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A Wegner-type Estimate for Correlated Potentials

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Abstract We propose a fairly simple and natural extension of Stollmann's lemma to correlated random variables. This extension allows to obtain Wegner-type estimates even in various problems of spectral analysis of random operators where the original Wegner's lemma is inapplicable, e.g., for correlated random potentials with singular marginal distributions and for multi-particle Hamiltonians.

Keywords Wegner estimate · Stollmann's lemma · Multi-scale analysis

Mathematics Subject Classifications (2000) Primary 35P20 · Secondary 47F05

1 Introduction

The regularity problem for the limiting distribution of eigenvalues of infinite dimensional self-adjoint operators appears in many problems of mathematical physics. Specifically, consider a lattice Schrödinger operator (LSO, for short) H: $\ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ given by

$$(H\psi)(x) = \sum_{y:\,|y-x|=1} \psi(y) + V(x)\psi(x);\;\; x, y \in \mathbb{Z}^d.$$

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For each finite subset $\Lambda \subset \mathbb{Z}^d$, let E_j^{Λ}, $j = 1, \dots, |\Lambda|$, be eigenvalues of H with Dirichlet b.c. in Λ. Consider the family of finite sets $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ and define the following quantity (which does not necessarily exist for an arbitrary LSO):

$$k(E) = \lim_{L\to\infty} \frac{1}{(2L+1)^d} \text{card}\left\{ j\colon\ E_j^{\Lambda_L} \leqslant E \right\}.$$

If the above limit exists, $k(E)$ is called the limiting distribution function (LDF) of e.v. of H. It is not difficult to construct various examples of a function V: $\mathbb{Z}^d \to \mathbb{R}$ (called the potential of the operator H) for which the LDF does not exist. One can prove the existence of LDF for periodic potentials V, but even in this, relatively simple situation the existence of $k(E)$ is not a trivial fact.

However, the existence of $k(E)$ can be proved for a large class of *ergodic random* potentials. Namely, consider an ergodic dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{T^x,\ x \in \mathbb{Z}^d\})$ with discrete time $\mathbb{Z}^d$ and a measurable function (sometimes called a hull) $v\colon \Omega \to \mathbb{R}$. Then we can introduce a family of sample potentials

$$V(x, \omega) = v(T^x \omega),\ \ x \in \mathbb{Z}^d,$$

labeled by $\omega \in \Omega$. Under the assumption of ergodicity of $\{T^x\}$ (and, for example, boundedness of function v), the quantity

$$k(E, \omega) = \lim_{L\to\infty} \frac{1}{(2L+1)^d} \text{card}\left\{ j\colon\ E_j^{\Lambda_L}(\omega) \leqslant E \right\}$$

is well-defined $\mathbb{P}$-a.s. Moreover, $k(E, \omega)$ is $\mathbb{P}$-a.s. independent of ω, so its value for a.e. ω is natural to take as $k(E)$. In such a context, $k(E)$ is usually called the *integrated density of states* (IDS, for short). It admits an equivalent definition:

$$k(E) = \mathbb{E}\left[\left(f, \Pi_{(-\infty, E]}(H(\omega))\ f\right) \right],$$

where $f \in \ell^2(\mathbb{Z}^d)$ is any vector of unit norm, and $\Pi_{(-\infty, E]}(H(\omega))$ is the spectral projection of $H(\omega)$ on $(-\infty, E]$. The reader can find a detailed discussion of the existence problem of IDS in excellent monographs by Carmona and Lacroix [5] and by Pastur and Figotin [16]. See also articles [3, 4, 6, 10, 15].

It is not difficult to see that $k(E)$ can be considered as the distribution function of a normalized measure, i.e. probability measure, on $\mathbb{R}$. If this measure $dk(E)$, called the measure of states, is absolutely continuous with respect to the Lebesgue measure dE, its density (or Radon–Nikodim derivative) $dk(E)/dE$ is called the density of states (DoS). In physical literature, it is customary to neglect the problem of existence of such density, for if $dk(E)/dE$ is not a function, then "it is simply a generalized function". However, the real problem is not terminological. The actual, explicit estimates of the probabilities of the form

$$\mathbb{P}\left\{ \exists\, \text{eigenvalue } E_j^{\Lambda_L} \in (a, a+\epsilon) \right\}$$

for an LSO H_{Λ_L} in a finite cube Λ_L of size L, for small ϵ, often depend essentially upon the existence and the regularity properties of the DoS $dk(E)/dE$.

Apparently, the first fairly general result relative to the existence and boundedness of the DoS is due to Wegner [21].

Lemma 1.1 (Wegner) *Assume that* $\{V(x,\omega), x \in \mathbb{Z}^d\}$ *are i.i.d. r.v. with a bounded density* $p_V(u)$ *of their common probability distribution:* $\|p_V\|_\infty = C < \infty$. *Then the DoS* $dk(E)/dE$ *exists and is bounded by the same constant* C.

The proof can be found, for example, in the monographs [5] and [16].

This estimate and some of its generalizations have been used in the multi-scale analysis (MSA) developed in the works by Fröhlich and Spencer [13], Fröhlich et al. [12], von Dreifus and Klein [19, 20], and in a number of more recent works where the so-called Anderson Localization phenomenon has been observed. Namely, it has been proven that all eigenfunctions of random LSOs decay exponentially at infinity with probability one (for $\mathbb{P}$-a.e. sample of random potential $V(\omega)$). Von Dreifus and Klein [20] proved an analog of Wegner estimate and used it in their proof of localization for Gaussian and some other correlated (but non-deterministic) potentials. The author of this paper recently proved, in a joint work with Suhov [8], an analog of Wegner estimate for a system of two or more interacting quantum particles on the lattice under the assumption of analyticity of the probability density $p_V(u)$, using a rigorous path integral formula by Molchanov (see a detailed discussion of this formula in the monograph [5]). In order to relax the analyticity assumption in a multi-particle context, Chulaevsky and Suhov [9] used later a more general and flexible result: a lemma proved by Stollmann (cf. [17] and [18]) which we discuss below.

In the present work, we propose a fairly simple and natural extension of Stollmann's lemma to correlated, but still non-deterministic random fields generating random potentials. Our main motivation here is to lay out a way to interesting applications to localization problems for multi-particle systems.

2 Stollmann's Lemma for Product Measures

Recall the Stollmann's lemma and its proof for independent random variables. Let $m \geqslant 1$ be a positive integer, and J an abstract finite set with $|J| (= \mathrm{card} J) = m$. Consider the Euclidean space $\mathbb{R}^J \cong \mathbb{R}^m$ with the standard basis $(e_1, \ldots, e_m)$, and its positive orthant

$$\mathbb{R}^J_+ = \left\{ q \in \mathbb{R}^J \colon\ q_j \geqslant 0,\ \ j = 1, 2, \ldots, m \right\}.$$

Definition 2.1 Let J be a finite set with $|J| = m$. Consider a function $\Phi\colon \mathbb{R}^J \to \mathbb{R}$. It is called diagonally monotone (DM, for short) if it satisfies the following conditions:

(1) for any $r \in \mathbb{R}^J_+$ and any $q \in \mathbb{R}^J$,

$$\Phi(q + r) \geqslant \Phi(q); \tag{1}$$

(2) moreover, for $e = e_1 + \cdots + e_m \in \mathbb{R}^J$, for any $q \in \mathbb{R}^J$ and for any $t > 0$

$$\Phi(q + t \cdot e) - \Phi(q) \geqslant t. \tag{2}$$

It is convenient to introduce the notion of DM operators considered as quadratic forms. In the following definition, we use the same notations as above.

Definition 2.2 Let $\mathcal{H}$ be a Hilbert space. A family of self-adjoint operators $B(q)\colon \mathcal{H} \to \mathcal{H}$, $q \in \mathbb{R}^J$, is called DM if,

$$\forall\, q \in \mathbb{R}^J \ \ \forall\, r \in \mathbb{R}^J_+ \ \ B(q + r) \geqslant B(q),$$

in the sense of quadratic forms, and for any vector $f \in \mathcal{H}$ with $\| f \| = 1$, the function $\Phi_f\colon \mathbb{R}^J \to \mathbb{R}$ defined by

$$\Phi_f(q) = (B(q) f,\ f)$$

is DM.

In other words, the quadratic form $Q_{B(q)}(f) := (B(q) f,\ f)$ as a function of $q \in \mathbb{R}^J$ is non-decreasing in any q_j, $j = 1, \ldots, |J|$, and

$$(B(q + t \cdot e) f,\ f) - (B(q) f,\ f) \geqslant t \cdot \| f \|^2.$$

Remark 2.3 By virtue of the min-max principle for self-adjoint operators, if an operator family $H(q)$ in a finite-dimensional Hilbert space $\mathcal{H}$ is DM, then each eigenvalue $E_k^{B(q)}$ of $B(q)$ is a DM function.

Remark 2.4 If $H(q)$, $q \in \mathbb{R}^J$, is a DM operator family in a Hilbert space $\mathcal{H}$, and $H_0 : \mathcal{H} \to \mathcal{H}$ is an arbitrary self-adjoint operator, then the family $H_0 + H(q)$ is also DM.

This explains why the notion of diagonal monotonicity is relevant to the spectral theory of random operators. Note also, that this property applies to physically interesting examples where $\dim \mathcal{H} = +\infty$, but $H(q)$ have, e.g., a compact resolvent, as in the case of Schrödinger operators in a finite cube with Dirichlet b.c. and with a bounded potential, so the respective spectrum is pure point, and even discrete.

For any measure μ on $\mathbb{R}$, we will denote by μ^J the product measure $\mu \times \cdots \times \mu$ on $\mathbb{R}^J$. Furthermore, for any probability measure μ and for any $\epsilon > 0$, define the following quantity:

$$s(\mu, \epsilon) = \sup_{a \in \mathbb{R}} \ \mu([a, a + \epsilon])$$

We will denote by μ_j^{m-1} the marginal probability distribution induced by μ^J on $q'_{\neq j} = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_m)$.

Lemma 2.5 (Stollmann [17]) *Let J be a finite index set, $|J| = m$, μ be a probability measure on $\mathbb{R}$, and μ^J be the product measure on $\mathbb{R}^J$ with marginal* *measures μ. If the function Φ: $\mathbb{R}^J \to \mathbb{R}$ is DM, then for any open interval $I \subset \mathbb{R}$ we have*

$$\mu^J\{q\colon\ \Phi(q) \in I\} \leqslant m \cdot s(\mu, |I|).$$

We provide below a proof of Stollmann's lemma; this will allow to extend it to the case of correlated potentials.

Proof Let $I = (a, b)$, $b - a = \epsilon > 0$, and consider the set

$$A = \{q\colon\ \Phi(q) \leqslant a\}.$$

Furthermore, define recursively sets A^ϵ_j, $j = 0, \dots, m$, by setting

$$A^\epsilon_0 = A,\ \ A^\epsilon_j = A^\epsilon_{j-1} + [0, \epsilon]e_j := \left\{q + te_j\colon\ q \in A^\epsilon_{j-1},\ \ t \in [0, \epsilon]\right\}.$$

Obviously, the sequence of sets A^ϵ_j, $j = 1, 2, \dots$, is increasing with j. The DM property implies

$$\{q\colon\ \Phi(q) < b\} \subset A^\epsilon_m.$$

Indeed, if $\Phi(q) < b$, then for the vector $r := q - \epsilon \cdot e$ we have by (2):

$$\Phi(r) \leqslant \Phi(r + \epsilon \cdot e) - \epsilon = \Phi(q) - \epsilon \leqslant b - \epsilon \leqslant a,$$

meaning that $r \in \{\Phi \leqslant a\} = A$ and, therefore,

$$q = r + \epsilon \cdot e \in A^\epsilon_m.$$

Now, we conclude that

$$\begin{aligned}\{q\colon\ \Phi(q) \in I\} &= \{q\colon\ \Phi(q) \in (a, b)\}\\ &= \{q\colon\ \Phi(q) < b\} \setminus \{q\colon\ \Phi(q) \leqslant a\} \subset A^\epsilon_m \setminus A.\end{aligned}$$

Furthermore,

$$\begin{aligned}\mu^m\{q\colon\ \Phi(q) \in I\} &\leqslant \mu^m\left(A^\epsilon_m \setminus A\right)\\ &= \mu^m\left(\bigcup_{j=1}^m \left(A^\epsilon_j \setminus A^\epsilon_{j-1}\right)\right) \leqslant \sum_{j=1}^m \mu^m\left(A^\epsilon_j \setminus A^\epsilon_{j-1}\right).\end{aligned}$$

For $q'_{\neq 1} = (q_2, \dots, q_m) \in \mathbb{R}^{m-1}$, set

$$I_1(q'_{\neq 1}) = \left\{q_1 \in \mathbb{R}\colon\ (q_1, q'_{\neq 1}) \in A^\epsilon_1 \setminus A\right\}.$$

By definition of the set A^ϵ_1, this is an interval of length not bigger than ϵ. Since μ^J is a product measure, we have

$$\mu^m(A^\epsilon_1 \setminus A) = \int d\mu^{m-1}(q'_{\neq 1}) \int_{I_1} d\mu(q_1) \leqslant s(\mu, \epsilon). \tag{3}$$

Similarly, we obtain for $j = 2, \dots, m$

$$\mu^m(A^\epsilon_j \setminus A^\epsilon_{j-1}) \leqslant s(\mu, \epsilon),$$

yielding

$$\mu^m\{q\colon\ \Phi(q)\in I\}\leqslant\sum_{j=1}^{m}\mu^m(A_j^{\epsilon}\setminus A_{j-1}^{\epsilon})\leqslant m\cdot c(\mu,\epsilon).\qquad\square$$

Now, taking into account the above Remark 2.4, Lemma 2.5 yields immediately the following estimate.

Lemma 2.6 *Let H_Λ be an LSO with a random potential $V(x;\omega)$ in a finite box $\Lambda\subset\mathbb{Z}^d$ with Dirichlet b.c., and $\Sigma(H_\Lambda)$ its spectrum, i.e. the collection of its eigenvalues $E_j^{(\Lambda)}$, $j=1,\dots,|\Lambda|$. Assume that r.v. $V(x;\cdot)$ are i.i.d. with the marginal distribution μ_V. Then*

$$\mathbb{P}\{\,\mathrm{dist}(\Sigma(H_\Lambda(\omega),E)\leqslant\epsilon\,\}\leqslant|\Lambda|^2 s(\mu_V,2\epsilon).$$

This is an analogue of Wegner bound. One visible distinction is the form of its volume dependence: the factor $|\Lambda|^2$ instead of $|\Lambda|$ in the conventional Wegner bound. One has to keep in mind, however, that

- Stollmann's lemma on monotone functions is *sharp* (cf. [17]); eigenvalues, however, are particular monotone functions, which explains why conventional Wegner bound has the factor of $|\Lambda|^1$;
- while Wegner's method requires the ensemble of random variables generating (or controlling) potential in a volume of cardinality $|\Lambda|$ to have $|\Lambda|$ degrees of freedom, correlated or not, the above approach works fine even for ensembles with *one* degree of freedom (the parameter m above may equal 1);
- in applications to the MSA, any upper bound of the form $Const\,|\Lambda|^N\epsilon$, or even $e^{|\Lambda|^\beta}\epsilon$, with $\beta\in(0,1)$ would be sufficient to make the MSA inductive scheme work;
- although the above version of Wegner bound does not allow to establish the existence of DoS even in models where the latter does exist (as can be shown with Wegner bound), the existence of the (limiting) DoS is not quite helpful *per se* for the finite-volume MSA, where upper bounds for the probability of high concentration of eigenvalues are vital;
- in applications to multi-particle localization problems with a short-range (or decaying) interaction between quantum particles, the existence of DoS for external potentials with regular marginal distributions can be proved by different methods. For example, Klopp and Zenk (2003, preprint) proved that the IDS for a multi-particle quantum system in $\mathbb{R}^d$ with a decaying particle interaction is the same as for the model without interaction. This result, quite natural from a physical point of view, was proved with the help of Helffer–Sjöstrand formula for almost analytic extensions. A similar result can be proved in a simpler way for lattice systems.

3 Extension to Multi-particle Systems

Results of this section have been obtained by the author and Suhov [9].

Let $N > 1$ and $d \geqslant 1$ be two positive integers and consider a random LSO $H = H(\omega)$ which can be used, in the framework of tight-binding approximation, as the Hamiltonian of a system of N quantum particles in $\mathbb{Z}^d$ with a random external potential V and an interaction potential U. Specifically, let $x_1, \ldots, x_N \in \mathbb{Z}^d$ be the positions of quantum particles in the lattice $\mathbb{Z}^d$, and $\underline{x} = (x_1, \ldots, x_N)$. Let $\{V(x; \omega),\ x \in \mathbb{Z}^d\}$ be a random field on $\mathbb{Z}^d$ describing the external potential acting on all particles, and U: $(x_1, \ldots, x_N) \mapsto \mathbb{R}$ be the interaction energy of the particles. In physics, U is usually to be a symmetric function of its N arguments $x_1, \ldots, x_N \in \mathbb{Z}^d$. We will assume in this section that the system in question obeys either Fermi or Bose quantum statistics, so it is convenient to assume U to be symmetric. Note, however, that the results of this section can be extended, with natural modifications, to more general interactions U. Further, U is assumed to be a finite-range interaction:

$$\operatorname{supp} U \subset \{\underline{x}\colon\ \max(|x_j - x_k| \leqslant r_0)\},\ r_0 < \infty.$$

Such an assumption is required in the proof of Anderson localization for multi-particle systems. However, it is irrelevant to the Wegner-type estimate we are going to discuss below.

Now, let H be as follows:

$$(H(\omega) f)(\underline{x}) = \sum_{j=1}^{N} \left(\Delta^{(j)} + V(x_j; \omega)\right) + U(\underline{x}),$$

where $\Delta^{(j)}$ is the lattice Laplacian acting on the j-th particle, i.e.

$$\Delta^{(j)} = \underset{1}{\mathbf{1}} \otimes \ldots \otimes \underset{j}{\Delta} \otimes \ldots \otimes \underset{N}{\mathbf{1}}$$

acting in the Hilbert space $\ell^2(\mathbb{Z}^{Nd})$. For any finite "box"

$$\Lambda = \Lambda^{(1)} \times \ldots \times \Lambda^{(N)} \subset \mathbb{Z}^{Nd}$$

one can consider the restriction, $H_\Lambda(\omega)$, of $H(\omega)$ on Λ with Dirichlet b.c. It is easy to see that the potential

$$W(\underline{x}) = \sum_{j=1}^{N} V(x_j; \omega) + U(\underline{x})$$

is no longer an i.i.d. random field on $\mathbb{Z}^{Nd}$, even if V is i.i.d. Therefore, neither version of the Wegner bound applies *directly*. But, in fact, Stollmann's lemma *does* apply to multi-particle systems, virtually in the same way as to single-particle ones.

Lemma 3.1 *Assume that r.v. $V(x; \cdot)$ are i.i.d. with marginal distribution μ_V. Then*

$$\mathbb{P}\{\operatorname{dist}(\Sigma(H_\Lambda(\omega), E) \leqslant \epsilon\} \leqslant |\Lambda| \cdot M(\Lambda) \cdot s(\mu_V, 2\epsilon),$$

with

$$M(\Lambda) = \sum_{j=1}^{N} \operatorname{card} \Lambda^{(j)}.$$

A reader familiar with the MSA method may notice that, in fact, the latter requires two different kinds of Wegner-type bounds: for individual finite volumes Λ and for couples of disjoint (or, more generally, distant) finite volumes Λ, Λ'. In the conventional, single-particle MSA the two-volume bound can be deduced (under certain conditions) from its single-volume counterpart. This is far from obvious for multi-particle (even two-particle) systems with an interaction. A detailed discussion of the multi-particle MSA scheme is beyond the scope of this short note (for details, see [9]). Recently, Kirsch [14] proved an analog of Wegner bound for single volumes (but not for couples of volumes) under a more restrictive assumption of existence and boundedness of the marginal probability distribution of the potential of a multi-particle lattice Anderson model with interaction.

4 Extension to Correlated Random Variables

Fix a positive integer $m \geqslant 1$ and consider the Euclidean space $\mathbb{R}^m$ with coordinates $q = (q_1, \dots, q_m)$. For a given point $q \in \mathbb{R}^m$, set $q'_{\neq j} := (q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_m)$. Let μ^m be a probability measure on $\mathbb{R}^m$ with marginal distributions $\mu_j^{m-1}(q'_{\neq j})$ of order $m-1$, and conditional distributions $\mu_j^1(q_j \,|\, q'_{\neq j})$ of order 1; here $j = 1, \dots, m$. In the case where the measure $\mu_j^1(q_j \,|\, q'_{\neq j})$ is absolutely continuous, we denote by $p(q_j | q'_{\neq j})$ its density with respect to the Lebesgue measure dq_j. The measure μ^m (unlike the measure μ^J in Section 2) is no longer assumed to be a product measure (we emphasize this fact by changing notation). For every $\epsilon > 0$, define the following quantities, measuring in different ways continuity properties of μ^m:

$$C_1(\mu^m, \epsilon) = \max_j \sup_{a \in \mathbb{R}} \int d\mu^{m-1}(q'_{\neq j}) \int_a^{a+\epsilon} d\mu(q_j | q'_{\neq j}), \tag{4}$$

$$C_2(\mu^m, \epsilon) = \max_j \operatorname{ess} \sup_{q'_{\neq j} \in \mathbb{R}^{m-1}} \sup_{a \in \mathbb{R}} \int_a^{a+\epsilon} d\mu(q_j | q'_{\neq j}), \tag{5}$$

and

$$C_3(\mu^m) = \operatorname{ess} \sup_{q'_{\neq j} \in \mathbb{R}^{m-1},\ q_j \in \mathbb{R}} p(q_j | q'_{\neq j}). \tag{6}$$

Since μ^m is a finite (even probability) measure, the quantities $C_1(\mu^m, \epsilon)$ and $C_2(\mu^m, \epsilon)$ are always finite, and bounded by 1, while $C_3(\mu^m, \epsilon)$ may be infinite (in which case, naturally, it is useless). If the density $p(q_j | q'_{\neq j})$ exists, we have

$$d\mu(q_j | q'_{\neq j}) = p(q_j | q'_{\neq j}) dq_j$$

and if $C_3(\mu^m, \epsilon) < \infty$, we can write

$$C_2(\mu^m, \epsilon) = \max_j \text{ ess} \sup_{q'_{\neq j} \in \mathbb{R}^{m-1}} \sup_{a \in \mathbb{R}} \int_a^{a+\epsilon} p(q_j | q'_{\neq j}) dq_j$$
$$\leqslant \max_j \text{ ess} \sup_{q'_{\neq j} \in \mathbb{R}^{m-1}} \sup_{a \in \mathbb{R}} \int_a^{a+\epsilon} C_3(\mu^m)\, dq_j \leqslant C_3(\mu^m)\, \epsilon.$$

Also, it is easy to see that

$$C_1(\mu^m, \epsilon) = \max_j \sup_{a \in \mathbb{R}} \int d\mu^{m-1}(q'_{\neq j}) \int_a^{a+\epsilon} d\mu(q_j | q'_{\neq j})$$
$$\leqslant \int C_2(\mu^m, \epsilon)\, d\mu^{m-1}(q'_{\neq j}) \; = C_2(\mu^m, \epsilon),$$

since μ^{m-1} is a probability measure. Therefore,

$$C_1(\mu^m, \epsilon) \leqslant C_2(\mu^m, \epsilon) \leqslant C_3(\mu^m, \epsilon)\, \epsilon. \tag{7}$$

Remark 4.1 In applications to localization problems, the aforementioned continuity moduli $C_1(\mu^m, \epsilon)$, $C_2(\mu^m, \epsilon)$ need to decay not too slowly as $\epsilon \to 0$. A power decay of order $O(\epsilon^\beta)$ with $\beta > 0$ is certainly sufficient, but it can be essentially relaxed. For example, it suffices to have an upper bound of the form

$$C_1\left(\mu^m, e^{-L^\beta}\right) \leqslant Const \cdot L^{-B},$$

uniformly for all sufficiently large $L > 0$ with some (arbitrarily small) $\beta > 0$ and with $B > 0$ which should sufficiently big, depending on the specific spectral problem.

Using notations of the previous section, one can formulate the following generalization of Stollmann's lemma.

Lemma 4.1 *Let J be a finite set with $|J| = m$, so that we can identify $\mathbb{R}^J$ with $\mathbb{R}^m$. Let Φ: $\mathbb{R}^J \to \mathbb{R}$ be a DM function and μ^m a probability measure on $\mathbb{R}^m \sim \mathbb{R}^J$ with $C_1(\mu^m, \epsilon) < \infty$. Then for any interval $I \subset \mathbb{R}$ of length $|I| = \epsilon > 0$, we have*

$$\mu^m\{q\colon\ \Phi(q) \in I\} \leqslant m \cdot C_1(\mu, \epsilon).$$

Proof We proceed as in the proof of Stollmann's lemma and introduce in $\mathbb{R}^m$ the sets $A = \{q\colon\ \Phi(q) \leqslant a\}$ and A^ϵ_j, $j = 0, \ldots, m$. Here, again, we have

$$\{q\colon\ \Phi(q) \in I\} =\subset A^\epsilon_m \setminus A$$

and

$$\mu^m\{q\colon\ \Phi(q) \in I\} \leqslant \sum_{j=1}^m \mu^m\left(A^\epsilon_j \setminus A^\epsilon_{j-1}\right).$$

For $q'_{\neq 1} \in \mathbb{R}^{m-1}$, we set

$$I_1(q'_{\neq 1}) = \left\{ q_1 \in \mathbb{R}\colon \ (q_1, q'_{\neq 1}) \in A_1^{\epsilon} \setminus A \right\}.$$

Furthermore, we come to the following upper bound which generalizes (3):

$$\mu^m \left(A_1^{\epsilon} \setminus A \right) = \int d\mu^{m-1}(q') \int_{I_1} d\mu(q_1 | q') \leqslant C_1(\mu, \epsilon). \tag{8}$$

Similarly, we obtain for $j = 2, \ldots, m$

$$\mu^m \left(A_j^{\epsilon} \setminus A_{j-1}^{\epsilon} \right) \leqslant C_1(\mu, \epsilon),$$

yielding

$$\mu^m \{ q\colon \ \Phi(q) \in I \} \leqslant \sum_{j=1}^{m} \mu^m \left(A_j^{\epsilon} \setminus A_{j-1}^{\epsilon} \right) \leqslant m \cdot C_1(\mu, \epsilon). \qquad \square$$

5 Application to Gibbs Fields with Continuous Spin

There exists a large variety of correlated random lattice fields for which the hypothesis of Lemma 4.2 can be easily verified. For example, conditional distributions of Gibbs fields are given explicitly in terms of their respective interaction potentials.

Gaussian fields can also be considered as a particular class of Gibbsian fields. The reader can find in the article by von Dreifus and Klein [20] a detailed discussion of such models and a proof of Wegner estimate for homogeneous non-deterministic Gaussian potentials. It suffices to notice, actually, that for such potentials the conditional density of a single-site value $V(x_0; \cdot)$ given all other values $\{V(y; \cdot),\ y \neq x_0\}$ exists and is bounded. Therefore, Lemma 4.2 applies, but so does the traditional Wegner's method.

Anderson localization for Gibbsian potentials on the lattice was proved by von Dreifus and Klein [20] under a rather strong assumption of *complete analyticity* in the sense of Dobrushin and Shlosman (see, e.g., [11]) of the Gibbsian field $V(x; \omega)$, with continuous spins, generating the potential of the respective LSO. We show in this section how Lemma 4.2 allows to relax the complete analyticity hypothesis to a quite general, single-site condition on the Hamiltonian generating the respective Gibbs state for a model of the classical statistical mechanics with *continuous* spins. Indeed, original Dobrushin–Shlosman techniques are adapted to models with a *finite* number of spin values. Bourgain and Kenig [2] considered continuous Anderson models where amplitudes determining the random potential take two values. Recently, Aizenman et al. [1] extended this result to a quite general case, including random variables taking with positive probability any finite number $n > 1$ of values. Unfortunately, no analog of such techniques is known so far for the lattice models.

It is worth mentioning that the condition described below can hold in some models where the marginal density does not exist, in which case more traditional methods do not apply.

Consider a lattice Gibbs field $S_x(\omega)$ with bounded continuous spins,

$$S\colon \Omega \times \mathbb{Z}^d \to \mathcal{S} = [a, b] \subset \mathbb{R}$$

generated by a short-range, bounded, two-body interaction potential $u_\bullet(\cdot, \cdot)$. The spin space is assumed to be equipped with a measure dS which may, in principle, be singular with respect to Lebesgue measure on $[a, b]$ (more general spin spaces $\mathcal{S}$ and measures dS can also be considered). In other words, consider the formal Hamiltonian

$$H(S) = \sum_{x\in\mathbb{Z}^d} h(S_x) + \sum_{x\in\mathbb{Z}^d} \sum_{|y-x|\leqslant R} u_{|x-y|}(S_x, S_y),$$

where $h\colon \mathcal{S} \to \mathbb{R}$ is the self-energy of a given spin. Assume that the interaction potentials $u_{|x-y|}(S_x, S_y)$ vanish for $|x - y| > R$ and are uniformly bounded:

$$\max_{l\leqslant R} \sup_{S,S'\in\mathcal{S}} |u_l(S, S')| < \infty.$$

Then for any lattice point x and any configuration $S' = S'_{\neq x}$ of spins outside $\{x\}$, the *single-site* conditional distribution of S_x given the external configuration S' admits a *bounded density* with respect to measure dS, namely,

$$p(S_x \mid S'_{\neq x}) = \frac{e^{-\beta U(S_x|S')}}{\Xi(\beta, S')} = \frac{e^{-\beta U(S_x|S')}}{\int_{\mathcal{S}} e^{-\beta U(S''|S')}\, dS''}$$

with

$$U(S_x|S') := \sum_{y\colon |y-x|\leqslant R} u_{|x-y|}(S_x, S'_y)$$

satisfying the upper bound

$$|U(S_x|S')| \leqslant (2R + 1)^d \sup_{S,S''\in\mathcal{S}} |u_l(S, S'')| < \infty.$$

A similar property is valid for sufficiently rapidly decaying long-range interaction potentials, for example, under the condition

$$\sup_{S,S''\in\mathcal{S}} |u_{|y|}(S, S'')| \leqslant \frac{Const}{|y|^{d+1+\delta}},\ \delta > 0. \tag{9}$$

as well as for more general, but still uniformly summable many-body interactions. Below we give one simple example of application of Wegner–Stollmann-type bound to such random potentials.

Lemma 5.1 *Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset of the lattice, $\Lambda' \subset \mathbb{Z}^d \setminus \Lambda$ any subset disjoint with Λ (Λ' may be empty), and let $S_x(\omega)$ be a Gibbs field in Λ with continuous spins $S \in \mathcal{S} = [a, b]$ generated by a two-body interaction potential $u_l(S, S'')$ satisfying condition (9), with any b.c. on $\mathbb{Z}^d \setminus \Lambda$. Consider a LSO*

H_Λ with the random potential $V(x,\omega) = S_x(\omega)$. Then for any interval $I \subset \mathbb{R}$ of length $\epsilon > 0$, we have

$$\mathbb{P}\left\{ \Sigma(H_\Lambda) \cap I \neq \emptyset \,|\, V(y,\cdot),\, y \in \Lambda' \right\} \leqslant C(V)\,|\Lambda|^2\,\epsilon,\;\; C(V) < \infty.$$

In the case of unbounded spins and/or interaction potentials, the uniform boundedness of conditional single-spin distributions does not necessarily hold, since the energy of interaction of a given spin S_0 with the external configuration S' may be arbitrarily large (depending on a particular form of interaction) and even *infinite*, if $S'_y \to \infty$ too fast. In such situations, our general condition (4) might still apply, provided that rapidly growing configurations S' have sufficiently small probability, so that the outer integral in the r.h.s. of (4) converges.

6 Conclusion

Wegner-type bounds of the IDS in finite volumes are a key ingredient of the MSA of spectra of random Schrödinger (and some other) operators. The proposed simple extension of Stollmann's lemma shows that a very general assumption on correlated random fields generating the potential rules out an abnormal accumulation of eigenvalues in finite volumes. This extension applies also to multi-particle systems with interaction.

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