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Multi-particle Anderson Localisation: Induction on the Number of Particles

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Abstract This paper is a follow-up of our recent papers Chulaevsky and Suhov (Commun Math Phys 283:479–489, 2008) and Chulaevsky and Suhov (Commun Math Phys in press, 2009) covering the two-particle Anderson model. Here we establish the phenomenon of Anderson localisation for a quantum N-particle system on a lattice $\mathbb{Z}^d$ with short-range interaction and in presence of an IID external potential with sufficiently regular marginal cumulative distribution function (CDF). Our main method is an adaptation of the multi-scale analysis (MSA; cf. Fröhlich and Spencer, Commun Math Phys 88:151–184, 1983; Fröhlich et al., Commun Math Phys 101:21–46, 1985; von Dreifus and Klein, Commun Math Phys 124:285–299, 1989) to multi-particle systems, in combination with an induction on the number of particles, as was proposed in our earlier manuscript (Chulaevsky and Suhov 2007). Recently, Aizenman and Warzel (2008) proved spectral and *dynamical* localisation for N-particle lattice systems with a short-range interaction, using an extension of the Fractional-Moment Method (FMM) developed earlier for single-particle models in Aizenman and Molchanov (Commun Math Phys 157:245–278, 1993) and Aizenman et al. (Commun Math Phys 224:219–253, 2001) (see also

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references therein) which is also combined with an induction on the number of particles.

Keywords Anderson localisation · Multi-particle systems

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1 Introduction and the Main Result

The status of the multi-particle Anderson localisation problem has been described in [3], Section 1.1; the reader is advised to consult this reference.

The configuration space of the N-particle lattice system is the Cartesian product $\mathbb{Z}^d \times \cdots \times \mathbb{Z}^d$ of N copies of a cubic lattice $\mathbb{Z}^d$, which we denote for brevity by $\mathbb{Z}^{Nd}$. The Hilbert space of the N-particle lattice system is $\ell_2(\mathbb{Z}^{Nd})$. The Hamiltonian $\mathbf{H}\left(= \mathbf{H}^{(N)}_{U,V,g}(\omega)\right)$ is a lattice Schrödinger operator acting on functions $\boldsymbol{\phi} \in \ell_2(\mathbb{Z}^{Nd})$ by

$$\begin{aligned}\mathbf{H}^{(N)}\boldsymbol{\phi}(\mathbf{x}) &= H^0\boldsymbol{\phi}(\mathbf{x}) + (U(\mathbf{x}) + gW(\mathbf{x};\omega))\boldsymbol{\phi}(\mathbf{x})\\ &= \sum_{\substack{\mathbf{y}\in\mathbb{Z}^{Nd}:\\ \|\mathbf{y}-\mathbf{x}\|=1}} \boldsymbol{\phi}(\mathbf{y}) + \left[U(\mathbf{x}) + gW(\mathbf{x};\omega)\right]\boldsymbol{\phi}(\mathbf{x}),\\ \text{where}\quad W(\mathbf{x};\omega) &= \sum\nolimits_{j=1}^{N} V(x_j;\omega),\\ &\quad \mathbf{x} = (x_1,\dots,x_N),\ \mathbf{y} = (y_1,\dots,y_N) \in \mathbb{Z}^{Nd}.\end{aligned} \tag{1.1}$$

Here and below we use boldface letters such as $\mathbf{x}$, $\mathbf{y}$, $\mathbf{H}$, etc., referring to a multi-particle system, where the particle number enters as an index or specified verbally. For example, small-case boldface letters $\mathbf{x}$, $\mathbf{y}$, etc., will stand for designate points in $\mathbb{Z}^{Nd}$, called N-particle configurations. Letters x, y will be systematically used for points in $\mathbb{Z}^d$ or $\mathbb{R}^d$, referred to as single-particle positions (or briefly, positions).

Our proof of N-particle Anderson localisation is organised as an induction in N, as has been explained in earlier presentations (see, e.g., [5]). Thus, we will have to deal with systems with smaller number of particles, $1 \leqslant n < N$. The respective objects, viz., points in $\mathbb{Z}^{nd}$, $n < N$, are still denoted by boldface letters: $\mathbf{x} \in \mathbb{Z}^{nd}$, $\mathbf{y} \in \mathbb{Z}^{nd}$, etc.

Next, $x_j = \left(\mathrm{x}_j^{(1)}, \dots, \mathrm{x}_j^{(d)}\right)$ and $y_j = \left(\mathrm{y}_j^{(1)}, \dots, \mathrm{y}_j^{(d)}\right)$ stand for the positions of individual particles in $\mathbb{Z}^d$, $j = 1, \dots, N$, and $\|\cdot\|$ denotes the sup-norm: for $\mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d = \mathbb{R}^{Nd}$,

$$\|\mathbf{v}\| = \max_{j=1,2} \|v_j\|, \tag{1.2}$$

where, for $v = (\mathrm{v}^{(1)}, \dots, \mathrm{v}^{(d)}) \in \mathbb{R}^d$,

$$\|v\| = \max_{i=1,\dots,d} \left|\mathrm{v}^{(i)}\right|. \tag{1.3}$$

We will consider the distance on $\mathbb{R}^{Nd}$, $\mathbb{Z}^{Nd}$ and $\mathbb{R}^d$, $\mathbb{Z}^d$ generated by the norm $\|\cdot\|$.

Throughout this paper, the random external potential $V(x;\omega)$, $x \in \mathbb{Z}^d$, is assumed to be real IID, with a common CDF F_V on $\mathbb{R}$. The condition on F_V guaranteeing the validity of our results is as follows:

$$\sup_{\epsilon\in(0,1)} \left[\frac{1}{\epsilon^A} \sup_{a\in\mathbb{R}} \left(\mathrm{F}_V(a+\epsilon) - \mathrm{F}_V(a)\right)\right] < +\infty, \tag{1.4}$$

for some $A > 0$. In other words, the marginal distribution of the random potential is Hölder-continuous.[1] Clearly, this does not require the absolute continuity of F_V.

Parameter $g \in \mathbb{R}$ is traditionally called the coupling, or amplitude, constant.

The interaction energy function U is assumed to be of the form

$$U(\mathbf{x}) = \sum_{1\leqslant j_1<j_2\leqslant N} \Phi\left(x_{j_1}, x_{j_2}\right), \qquad \mathbf{x} = (x_1, \dots, x_N) \in \mathbb{Z}^{Nd}, \tag{1.5}$$

where function $\Phi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ (the two-body interaction potential) satisfies the following properties.

(i) *Φ is a bounded symmetric function:*

$$\sup\left[|\Phi\left(x, x'\right)| : x, x' \in \mathbb{Z}^d\right] < +\infty, \qquad \Phi\left(x, x'\right) = \Phi\left(x', x\right), \ x, x' \in \mathbb{Z}^d. \tag{1.6}$$

(ii) *Φ has a finite range:*

$$\Phi\left(x, x'\right) = 0, \qquad \text{if } \|x - x'\| > r_0, \tag{1.7}$$

where $r_0 \in [0, +\infty)$ is a given value.

It is then obvious that function $U : \mathbb{Z}^{Nd} \to \mathbb{R}$ is symmetric under any permutation of positions x_j: $U(\mathbf{x}) = U(\mathcal{S}_\sigma \mathbf{x})$. Here σ is an arbitrary element of the symmetric group $\mathfrak{S}_N$, and, given $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{Z}^{Nd}$,

$$\mathcal{S}_\sigma \mathbf{x} = \left(x_{\sigma(1)}, \dots, x_{\sigma(N)}\right).$$

The same is true for function W (see Eq. 1.1).

We consider binary interaction potentials in order not to make our notations excessively cumbersome. The reader will see that, actually, more general bounded short-range many-body interactions can be treated in the same way. The symmetry does not play an important role, but is convenient technically (and natural from the physical point of view).

[1] One can easily show that the main result of this paper remains valid for log-Hölder continuous CDF F_v, satisfying $|F_V(a+\epsilon) - F_V(a)| \leqslant C \ln^{-A} |\epsilon|^{-1}$ with $A > 0$ large enough.

Throughout the paper, $\mathbb{P}$ stands for the joint probability distribution of RVs $\{V(x;\omega), x \in \mathbb{Z}^d\}$. The main assertion of this paper is

Theorem 1 *Consider the random Hamiltonian* $\mathbf{H}^{(N)}(\omega)$ *given by Eq. 1.1. Suppose that* U *satisfies conditions (1.4) and (1.5), and the random potential* $\{V(x;\omega), x \in \mathbb{Z}^d\}$ *is* IID *obeying Eq. 1.3. Then there exists* $g^* \in (0, +\infty)$ *such that for any* g *with* $|g| \geqslant g^*$, *the spectrum of operator* $\mathbf{H}^{(N)}(\omega)$ *is* $\mathbb{P}$*-a.s. pure point. Furthermore, there exists a nonrandom constant* $m_+ = m_+(g) > 0$ *such that all eigenfunctions* $\boldsymbol{\Psi}_j(\mathbf{x};\omega)$ *of* $\mathbf{H}^{(N)}(\omega)$ *admit an exponential bound:*

$$|\boldsymbol{\Psi}_j(\mathbf{x};\omega)| \leqslant C_j(\omega)\, e^{-m_+\|\mathbf{x}\|}. \tag{1.8}$$

The assertion of Theorem 1 can also be stated in the form where $\forall$ given $m_+ > 0$, $\exists\ g_* = g_*(m_+) \in (0, +\infty)$ such that $\forall\ g$ with $|g| \geqslant g_*$, the eigenfunctions $\boldsymbol{\Psi}_j(\mathbf{x};\omega)$ of $\mathbf{H}^{(N)}(\omega)$ admit exponential bound (1.6).

Remarks

1. The threshold value g^* in Theorem 1 depends on N: $g^* = g^*(N)$. (It also depends on $\mathrm{F_V}$ and Φ.) The important question is how g^* grows with N. We plan to address this problem in a separate paper.
2. It suffices to prove Theorem 1 for any bounded interval $I \subset \mathbb{R}$ of length $\geqslant \delta_0$ with a given, suitably chosen $\delta_0 > 0$. This is convenient (albeit not crucial) in some arguments used below.

The conditions of Theorem 1 are assumed throughout the paper. As was said earlier, the Proof of Theorem 1 uses mainly MSA, in its N-particle version. The MSA scheme for N particles does not differ in principle from that for two particles; for that reason, we will often refer to paper [7].

Most of the time we work with finite-volume approximation operators $H^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})}\left(= H^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})}(\omega)\right)$ given by

$$\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})} = \mathbf{H}^{(N)}\restriction_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})} + \text{Dirichlet boundary conditions on } \partial\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u}) \tag{1.9}$$

and acting on vectors $\boldsymbol{\phi} \in \mathbb{C}^{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})}$ by

$$\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})}\boldsymbol{\phi}(\mathbf{x}) = \sum_{\substack{\mathbf{y}\in\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u}):\\ \|\mathbf{y}-\mathbf{x}\|=1}} \boldsymbol{\phi}(\mathbf{y}) + \left[U(\mathbf{x}) + gW(\mathbf{x};\omega)\right]\boldsymbol{\phi}(\mathbf{x}), \tag{1.10}$$

with the external N-particle random potential $W(\mathbf{x};\omega)$ as in Eq. 1.1. Here and below, $\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})$ stands for an 'N-particle lattice box' (a box, for short) of size L around $\mathbf{u} = (u_1, \dots, u_N)$, where $u_j = (\mathrm{u}^{(1)}_j, \dots, \mathrm{u}^{(d)}_j) \in \mathbb{Z}^d$:

$$\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u}) = \mathop{\times}_{j=1}^{N} \Lambda_L\left(u_j\right) \tag{1.11}$$

where $\Lambda_L(u_j)$ is a 'single-particle box' around $u_j = \left(\mathrm{u}_j^{1)}, \ldots, \mathrm{u}_j^{d)}\right) \in \mathbb{Z}^d$:

$$\Lambda_L\left(u_j\right) = \left(\bigtimes_{i=1}^{d}\left[\mathrm{u}_j^{(i)} - L/2, \mathrm{u}_j^{(i)} + L/2\right]\right) \cap \mathbb{Z}^d. \tag{1.12}$$

For a box $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$ as in Eq. 1.11, we will also use the notation:

$$\Pi_j \boldsymbol{\Lambda}_L^{(N)}(\mathbf{u}) = \Lambda_L\left(u_j\right)$$

and

$$\Pi \boldsymbol{\Lambda}_L^{(N)}(\mathbf{u}) = \cup_{j=1}^{N} \Pi_j \boldsymbol{\Lambda}_L^{(N)}(\mathbf{u}); \tag{1.13}$$

set $\Pi \boldsymbol{\Lambda}_L^{(N)}(\mathbf{u}) \subset \mathbb{Z}^d$ describes the single-particle 'base' of $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$.

Next, $\partial \boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$ in Eq. 1.7 stands for the interior boundary (or briefly, the boundary) of box $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$: $\partial \boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$ is formed by points $\mathbf{y} \in \boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$ such that $\exists$ a site $\mathbf{v} \in \left(\mathbb{Z}^{Nd}\right) \setminus \boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$ with $\|\mathbf{y} - \mathbf{v}\| = 1$. These definitions remain valid if we replace N with $n = 1, \ldots, N-1$.

As follows from Eqs. 1.7, 1.8, $H^{(N)}_{\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})}$ is a Hermitian operator in the Hilbert space $\ell_2(\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u}))$. In fact, the approximation (1.7) can be used for any finite subset $\boldsymbol{\Lambda}^{(N)} \subset \mathbb{Z}^{Nd}$ of cardinality $|\boldsymbol{\Lambda}^{(N)}|$ and with boundary $\partial \boldsymbol{\Lambda}^{(N)}$, producing Hermitian operator $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}}$ in $\ell_2(\boldsymbol{\Lambda}^{(N)})$.

Hamiltonian $\mathbf{H}^{(N)}$ and its approximants $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}}$ admit the permutation symmetry. Namely, let $\mathbf{S}_\sigma$ be the unitary operator in $\ell_2(\mathbb{Z}^{Nd})$ induced by map $\mathcal{S}_\sigma$:

$$\mathbf{S}_\sigma \boldsymbol{\phi}(\mathbf{x}) = \boldsymbol{\phi}(\mathcal{S}_\sigma \mathbf{x}). \tag{1.14}$$

Then $\mathbf{S}_\sigma^{-1} \mathbf{H}^{(N)} \mathbf{S}_\sigma = \mathbf{H}^{(N)}$ and $\mathbf{S}_\sigma^{-1} \mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}} \mathbf{S}_\sigma = \mathbf{H}^{(N)}_{\mathcal{S}_\sigma \boldsymbol{\Lambda}^{(N)}}$. This implies, in particular, that for any finite $\boldsymbol{\Lambda}^{(N)} \subset \mathbb{Z}^{Nd}$, the eigenvalues of operators $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}}$ and $\mathbf{H}^{(N)}_{\mathcal{S}_\sigma \boldsymbol{\Lambda}^{(N)}}$ are identical.

Like its two-particle counterpart (see [6, 7]), the N-particle MSA scheme involves a number of technical parameters borrowed from the single-particle MSA; see [8]. Following [8] and [6, 7], given a number $\alpha \in (1, 2)$ and starting with $L_0 \geqslant 2$ and $m_0 > 0$, we define an increasing positive sequence L_k:

$$L_k = L_0^{\alpha^k}, \qquad k \geqslant 1, \tag{1.15}$$

and a decreasing positive sequence m_k (depending on a positive number γ):

$$m_k = m_0 \prod_{j=1}^{k}\left(1 - \gamma L_k^{-1/2}\right), \qquad k \geqslant 1. \tag{1.16}$$

In fact, it suffices to set $\alpha = 3/2$, albeit we will use the symbolic form of parameter α instead of its value: this makes our notations less cumbersome. Besides, it will make our notation agreed with that of [8].

We will also make use of parameters

$$p = p(N, g) > d \text{ and } q = q(N, p(N, g)) > p, \tag{1.17}$$

varying with the number of particles N. The roles of parameters p and q (and the choice of their values) have been discussed in [7]: they appear systematically in the exponents of power-law bounds for probabilities of "unwanted", or "unlikely" events defined in terms of finite-volume Hamiltonians $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}}$. These bounds also depend on d, α and γ (which could be added to the list of arguments for p and q) and are specified, for a given value of N, recursively, depending on the values $\{p(n)$ and $q(n, p(n))$ for n-particle systems, where $n = 1, \dots, N-1\}$. In the course of presentation, it will be made clear (and used in various places) that, for any $N \geqslant 1$,

$$p(n, g), q(n, g) \to +\infty \text{ as } |g| \to \infty, \qquad n = 1, \dots, N. \tag{1.18}$$

Note that sequence m_k in Eq. 1.12 is indeed positive, and the limit $\lim\limits_{k\to\infty} m_k \geqslant m_0/2$ when L_0 is sufficiently large. (A similar observation was, in fact, made in the Appendix in [8].) We will also assume that $L_0 > r_0$.

The single-particle MSA scheme was used in [8] to check, for IID potentials, decay properties of the Green's functions (GFs) for single-particle Hamiltonians with IID external potentials. As was said before, for a two-particle model, the MSA scheme was established in [6, 7]. In this paper we adopt a similar strategy for the N-particle model. Here, the GFs in a box $\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})$ are defined by:

$$G^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})}(E; \mathbf{x}, \mathbf{y}) = \left\langle \left(\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})} - E \right)^{-1} \delta_{\mathbf{x}}, \delta_{\mathbf{y}} \right\rangle, \qquad \mathbf{x}, \mathbf{y} \in \boldsymbol{\Lambda}^{(N)}_L(\mathbf{u}), \tag{1.19}$$

where $\delta_{\mathbf{x}}(\mathbf{v})$ is the lattice delta-function and $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\ell_2(\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u}))$.

Definition 1 Fix $E \in \mathbb{R}$ and $m > 0$. An N-particle box $\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})$ is said to be (E, m)-non-singular (in short: (E, m)-NS) if the GFs $\mathbf{G}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})}(E; \mathbf{u}, \mathbf{u}')$ defined by Eq. 1.15 for the Hamiltonian $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})}$ from Eq. 1.8 satisfy

$$\max_{\mathbf{y} \in \partial \boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})} \left| \mathbf{G}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{u})}(E; \mathbf{u}, \mathbf{y}) \right| \leqslant e^{-mL}. \tag{1.20}$$

Otherwise, it is called (E, m)-singular (or (E, m)-S).

A similar concept can be introduced for any finite set $\boldsymbol{\Lambda}^{(N)} \subset \mathbb{Z}^{Nd}$.

Definition 2 Let n be a positive integer and $\mathcal{J}$ be a non-empty subset of $\{1, \dots, n\}$. We say that box $\boldsymbol{\Lambda}^{(n)}_L(\mathbf{y})$ is $\mathcal{J}$-separable from a box $\boldsymbol{\Lambda}^{(n)}_L(\mathbf{x})$ (or, equivalently, a point $\mathbf{y} \in \mathbb{Z}^d$ is called $(\mathcal{J}, L)$-separable from a point $\mathbf{x}$) if

$$\left(\bigcup_{j \in \mathcal{J}} \Pi_j \boldsymbol{\Lambda}^{(n)}_L(\mathbf{y}) \right) \cap \left(\bigcup_{i \notin \mathcal{J}} \Pi_i \boldsymbol{\Lambda}^{(n)}_L(\mathbf{y}) \, \cup \Pi \boldsymbol{\Lambda}^{(n)}_L(\mathbf{x}) \right) = \emptyset. \tag{1.21}$$

A pair of boxes $\mathbf{\Lambda}_L^{(n)}(\mathbf{x})$, $\mathbf{\Lambda}_L^{(n)}(\mathbf{y})$ is said to be separable (or, equivalently, a pair of points $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{nd}$ is called L-separable) if, for some $\mathcal{J} \subseteq \{1, \ldots, n\}$, either $\mathbf{\Lambda}_L^{(n)}(\mathbf{y})$ is $\mathcal{J}$-separable from a box $\mathbf{\Lambda}_L^{(n)}(\mathbf{x})$, or $\mathbf{\Lambda}_L^{(n)}(\mathbf{x})$ is $\mathcal{J}$-separable from a box $\mathbf{\Lambda}_L^{(n)}(\mathbf{y})$.

The notion of separability of boxes is designed so as to enable us to establish Wegner–Stollmann type bounds[2] (cf. [5, 12, 13]); see Eqs. 2.2, 2.3.

In Lemma 1 we give a geometrical upper bound for the set of points $\mathbf{y}$ which are *not* separable from a given point $\mathbf{x}$.

Lemma 1 *Given an $n \geqslant 2$, let $\mathbf{x} \in \mathbb{Z}^{nd}$ be an n-particle configuration. For any $L > 1$, there exists a finite collection of n-particle boxes $\mathbf{\Lambda}_{\widetilde{L}^{(l)}}(\widetilde{\mathbf{x}}^{(l)})$, $l = 1, \ldots, K(\mathbf{x}, n)$, $K(\mathbf{x}, n) \leqslant \widehat{K}(n) < \infty$, of sides $\widetilde{L}^{(l)} \leqslant 5nL$ such that if a configuration $\mathbf{y} \in \mathbb{Z}^{nd}$ satisfies*

$$\mathbf{y} \notin \bigcup_{\ell=1}^{K(\mathbf{x},n)} \widetilde{\mathbf{\Lambda}}^{(l)} \tag{1.22}$$

then the boxes $\mathbf{\Lambda}_L^{(n)}(\mathbf{x})$ and $\mathbf{\Lambda}_L^{(n)}(\mathbf{y})$ are separable.

The Proof of Lemma 1 is given in Appendix A.

The following Theorem 2 is completely analogous to Theorem 2.3 in [8] and to Theorem 2 in [6], and so is its proof, which we omit. The reader can check, by inspecting the proofs in the single-particle case [8] and in the two-particle case [6] that the only modification which causes concern is the choice of intermediate constants, depending on N. However, the core argument of the proof remains unchanged.

Theorem 2 *Let $I \subseteq \mathbb{R}$ be a bounded interval. Assume that for some $m_0 > 0$ and $L_0/2 > 1$, $\lim\limits_{k\to\infty} m_k \geqslant m_0/2$, and for any $k \geqslant 0$ the following properties hold:*

If two boxes $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u})$, $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{v})$ are separable, then

$$(\mathbf{DS}.\boldsymbol{k}, \boldsymbol{I}, \boldsymbol{N}) \quad \mathbb{P}\left\{\forall\, E \in I : \mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u}) \text{ or } \mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{v}) \text{ is } (m_k, E)\text{–NS}\right\} \geqslant 1 - L_k^{-2p(N)}. \tag{1.23}$$

Here L_k and m_k are defined in Eqs. 1.11, 1.12, with p, α and γ satisfying Eq. 1.14. Then, for $|g|$ large enough, with probability one, the spectrum of operator $\mathbf{H}^{(N)}(\omega)$ in I is pure point. Furthermore, there exists a constant $m_+ \geqslant$

[2]For random potentials admitting a (bounded) marginal probability density, W. Kirsch has proved an analog of Eq. 2.2, as well as the existence of the DoS for multi-particle systems.

$m_0/2$ such that all eigenfunctions $\Psi_j(\mathbf{x};\omega)$ of $\mathbf{H}^{(N)}(\omega)$ with eigenvalues $E_j(\omega) \in I$ decay exponentially fast at infinity, with the effective mass m_+:

$$|\Psi_j(\mathbf{x};\omega)| \leqslant C_j(\omega)\, e^{-m_+\|\mathbf{x}\|}. \tag{1.24}$$

In future, the eigenvectors of finite-volume Hamiltonians appearing in arguments and calculations, will be assumed normalised.

We stress that it is the property **(DS.*k*, *I*, *N*)** encapsulating decay of the GFs which enables the N-particle MSA scheme to work. (Here and below, DS stands for 'double singularity').

Clearly, Theorem 1.1 would be proved, once the validity of property **(DS.*k*, *I*, *N*)** is established for all $k \geqslant 0$.

Our strategy, as indicated in the title of this paper and mentioned earlier in this section, is an induction on the number of particles $N \geqslant 1$. The base of this induction had been established earlier, starting from papers [8–10], with the help of the MSA, and also in [1, 2], in a different way, with the help of the FMM. This allows us to use results of the single-particle localisation theory. We show in this paper that, assuming a certain number of facts established for systems with $n = 1, \dots, N-1$ particles, one can establish similar facts for N-particle systems. Once these facts, mostly concerning the decay properties of Green's functions in finite boxes, are established for N-particle systems, they imply, in a fairly standard way (essentially, in the same way as in the single-particle and in the two-particle [7] theories) the spectral localization for N-particle systems. So, according to this plan, we assume established all necessary properties of n-particle systems, $1 \leqslant n \leqslant N-1$, and use them whenever necessary. Of course, these properties have to be re-established for $n = N$. When appropriate, we discuss technical details of proofs in previous works, where the required properties have been proved for $n = 1$.

In other words, our paper is organised as a proof of the induction step from $N-1$ to N particles. Within this induction step, we use another inductive scheme—the MSA—where some properties of Green's functions are proved first at an initial scale L_0, and then recursively derived for N-particle boxes of sizes L_k, $k \geqslant 1$.

The main property that we have to verify for a given N and for all L_k, $k \geqslant 0$, is **(DS.*k*, *I*, *N*)**. Further, the main technical parameter is the exponent $p = p(N) = p(N, g)$ figuring in the RHS of **(DS.*k*, *I*, *N*)**. At the initial step of induction in N, we use an important fact from the single-particle theory [8]: one can guarantee any (arbitrarily large) value $p(1, g)$, provided that $|g|$ is large enough. Cf. Eq. 1.14. Then we show that a similar property holds for any N and for $k = 0$, i.e., for the scale L_0 (cf. Theorem 3). Therefore, in our double induction scheme (on N and, for a given N, on k), we require $|g|$ to be sufficiently large so as to guarantee:

(i) property **(DS.*k*, *I*, *n*)** for all $k \geqslant 0$ and for $n = 1, \dots, N-1$ (this property is defined verbatim, following Eq. 1.20 mutatis mutandis);
(ii) property **(DS.*k*, *I*, *N*)** for $k = 0$.

Parameter $q = q(N) = q(N, g)$ is controlled via Wegner–Stollmann type bounds **(WS1.n)**, **(WS2.n)** in Eqs. 2.2, 2.3, which are proved for all scales L_k at once, without induction in k.

2 The N-Particle MSA Scheme

In view of Theorem 2, our aim is to check property **(DS.k, I, N)** in Eq. 1.20. We now outline the N-particle MSA which is used for this purpose. In both single- and N-particle versions, the MSA scheme is an elaborate scale induction in k dealing with GFs $\mathbf{G}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})} = \mathbf{G}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})}$ and involving several mutually related parameters; some of them have been used in Sections 1 and 2. For a detailed discussion of the role of each parameter, see [7].

We will focus in the rest of the paper on the aforementioned *scale* induction in k, along sequences $\{(L_k, m_k)\}$ outlined in Eqs. 1.11, 1.12. Consequently, in some definitions below we refer to the particle number parameter $n \geqslant 1$, whereas in other definitions - where we want to stress the passage from $N - 1$ to N - we will use the capital letter.

Definition 3 Given $n \geqslant 1$, $E \in \mathbb{R}$, $\mathbf{v} \in \mathbb{Z}^{nd}$ and $L \geqslant 2$, we call the n-particle box $\boldsymbol{\Lambda}^{(n)}_L(\mathbf{v})$ E-resonant (briefly: E-R) if the spectrum of the Hamiltonian $H^{(n)}_{\boldsymbol{\Lambda}^{(n)}_L(\mathbf{v})}$ satisfies

$$\operatorname{dist}\left[E, \operatorname{spec}\left(H^{(n)}_{\boldsymbol{\Lambda}^{(n)}_L(\mathbf{v})}\right)\right] < e^{-L^{\beta}}, \quad \text{where } \beta = 1/2. \tag{2.1}$$

Box $\boldsymbol{\Lambda}^{(n)}_L(\mathbf{v})$ is called E-completely non-resonant (briefly: E-CNR) if it is E-NR and does not contain any E-R box of size $\geqslant L^{1/\alpha}$.

Throughout this paper, we use parameter β instead of its value, 1/2. As with $\alpha = 3/2$, this may be helpful to readers familiar with [8] and make our notations less cumbersome.

Given $n \geqslant 1$ and $L_0 \geqslant 2$, introduce the following properties **(WS1.n)** and **(WS2.n)** of random Hamiltonians $H^{(n)}_{\boldsymbol{\Lambda}^{(n)}_l}$, $l \geqslant L_0$.

$$\textbf{(WS1.}n\textbf{)} \qquad \forall\, l \geqslant L_0, \text{ box } \boldsymbol{\Lambda}^{(n)}_l(\mathbf{x}) \text{ and } E \in \mathbb{R}\text{: } \mathbb{P}\left\{ \boldsymbol{\Lambda}^{(n)}_l(\mathbf{x}) \text{ is not } E\text{-CNR} \right\} < l^{-q}. \tag{2.2}$$

$$\textbf{(WS2.}n\textbf{)} \qquad \begin{array}{l} \forall\, l \geqslant L_0 \text{ and separable boxes } \boldsymbol{\Lambda}^{(n)}_\ell(\mathbf{x}) \text{ and } \boldsymbol{\Lambda}^{(n)}_\ell(\mathbf{y}), \\ \mathbb{P}\left\{ \exists\, E \in \mathbb{R} : \text{neither } \boldsymbol{\Lambda}^{(n)}_l(\mathbf{x}) \text{ nor } \boldsymbol{\Lambda}^{(n)}_l(\mathbf{y}) \text{ is } E\text{–CNR} \right\} < l^{-q}. \end{array} \tag{2.3}$$

Here $q = q(n)$ is the parameter mentioned in Eqs. 1.13, 1.14.

As we already said, the initial step of the N-particle MSA scheme consists in establishing properties **(S.0, I, N)** and **(DS.0, I, N)**; see Eqs. 2.4 and 1.20. The inductive step of the N-particle MSA consists in deducing property **(DS.$k+1$, I, N)** from property **(DS.k, I, N)**; again see Eq. 1.20. Both the initial and the inductive step will be done with the assistance of properties **(WS1.n)** and/or **(WS2.n)**, $n = 1, \dots, N$, which have to be proved independently of the scale induction. In our context, properties **(WS1.n)** and **(WS2.n)** have been established in [6], Theorems 1, 2. (Despite the fact that properties **(WS1.n)** and **(WS2.n)** had been stated [6] for $n = 2$, their proof is automatically extended to the case of a general n.) A bound similar to **(WS1.n)** was independently proved by Kirsch [11]. The assumption of independence of potential V can be relaxed; see [4]. For reader's convenience we repeat the corresponding assertion from [6]:

Lemma 2 *Under the above assumptions on* $\{V(x;\omega)\}$ *and* U (*see Eqs.* 1.3–1.5), *properties* **(WS1.n)**, **(WS2.n)** *hold true* $\forall$ *positive integer* n.

Let $I \subseteq \mathbb{R}$ be an interval. Given $m_0 > 0$ and $L_0 \geqslant 2$, consider property **(S.0, I, N)**:

$$(\mathbf{S.0}, \boldsymbol{I}, \boldsymbol{N}) \quad \forall\, \mathbf{x} \in \mathbb{Z}^{Nd}, \qquad \mathbb{P}\left\{ \exists\, E \in I : \boldsymbol{\Lambda}_{L_0}^{(N)}(\mathbf{x}) \text{ is } (E, m_0)\text{–S} \right\} < L_0^{-2p}. \quad (2.4)$$

Here $p = p(N)$ is the parameter mentioned in Eqs. 1.13, 1.14.

The initial MSA step is summarised in the Theorem 3 below. It is completely analogous to Proposition A.1.2 in [8], and so is its proof. Note as well that multi-particle analogs of Propositions A.1.1 and A.1.3 from [8] can also be proved in the same way as in [8]. The reason for that is that the multi-particle structure of the external potential $W(\mathbf{x};\omega)$ and the presence of a bounded interaction potential $U(\mathbf{x})$ (as well as the form of $U(\mathbf{x})$ in Eq. 1.4) are virtually irrelevant for these statements.

Theorem 3 $\forall$ *given* m_0 *and* $L_0 \geqslant 2$ *and* $\forall$ *bounded interval* $I \subset \mathbb{R}$, *there exists* $g_0^* = g_0^*(N, m_0, L_0, I) \in (0, +\infty)$ *such that for* $|g| \geqslant g_0^*$:

(A) *Properties* **(S.0, I, N)** *and* **(DS.0, I, N)** *hold true.*
(B) *Moreover, there exists a function* $\widetilde{g} : \widetilde{p} \in (d, +\infty) \mapsto \widetilde{g}(\widetilde{p}) \in [g_0^*, +\infty)$ *such that if* $|g| \geqslant \widetilde{g}(\widetilde{p})$, *then Eq.* 2.4 *is satisfied with* $p = \widetilde{p}$. *Equivalently, there exists a function* $p(N, g)$ *of parameter* $g \in [g_0^*, +\infty)$ (*referred to in Eqs.* 1.13, 1.14) *such that* $p(N, g) \to \infty$ *as* $|g| \to \infty$ *and Eq.* 2.4 *is satisfied with* $p = p(N, g)$.

To complete the inductive MSA step, we will prove

Theorem 4 $\forall$ *given* $m_0 > 0$, *there exist* $g_1^* \in (0, +\infty)$ *and* $L_1^* \in (0, +\infty)$ *such that the following statement holds. Suppose that* $|g| \geqslant g_1^*$ *and* $L_0 \geqslant L_1^*$. *Then,* $\forall\ k = 0, 1, \dots$ *and* $\forall$ *interval* $I \subseteq \mathbb{R}$, *property* **(DS.k, I, N)** *implies* **(DS.$k+1$, I, N)**.

The Proof of Theorem 4 occupies the rest of the paper. Before we proceed further, let us repeat that the property **(DS.*k*, *I*, *N*)** for $\forall\ k \geqslant 0$ and $\forall$ unit interval $I \subset \mathbb{R}$, follows directly from Theorems 3 and 4.

To deduce property **(DS.*k* + 1, *I*, *N*)** from **(DS.*k*, *I*, *N*)**, we introduce

Definition 4 Given $R > 0$, consider the following set in $\mathbb{Z}^{Nd}$:

$$\mathbb{D}_R = \left\{ \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{Z}^{Nd} : \max_{1 \leqslant j_1,\, j_2 \leqslant N} \|x_{j_1} - x_{j_2}\| \leqslant NR \right\} \tag{2.5}$$

It is plain that, with $R = r_0 + 2L$, if $\mathbf{u}$ is not in D_R and $\mathbf{x}$ is in $\Lambda_L(\mathbf{u})$, then there is a subset $\mathcal{J}$ of $\{1, \ldots, N\}$ with $1 \leqslant$ card $\mathcal{J} < N$ and

$$\min_{j_1 \in \mathcal{J},\, j_2 \notin \mathcal{J}} \|x_{j_1} - x_{j_2}\| > r_0.$$

An N-particle box $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$ is called fully interactive when $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u}) \cap \mathbb{D}_{r_0} \neq \emptyset$, and partially interactive if $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u}) \cap \mathbb{D}_{r_0} = \emptyset$. For brevity, we use the terms an FI-box and a PI-box, respectively.

The procedure of deducing property **(DS.*k* + 1, *I*, *N*)** from **(DS.*k*, *I*, *N*)** is done here separately for the following three cases.

(I) Both $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$ and $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y})$ are PI-boxes.
(II) Both $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$ and $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y})$ are FI-boxes.
(III) One of the boxes is FI, while the other is PI.

These three cases are treated in Sections 3, 4 and 5, respectively. The end of Section 5 will mark the end of the Proof of Theorem 4. We repeat that all cases require the use of property **(WS1.*N*)** and/or **(WS2.*N*)**.

3 Case I: Partially Interactive Pairs of Singular Boxes

In this section, we aim to derive property **(DS.*k* + 1, *I*, *N*)** for a pair of partially interactive and separable boxes $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$, $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y})$. Recall, we are allowed to assume property **(DS.*k*, *I*, *N*)** for every pair of separable boxes $\boldsymbol{\Lambda}_{L_k}^{(N)}(\widetilde{\mathbf{x}})$, $\boldsymbol{\Lambda}_{L_k}^{(N)}(\widetilde{\mathbf{y}})$, where $\mathbf{x}, \mathbf{y}, \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \in \mathbb{Z}^{Nd}$. In fact, we will be able to establish property **(DS.*k* + 1, *I*, *N*)** for partially interactive separable boxes $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$, $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y})$ directly, without referring to **(DS.*k*, *I*, *N*)**. (However, in cases (II) and (III) such a reference will be needed.)

Let $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$ be an PI-box and write $\mathbf{u} = (u_1, \ldots, u_N)$ as a pair $(\mathbf{u}', \mathbf{u}'')$ where $\mathcal{J}$ is a non-empty subset of $\{1, \ldots, N\}$ figuring in Definition 4, and $\mathbf{u}' = \mathbf{u}_{\mathcal{J}} \in \mathbb{Z}^{\mathcal{J}}$ and $\mathbf{u}'' = \mathbf{u}_{\mathcal{J}^c} \in \mathbb{Z}^{\mathcal{J}^c}$ are the corresponding sub-configurations in $\mathbf{u}$: $\mathbf{u}' = (u_j,\ j \in \mathcal{J})$ and $\mathbf{u}'' = (u_j,\ j \notin \mathcal{J})$. Set: $n' =$ card $\mathcal{J}$ and $n'' = N - n'$. It is convenient to represent $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$ as the Cartesian product

$$\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u}) = \boldsymbol{\Lambda}_{L_{k+1}}^{(n')}\left(\mathbf{u}'\right) \times \boldsymbol{\Lambda}_{L_{k+1}}^{(n'')}\left(\mathbf{u}''\right)$$

and write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ in the same fashion as $(\mathbf{u}', \mathbf{u}'')$. Correspondingly, the Hamiltonian $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})}$ can be written in the form

$$\mathbf{H}\boldsymbol{\phi}(\mathbf{x}) = \sum_{\substack{\mathbf{y}\in\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}):\\ \|\mathbf{y}-\mathbf{x}\|=1}} \boldsymbol{\phi}(\mathbf{y}) + \left[U\left(\mathbf{x}'\right) + gW\left(\mathbf{x}';\omega\right) + U\left(\mathbf{x}''\right) + gW\left(\mathbf{x}'';\omega\right)\right]\boldsymbol{\phi}\left(\mathbf{x}\right), \tag{3.1}$$

or, algebraically,

$$H^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})} = H^{(n')}_{1;\boldsymbol{\Lambda}^{(N)}_{L_{k+1}(\mathbf{u}')}} \otimes \mathbf{I} + \mathbf{I} \otimes H^{(n'')}_{2;\boldsymbol{\Lambda}^{(N)}_{L_{k+1}(\mathbf{u}'')}}. \tag{3.2}$$

Here $\mathbf{I}$ is the identity operator on the complementary variable.

Due to the symmetry of terms U and W, in the forthcoming argument we can assume, without loss of generality, that

$$\mathcal{J} = \{1, \dots, n'\}, \qquad \mathcal{J}^{\mathrm{c}} = \{n'+1, \dots, N\}.$$

Definition 5 Let be $n \in \{1, \dots, N-1\}$, $k \geqslant 0$ and $\mathbf{u}' = (u_1, \dots, u_n) \in \mathbb{Z}^{nd}$. Given a bounded interval $I \subset \mathbb{R}$ and $m > 0$, the n-particle box $\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{u}')$ is called m-tunneling (m-T, for short) if $\exists\, E \in I$ and disjoint n-particle boxes $\boldsymbol{\Lambda}^{(n)}_{L_{k-1}}(\mathbf{v}_1)$, $\Lambda^{(n)}_{L_{k-1}}(\mathbf{v}_2) \subset \boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{u}')$ which are (E, m)-S. An N-particle box of the form $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}) = \Lambda^{(n')}_{L_{k-1}}(\mathbf{u}') \times \Lambda^{(n'')}_{L_{k-1}}(\mathbf{u}'')$, with $n' + n'' = N$, $\mathbf{u} = (\mathbf{u}', \mathbf{u}'')$, $\mathbf{u}' = (u_1, \dots, u_{n'})$, $\mathbf{u}'' = (u_{n'+1}, \dots, u_N)$, is called (m, n', n'')-partially tunelling ((m, n', n'')-PT) if either $\Lambda^{(n'}_{L_{k-1}}(\mathbf{u}')$ or $\Lambda^{(n'')}_{L_{k-1}}(\mathbf{u}'')$ is m-T. Otherwise, it is called (m, n', n'')-NPT. Finally, a box $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$ is called m-PT if it is (m, n', n'')-PT for some $n', n'' \geqslant 1$ with $n' + n'' = N$, and m-NPT, otherwise.

The following statement will be sometimes referred to as the **NITRoNS** property of PI-boxes: *Non-Interacting boxes are Tunneling, Resonant or* (*otherwise*) *Non-Singular*. Cf. [7].

Lemma 3 *Consider an N-particle box $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$ of the form $\boldsymbol{\Lambda}^{(n')}_{L_k}(\mathbf{u}') \times \boldsymbol{\Lambda}^{(n'')}_{L_k}(\mathbf{u}'')$, where $\mathbf{u} = (\mathbf{u}', \mathbf{u}'')$, $\mathbf{u}' = (u_1, \dots, u_{n'}) \in \mathbb{Z}^{n'd}$, $\mathbf{u}'' = (u_{n'+1}, \dots, u_N) \in \mathbb{Z}^{n''d}$. Assume that $\forall\, j_1, j_2$ with $1 \leqslant j_1 \leqslant n'$, $n'+1 \leqslant j_2 \leqslant N$, we have $\|u_{j_1} - u_{j_2}\| > r_0$, so that $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$ is* PI. *Assume also that $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$ is E-*CNR *and m-*NPT. *Let*

$$\left\{(\lambda_a, \varphi_a),\ a = 1, \dots, |\boldsymbol{\Lambda}^{(n')}_{L_k}\left(\mathbf{u}'\right)|\right\}, \qquad \left\{(\mu_b, \psi_b),\ b = 1, \dots, |\boldsymbol{\Lambda}^{(n'')}_{L_k}\left(\mathbf{u}''\right)|\right\},$$

be the eigenvalues and eigenvectors of $\mathbf{H}^{(n')}_{\boldsymbol{\Lambda}^{(n')}_{L_k}(\mathbf{u}')}$ and $\mathbf{H}^{(n'')}_{\boldsymbol{\Lambda}^{(n'')}_{L_k}(\mathbf{u}'')}$, respectively. Set

$$m' = m\left(1 - L_k^{-(1-\beta)} - L_k^{-1}\ln L_k^{N(d-1)}\right).$$

Then we have

$$\max_{1\leqslant a\leqslant|\boldsymbol{\Lambda}^{(n')}_{L_k}(\mathbf{u}')|}\ \max_{\mathbf{v}''\in\partial\boldsymbol{\Lambda}^{(n'')}_{L_k}(\mathbf{u}'')} |G^{(n'')}\left(\mathbf{u}'', \mathbf{v}''; E - \lambda_a\right)| \leqslant e^{-m'L_k}$$

and, similarly,

$$\max_{1\leqslant b\leqslant|\boldsymbol{\Lambda}^{(n'')}_{L_k}(\mathbf{u}'')|}\ \max_{\mathbf{v}'\in\partial\boldsymbol{\Lambda}^{(n')}_{L_k}(\mathbf{u}')} |G^{(n')}\left(\mathbf{u}',\mathbf{v}';\,E-\mu_b\right)| \leqslant e^{-m'L_k}.$$

Moreover, this implies that the N-particle box $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$ is (E,m')-NS.

The Proof of Lemma 3 is given in Appendix B. (It is fairly straightforward and based on the representations (7.1)–(7.3).)

Lemma 4 *Let n,k be positive integers and suppose that* **(DS.$\boldsymbol{k}$, $\boldsymbol{I}$, $\boldsymbol{n}$)** *holds true. Then*

$$\mathbb{P}\left\{\,\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{y})\ \textit{is}\ m\text{-PT}\,\right\} \leqslant \frac{1}{2}|\boldsymbol{\Lambda}^{(n)}_{L_k(\mathbf{y})}|^2\,L_{k-1}^{-2p(n)} = \frac{1}{2}L_k^{-\frac{2p(n)}{\alpha}+2d}. \tag{3.3}$$

Here $p(n)$ is the parameter figuring in Eqs. 1.13, 1.14.

Proof Combine **(DS.$\boldsymbol{k}$, $\boldsymbol{I}$, $\boldsymbol{n}$)** with a straightforward (albeit not sharp) upper bound $\frac{1}{2}|\boldsymbol{\Lambda}^{(n)}_{L_k(\mathbf{y})}|^2$ for the number of pairs of centers $\mathbf{v}_1,\mathbf{v}_2$ of boxes $\boldsymbol{\Lambda}^{(n)}_{L_{k-1}}(\mathbf{v}_1),\boldsymbol{\Lambda}^{(n)}_{L_{k-1}}(\mathbf{v}_2)\subset\boldsymbol{\Lambda}^{(n)}_{L_k}(\mathbf{y})$. ◻

In Lemma 5 we assume for simplicity that a PI box $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y})$ corresponds to an N-particle system that splits into two subsystems, with particles $1,\dots,n'$ and $n'+1,\dots,n'+n''=N$, respectively, and the two subsystems do not interact with each other.

Lemma 5 *Let $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y})$ be an N-particle* PI *box, with*

$$\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y}) = \boldsymbol{\Lambda}^{(n')}_{L_k}\left(\mathbf{y}'\right)\times\boldsymbol{\Lambda}^{(n'')}_{L_k}\left(\mathbf{y}''\right),$$

where $n',n''\geqslant 1,\quad n'+n''=N;\quad \mathbf{y}=(\mathbf{y}',\mathbf{y}''),\quad \mathbf{y}'=(y_1,\dots y_{n'})\in\mathbb{Z}^{n'd},\quad \mathbf{y}''=(y_{n'+1},\dots y_N)\in\mathbb{Z}^{n''d}$*, and*

$$\min_{1\leqslant i\leqslant n'}\ \min_{n'+1\leqslant j\leqslant N}\|y_i-y_j\|>r_0.$$

Then for any given value $p(N)>0$ there exists $g_2^*\in(0,+\infty)$ such that if $|g|\geqslant g_2^*$, then

$$\mathbb{P}\left\{\,\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y})\ \textit{is}\ m\text{-PT}\,\right\} \leqslant \frac{1}{4}L_k^{-2p(N)}. \tag{3.4}$$

Proof By Definition 5, box $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y})$ is m-PT iff at least one of constituent boxes $\boldsymbol{\Lambda}^{(n')}_{L_k}(\mathbf{y}')$, $\boldsymbol{\Lambda}^{(n'')}_{L_k}(\mathbf{y}'')$ is m-T. By Lemma 4, inequality (3.3) holds for both $n=n'$ and $n=n''$. Since $\forall\,n<N\ p(n,g)\to\infty$, this leads to the assertion of Lemma 5. ◻

Remark The assertion of Lemma 5 remains true for a general type of interaction (with appropriate modifications), but is simpler and more transparent in the case of two-body interaction of the form (1.4). This explains our choice of the interaction energy function $U(\mathbf{x})$. Besides, in applications to the electron transport problems, such a choice is perfectly justified: here, a commonly accepted form of interaction is two-body Coulomb.

We repeat that, according to the structure of the MSA scheme, for any given number of particles $n = 1, \dots, N$, any (i.e., arbitrarily large) values $p(n), q(n)$ can be used, provided that $|g|$ is sufficiently large. In other words, parameters $p(n), q(n)$ follow Eq. 1.14. Indeed, for $p(n)$ this can be guaranteed, by direct inspection, for the boxes of initial size L_0. Cf. Appendix in [8]. The same property is then reproduced inductively at any scale L_k, $k \geqslant 1$. As to $q(n)$, one can actually obtain a stronger bound:

$$\mathbb{P}\left\{ \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{u}) \text{ is } E\text{-R} \right\} \leqslant e^{-L_k^{\beta}} \ll L_k^{-s}$$

for any a priori given $s \in (0, \infty)$ including $s = q(N)$, provided that $\beta > 0$ and L_0 (hence, any L_k) is large enough.

Lemma 6 *Assume that property* **(WS2.N)** *and Eqs.* 3.3, 3.4 *hold true. Suppose also that $|g|$ is sufficiently large, so that for all $n = 1, \dots, N-1$ the bound* (3.3) *holds with $p(n) \geqslant 2p(N) + 2d$, and that L_0 is sufficiently large, so that for any $k \geqslant 0$ we have*

$$L_k^{-\frac{2p(n)}{\alpha}+2d} \leqslant \frac{1}{4} L_k^{-2p(N)}.$$

Then, $\forall$ interval $I \subseteq \mathbb{R}$, $\forall$ integer $k \geqslant 0$ and $\forall$ pair of separable PI *N-particle boxes $\boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{x})$ and $\boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{y})$,*

$$\mathbb{P}\left\{ \exists\ E \in I : \ \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{x}) \text{ and } \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{y}) \text{ are } (E, m_k)\text{−S} \right\} \leqslant \frac{1}{2} L_k^{-2p(N)} + L_k^{-q(N)}. \tag{3.5}$$

Here $p(N), q(N)$ are the parameters from Eq. 1.13.

Proof of Lemma 6 By virtue of Lemma 3 (**NITRoNS** property),

$$\begin{aligned} &\mathbb{P}\left\{ \exists\ E \in I : \ \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{x}) \text{ and } \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{y}) \text{ are } (E, m_k)\text{−S} \right\} \\ &\quad \leqslant \mathbb{P}\left\{ \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{x}) \text{ or } \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{y}) \text{ is } m_k\text{−PT} \right\} + \\ &\quad\quad + \mathbb{P}\left\{ \exists\ E \in I : \ \text{ neither } \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{x}) \text{ nor } \boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{y}) \text{ is } E\text{−CNR} \right\}. \end{aligned} \tag{3.6}$$

Now the assertion of Lemma 6 follows from Eqs. 3.6, 2.3 and Lemma 5. □

Remark It is readily seen that the RHS of Eq. 3.5 is bounded by $L_k^{-2p(N)}$, provided that $L_k^{-q(N)} < L_k^{-2p(N)}/2$, i.e., for $q(N)$ large enough.

An immediate corollary of Lemma 6 is the following

Theorem 5 $\forall$ *given interval* $I \subseteq \mathbb{R}$ *and* $k = 0, 1, \ldots,$ *property* **(DS.*k*, *I*, *N*)** *holds for all pairs of separable* PI-*boxes* $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{x})$, $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{y})$.

Summarising the above argument: as was said earlier, verifying property **(DS.*k* + 1, *I*, *N*)** for a pair of N-particle PI-boxes did not force us to assume **(DS.*k*, *I*, *N*)**. However, in the course of deriving **(DS.*k* + 1, *I*, *N*)** for PI-boxes we used property **(WS2.*N*)**.

This completes the analysis of the case (I) where both boxes $\mathbf{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$ and $\mathbf{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y})$ are PI.

For future use, we also give

Lemma 7 *Consider a* N*-particle box* $\mathbf{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$. *Let* $M = M(\mathbf{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u}); E)$ *be the maximal number of* (E, m_k)-S, *pair-wise* separable PI-*boxes* $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u}^{(l)}) \subset \Lambda_{L_{k+1}}(\mathbf{u})$. *The following property holds*

$$\mathbb{P}\left\{ \exists E \in I: \quad M(\mathbf{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u}); E) \geqslant 2 \right\} \leqslant L_k^{2d\alpha} \cdot \left(\frac{1}{2} L_k^{-2p'(N-1)} + L_k^{-q(N)} \right), \tag{3.7}$$

where

$$p'(N-1, g) := \min\{p(n, g), \qquad 1 \leqslant n \leqslant N-1\} \underset{|g| \to \infty}{\longrightarrow} +\infty. \tag{3.8}$$

As before, $p(N), q(N)$ *are the parameters from in Eqs.* 1.13, 1.14.

Proof of Lemma 6 The number of possible pairs of centres $(\mathbf{u}^{(l_1)}, \mathbf{u}^{(l_2)})$, $1 \leqslant l_1 < l_2 \leqslant M$, is bounded by $L_{k+1}^{2d}/2$, while for a given pair of centres one can apply Lemma 6. This leads to the assertion of Lemma 7. □

4 Fully Interactive Pairs of Singular Boxes

The main outcome in case (II) is Theorem 6 placed at the end of this section. Before we proceed further, let us state a geometric assertion (see Lemma 8 below) which we prove in Section 6.

Lemma 8 *Let be* $n \geqslant 1$, $L > r_0$ *and consider two separable* n*-particle* FI-*boxes* $\mathbf{\Lambda}_L^{(n)}(\mathbf{u}')$ *and* $\mathbf{\Lambda}_L^{(n)}(\mathbf{u}'')$, *with* $\operatorname{dist}\left[\mathbf{\Lambda}_L^{(n)}(\mathbf{u}'), \mathbf{\Lambda}_L^{(n)}(\mathbf{u}'')\right] > 8L$. *Then*

$$\Pi \mathbf{\Lambda}_L^{(n)}\left(\mathbf{u}'\right) \cap \Pi \mathbf{\Lambda}_L^{(n)}\left(\mathbf{u}''\right) = \emptyset. \tag{4.1}$$

Lemma 8 is used in the Proof of Lemma 9 which, in turn, is important in establishing Theorem 6. In fact, Lemma 8 is a natural development of Lemma 2.2 in [6]. Let $I \subseteq \mathbb{R}$ be an interval. Consider the following assertion

$$
(\mathbf{IS}.\boldsymbol{k}.\boldsymbol{N}): \quad \begin{array}{l} \forall \text{ pair of interactive separable boxes } \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{x}) \text{ and } \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y}): \\ \mathbb{P}\left\{ \exists\, E \in I: \text{ both } \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{x}),\ \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{y}) \text{ are } (E, m_k)\text{-S} \right\} \leqslant L_k^{-2p(N)}, \end{array} \tag{4.2}
$$

with $p(N)$ as in Eqs. 1.13, 1.14. (This is a particular case of **(DS.*k*, *I*, *N*)**).

Lemma 9 *Given $k \geqslant 0$, assume that property* **(IS.*k*.*N*)** *holds true. Consider a box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ and let $\widetilde{N}(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}); E)$ be the maximal number of (E, m_k)-S, pair-wise* separable FI-*boxes $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(j)}) \subset \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$. Then $\forall\, \ell \geqslant 1$,*

$$
\mathbb{P}\left\{ \exists\, E \in I: \quad \widetilde{N}(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}); E) \geqslant 2\ell \right\} \leqslant L_k^{2\ell(1+d\alpha)} \cdot L_k^{-2\ell p(N)}. \tag{4.3}
$$

Proof of Lemma 9 Suppose $\exists$ FI-boxes $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(1)}), \ldots, \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2n)})$ $\subset \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$ such that any two of them are separable. By virtue of Lemma 8, it is readily seen that

(a) $\forall$ pair $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i-1)})$, $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i)})$, the respective (random) operators $H^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i-1)})}(\omega)$ and $H^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i)})}(\omega)$ are mutually independent, and so are their spectra and Green's functions $\mathbf{G}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i-1)})}$ and $\mathbf{G}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i)})}$.

(b) Moreover, the following pairs of operators form an independent family:

$$
\left(\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i-1)})}(\omega),\, H^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i)})}(\omega) \right), \qquad i = 1, \ldots, \ell, \tag{4.4}
$$

Indeed, operator $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(i)})}$, with $i \in \{1, \ldots, 2n\}$, is measurable relative to the sigma-algebra $\mathcal{B}(\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(i)})$ generated by $\{V(x),\ x \in \Pi\, \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(i)})\}$, $i = 1, \ldots, 2\ell$. Now, by Lemma 4.2, the sets $\Pi\, \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(i)})$, $i \in \{1, \ldots, 2\ell\}$, are pairwise disjoint, so that all sigma-algebras $\mathcal{B}(\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(i)})$, $i \in \{1, \ldots, 2\ell\}$, are independent.

Thus, any collection of events $\mathrm{A}_1, \ldots, \mathrm{A}_\ell$ related to the corresponding pairs

$$
\left(\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i-1)})},\, \mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}^{(2i)})} \right), \qquad i = 1, \ldots, \ell,
$$

also form an independent family. Now, for $i = 1, \ldots, \ell - 1$, set

$$
\mathrm{A}_i = \left\{ \exists\, E \in I: \boldsymbol{\Lambda}^{(N)}_{L_k}\left(\mathbf{u}^{(2i+1)}\right) \text{ and } \boldsymbol{\Lambda}^{(N)}_{L_k}\left(\mathbf{u}^{(2i+2)}\right) \text{ are } (E, m_k)\text{-S} \right\}. \tag{4.5}
$$

Then, by virtue of **(IS.*k*.*N*)**(see Eq. 4.3),

$$
\mathbb{P}\left\{ \mathrm{A}_j \right\} \leqslant L_k^{-2p(N)}, \qquad 0 \leqslant j \leqslant \ell - 1, \tag{4.6}
$$

and by virtue of independence of events $\mathrm{A}_0, \ldots, \mathrm{A}_{n-1}$, we obtain

$$\mathbb{P}\left\{\bigcap_{j=0}^{\ell-1} \mathrm{A}_j\right\} = \prod_{j=0}^{\ell-1} \mathbb{P}\ \{\mathrm{A}_j\} \leqslant \left(L_k^{-2p(N)}\right)^{\ell}. \tag{4.7}$$

To complete the proof, note that the total number of different families of 2ℓ boxes $\boldsymbol{\Lambda}_{L_k}^{(N)} \subset \boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$ with required properties is bounded from above by

$$\frac{1}{(2\ell)!}\left[2\left(L_k/2 + r_0 + 1\right) L_{k+1}^d\right]^{2\ell} \leqslant \frac{1}{(2\ell)!}\left(2L_k L_{k+1}^d\right)^{2\ell} \leqslant L_k^{2\ell(1+d\alpha)},$$

since their centres must belong to the subset $\mathbb{D}_{L_k+r_0} \cap \boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$ (see Eq. 2.5). Recall also that $r_0 < L_0 \leqslant L_k\ \forall\, k \geqslant 0$, by our assumption and by construction. This yields Lemma 9. □

Lemma 10 *Let $K(\mathbf{u}, L_{k+1}; E)$ be the maximal number of (E, m_k)-S, pair-wise* separable *boxes $\boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{u}^{(j)}) \subset \boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u})$ (fully or partially interactive). Then $\forall\ \ell \geqslant 1$,*

$$\mathbb{P}\{\exists E \in I : K(\mathbf{u}, L_{k+1}; E) \geqslant 2\ell+2\} \leqslant L_k^{4d\alpha} \cdot L_k^{-2p(N-1)} + L_k^{2\ell(1+d\alpha)} \cdot L_k^{-2\ell p(N)}, \tag{4.8}$$

where $p(N-1)$ and $p(N)$ are parameters from Eqs. 1.13, 1.14, for the system with $N-1$ and N particles, respectively.

Proof of Lemma 10 Assume that $K(\mathbf{u}, L_{k+1}; E) \geqslant 2\ell + 2$. Let $M(\boldsymbol{\Lambda}_{L_{k+1}(\mathbf{u})}^{(N)}; E)$ be as in Lemma 7 and $N(\boldsymbol{\Lambda}_{L_{k+1}(\mathbf{u})}^{(N)}; E)$ as in Lemma 9. Obviously,

$$K\left(\mathbf{u}, L_{k+1}; E\right) \leqslant M\left(\boldsymbol{\Lambda}_{L_{k+1}(\mathbf{u})}^{(N)}; E\right) + N\left(\boldsymbol{\Lambda}_{L_{k+1}(\mathbf{u})}^{(N)}; E\right).$$

Then either $M(\boldsymbol{\Lambda}_{L_{k+1}(\mathbf{u})}^{(N)}; E) \geqslant 2$ or $N(\boldsymbol{\Lambda}_{L_{k+1}(\mathbf{u})}^{(N)}; E) \geqslant 2\ell$. Therefore,

$$\begin{aligned}
&\mathbb{P}\{\exists E \in I :\quad K(\mathbf{u}, L_{k+1}; E) \geqslant 2\ell + 2\} \\
&\leqslant \mathbb{P}\left\{\exists E \in I :\quad M(\boldsymbol{\Lambda}_{L_{k+1}(\mathbf{u})}^{(N)}; E) \geqslant 2\right\} + \\
&\quad + \mathbb{P}\left\{\exists E \in I :\quad N(\boldsymbol{\Lambda}_{L_{k+1}(\mathbf{u})}^{(N)}; E) \geqslant 2\ell\right\} \\
&\leqslant L_k^{4d\alpha} \cdot L_k^{-2p(N-1)} + L_k^{2\ell(1+d\alpha)} \cdot L_k^{-2\ell p(N)},
\end{aligned}$$

by virtue of Eqs. 3.8 and 4.3. □

An elementary calculation now gives rise to the following

Corollary 1 *Under assumptions of Lemma* 10*, with $\ell \geqslant 4$, $p(N-1)$ and $p(N)$ large enough and for L_0 large enough, we have, $\forall$ integer $k \geqslant 0$,*

$$\mathbb{P}\{\exists E \in I :\quad K(\mathbf{u}, L_{k+1}; E) \geqslant 2\ell + 2\} \leqslant L_{k+1}^{-2p(N)-1}. \tag{4.9}$$

Now the Wegner–Stollmann bound **(WS2.N)** implies

Lemma 11 *If N-particle boxes $\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}')$, $\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}'')$ (fully or partially interactive) are separable, then $\forall\, L_0 > (J+1)^2$,*

$$\mathbb{P}\left\{ \forall\, E \in I: \ \textit{either } \mathbf{\Lambda}^{(N)}_{L_{k+1}}\left(\mathbf{u}'\right) \textit{ or } \mathbf{\Lambda}^{(N)}_{L_{k+1}}\left(\mathbf{u}''\right) \textit{ is } (E,J){-}\mathrm{CNR} \right\}$$

$$\geqslant 1-(J+1)^2 L_{k+1}^{-(q(N)\alpha^{-1}-2\alpha)} > 1 - L_{k+1}^{-\left(q'(N)-4\right)}. \tag{4.10}$$

Here $q(N)$ is the parameter from Eq. 1.13 and $q'(N) := q(N)/\alpha$.

The statement of Lemma 12 below is a simple reformulation of Lemma 4.2 from [8], adapted to our notations. Indeed, the reader familiar with the proof given in [8] can see that the structure of the external potential is irrelevant to this completely deterministic statement. So it applies directly to our model with potential energy $U(\mathbf{x}) + gW(\mathbf{x};\omega)$. For that reason, the Proof of Lemma 12 is omitted.

Lemma 12 *Fix an odd positive integer J and suppose that the following properties are fulfilled:*

$$\text{(i)}\ \ \mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v}) \ \textit{is}\ (E,J){-}\mathrm{CNR}, \ \ \textit{and}\ \ \text{(ii)}\ \ K\left(\mathbf{\Lambda}^{(N)}_{L_{k+1}(\mathbf{u})};E\right) \leqslant J.$$

Then for sufficiently large L_0, box $\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{v})$ is (E, m_{k+1})-NS with

$$m_{k+1} \geqslant m_k\left(1-\frac{5J+6}{L_k^{1/2}}\right) > m_0/2 > 0. \tag{4.11}$$

Taking into account Corollary 1, we set $J = 2\ell+1$.
Now the main result of this section:

Theorem 6 *Fix a bounded interval $I \subset \mathbb{R}$. For $p(N)$ large enough there exists $L_0^* \in (0,+\infty)$ such that if $L_0 \geqslant L_0^*$ and $p(N-1)$ is large enough, then, $\forall\, k \geqslant 0$, property* **(IS.k.N)** *in Eq. 4.2 implies* **(IS.$k+1$.N)**, *with the same $p(N)$.*

Proof of Theorem 6 Let $\mathbf{x},\mathbf{y} \in \mathbb{Z}^{Nd}$ and assume that $\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y})$ are separable FI-boxes. Consider the following two events:

$$\mathrm{B} = \left\{ \exists\, E \in I: \ \text{both } \mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}) \text{ and } \mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y}) \ \text{ are } (E, m_{k+1})\text{-S} \right\},$$

and, for a given odd integer J,

$$\mathrm{R} = \left\{ \exists\, E \in I: \ \text{ neither } \mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y}) \text{ is } (E,J){-}\mathrm{CNR} \right\}.$$

By virtue of Lemma 11, for $L_0 \geqslant (J+1)^2$ and $\alpha = 3/2$, we have:

$$\mathbb{P}\{\,\mathrm{R}\,\} < L_{k+1}^{-(q'(N)-4)}, \qquad q'(N) := q(N)/\alpha. \tag{4.12}$$

Further, $\mathbb{P}\{\mathrm{B}\} \leqslant \mathbb{P}\{\mathrm{R}\} + \mathbb{P}\{\mathrm{B} \cap \mathrm{R}^{\mathrm{c}}\}$, and we know that $\mathbb{P}\{\mathrm{R}\} \leqslant L_{k+1}^{-q'(N)+4}$. So, it suffices now to estimate $\mathbb{P}\{\mathrm{B} \cap \mathrm{R}^{\mathrm{c}}\}$. Within the event $\mathrm{B} \cap \mathrm{R}^{\mathrm{c}}$, for any $E \in I$, either $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$ or $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y})$ must be (E, J)-CNR. Without loss of generality, assume that for some $E \in I$, $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$ is (E, J)-CNR and (E, m_{k+1})-S. By Lemma 12, for such value of E, $K(\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x}); E) \geqslant J+1$. We see that

$$\mathrm{B} \cap \mathrm{R}^{\mathrm{c}} \subset \left\{ \exists E \in I : \quad K(\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x}); E) \geqslant J+1 \right\}$$

and, therefore, by Lemma 10, with $J = 2\ell + 2$, and Corollary 1,

$$\mathbb{P}\left\{ \mathrm{B} \cap \mathrm{R}^{\mathrm{c}} \right\} \leqslant \mathbb{P}\left\{ \exists E \in I : \quad K(\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x}); E) \geqslant J+1 \right\} \leqslant L_k^{-2p(N)}. \tag{4.13}$$

□

Remark The integer J figuring throughout Section 4 depends on N, d, and the choice of parameter $p(N)$. In turn, $p(N)$ is determined by dimension d and the choice of value ℓ from Lemma 9. In addition, parameter $p(N-1)$ should be large enough (as was stated in Theorem 6).

5 Mixed Pairs of Singular N-Particle Boxes

It remains to derive the property **(DS.$\boldsymbol{k+1}$, $\boldsymbol{I}$, $\boldsymbol{N}$)** in case (III), i.e., for mixed pairs of N-particle boxes (where one is FI and the other PI). Here we use several properties which have been established earlier in this paper for all scale lengths, namely, **(WS1.n)**, **(WS2.n)** for $n = 1, \ldots, N$, **NITRoNS**, and the inductive assumption **(IS.$\boldsymbol{k+1}$.$\boldsymbol{N}$)** which we have already derived from **(IS.$\boldsymbol{k}$.$\boldsymbol{N}$)** in Section 4.

A natural counterpart of Theorem 6 for mixed pairs of boxes is the following

Theorem 7 *$\forall$ given interval $I \subseteq \mathbb{R}$, there exists a constant $L_1^* \in (0, +\infty)$ with the following property. Assume that $L_0 \geqslant L_1^*$ and, for a given $k \geqslant 0$, the property* **(DS.$\boldsymbol{k}$, $\boldsymbol{I}$, $\boldsymbol{N}$)** *holds* (i) *$\forall$ pair of* separable PI-*boxes $\boldsymbol{\Lambda}_{L_k}^{(N)}(\widetilde{\mathbf{x}})$, $\boldsymbol{\Lambda}_{L_k}^{(N)}(\widetilde{\mathbf{y}})$, and* (ii) *$\forall$ pair of separable* FI-*boxes $\boldsymbol{\Lambda}_{L_k}^{(N)}(\widetilde{\mathbf{x}})$, $\boldsymbol{\Lambda}_{L_k}^{(N)}(\widetilde{\mathbf{y}})$.*

Let $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$, $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y})$ be a pair of separable *boxes, where $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$ is* FI *and $\boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y})$* PI. *Then*

$$\mathbb{P}\left\{ \exists\, E \in I : \text{ both } \boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x}), \quad \boldsymbol{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{y}) \text{ are } (E, m_{k+1})\text{–S} \right\} \leqslant L_{k+1}^{-2p(N)}. \tag{5.1}$$

Proof of Theorem 7 Recall that the Hamiltonian $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y})}$ is decomposed as in Eqs. 3.1, 3.2. Consider the following three events:

$$\mathrm{B} = \left\{ \exists\, E \in I : \text{ both } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}),\ \ \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y}) \text{ are } (E, m_{k+1})\text{-S} \right\},$$

$$\mathrm{T} = \left\{ \boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{y}) \text{ is } m_0\text{-PT} \right\},$$

$$\mathrm{R} = \left\{ \exists\, E \in I : \text{ neither } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}) \text{ nor } \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y}) \text{ is } (E, J)\text{-CNR} \right\}.$$

Recall that by virtue of Eq. 3.4, we have

$$\mathbb{P}\{\mathrm{T}\} \leqslant \frac{1}{4} L_{k+1}^{-2p(N)} \tag{5.2}$$

For the event R we have, by virtue of Lemma 11 and inequality (4.13),

$$\mathbb{P}\{\mathrm{R}\} \leqslant L_{k+1}^{-q(N)+2}; \tag{5.3}$$

as before, $q(N)$ is the parameter from Eq. 1.13. Further, $\mathbb{P}\{\mathrm{B}\} \leqslant \mathbb{P}\{\mathrm{T}\} + \mathbb{P}\{\mathrm{B} \cap \mathrm{T}^{\mathrm{c}}\} \leqslant \frac{1}{4} L_{k+1}^{-2p(N)} + \mathbb{P}\{\mathrm{B} \cap \mathrm{T}^{\mathrm{c}}\}$, and we have

$$\mathbb{P}\left\{\mathrm{B} \cap \mathrm{T}^{\mathrm{c}}\right\} \leqslant \mathbb{P}\{\mathrm{R}\} + \mathbb{P}\left\{\mathrm{B} \cap \mathrm{T}^{\mathrm{c}} \cap \mathrm{R}^{\mathrm{c}}\right\} \leqslant L_{k+1}^{-q(N)+2} + \mathbb{P}\left\{\mathrm{B} \cap \mathrm{T}^{\mathrm{c}} \cap \mathrm{R}^{\mathrm{c}}\right\}.$$

Within the event $\mathrm{B} \cap \mathrm{T}^{\mathrm{c}} \cap \mathrm{R}^{\mathrm{c}}$, either $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x})$ or $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y})$ is E-CNR. It must be the FI-box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x})$. Indeed, by **NITRoNS** (Lemma 3), had box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{y})$ been both E-CNR and m_0-NPT, it would have been (E, m_{k+1})-NS, which is not allowed within the event B. Thus, the box $\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x})$ must be E-CNR, but (E, m_{k+1})-S:

$$\mathrm{B} \cap \mathrm{T}^{\mathrm{c}} \cap \mathrm{R}^{\mathrm{c}} \subset \{\exists\, E \in I :\ \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}) \text{ is } (E, m_{k+1})\text{-S and } E\text{-CNR}\}.$$

However, applying Lemma 12, we see that

$$\left\{\exists\, E \in I :\ \boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}) \text{ is } (E, m_{k+1})\text{-S and } E\text{-CNR}\right\} \subset \left\{\exists\, E \in I :\ K(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}); E) \geqslant J+1\right\}.$$

Therefore, with the same values of parameters as in Corollary 1 ($J = 2\ell + 1$, $\ell \geqslant 4$),

$$\mathbb{P}\left\{\mathrm{B} \cap \mathrm{T}^{\mathrm{c}} \cap \mathrm{R}^{\mathrm{c}}\right\} \leqslant \mathbb{P}\left\{\exists\, E \in I :\ K\left(\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x}); E\right) \geqslant 2\ell + 2\right\} \leqslant 2 L_{k+1}^{-1}\, L_{k+1}^{-2p(N)}. \tag{5.4}$$

Finally, we get, with $q'(N) := q(N)/\alpha$,

$$\mathbb{P}\{\mathrm{B}\} \leqslant \mathbb{P}\{\mathrm{B} \cap \mathrm{T}\} + \mathbb{P}\{\mathrm{R}\} + \mathbb{P}\left\{\mathrm{B} \cap \mathrm{T}^{\mathrm{c}} \cap \mathrm{R}^{\mathrm{c}}\right\} \leqslant \frac{1}{2} L_{k+1}^{-2p(N)} + L_{k+1}^{-q'(N)+4} + 2 L_{k+1}^{-1}\, L_{k+1}^{-2p(N)} \leqslant L_{k+1}^{-2p(N)}, \tag{5.5}$$

for sufficiently large L_0, if we can guarantee, by taking $|g|$ large enough, that $q'(N) > 2p(N) + 5$. This completes the Proof of Theorem 7. □

Remark The Proof of Theorem 7 practically repeats that of Theorem 5.1 from [7]; the only difference is in specification of constants in the exponents.

Therefore, Theorem 4 is also proven.

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Appendix A

Proof of Lemma 1 Consider two N-particle configurations $\mathbf{x}$ and $\mathbf{y}$ and introduce the following notion: we shall say that the set of positions $\{x_j,\ j \in \mathcal{J}\}$, $\mathcal{J} \subseteq \{1, \ldots, N\}$, form an R-connected cluster (or simply an R-cluster) iff the set

$$\bigcup_{j \in \mathcal{J}} \Lambda_R\left(y_j\right) \subset \mathbb{Z}^d \tag{6.1}$$

is connected. Otherwise, this set of particles is called R-disconnected, in which case it can be decomposed into two or more R-clusters. Now, we proceed as follows.

(1) Decompose the configuration $\mathbf{y}$ into L-clusters (of diameter $\leqslant 2NL$).

(2) To each position y_j there corresponds precisely one cluster, denoted by $\Gamma(j)$. Let $\mathbb{Y} = \{\Gamma(j) :\ j \in \mathcal{J}\}$ stand for the collection of clusters, with $\operatorname{card} \mathbb{Y} \leqslant N$.

(3) Consider any of the clusters $\Gamma(j) \in \mathbb{Y}$. By definition, $\Gamma(j)$ is disjoint from all other clusters:

$$\Gamma(j) \cap \Gamma(i) = \begin{cases} \Gamma(j), & \text{if } \Gamma(i) = \Gamma(j), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{6.2}$$

Therefore, for any two distinct clusters $\Gamma', \Gamma'' \in \mathbb{Y}$, the respective sigma-algebras $\mathfrak{B}(\Gamma')$, $\mathfrak{B}(\Gamma'')$ are independent.

(4) Suppose that $\exists\, j \in \{1, \dots, N\} : \Gamma(j) \cap \Pi \boldsymbol{\Lambda}_L^{(N)}(\mathbf{x}) = \emptyset$. Set

$$\bar{\mathfrak{B}}_j(\mathbf{y}) := \mathfrak{B}\left(\cup_{\Gamma(i) \neq \Gamma(j)} \Gamma(i)\right).$$

Then the sigma-algebra $\mathfrak{B}(\Gamma(j))$ is independent of $\mathfrak{B}(\boldsymbol{\Lambda}_L^{(N)}(\mathbf{x}))$ and of $\bar{\mathfrak{B}}_j(\mathbf{y})$:

$$\mathfrak{B}(\Gamma(j)) \coprod \mathfrak{B}\left(\boldsymbol{\Lambda}_L^{(N)}(\mathbf{x})\right), \qquad \mathfrak{B}(\Gamma(j)) \coprod \bar{\mathfrak{B}}_j(\mathbf{y}). \tag{6.3}$$

In other words, the box $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{y})$ is separable from $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{x})$.

(5) Suppose (4) is wrong, and let's deduce from the negation of (4) a necessary condition on possible locations of the configuration $\mathbf{y}$, so as to show that the number of possible choices is finite. Indeed our hypothesis reads as follows:

$$\forall\, j \in \{1, \dots, N\} \;\; Y(j) \cap \Pi \boldsymbol{\Lambda}_L^{(N)}(\mathbf{x}) \neq \emptyset. \tag{6.4}$$

Therefore,

$$\forall\, j \in \{1, \dots, N\} \;\; \exists\, i : \; \|y_j - x_i\| \leqslant 4NL + L = (4N+1)L \leqslant 5NL$$
$$\Rightarrow \forall\, j \in \{1, \dots, N\} \;\; y_j \in \Pi \boldsymbol{\Lambda}_{AL}^{(N)}(\mathbf{x}), \;\; A = A(N) = 5N.$$

We see that if a configuration $\mathbf{y}$ is not separable from $\mathbf{x}$, then every position y_j must belong to one of the boxes $\Pi_i \boldsymbol{\Lambda}_{AL}^{(N)}(\mathbf{x}) = \Lambda_{AL}(x_i) \subset \mathbb{Z}^d$. The total number k of these boxes is bounded by N. There are at most $k^N/k!$ choices of the boxes $\Lambda_{AL}(x_i)$ for the N positions $y_1, \dots, y_N$. For any given choice among $J(N) = J(N, \widehat{K}) \leqslant k^N/k! \leqslant \widehat{K}(N)$ possibilities, with $\widehat{K}(N) < \infty$, the point $\mathbf{y} = (y_1, \dots, y_N)$ must belong to the Cartesian product of N boxes of size AL, i.e. to an Nd-dimensional box of size AL. The assertion of Lemma 1 now follows. □

Appendix B: Finite-Volume Localisation Bounds

Here we give the proof of Lemma 3. Recall, we consider operator $\mathbf{H}^{(N)}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{u})}$ in a box $\boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{u})$. Let $\boldsymbol{\Psi}_j$, $j = 1, \dots, |\boldsymbol{\Lambda}_{L_k}^{(N)}|$, be its normalised EFs and E_j the respective EVs. Fix j and consider the GFs $\mathbf{G}^{(N)}(\mathbf{v}, \mathbf{y}; E_j)$, $\mathbf{v}, \mathbf{y} \in \boldsymbol{\Lambda}^{(N)}$.

Proof of Lemma 3 Recall that the CNR property implies NR. Observe that $E - \lambda_a - \mu_b = (E - \lambda_a) - \mu_b$. Further, by the hypothesis of the lemma, $\boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{u})$ is E-CNR. Therefore, for all λ_a, the n''-particle box $\boldsymbol{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')$ is $(E - \lambda_a)$-NR. By the assumption of m-NPT, $\forall\, E \in I$ box $\boldsymbol{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')$ must not contain two disjoint $(E - \lambda_a, m)$-S sub-boxes of size L_{k-1}. Therefore, the MSA procedure proves that $\boldsymbol{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')$ is $(E - \lambda_a)$-NS, yielding the required upper bound.

Let us now prove the second assertion of the lemma. If $\mathbf{v} = (\mathbf{v}', \mathbf{v}'') \in \partial \boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$, then either $\|\mathbf{u}' - \mathbf{v}'\| = L_k$, or $\|\mathbf{u}'' - \mathbf{v}''\| = L_k$. In the former case we can write

$$\begin{aligned}\mathbf{G}^{(N)}(\mathbf{u}, \mathbf{v}; E) &= \sum_a \varphi_a\left(\mathbf{u}'\right)\varphi_a\left(\mathbf{v}'\right) \sum_b \frac{\psi_b\left(\mathbf{u}''\right)\psi_b\left(\mathbf{v}''\right)}{(E-\lambda_a)-\mu_b} \\ &= \sum_a \varphi_a\left(\mathbf{u}'\right)\varphi_a\left(\mathbf{v}'\right)\ \mathbf{G}^{(n'')}_{\boldsymbol{\Lambda}^{(n'')}_{L_k}(\mathbf{u}'')}\left(\mathbf{u}'', \mathbf{v}''; E-\lambda_a\right). \qquad (7.1)\end{aligned}$$

Since $\|\varphi_a\| = 1$, we see that

$$|\mathbf{G}^{(N)}(\mathbf{u}, \mathbf{v}; E)| \leqslant \left|\Lambda^{(n')}_{L_k}\left(\mathbf{u}'\right)\right| \max_{\lambda_a} |\mathbf{G}^{(n'')}_{\Lambda^{(n'')}_{L_k}(\mathbf{u}'')}\left(\mathbf{u}'', \mathbf{v}''; E-\lambda_a\right)|. \qquad (7.2)$$

In the case where $\|\mathbf{u}'' - \mathbf{v}''\| = L$, we can use the representation

$$\mathbf{G}^{(N)}(\mathbf{u}, \mathbf{v}; E) = \sum_b \psi_b\left(\mathbf{u}''\right)\psi_b\left(\mathbf{v}''\right)\ \mathbf{G}^{(n')}_{\Lambda^{(n')}_{L_k}(\mathbf{u}')}\left(\mathbf{u}', \mathbf{v}'; E-\mu_b\right). \qquad (7.3)$$

□

Now, as was said before, Lemma 4 follows from Lemma 3 combined with the bounds **(DS.*k*, *I*, *n*′)**, **(DS.*k*, *I*, *n*″)**, for $1 \leqslant n', n'' < N$.

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