big(\theta_2^4 - 1\big) - 4\theta_1^2 + 4\theta_3^2\big)\big).$$

Denote

$$A = 2\theta_1\theta_2\big(\theta_1^2\theta_3^2 - 1\big); \qquad B = 3\theta_2^2\theta_3\big(\theta_1^4 - 1\big);$$

$$C = 6\theta_1\theta_2\big(\theta_3^2 - \theta_1^2\big); \qquad D = \theta_1^2\theta_3\big(\theta_2^2\big(\theta_2^4 - 1\big) - 4\theta_1^2 + 4\theta_3^2\big).$$

It is easy to see that under conditions of the proposition we have $A > 0$, $B > 0$, $C > 0$, $D > 0$. Equation $f''(u) = 0$ is equivalent to $Au^3 + Bu^2 - Cu - D = 0$, one can easily prove that the last equation has unique positive solution, say u_*. Thus f is convex for $u < u_*$ and concave for $u > u_*$. Consequently there are at most three solutions. On the other hand, it is easy to see that (9) has more than one solution if and only if there is more than one solution of the equation $uf'(u) = f(u)$ which is equivalent to Eq. 10. Now consider (10), which can be rewritten as

$$\theta_1^2\theta_2^2\left(u + \frac{1}{u}\right)^2 + 4\theta_1\theta_2\left(\frac{u}{\theta_3} + \frac{\theta_3}{u}\right) + 3\theta_2^2 - \theta_1^4\theta_2^2 + 4\theta_1^2 - 2\theta_1^2\theta_2^2 = 0.$$

Denote

$$\varphi(u) = 4\theta_1\theta_2\left(\frac{u}{\theta_3} + \frac{\theta_3}{u}\right),$$

$$\psi(u) = \theta_2^2\big(\theta_1^4 + 2\theta_1^2 - 3\big) - \theta_1^2\theta_2^2\left(u + \frac{1}{u}\right)^2 - 4\theta_1^2.$$

A simple analysis of these functions show that under conditions of the proposition the Eq. 9 has three positive solutions. This completes the proof. □

Thus by Propositions 1 and 2 we can formulate the following

Theorem 3 *Assume the conditions of the Proposition* 2 *are satisfied then for the model* (1) *there are three translation-invariant Gibbs measures* μ_1, μ_2, μ_3 *i.e. there is phase transition.*

Note that μ_1 (μ_3) corresponds to positive (resp. negative) boundary condition. The boundary condition corresponding to μ_2 unclear.

The following Proposition 4 describes a useful property of general (non translation-invariant) solutions h_x to (5).

Proposition 4 *Assume the conditions of the Proposition* 2 *are satisfied and* h_x *is a solution of* (5)*, with* $u_x = e^{2h_x}$*, then*

$$u_1^* \leqslant u_x \leqslant u_3^*, \qquad x \in V \tag{11}$$

where $u_1^* < u_3^*$ *are solutions of* (9).

Proof It is clear that $u_x > 0$, for any $x \in V$. For $u, v > 0$ denote

$$F(u, v) = \theta_4 \frac{\theta_1^2\theta_2\theta_3 uv + \theta_1(u+v) + \theta_2\theta_3}{\theta_1^2\theta_2 + \theta_1\theta_3(u+v) + \theta_2 uv}.$$

The Eq. 5 can be rewritten as $u_x = F(u_y, u_z)$.

Observe that under conditions of the Proposition 2 the function $F(u, v)$ is increasing with respect to u and v on $(0, \infty)$. Hence we conclude that

$$\frac{\theta_3\theta_4}{\theta_1^2} < F(u, v) < \theta_1^2\theta_3\theta_4,$$

for all $u, v > 0$. Now we consider the function with $u, v \in \left(\frac{\theta_3\theta_4}{\theta_1^2}, \theta_1^2\theta_2\theta_3\right)$. By similar reason as above we get

$$f\left(\frac{\theta_3\theta_4}{\theta_1^2}\right) < F(u, v) < f\left(\theta_1^2\theta_3\theta_4\right),$$

where $f(u) = F(u, u)$. Repeating this argument one gets

$$f^{(n)}\left(\frac{\theta_3\theta_4}{\theta_1^2}\right) < F(u, v) < f^{(n)}\left(\theta_1^2\theta_3\theta_4\right),$$

for all $n \geqslant 1$. Here $f^{(n)}$ is n-th iteration of the map $x \to f(x)$. The sequence $f^{(n)}(\theta_1^2\theta_3\theta_4)$ is decreasing and bounded below by u_3^*. Its limit is a fixed point of f and thus equal to u_3^*. This proves that $u_x \leqslant u_3^*$. The lower bound for u_x is similar and gives u_1^*. □

Using Proposition 4 by similar argument as in the proof of Theorem 12.31 of [10] one can prove the following

Theorem 5 *Assume conditions of Proposition* 2 *are satisfied then translation-invariant measures* μ_1, μ_3 (*see Theorem* 3) *are extreme.*

Remark The problem of extremality for measure μ_2 is a difficult problem. Usually (see [3, 24]) such measure which corresponds to unordered phase is extreme for the temperature $T \in (T'_c, T_c]$ where T_c is the critical temperature of phase transition and T'_c is a (second) critical temperature ($0 < T'_c < T_c$).

5 Periodic Gibbs Measures

In this section we study a periodic (see Definition 6) solutions of (5).

Definition 6 Let K be a subgroup of G_k, $k \geqslant 1$. We say that a collection (of functions) $h = \{h_x \in R^1 : x \in G_k\}$ is K-*periodic* if $h_{yx} = h_x$ for all $x \in G_k$ and $y \in K$.

Definition 7 A Gibbs measure is called K-periodic if it corresponds to K-periodic collection h.

Observe that a translation-invariant Gibbs measure is G_k-periodic.

We give a complete description of periodic Gibbs measures i.e. a characterization of such measures with respect to any normal subgroup of finite index in G_k.

Let K be a subgroup of index r in G_k, and let $G_k/K = \{K_0, K_1, ..., K_{r-1}\}$ be the quotient group, with the coset $K_0 = K$. Let $q_i(x) = |S_1(x) \cap K_i|$, $i = 0, 1, ..., r-1$; $N(x) = |\{j : q_j(x) \neq 0\}|$, where $S_1(x) = \{y \in G_k : \langle x, y\rangle\}$, $x \in G_k$ and $|\cdot|$ is the number of elements in the set. Denote $Q(x) = (q_0(x), q_1(x), ..., q_{r-1}(x))$.

We note (see [19]) that for every $x \in G_k$ there is a permutation π_x of the coordinates of the vector $Q(e)$ (where e is the identity of G_k) such that

$$\pi_x Q(e) = Q(x). \tag{12}$$

It follows from (12) that $N(x) = N(e)$ for all $x \in G_k$.

Each $K-$ periodic collection is given by

$$\{h_x = h_i \ \text{ for } \ x \in K_i, \ \ i = 0, 1, ..., r-1\}.$$

For $k = 2$ by Proposition 1 and (12), h_n, $n = 0, 1, ..., r-1$, satisfies

$$h_n = \frac{1}{2} \log F\left(e^{2h_{\pi_n(i)}}, e^{2h_{\pi_n(j)}}\right), \tag{13}$$

where $F(u, v)$ is defined in proof of the Proposition 4 and π_n is permutation of $Q(e)$ for $x \in K_n, i, j \in Q(e)$.

Proposition 8 *Suppose the conditions of Proposition 2 are satisfied then* $F(u, v) = F(h, v)$ *if and only if* $u = h$ $\quad (F(u, v) = F(u, h)$ *if and only if* $v = h)$.

Proof Follows from monotonity of F with respect to u (resp. v). □

Let G_2^* be the subgroup in G_2 consisting of all words of even length. Clearly, G_2^* is a subgroup of index 2.

Theorem 9 *Let K be a normal subgroup of finite index in G_2. Then each $K-$ periodic Gibbs measure for model* (1) *is either translation-invariant or G_2^*- periodic.*

Proof We see from (13) that

$$F\left(e^{h_{\pi_n(i)}}, e^{h_{\pi_n(j)}}\right) = F\left(e^{h_{\pi_n(i')}}, e^{h_{\pi_n(j')}}\right), \tag{14}$$

For any $i, j, i', j' \in Q(e), n = 0, 1, ..., r-1$. Hence from Proposition 8 we have

$$h_{\pi_n(i_1)} = h_{\pi_n(i_2)} = ... = h_{\pi_n(i_{N(e)})}.$$

Therefore,

$$h_x = h_y = h, \quad \text{if } x, y \in S_1(z), \qquad z \in G_2^*;$$
$$h_x = h_y = l, \quad \text{if } x, y \in S_1(z), \qquad z \in G_2 \setminus G_2^*.$$

Thus the measures are translation-invariant (if $h = l$) or G_2^*- periodic (if $h \neq l$). This completes the proof of the theorem. □

Let K be a normal subgroup of finite index in G_2. What condition on K will guarantee that each $K-$periodic Gibbs measure is translation-invariant? We put $I(K) = K \cap \{a_1, a_2, a_3\}$, where $a_i, \;\; i = 1, 2, 3$ are generators of G_2.

Theorem 10 *If $I(K) \neq \emptyset$, then each $K-$ periodic Gibbs measure for model* (1) *is translation-invariant.*

Proof Take $x \in K$. We note that the inclusion $xa_i \in K$ holds if and only if $a_i \in K$. Since $I(K) \neq \emptyset$, there is an element $a_i \in K$. Therefore K contains the subset $Ka_i = \{xa_i : x \in K\}$. By Theorem 9 we have $h_x = h$ and $h_{xa_i} = l$. Since x and xa_i belong to K, it follows that $h_x = h_{xa_i} = h = l$. Thus each $K-$ periodic Gibbs measure is translation-invariant. This proves Theorem 10. □

Theorems 9 and 10 reduce the problem of describing $K-$ periodic Gibbs measure with $I(K) \neq \emptyset$ to describing the fixed points of $f(u) = F(u, u)$ (see

(9)) which describes translation-invariant Gibbs measures. If $I(K) = \emptyset$, this problem is reduced to describing the solutions of the system:

$$\begin{cases} u = f(v), \\ v = f(u) \end{cases} \tag{15}$$

with

$$f(u) = \theta_4 \frac{\theta_1^2\theta_2\theta_3 u^2 + 2\theta_1 u + \theta_2\theta_3}{\theta_1^2\theta_2 + 2\theta_1\theta_3 u + \theta_2 u^2}.$$

Evidently the positive roots of the equation

$$\frac{f(f(u)) - u}{f(u) - u} = 0 \tag{16}$$

describe the periodic (non translation-invariant) Gibbs measures.

As we are looking for positive roots (16) has the following form:

$$\theta_1^2\theta_2\big(\theta_1^2\theta_2\theta_3^2\theta_4^2 + 2\theta_1\theta_3^2\theta_4 + \theta_2\big)u^2 + \theta_3\big(\theta_1^4\theta_2^2\theta_4 + 2\theta_1^3\theta_2\theta_4^2 + 2\theta_1^2\theta_2 + 4\theta_1^2\theta_4 - \theta_2^2\theta_4\big)u + \\ + \theta_1^2\theta_2\big(\theta_2\theta_3^2\theta_4^2 + 2\theta_1\theta_4 + \theta_1^2\theta_2\big) = 0, \tag{17}$$

The discriminant Δ of (17) is equal to

$$\Delta = -4\theta_1^5\theta_2^3\theta_4^3(\theta_1\theta_2\theta_4 + 2)\theta_3^4 + A\theta_3^2 - 4\theta_1^5\theta_2^3(\theta_1\theta_2 + 2\theta_4),$$

where

$$A = -\theta_4^2\big(3\theta_1^8 + 6\theta_1^4 - 1\big)\theta_2^4 - 4\theta_1^3\theta_4\big(1 + \theta_4^2\big)\big(1 + \theta_1^4\big)\theta_2^3 + \\ + 4\theta_1^2\big(\theta_1^4\theta_4^4 + \theta_1^4 - 2\theta_4^2\big)\theta_2^2 + 16\theta_1^5\theta_4\big(1 + \theta_4^2\big)\theta_2 + 16\theta_1^4\theta_4^2.$$

Using simple analysis one can see that (17) has two positive solutions if

$$\theta_1 < 1, \ \ \theta_2 > \frac{2\theta_1}{1 - \theta_1^2}, \qquad \frac{1}{\theta_4^*} < \theta_4 < \theta_4^*, \tag{18}$$

where

$$\theta_4^* = \frac{\theta_2^2 - 4\theta_1^2 - \theta_1^4\theta_2^2 + \sqrt{\big(4\theta_1^2 + \theta_1^4\theta_2^2 - \theta_2^2\big)^2 - 16\theta_1^6\theta_2^2}}{4\theta_1^3\theta_2}$$

and

$$A^2 > 64\theta_1^{10}\theta_2^6\theta_4(\theta_1\theta_2\theta_4 + 2)(\theta_1\theta_2 + 2\theta_4), \tag{19}$$

$$\theta_3^- < \theta_3^2 < \theta_3^+, \tag{20}$$

where $\theta_3^{\mp}$ are solutions of $\Delta = 0$.

Therefore, the following theorem is proved:

Theorem 11 *Assume* $(\theta_1, \theta_2, \theta_3, \theta_4)$ *satisfied conditions* (18)–(20) *then for the model* (1) *there are two* G_2^*- *periodic Gibbs measures* $\mu_1^{\text{per}}, \mu_2^{\text{per}}$.

Remark

1. By construction measures $\mu_1^{\mathrm{per}}, \mu_2^{\mathrm{per}}$ are non translation-invariant, but periodic with period 2 (= index of normal subgroup).
2. For $\theta_4 = 1$ the condition (19) can be rewritten as (see [9])

$$\left(1-3\theta_1^2\right)\left(\theta_1^2+1\right)\left(\theta_2^2-\frac{2\theta_1}{1-3\theta_1^2}\right)\left(\theta_2^2-\frac{2\theta_1}{1+\theta_1^2}\right)^2\left(\theta_2^2+\frac{2\theta_1}{1+\theta_1^2}\right)>0.$$

This factorization gives more simple formulation of the conditions (18)–(20) i.e. for $\theta_4 = 1$ conditions (18)–(20) can be reduced to

$$0<\theta_1<\frac{1}{\sqrt{3}}, \qquad \theta_2>\frac{2\theta_1}{1-3\theta_1^2}, \qquad \theta_3^- <\theta_3<\theta_3^+.$$

6 Non Periodic Gibbs Measures

In this section we consider the case of phase transition (i.e. assume that the conditions of Proposition 2 are satisfied). We show that functional Eq. 5 admits uncountably many non periodic solutions.

Take an arbitrary infinite path $\pi = \{x^0 = x_0, x_1, ...\}$ on the Cayley tree of order 2. There is (see [4, 20]) one-to-one correspondence between such paths and real numbers $t \in [0; 1]$. We will map the path π to a function $h^\pi : x \in V \to h_x^\pi$ satisfying (5). Path π splits Cayley tree Γ^2 into two parts Γ_1^2 and Γ_2^2.

Function h^π is defined by

$$h_x^\pi = \begin{cases} \log u_1^*, & \text{if } x \in \Gamma_1^2 \\ \log u_3^*, & \text{if } x \in \Gamma_2^2 \end{cases} \tag{21}$$

Denote

$$\Phi(x, y) = \frac{1}{2}\log\left(\theta_4\frac{\theta_1^2\theta_2\theta_3 e^{2(x+y)}+\theta_1(e^{2x}+e^{2y})+\theta_2\theta_3}{\theta_1^2\theta_2+\theta_1\theta_3(e^{2x}+e^{2y})+\theta_2 e^{2(x+y)}}\right)$$

Proposition 12 *Following inequality holds:*

$$|\Phi(x_1, y)-\Phi(x_2, y)| \leqslant \gamma(\theta_1, \theta_2, \theta_3)|x_1-x_2|,$$

where

$$\gamma(\theta_1,\theta_2,\theta_3) = \max_{t\in[u_1^*, u_3^*]}\frac{|\sqrt{(\theta_1 t+\theta_2\theta_3)(\theta_2 t+\theta_1\theta_3)}-\theta_1\sqrt{(\theta_1\theta_2\theta_3 t+1)(\theta_3 t+\theta_1\theta_2)}|}{\sqrt{(\theta_1 t+\theta_2\theta_3)(\theta_2 t+\theta_1\theta_3)}+\theta_1\sqrt{(\theta_1\theta_2\theta_3 t+1)(\theta_3 t+\theta_1\theta_2)}}<1$$

Proof The function $\Phi(x, y)$ can be rewritten as

$$\Phi(x, y) = \frac{1}{2}\log\theta_4 + \frac{1}{2}\log\frac{Ae^{2x}+B}{Ce^{2x}+D},$$

where A, B, C, D depends on $\theta_1, \theta_2, \theta_3$ and y. It is easy to see that

$$|\Phi'_x(x, y)| \leqslant \frac{|\sqrt{AD} - \sqrt{BC}|}{\sqrt{AD} + \sqrt{BC}}.$$

This completes the proof. □

With the help of Proposition 12 it is easy to prove the following Theorem 13, similar to Theorem 3 of [20]:

Theorem 13 *For any infinite path π, there exists a unique function h^{π} satisfying* (5) *and* (21).

In the standard way (see e.g. [4, 20]) one can prove that functions $h^{\pi(t)}$ are different for different $t \in [0; 1]$.

Now let $\mu(t)$ denote the Gibbs measure corresponding to function $h^{\pi(t)}$, $t \in [0; 1]$.

Using Theorem 5, similar to analogous theorem of [4] we obtain the following

Theorem 14 *For any $t \in [0; 1]$, there exists a unique extreme Gibbs measure $\mu(t)$. Moreover, the above Gibbs measures μ_i, $i = 1, 3$, are specified as $\mu(0) = \mu_3$, $\mu(1) = \mu_1$.*

Because measures $\mu(t)$ are different for different $t \in [0; 1]$ we obtain a continuum of distinct extreme Gibbs measures.

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