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Cuspons and Smooth Solitons of the Degasperis–Procesi Equation Under Inhomogeneous Boundary Condition

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Abstract This paper is contributed to explore all possible single peakon solutions for the Degasperis–Procesi (DP) equation $m_t + m_x u + 3mu_x = 0,\ m = u - u_{xx}$. Our procedure shows that the DP equation *either* has cusp soliton and smooth soliton solutions only under the inhomogeneous boundary condition $\lim_{|x|\to\infty} u = A \neq 0$, *or* possesses the regular peakon solutions $ce^{-|x-ct|} \in H^1$ (c is the wave speed) only when $\lim_{|x|\to\infty} u = 0$ (see Theorem 4.1). In particular, we first time obtain the stationary cuspon solution $u = \sqrt{1 - e^{-2|x|}} \in W^{1,1}_{loc}$ of the DP equation. Moreover we present new cusp solitons (in the space of $W^{1,1}_{loc}$) and smooth soliton solutions in an explicit form. Asymptotic analysis and numerical simulations are provided for smooth solitons and cusp solitons of the DP equation.

Keywords Soliton · Integrable system · Analysis · Traveling wave

Mathematics Subject Classifications (2000) 35D05 · 35G30 · 35Q53 · 37K10 · 37K40 · 76B15 · 76B25

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1 Introduction

The b-weight-balanced wave equation

$$m_t + m_x u + b m u_x = 0, \qquad m = u - u_{xx} \tag{1.1}$$

was proposed in [8] in 2003, and recently has arisen a lot of attractive attention. This family is proved to be integrable when $b = 2, 3$ by using symmetry approach [11]. More deeper mathematical attributes on the CH equation was shown by the constraint method and r-matrix structure [14, 15], inverse spectral theory [3], Riemann–Hilbert problem [12], and the global conservative solution [1].

In an earlier paper [18], Qiao and Zhang discussed the traveling wave solutions for the $b = 2$ equation – the Camassa–Holm (CH) equation [2] under the inhomogeneous boundary condition $\lim_{|x|\to\infty} u = A$ (A is a non-zero constant), and found new soliton solutions both smooth and cusped. In the paper [19], Vakhnenko and Parkes presented the periodic and loop soliton solutions for the $b = 3$ equation – the Degasperis–Procesi (DP) equation [6] from the mathematical point of view. Their solutions were expressed in a implicit form. Later in [10], Lenells also studied the traveling wave solitary solutions of the DP equation, which decay to zero at both infinities, but did not give explicit soliton solutions, either.

An important issue to study both the CH equation and the DP equation is to find their new solutions through investigating their intrinsic mathematical structures. The DP equation has Lax pair [7] (therefore is integrable), and is able to be extended to a whole integrable hierarchy of equations with parametric solutions under some constraints [13, 17]. Also, the DP equation admits the global weak solution and blow-up structure [9].

In this paper, we give all possible single peak soliton solutions of the DP equation

$$m_t + m_x u + 3 m u_x = 0, \qquad m = u - u_{xx}, \tag{1.2}$$

through setting the traveling wave mode under the boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. Our procedure shows that the DP equation *either* has cusp soliton and smooth soliton solutions only under the boundary condition $\lim_{|x|\to\infty} u = A \neq 0$, *or* possesses the regular peakon solutions only when $\lim_{|x|\to\infty} u = 0$ (see Theorem 4.1). In particular, we first time obtain a stationary cuspon solution of the DP equation. Moreover we present new cusp solitons and smooth soliton solutions in an explicit form [see formulas (3.7), (5.11) and (5.16)]. Asymptotic analysis and numerical simulations are provided for smooth solitons and cusp solitons of the DP equation. In the literature [4], cusp soliton was also called breaking wave. Due to some complex notations and definitions, we shift the statement of our main result to Section 4 (see Theorem 4.1).

2 Traveling Wave Setting

Let us consider the traveling wave solution of the DP equation (1.2) through a generic setting $u(x, t) = U(x - ct)$, where c is the wave speed. Let $\xi = x - ct$, then $u(x, t) = U(\xi)$. Substituting it into the DP equation (1.2) yields

$$(U - c)(U - U'')' + 3U'(U - U'') = 0, \tag{2.1}$$

where $U' = U_\xi,\ U'' = U_{\xi\xi},\ U''' = U_{\xi\xi\xi}$.

If $U - U'' = 0$, then (2.1) has general solutions of $U(\xi) = c_1 e^\xi + c_2 e^{-\xi}$ with any real constants $c_1,\ c_2$. Of course, they are the solutions of the DP equation (1.2). This result is not so interesting for us. On the other hand, the DP equation has the well-known peakon solution [8] $u(x, t) = U(\xi) = ce^{-|x-ct-\xi_0|}$ ($\xi_0 = x_0 - ct_0$) with the following properties

$$U(\xi_0) = c, \qquad U(\pm\infty) = 0, \qquad U'(\xi_0-) = c, \qquad U'(\xi_0+) = -c, \tag{2.2}$$

where $U'(\xi_0-)$ and $U'(\xi_0+)$ represent the left-derivative and the right-derivative at ξ_0, respectively.

Let us now assume that U is neither a constant function nor satisfies $U - U'' = 0$. Then (2.1) can be changed to

$$\frac{(U - U'')'}{U - U''} = \frac{3U'}{c - U}. \tag{2.3}$$

Taking the integration twice on both sides leads to

$$(U')^2 = U^2 - \frac{C_2}{(c - U)^2} + C_1, \tag{2.4}$$

where $C_2 \neq 0$, $C_1 \in \mathbb{R}$ are two integration constants.

Let us solve (2.4) with the following boundary condition

$$\lim_{\xi \to \pm\infty} U = A, \tag{2.5}$$

where A is a constant. Equation (2.4) can be cast into the following ODE:

$$(U')^2 = \frac{(U - A)^2[(U - c + A)^2 - cA]}{(U - c)^2}. \tag{2.6}$$

The fact that both sides of (2.6) are nonnegative implies

$$(U - c + A)^2 - cA \geqslant 0. \tag{2.7}$$

If $cA \geqslant 0$, we denote

$$B_1 = c - A + \sqrt{cA}, \qquad B_2 = c - A - \sqrt{cA}. \tag{2.8}$$

Apparently, $B_1 \geqslant B_2$.

3 Smooth Solution and Weak Solution

Let $C^k(\Omega)$ denote the set of all k times continuously differentiable functions on the open set Ω. $L^p_{loc}(\mathbb{R})$ refers to be the set of all functions whose restriction on any compact subset is L^p integrable. $H^1_{loc}(\mathbb{R})$ and $W^{1,1}_{loc}(\mathbb{R})$ stand for $H^1_{loc}(\mathbb{R}) = \{u \in L^2_{loc}(\mathbb{R}) | \, u' \in L^2_{loc}(\mathbb{R})\}$ and $W^{1,1}_{loc}(\mathbb{R}) = \{u \in L^1_{loc}(\mathbb{R}) | \, u' \in L^1_{loc}(\mathbb{R})\}$, respectively.

Definition 3.1 A function $u(x, t) = U(x - ct)$ is said to be a single peak soliton solution for the DP equation (1.2) if U satisfies the following conditions

(C1) $U(\xi)$ is continuous on $\mathbb{R}$ and has a unique peak point, denoted by ξ_0, where $U(\xi)$ attains its global maximum or minimum value;
(C2) $U(\xi) \in C^3(\mathbb{R} - \{\xi_0\})$ satisfies (2.6) on $\mathbb{R} - \{\xi_0\}$;
(C3) $U(\xi)$ satisfies the boundary condition (2.5).

Without losing the generality, from now on we assume $\xi_0 = 0$.

Lemma 3.2 *Assume that $u(x, t) = U(x - ct)$ is a single peak soliton solution of the DP equation (1.2) at the peak point $\xi_0 = 0$. Then we have*

a) *if $cA < 0$, then $U(0) = c$;*
b) *if $cA \geqslant 0$, then $U(0) = c$ or $U(0) = B_1$ or $U(0) = B_2$.*

Moreover, we have the following solutions classification:

(1) *if $U(0) \neq c$, then $U(\xi) \in C^\infty(\mathbb{R})$, and u is a smooth soliton solution.*
(2) *if $U(0) = c$ and $A \neq 0$, then u is a cusp soliton and U has the following asymptotic behavior*

$$U(\xi) - c = \lambda|\xi|^{1/2} + O(|\xi|), \qquad \xi \to 0;$$

$$U'(\xi) = 1/2\lambda|\xi|^{-1/2} sign(\xi) + O(1), \qquad \xi \to 0;$$

where $\lambda = \pm\sqrt{2|c - A|\sqrt{A^2 - cA}}$. Thus $U(\xi) \notin H^1_{loc}(\mathbb{R})$.
(3) *if $U(0) = c$ and $A = 0$, then u gives the regular peaked soliton $ce^{-|x-ct|}$.*

Proof (3) is obvious. Let us prove (1), a), b) and (2) in order.

(1) If $U(0) \neq c$, then $U(\xi) \neq c$ for any $\xi \in \mathbb{R}$ since $U(\xi) \in C^3(\mathbb{R} - \{0\})$. Differentiating both sides of (2.4) yields $U \in C^\infty(\mathbb{R})$.

a) When $cA < 0$, if $U(0) \neq c$, by (1) we know that U is smooth and $U'(0) = 0$. However, by (2.6) we must have $U(0) = A$, which contradicts the fact that 0 is the unique peak point.

b) When $cA \geqslant 0$, if $U(0) \neq c$, by (2.4) we know $U'(0)$ exists. So, $U'(0) = 0$ since 0 is a peak point. But, by (2.6) we obtain $U(0) = B_1$ or $U(0) = B_2$, since $U(0) = A$ contradicts the fact that 0 is the unique peak point.

(2) If $U(0) = c$ and $A \neq 0$, then by the definition of the single peak soliton we have $A \neq c$, thus $(U - c + A)^2 - cA$ doesn't contain the factor $U - c$. From (2.6) we obtain

$$U' = sign(c - A)\frac{U - A}{U - c}\sqrt{(U - c + A)^2 - cA sign(\xi)}. \tag{3.1}$$

Let $h(U) = \frac{sign(c-A)}{(U-A)\sqrt{(U-c+A)^2-cA}}$, then $h(c) = \frac{1}{|c-A|\sqrt{A^2-cA}}$, and

$$\int h(U)(U-c)dU = \int sign(\xi)d\xi. \tag{3.2}$$

Inserting $h(U) = h(c) + O(U-c)$ into (3.2) and using the initial condition $U(0) = c$, we obtain

$$\frac{h(c)}{2}(U-c)^2(1+O(U-c)) = |\xi|, \tag{3.3}$$

thus

$$U-c = \pm\sqrt{\frac{2}{h(c)}}|\xi|^{1/2}(1+O(U-c))^{-1/2} = \pm\sqrt{\frac{2}{h(c)}}|\xi|^{1/2}(1+O(U-c)) \tag{3.4}$$

which implies $U-c = O(|\xi|^{1/2})$. Therefore we have

$$U(\xi) = c \pm \sqrt{\frac{2}{h(c)}}|\xi|^{1/2} + O(|\xi|) = c + \lambda|\xi|^{1/2} + O(|\xi|), \qquad \xi \to 0,$$

$$\lambda = \pm\sqrt{\frac{2}{h(c)}} = \pm\sqrt{2|c-A|\sqrt{A^2-cA}},$$

and

$$U'(\xi) = (1/2)\lambda|\xi|^{-1/2}sign(\xi) + O(1), \qquad \xi \to 0.$$

So, $U \notin H^1_{loc}(\mathbb{R})$. □

Let us rewrite (2.6) in the following form

$$[U'(U-c)]^2 = (U-A)^2[(U-c+A)^2 - cA]. \tag{3.5}$$

Then, we have the following proposition.

Proposition 3.3 *If $u(x,t) = U(x-ct)$ is a single peak soliton for the DP equation* (1.2)*, then U must be a weak solution of* (3.5) *in the distribution sense. In this sense we say u is a weak solution for the DP equation* (1.2).

Proof By the asymptotic estimates in Lemma 3.2, we have $U'(U-c)$ is bounded, which implies $[U'(U-c)]^2 \in L^1_{loc}$, thus the left hand side of (3.5) does make sense. Since U is bounded, the right hand side of (3.5) is also in L^1_{loc}. Thus we may define the distribution function $L(U) = [U'(U-c)]^2 - (U-A)^2[(U-c+A)^2 - cA]$. By the definition condition (C2), we know that suppL$(U) \subset \{0\}$. Thus $L(U)$ must be a linear combination of Dirac function $\delta(\xi)$ and its derivatives. However the previous analysis shows $L(U) \in L^1_{loc}(\mathbb{R})$. Therefore $L(U) = 0$. □

If $u \in H^1$, the DP equation (1.2) can be equivalently cast into the following nonlocal conservation law form [1, 4]

$$\mathcal{L}(u) \equiv u_t + uu_x + \partial_x(1-\partial_x^2)^{-1}\left(\frac{3}{2}u^2\right) = 0. \tag{3.6}$$

However, if $u \notin H^1$, (3.6) is no longer equivalent to the DP equation (1.2). Actually, through a direct calculation we can verify that the following function (a stationary cusp soliton)

$$u(x,t) = \sqrt{1-e^{-2|x|}} \in W_{loc}^{1,1} \quad (\text{but} \notin H^1) \tag{3.7}$$

satisfies the DP equation (1.2) for any $x \neq 0$. However, the solution (3.7) does not solve the nonlocal conservation system (3.6) because

$$\mathcal{L}(u) = e^{-|x|}sign(x).$$

Therefore it is impossible to find the cusp soliton of the DP equation (1.2) through investigating (3.6). Therefore, we conclude this section with the following remark.

Remark 3.4 The nonlocal DP equation (3.6) is equivalent to the standard DP equation (1.2) under the H^1-norm instead of $W_{loc}^{1,1}$ norm. In fact, the regular peakon $u(x,t) = ce^{-|x-ct|} \in H^1$ (c is the wave speed), which is a stable solution of the CH equation [5], also satisfies both the nonlocal equation (3.6) and the DP equation (1.2). But $u(x,t) = \sqrt{1-e^{-2|x|}} \in W_{loc}^{1,1}$ is a solution of the DP equation (1.2) in the sense of our definition, but does not satisfy the nonlocal DP equation (3.6).

4 New Single Peak Solitons

Lemma 3.2 (3) gives a classification for all single peak soliton solutions for the DP equation (1.2). In this section we will present all possible single peak soliton solutions and find some explicit solution in the case of specific c and A.

We will discuss three cases: $cA = 0$, $cA > 0$ and $cA < 0$.

4.1 Case I: $cA = 0$

(1) If $A = 0$, then the only possible single peak soliton is the regular peakon soliton.

(2) If $A \neq 0$ and $c = 0$, there is the following stationary cusp soliton solution

$$u(x,t) = A\sqrt{1-e^{-2|x|}} \in W_{loc}^{1,1}.$$

4.2 Case II: $cA > 0$

By virtue of Lemma 3.2 any single peak soliton for the DP equation (1.2) must satisfy the following initial and boundary values problem (IBVP)

$$\begin{cases} (U')^2 = g(U) = \dfrac{(U-A)^2(U-B_1)(U-B_2)}{(U-c)^2}; \\ U(0) \in \{c, B_1, B_2\}; \\ \lim\limits_{|\xi|\to\infty} U(\xi) = A. \end{cases} \tag{4.1}$$

$g(U) \geqslant 0$ and the boundary condition (2.5) imply

$$U \geqslant B_1, \quad \text{or} \quad U \leqslant B_2,$$

and

$$(A-B_1)(A-B_2) \geqslant 0. \tag{4.2}$$

By (2.8), (4.2) is equivalent to

$$(c-A)(c-4A) \geqslant 0. \tag{4.3}$$

Since $A \neq 0$, introducing the constant $\alpha = c/A$ yields

$$(\alpha-1)(\alpha-4) \geqslant 0, \tag{4.4}$$

which implies:

$$0 < \alpha < 1; \qquad \alpha > 4; \qquad \alpha = 1; \qquad \alpha = 4.$$

From the standard phase analysis and Lemma 3.2 we know that if U is a single peak soliton of the DP equation, then

$$U' = -\frac{U-A}{U-c}(U-B_1)\sqrt{\frac{U-B_2}{U-B_1}}\,sign(\xi), \tag{4.5}$$

and

$$U(0) = \begin{cases} \max(c,\ B_1), & \text{if} \quad U(0) \text{ is a minimum}, \\ \min(c,\ B_2), & \text{if} \quad U(0) \text{ is a maximum}. \end{cases} \tag{4.6}$$

Let

$$h(U) = \frac{U-c}{(U-A)(U-B_1)}\sqrt{\frac{U-B_1}{U-B_2}}, \tag{4.7}$$

then taking the integration of both sides of (4.5) leads to

$$\int h(U)dU = -|\xi|.$$

After a lengthy calculation of integral, we obtain ($\alpha \neq 4$, i.e. $c \neq 4A$)

$$\int h(U)dU = sign(A-c)\mathrm{INT}_1(U) - \sqrt{\frac{c-A}{c-4A}}\mathrm{INT}_2(U) - K \equiv H(U) - K, \tag{4.8}$$

where

$$\mathrm{INT}_1(x) = \ln\left|\frac{B_1 + B_2}{2} - x - \sqrt{(x - B_1)(x - B_2)}\right|, \tag{4.9}$$

$$\mathrm{INT}_2(x) = \ln\left|\frac{(A - B_1)(x - B_2) + (A - B_2)(x - B_1) + 2\sqrt{(A - B_1)(A - B_2)}\sqrt{(x - B_1)(x - B_2)}}{x - A}\right|, \tag{4.10}$$

and K is an arbitrary integration constant. Thus we obtain the implicit solution U defined by

$$H(U) = -|\xi| + K, \tag{4.11}$$

where $H(U)$ is very complicated. But its derivative $h(U)$ is simple so that we may get all single peak soliton through our monotonicity analysis [18].

Apparently,

$$\mathrm{INT}_1(B_1) = \mathrm{INT}_1(B_2) = \ln\left|\frac{B_1 - B_2}{2}\right|, \qquad \mathrm{INT}_2(B_1) = \mathrm{INT}_2(B_2) = \ln|B_1 - B_2|.$$

So, for $U(0) = B_1$ or B_2, the constant $K_0 = H(U(0))$ is defined by

$$K_0 = -\left(sign(A) + \sqrt{\frac{c - A}{c - 4A}}\right)\ln\sqrt{cA} - \sqrt{\frac{c - A}{c - 4A}}\ln 2 \in \mathbb{R}, \tag{4.12}$$

and for $U(0) = c$,

$$K_0 = sign(A - c)\mathrm{INT}_1(c) - \sqrt{\frac{c - A}{c - 4A}}\mathrm{INT}_2(c) \in \mathbb{R}. \tag{4.13}$$

4.2.1 Case II.1: $0 < \alpha < 1$

1. If $A > 0$, then $B_2 < B_1 < c < A$ and $U \geqslant B_1$. By standard phase analysis, we have

$$U(0) = c, \qquad c < U < A,$$

and

$$H(U) = \mathrm{INT}_1(U) - \sqrt{\frac{c - A}{c - 4A}}\mathrm{INT}_2(U). \tag{4.14}$$

$H(U)$ is strictly decreasing on the interval $[c, A)$, thus

$$H_1(U) = H|_{[c,A)}(U) \tag{4.15}$$

will give an single peak soliton. Apparently,

$$H_1(c) = K_0, \qquad \lim_{U \to A} H_1(U) = -\infty. \tag{4.16}$$

Therefore $U_1(\xi) = H_1^{-1}(-|\xi| + K_0)$ is the solution satisfying

$$U_1(0) = c, \qquad \lim_{\xi \to \pm\infty} U_1(\xi) = A,$$

and

$$U_1'(\pm 0) = \pm\infty.$$

So, $U_1(\xi)$ is a kind of cusp soliton solution.

2. If $A < 0$, then $B_1 > B_2 > c > A$ and $U \leqslant B_2$. A similar analysis gives

$$H(U) = -\mathrm{INT}_1(U) - \sqrt{\frac{c-A}{c-4A}}\mathrm{INT}_2(U) \tag{4.17}$$

is strictly decreasing on the interval $(A, c]$. Thus

$$H_1(U) = H|_{(A,c]}(U) \tag{4.18}$$

has the inverse denoted by $U_1(\xi) = H_1^{-1}(-|\xi| + K_0)$. $U_1(\xi)$ gives a kind of cusp soliton solution satisfying

$$U_1(0) = c, \qquad \lim_{\xi\to\pm\infty} U_1(\xi) = A, \qquad U_1'(\pm 0) = \mp\infty.$$

4.2.2 Case II.2: $\alpha > 4$

1. If $A > 0$, then $c > 4A$ and

$$A < B_2 < c < B_1, \qquad U(0) = B_2, \qquad A < U \leqslant B_2.$$

Since $H(U)$ is strictly increasing on the interval $(A, B_2]$,

$$H_2(U) = H|_{(A,B_2]}(U) \tag{4.19}$$

gives a smooth soliton solution. Moreover, $U_2(\xi) = H_2^{-1}(-|\xi| + K_0)$ is the unique soliton satisfying the IBVP (4.1) with $U_2(0) = B_2$ and $U_2'(0) = 0$.

2. If $A < 0$, then $c < 4A$ and

$$B_2 < c < B_1 < A, \qquad U(0) = B_1, \qquad B_1 \leqslant U < A.$$

Through a similar analysis, we know that the strictly increasing on the interval $[B_1, A)$

$$H_2(U) = H|_{[B_1,A)}(U) \tag{4.20}$$

gives a smooth soliton solution $U_2(\xi) = H_2^{-1}(-|\xi| + K_0)$ satisfying the IBVP (4.1) with $U_2(0) = B_2$ and $U_2'(0) = 0$.

4.2.3 Case II.3: $\alpha = 1$

In this case $A = c$, (2.6) becomes

$$U' = -\sqrt{U^2 - A^2}\,sign(A)sign(\xi), \qquad U(\pm\infty) = A.$$

A direct calculation shows that there is no solution for the above boundary condition.

4.2.4 Case II.4: $\alpha = 4$

If $A > 0$, then $B_1 = 5A$, $B_2 = A$, $B_2 < c < B_1$ and if $A < 0$, then $B_1 = A$, $B_2 = 5A$, $B_2 < c < B_1$. For both subcases, there is no single peak soliton.

4.3 Case III: $cA < 0$

In this case, $U(0) = c$ (see Lemma 3.2). Let us separate two subcases to discuss: (1) $c < 0 < A$ and (2) $c < 0 < A$.

(1) If $A < 0 < c$, we have $c \leqslant U < A$ and

$$U' = -\frac{U - A}{U - c}\sqrt{(U - c + A)^2 - cA}sign(\xi). \tag{4.21}$$

Let

$$X = U - c + A, \qquad p = A, \qquad q = 2A - c, \qquad r = \sqrt{-cA},$$

then (4.21) becomes

$$f(X)dX \equiv \frac{X - p}{X - q}\frac{\mathrm{dX}}{\sqrt{X^2 + r^2}} = -sign(\xi)d\xi. \tag{4.22}$$

Integration of both sides of (4.22) gives

$$F(X) = -|\xi| + K \tag{4.23}$$

where

$$F(X) = \ln(X + \sqrt{X^2 - cA}) - \\ -\sqrt{\frac{(c-A)}{(c-4A)}}\left[\ln\left|\frac{(2A-c)X - cA + \sqrt{(c-A)(c-4A)}\sqrt{X^2 - cA}}{X + c - 2A}\right| + \ln 2\right] \tag{4.24}$$

$F(X)$ is strictly decreasing on the interval $[A, 2A - c)$ and $\lim_{X\to 2A-c} F(X) = -\infty$. Define

$$F_1(X) = F|_{[A,2A-c)}(X). \tag{4.25}$$

Then

$$F_1(X) = K_0 - |\xi|, \tag{4.26}$$

where

$$K_0 = F(X(0)) = F(A) = \ln(A + \sqrt{A^2 - cA}\,) + \\ + \frac{c - A}{\sqrt{(c - A)(c - 4A)}}\left[\ln(2A + \sqrt{4A^2 - cA}\,) + \ln 2\right] \in \mathbb{R}. \tag{4.27}$$

Since F_1 is a strictly decreasing function from $[A, 2A - c)$ onto $(-\infty, K_0]$ we can solve for X uniquely from (4.26) and obtain

$$U(\xi) = F_1^{-1}(K_0 - |\xi|) + c - A. \tag{4.28}$$

It is easy to check that U satisfies

$$U(0) = c, \qquad \lim_{|\xi|\to\infty} U(\xi) = A, \qquad U'(0+) = \infty, \qquad U'(0-) = -\infty.$$

Therefore, the solution U defined by (4.28) is a cusp soliton solution for the DP equation.

(2) If $A < 0 < c$, we have $A < U \leqslant c$ and

$$U' = \frac{U - A}{U - c}\sqrt{(U - c + A)^2 - cAsign(\xi)}. \tag{4.29}$$

Similarly, we get a strictly decreasing function $F(X)$ on the interval $(2A - c, A]$ satisfying:

$$F(X) = |\xi| + K \tag{4.30}$$

where $F(X)$ is defined by equation (4.24). Let

$$F_1(X) = F|_{(2A-c,A]}(X), \tag{4.31}$$

then F_1 is a strictly decreasing function from $(2A - c, A]$ onto $[K_0, \infty)$ so that we can solve for X and obtain

$$U(\xi) = F_1^{-1}(|\xi| + K_0) + c - A. \tag{4.32}$$

It is easy to check that U satisfies

$$U(0) = c, \qquad \lim_{|\xi|\to\infty} U(\xi) = A, \qquad U'(0+) = -\infty, \qquad U'(0-) = +\infty.$$

Therefore, the solution U defined by (4.32) is also a cusp soliton solution for the DP equation.

Let us summarize our results in the following theorem.

Theorem 4.1 *Assume that the single peak soliton $u(x, t) = U(x - ct)$ (without losing the generality, we assume 0 is the unique peak point of U) of the DP equation (1.2) satisfies the boundary condition (2.5). Then we have*

(1) *if $A = 0$, the single peak soliton $u(x, t)$ is only the following peakon*

$$u(x, t) = U(x - ct) = ce^{-|x-ct|},$$

with the properties:

$$U(0) = c, \qquad U(\pm\infty) = 0, \qquad U'(0+) = -c, \qquad U'(0-) = c;$$

(2) *if $A \neq 0$, let $\alpha = c/A$, then*

- (a) *if $1 \leqslant \alpha \leqslant 4$, there is no soliton for the DP equation (1.2);*
- (b) *if $\alpha < 0$ ($cA < 0$), the single peak soliton can be uniquely expressed as (see Figs. 1 and 2)*

$$u(x, t) = U(x - ct) = F_1^{-1}(K_0 - sign(A)|x - ct|),$$

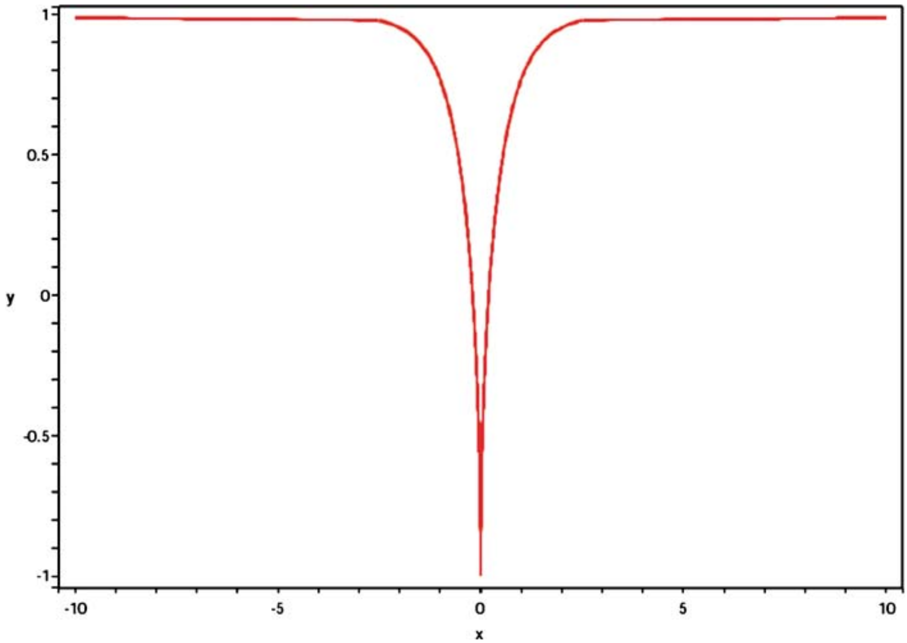

Fig. 1 2D graphic for a cusp soliton with $A = 1,\ c = -1$

with the property:

$$U(0) = c, \qquad U'(0+) = sign(A)\infty,$$
$$U(\pm\infty) = A, \qquad U'(0-) = -sign(A)\infty,$$

where F_1 and K_0 are defined by (4.25) *(if $A > 0$),* (4.31) *(if $A < 0$) and* (4.27) *respectively. In this case, the single peak soliton is a cusp soliton.*

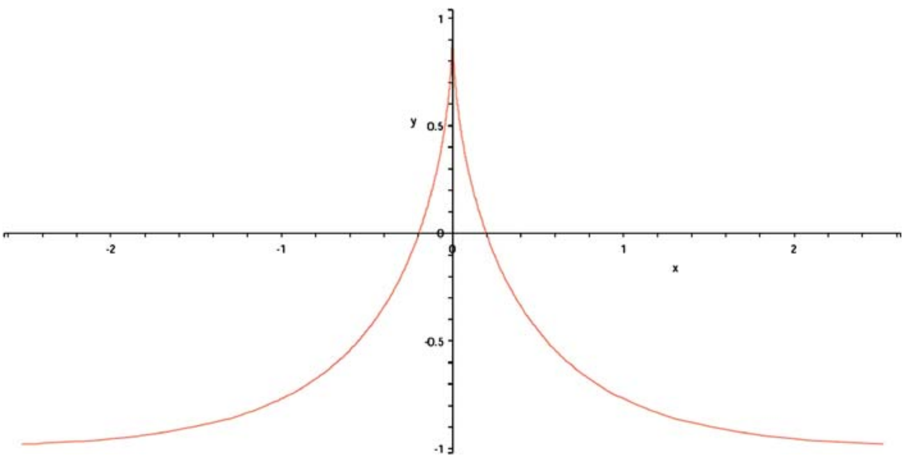

Fig. 2 2D graphic for a cusp soliton with $A = -1,\ c = 1$

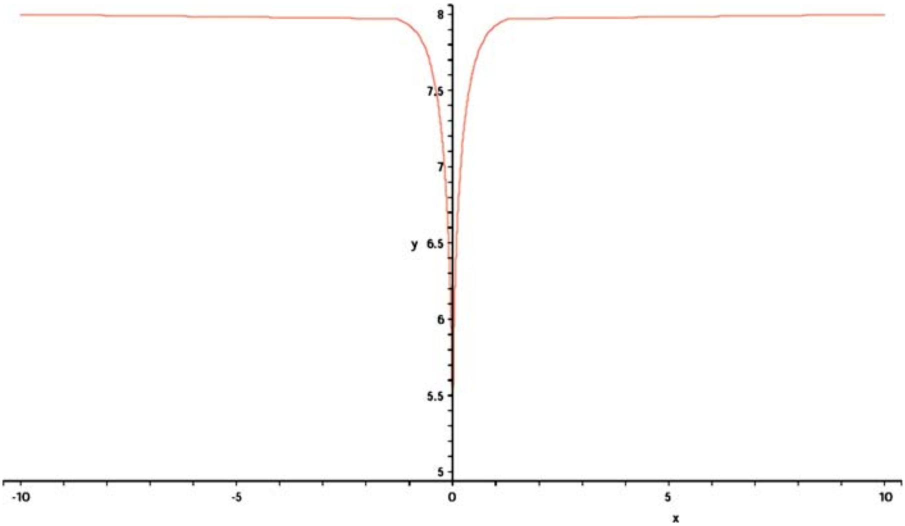

Fig. 3 2D graphic for a cusp soliton with $A = 8,\ c = 5$

(c) *if* $\alpha = 0$ $(c = 0)$*, there is the following stationary cusp soliton*

$$u(x,t) = A\sqrt{1 - e^{-2|x|}} \in W^{1,1}_{loc};$$

(d) *if* $0 < \alpha < 1$*, the single peak soliton (see Figs. 3 and 4) of the DP equation* (1.2) *can be uniquely expressed as*

$$u(x,t) = U(x - ct) = H_1^{-1}(-|x - ct| + K_0),$$

with the property:

$$U(0) = c, \qquad U'(0+) = sign(A)\infty,$$
$$U(\pm\infty) = A, \qquad U'(0-) = -sign(A)\infty,$$

where H_1 *and* K_0 *are defined by* (4.15) *(if* $A > 0$*),* (4.18) *(if* $A < 0$*) and* (4.13) *respectively. In this case, the single peak soliton is a cusp soliton.*

(e) *if* $\alpha > 4$*, the DP equation* (1.2) *has the following traveling soliary wave solutions (see Figs. 5 and 6)*

$$u(x,t) = U(x - ct) = H_2^{-1}(-|x - ct| + K_0)$$

with the properties:

$$U(0) = c - A + sign(A)\sqrt{cA}, \qquad U(\pm\infty) = A, \qquad U'(0) = 0,$$

where H_2 *and* K_0 *are defined by* (4.19)*(if* $A > 0$*),* (4.20)*(if* $A > 0$*) and* (4.12) *respectively. In this case, the soliton solution is smooth.*

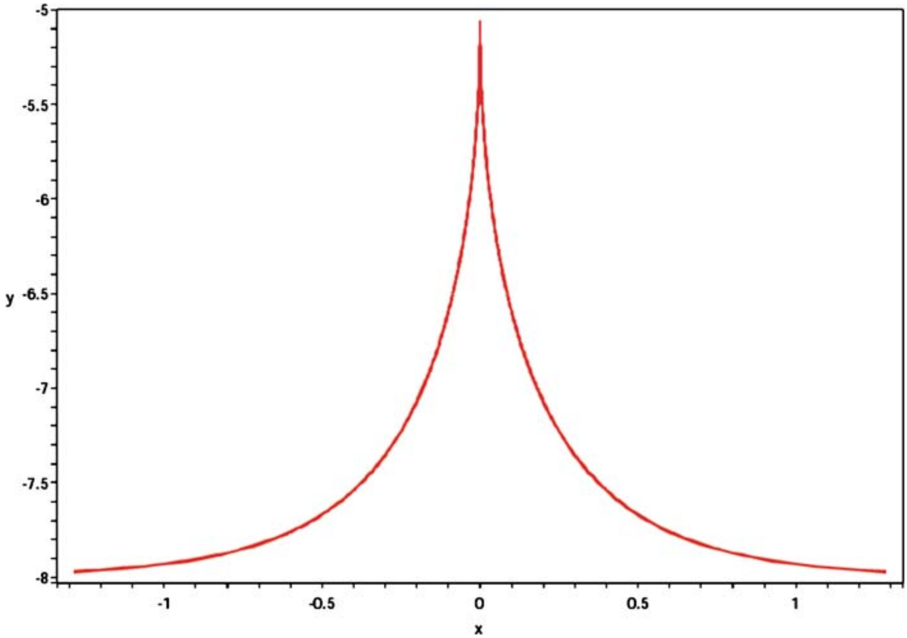

Fig. 4 2D graphic for a cusp soliton with $A = -8, \ c = -5$

5 Explicit Solutions

In the previous section we constructed all possible single peak soliton solutions in our main theorem (Theorem 4.1). But, usually it is very hard to find an explicit formula of the solution based on the implicit functions $H(U)$ we

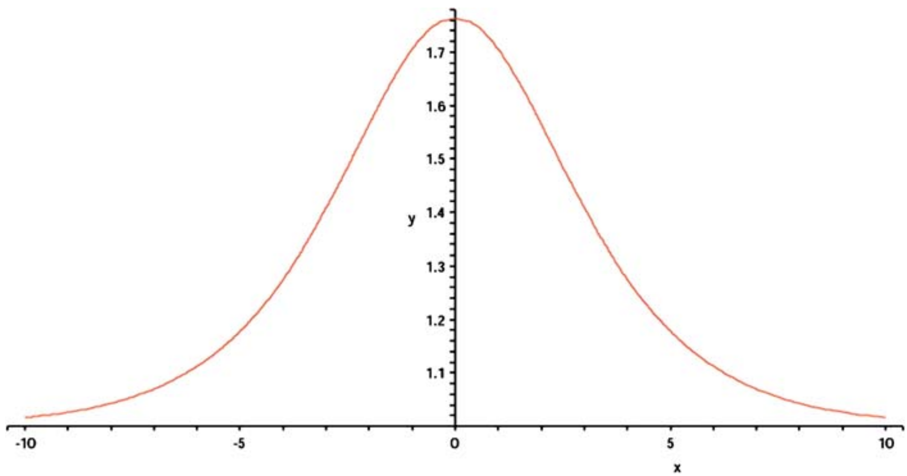

Fig. 5 2D graphic for a smooth soliton with $A = 1, \ c = 5$

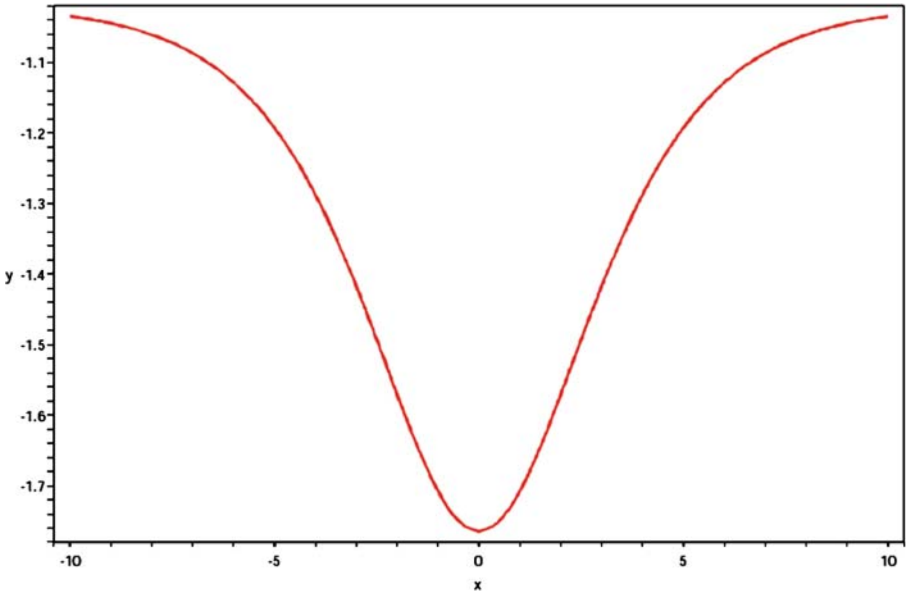

Fig. 6 2D graphic for a smooth soliton with $A = -1,\ c = -5$

obtained. However, in some specific cases, we do have explicit solutions. For this purpose, let us rewrite (4.5) as follows

$$\frac{U-c}{(U-A)(U-B_1)}\sqrt{\frac{U-B_1}{U-B_2}}dU = -sign(\xi)d\xi. \tag{5.1}$$

Let

$$X = \sqrt{\frac{U-B_1}{U-B_2}}, \qquad a = \sqrt{\frac{A-B_1}{A-B_2}},$$

then

$$U = B_2 - \frac{B_1-B_2}{X^2-1}, \qquad dU = \frac{2(B_1-B_2)XdX}{(X^2-1)^2},$$

and (5.1) is converted to

$$r\left(\frac{dX}{X-a} - \frac{dX}{X+a}\right) + \frac{dX}{X+1} - \frac{dX}{X-1} = -sign(\xi)d\xi, \tag{5.2}$$

where

$$r = \frac{c-A}{a(B_2-A)}.$$

Taking integration on both sides, we arrive at

$$\left|\frac{X-a}{X+a}\right|^r \left|\frac{X+1}{X-1}\right| = C_0 e^{-|\xi|}, \tag{5.3}$$

where C_0 is a positive constant.

In the following, we try to find explicit formulas of cusp soliton and smooth soliton for some specific number r.

Case 1 $0 < \alpha = c/A < 1$ In this case, the solution U is a cusp soliton.

1. If $A > 0$, then we have

$$0 < c < A, \qquad B_2 < B_1 < c < U < A, \qquad U(0) = c,$$

therefore

$$0 < X < a < 1.$$

Equation (5.3) becomes

$$\left(\frac{a-X}{a+X}\right)^r \left(\frac{1+X}{1-X}\right) = C_0 e^{-|\xi|}, \tag{5.4}$$

where

$$C_0 = \left(\frac{a-X(0)}{a+X(0)}\right)^r \left(\frac{1+X(0)}{1-X(0)}\right), \qquad X(0) = \sqrt{\frac{A-\sqrt{cA}}{A+\sqrt{cA}}}.$$

For general r, (5.4) is not algebraically solvable. But, a specific value of r may make (5.4) solvable for X. To this end let us write r as follows

$$r = \frac{A-c}{\sqrt{(c-A)(c-4A)}} = \sqrt{\frac{1-\alpha}{4-\alpha}}, \qquad 0 < \alpha < 1.$$

A simple analysis shows that the range of r is $0 < r < 1/2$.

2. If $A < 0$, a similar analysis yields

$$X > a > 1,$$

and

$$\left(\frac{X-a}{X+a}\right)^r \left(\frac{X+1}{X-1}\right) = C_0 e^{-|\xi|}, \tag{5.5}$$

where

$$C_0 = \left(\frac{X(0)-a}{X(0)+a}\right)^r \left(\frac{X(0)+1}{X(0)-1}\right), \qquad X(0) = \sqrt{\frac{A-\sqrt{cA}}{A+\sqrt{cA}}}$$

The range of r is $0 < r < 1/2$.

Notice that $r = 1/2$ corresponds $\alpha = 0$ $(c = 0)$. In this case, there is an explicit stationary cusp soliton

$$u(x,t) = A\sqrt{1-e^{-2|x|}}.$$

If taking $r = 1/3$, then we obtain

$$c = 5/8A, \qquad a = \frac{11-2\sqrt{10}sign(A)}{9}, \qquad X(0) = \frac{2\sqrt{6}+\sqrt{15}sign(A)}{3},$$

and

$$(X-a)(X+1)^3 = b(\xi)(X+a)(X-1)^3, \tag{5.6}$$

where

$$b(\xi) = C_0^3 e^{-3|\xi|}.$$

Equation (5.6) is able to be algebraically solvable. But that is a very complicated procedure. We omit it here.

Case 2 $\alpha = c/A > 4$ In this case, the solution U is a smooth soliton.

1. If $A > 0$, then we have

$$c > 4A > 0, \qquad A < U < B_2 < c < B_1, \qquad U(0) = B_2,$$
$$X > a > 1, \qquad X(0) = \infty,$$

and

$$\left(\frac{X-a}{X+a}\right)^r \left(\frac{X+1}{X-1}\right) = e^{-|\xi|}, \qquad r = \sqrt{\frac{\alpha-1}{\alpha-4}}, \qquad 1 < r < \infty. \tag{5.7}$$

Let us choose $r = 2$, then

$$c = 5A, \quad a = \frac{3+\sqrt{5}}{2},$$

and (5.7) becomes

$$(X-a)^2(X+1) = e^{-|\xi|}(X+a)^2(X-1). \tag{5.8}$$

Substituting $a = \frac{3+\sqrt{5}}{2}$ into (5.8), we obtain

$$X^3 - (2+\sqrt{5})bX^2 + \frac{1+\sqrt{5}}{2}X + \frac{7+3\sqrt{5}}{2}b = 0, \qquad b = \frac{1+e^{-|\xi|}}{1-e^{-|\xi|}}. \tag{5.9}$$

With the help of Maple it is easy to check (5.9) has the following real root

$$X(\xi)=\left[-\frac{7+3\sqrt{5}}{3}b+\right.$$
$$\left.+\frac{38+17\sqrt{5}}{27}b^3+\sqrt{\frac{2+\sqrt{5}}{27}+\frac{517+231\sqrt{5}}{54}b^2-\frac{521+233\sqrt{5}}{54}b^4}\right]^{1/3}+$$
$$+\left[-\frac{7+3\sqrt{5}}{3}b+\frac{38+17\sqrt{5}}{27}b^3-\right.$$
$$\left.-\sqrt{\frac{2+\sqrt{5}}{27}+\frac{517+231\sqrt{5}}{54}b^2-\frac{521+233\sqrt{5}}{54}b^4}\right]^{1/3}+\frac{2+\sqrt{5}}{3}b. \tag{5.10}$$

Hence, we obtain an explicit formula of the smooth soliton solution (see Fig. 7)

$$U(\xi)=A\left[(4-\sqrt{5})-\frac{2\sqrt{5}}{X(\xi)^2-1}\right],\qquad A>0, \tag{5.11}$$

where $X(\xi)$ is defined by (5.10).

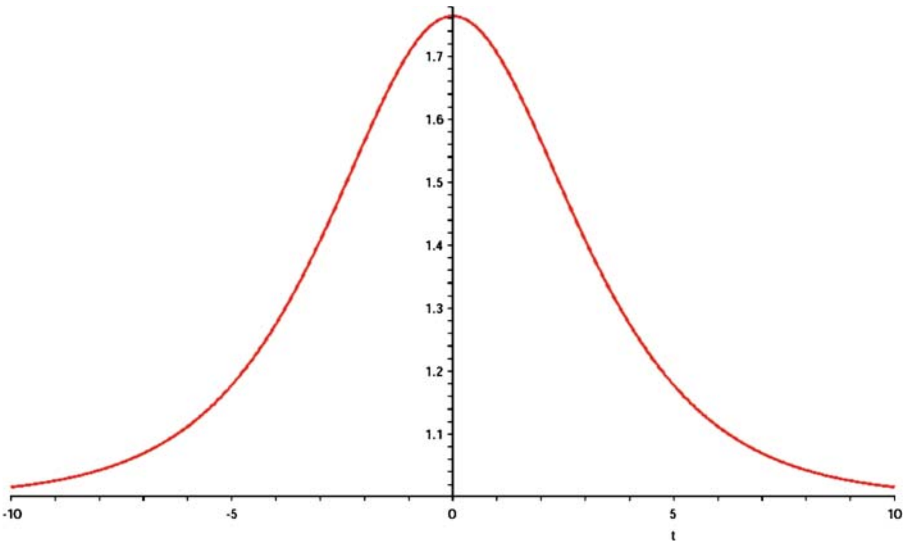

Fig. 7 2D graphic for a smooth soliton with $A=1,\ c=5$

2. If $A < 0$, then we have

$$c < 4A < 0, \qquad A > U > B_1 > c > B_2, \qquad U(0) = B_1,$$
$$0 < X < a < 1, \qquad X(0) = 0,$$

and

$$\left(\frac{a-X}{a+X}\right)^r \left(\frac{1+X}{1-X}\right) = e^{-|\xi|}, \qquad r = \sqrt{\frac{\alpha-1}{\alpha-4}}, \qquad 1 < r < \infty. \quad (5.12)$$

Let us take $r = 2$, then

$$c = 5A, \qquad a = \frac{3-\sqrt{5}}{2}.$$

In a similar way, we obtain

$$X^3 + (\sqrt{5}-2)bX^2 + \frac{1-\sqrt{5}}{2}X + \frac{7-3\sqrt{5}}{2}b = 0, \quad (5.13)$$

where

$$b = \frac{1-e^{-|\xi|}}{1+e^{-|\xi|}}. \quad (5.14)$$

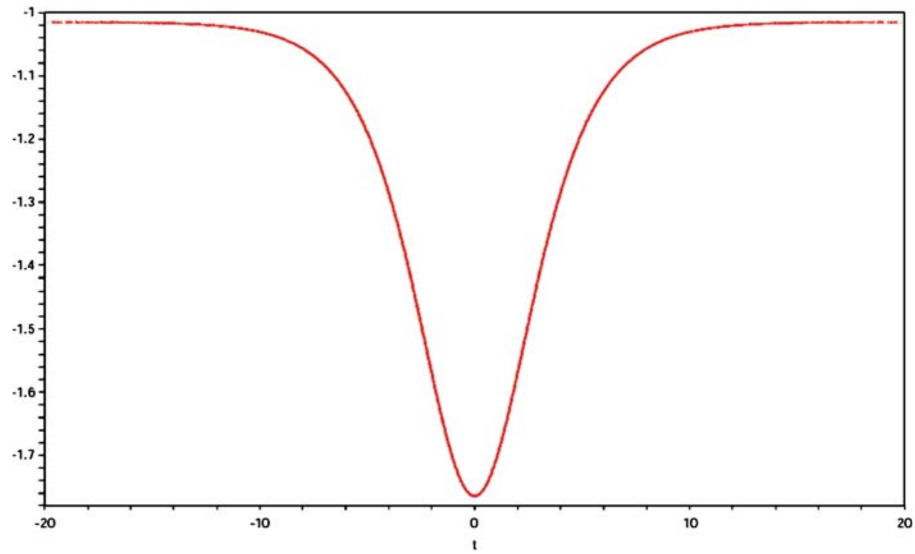

Fig. 8 2D graphic for a smooth soliton with $A = -1,\ c = -5$

Solving (5.13) leads to

$$X(\xi) = \omega^2 \left[-\frac{7-3\sqrt{5}}{3}b + \frac{38-17\sqrt{5}}{27}b^3 + \right. \\ \left. + \sqrt{\frac{2-\sqrt{5}}{27} + \frac{517-231\sqrt{5}}{54}b^2 - \frac{521-233\sqrt{5}}{54}b^4} \right]^{1/3} + \\ + \omega \left[-\frac{7-3\sqrt{5}}{3}b + \frac{38-17\sqrt{5}}{27}b^3 - \right. \\ \left. - \sqrt{\frac{2-\sqrt{5}}{27} + \frac{517-231\sqrt{5}}{54}b^2 - \frac{521-233\sqrt{5}}{54}b^4} \right]^{1/3} + \\ + \frac{2-\sqrt{5}}{3}b, \tag{5.15}$$

where b is defined by (5.14) and $\omega = \frac{-1+\sqrt{3}i}{2}$.

Thus we obtain another explicit form of smooth soliton solution (see Fig. 8)

$$U(\xi) = A\left[(4+\sqrt{5}) + \frac{2\sqrt{5}}{X(\xi)^2 - 1} \right], \qquad A < 0, \tag{5.16}$$

where $X(\xi)$ is defined by equation (5.15).

If we take $r = 3$, then

$$c = \frac{35}{8}A, \qquad a = \frac{19 + 2\sqrt{70}sign(A)}{9}.$$

We can repeat the above procedure to get explicit soliton solutions corresponding to $r = 3$. This is left for reader's practice.

6 Conclusions

In this paper, we investigate the DP equation under the inhomogeneous boundary condition. Through the traveling wave setting, the DP equation is converted to the ODE (2.6), which we solve for all possible single soliton solutions of the DP equation. Actually, the ODE (2.6) has a physical meaning and can be cast into the Newton equation $U'^2 = V(U) - V(A)$ of a particle with a new potential $V(U)$

$$V(U) = U^2 + \frac{4cA(c-A)}{U-c} + \frac{A(c+A)(c-A)^2}{(U-c)^2}.$$

In the paper, we successfully solve the Newton equation $U'^2 = V(U) - V(A)$ and give a single peak cusp [including a stationary cusp soliton, see (3.7)] and smooth soliton solutions in an explicit formula [see (5.11) and (5.16)]. Our smooth solutions (5.11) and (5.16) are orbitally stable, but we do not know if our new cuspon (4.32) and the cuspon, defined by equation (4.18), are stable. Very recently, we found a new integrable equation with no classical (smooth) soliton, only possessing weak solutions, such as cuspons and W/M-shape peak solitons, see the details in paper [16].

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References

1. Bressan, A., Constantin, A.: Global conservative solutions of the Camassa–Holm equation. Arch. Rational Mech. Anal. **183**, 215–239 (2007)
2. Camassa, R., Holm, D.D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661–1664 (1993)
3. Constantin, A., Gerdjikov, V., Ivanov, R.: Inverse scattering transform for the Camassa-Holm equation. Inverse Problems **22**, 2197–2207 (2006)
4. Constantin, A., Strauss, W.: Stability of the Camassa–Holm solitons. J. Nonlinear Sci. **12**, 415–522 (2002)
5. Constantin, A., Strauss, W.: Stability of peakons. Comm. Pure Appl. Math. **53**, 603–610 (2000)
6. Degasperis, A., Procesi, M.: Asymptotic integrability. In: Degasperis, A., Gaeta, G. (eds.) Symmetry and Perturbation Theory, World Scientific, pp. 23–37 (1999)
7. Degasperis, A., Holm, D.D., Hone, A.N.W.: A new integrable equation with peakon solutions. Theoret. and Math. Phys. **133**, 1463–1474 (2002)
8. Holm, D.D., Staley, M.F.: Nonlinear balance and exchange of stability in dynamics of solitons, peakons, ramps/cliffs and leftons in a 1+1 nonlinear evolutionary PDE. Phys. Lett. A **308**, 437–444 (2003)
9. Escher, J., Liu, Y., Yin, Z.: Global weak solutions and blow-up structure for the Degasperis–Procesi equation. J. Funct. Anal. **241**, 457–485 (2006)
10. Lenells, J.: Traveling wave solutions of the Degasperis–Procesi equation. J. Math. Anal. Appl. **306**, 72–82 (2005)
11. Mikhailov, A.V., Novikov, V.S.: Perturbative symmetry approach. J. Phys. A, Math. Gen. **35**, 4775–4790 (2002)
12. Boutet de Monvel, A., Shepelsky, D.: Riemann–Hilbert approach for the Camassa-Holm equation on the line. C. R. Math. Acad. Sci. Paris **343**, 627–632 (2006)
13. Qiao, Z.J.: Integrable hierarchy, 3×3 constrained systems, and parametric and stationary solutions. Acta Appl. Math. **83**, 199–220 (2004)
14. Qiao, Z.J.: The Camassa–Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold. Comm. Math. Phys. **239**, 309–341 (2003)
15. Qiao, Z.J.: Generalized r-matrix structure and algebro-geometric solutions for integrable systems. Rev. Math. Phys. **13**, 545–586 (2001)
16. Qiao, Z.J.: A new integrable equation with cuspons and W/M-shape-peaks solitons. J. Math. Phys. **47**, 112701–9 (2006)
17. Qiao, Z.J., Li, S.: A new integrable hierarchy, parametric solution, and traveling wave solution. Math. Phys. Anal. Geom. **7**, 289–308 (2004)
18. Qiao, Z.J., Zhang, G.: On peaked and smooth solitons for the Camassa–Holm equation. Europhys. Lett. **73**, 657–663 (2006)
19. Vakhnenko, V., Parkes, E.: Periodic and solitary-wave solutions of the Degasperis–Procesi equation. Chaos Solitons Fractals **20**, 1059–1073 (2004)