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On the Two Spectra Inverse Problem for Semi-infinite Jacobi Matrices

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Abstract We present results on the unique reconstruction of a semi-infinite Jacobi operator from the spectra of the operator with two different boundary conditions. This is the discrete analogue of the Borg–Marchenko theorem for Schrödinger operators on the half-line. Furthermore, we give necessary and sufficient conditions for two real sequences to be the spectra of a Jacobi operator with different boundary conditions.

Key words semi-infinite Jacobi matrices · two-spectra inverse problem

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1 Introduction

In the Hilbert space $l_2(\mathbb{N})$ let us single out the dense subset $l_{fin}(\mathbb{N})$ of sequences which have a finite number of non-zero elements. Consider the

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operator J defined for every $f = \{f_k\}_{k=1}^{\infty}$ in $l_{fin}(\mathbb{N})$ by means of the recurrence relation

$$(Jf)_k := b_{k-1} f_{k-1} + q_k f_k + b_k f_{k+1} \qquad k \in \mathbb{N} \setminus \{1\} \tag{1.1}$$

$$(Jf)_1 := q_1 f_1 + b_1 f_2 \,, \tag{1.2}$$

where, for every $n \in \mathbb{N}$, b_n is positive, while q_n is real. J is symmetric, therefore closable, and in the sequel we shall consider the closure of J and denote it by the same letter.

Notice that we have defined the Jacobi operator J in such a way that

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} \tag{1.3}$$

is the matrix representation of J with respect to the canonical basis in $l_2(\mathbb{N})$ (we refer the reader to [2] for a discussion on matrix representation of unbounded symmetric operators).

It is known that the symmetric operator J has deficiency indices $(1, 1)$ or $(0, 0)$ [1, Chap. 4, Sec. 1.2] and [23, Corollary 2.9]. In the case $(1, 1)$ we can always define a linear set $D(g) \subset \operatorname{dom}(J^*)$ parametrized by $g \in \mathbb{R} \cup \{+\infty\}$ such that

$$J^* \upharpoonright D(g) =: J(g)$$

is a self-adjoint extension of J. Moreover, for any self-adjoint extension (von Neumann extension) $\widetilde{J}$ of J, there exists a $\widetilde{g} \in \mathbb{R} \cup \{+\infty\}$ such that

$$J(\widetilde{g}) = \widetilde{J}\,,$$

[25, Lemma 2.20]. We shall show later (see the Appendix) that g defines a boundary condition at infinity.

To simplify the notation, even in the case of deficiency indices $(0, 0)$, we shall use $J(g)$ to denote the operator $J = J^*$. Thus, throughout the paper $J(g)$ stands either for a self-adjoint extension of the nonself-adjoint operator J, uniquely determined by g, or for the self-adjoint operator J.

In what follows we shall consider the inverse spectral problem for the self-adjoint operator $J(g)$.

It turns out that if $J \neq J^*$ (the case of indices $(1, 1)$), then for all $g \in \mathbb{R} \cup \{+\infty\}$ the Jacobi operator $J(g)$ has discrete spectrum with eigenvalues of multiplicity one, i. e., the spectrum consists of eigenvalues of multiplicity one that can accumulate only at $\pm\infty$, [25, Lemma 2.19]. Throughout this work we shall always require that the spectrum of $J(g)$, denoted $\sigma(J(g))$, be discrete, which is not an empty assumption only for the case $J(g) = J$. Notice that the discreteness of $\sigma(J(g))$ implies that $J(g)$ has to be unbounded.

For the Jacobi operators $J(g)$ one can define boundary conditions at the origin in complete analogy to those of the half-line Sturm–Liouville operator (see the Appendix). Different boundary conditions at the origin define different self-adjoint operators $J_h(g)$, $h \in \mathbb{R} \cup \{+\infty\}$. $J_0(g)$ corresponds to the Dirichlet boundary condition, while the operator $J_\infty(g)$ has Neumann boundary condition. If $J(g)$ has discrete spectrum, the same is true for $J_h(g)$, $\forall h \in \mathbb{R} \cup \{+\infty\}$ (for the case of h finite see Section 2 and for $h = \infty$, Section 4).

In this work we prove that a Jacobi operator $J(g)$ with discrete spectrum is uniquely determined by $\sigma(J_{h_1}(g))$, $\sigma(J_{h_2}(g))$, with $h_1, h_2 \in \mathbb{R}$ and $h_1 \neq h_2$, and either h_1 or h_2. If h_1, respectively, h_2 is given, the reconstruction method also gives h_2, respectively, h_1. Saying that $J(g)$ is determined means that we can recover the matrix (1.3) and the boundary condition g at infinity, in the case of deficiency indices $(1, 1)$. We will also establish (the precise statement is in Theorem 3.2) that if two infinite real sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ that can accumulate only at $\pm\infty$ satisfy

(a) $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ interlace, i. e., between two elements of a sequence there is one and only one element of the other. Thus, we assume below that $\lambda_k < \mu_k < \lambda_{k+1}$.

(b) The series $\sum_k (\mu_k - \lambda_k)$ converges, so

$$\sum_k (\mu_k - \lambda_k) =: \Delta < \infty\,.$$

By (b) the product $\prod_{k \neq n} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}$ is convergent, so define

$$\tau_n^{-1} := \frac{\mu_n - \lambda_n}{\Delta} \prod_{k \neq n} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}\,.$$

(c) The sequence $\{\tau_n\}_n$ is such that, for $m = 0, 1, 2, \ldots,$

$$\sum_k \frac{\lambda_k^{2m}}{\tau_k} \quad \text{converges.}$$

(d) For a sequence of complex numbers $\{\beta_k\}_k$, such that the series

$$\sum_k \frac{|\beta_k|^2}{\tau_k} \quad \text{converges}$$

and

$$\sum_k \frac{\beta_k \lambda_k^m}{\tau_k} = 0, \qquad m = 0, 1, 2, \ldots$$

it must hold true that $\beta_k = 0$ for all k.

Then, for any real number h_1, there exists a unique Jacobi operator J, a unique $h_2 > h_1$, and if $J \neq J^*$, a unique $g \in \mathbb{R} \cup \{+\infty\}$, such that $\sigma(J_{h_2}(g)) = \{\lambda_k\}_k$ and $\sigma(J_{h_1}(g)) = \{\mu_k\}_k$. Moreover, we show that if the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are the spectra of a Jacobi operator $J(g)$ with two different boundary conditions $h_1 < h_2$ ($h_2 \in \mathbb{R}$), then (a), (b), (c), (d) hold for $\Delta = h_2 - h_1$.

Necessary and sufficient conditions for two sequences to be the spectra of a Jacobi operator $J(g)$ with Dirichlet and Neumann boundary conditions are also given. Conditions (b) and (c) differ in this case (see Section 4).

Our necessary and sufficient conditions give a characterization of the spectral data for our two spectra inverse problem.

Our proofs are constructive and they give a method for the unique reconstruction of the operator J, the boundary condition at infinity, g, and either h_1 or h_2.

The two-spectra inverse problem for Jacobi matrices has also been studied in several papers [10, 14, 15, 24]. There are also results on this problem in [9]. We shall comment on these results in the following sections.

The problem that we solve here is the discrete analogue of the two-spectra inverse problem for Sturm–Liouville operators on the half-line. The classical result is the celebrated Borg–Marchenko theorem [6, 20]. Let us briefly explain this result. Consider the self-adjoint Schrödinger operator,

$$\mathcal{B}f = -f''(x) + Q(x)f(x), \qquad x \in \mathbb{R}_+ , \tag{1.4}$$

where $Q(x)$ is real-valued and locally integrable on $[0, \infty)$, and the following boundary condition at zero is satisfied,

$$\cos\alpha f(0) + \sin\alpha f'(0) = 0, \qquad \alpha \in [0, \pi).$$

Moreover, the boundary condition at infinity, if any, is considered fixed. Suppose that the spectrum is discrete for one (and then for all) α, and denote by $\{\lambda_k(\alpha)\}_{k\in\mathbb{N}}$ the corresponding eigenvalues.

The Borg–Marchenko theorem asserts that the sets $\{\lambda_k(\alpha_1)\}_{k\in\mathbb{N}}$ and $\{\lambda_k(\alpha_2)\}_{k\in\mathbb{N}}$ for some $\alpha_1 \neq \alpha_2$ uniquely determine α_1, α_2, and Q. Thus, the differential expression and the boundary conditions are determined by two spectra. Other results here are the necessary and sufficient conditions for a pair of sequences to be the eigenvalues of a Sturm–Liouville equation with different boundary conditions found by Levitan and Gasymov in [19].

Other settings for two-spectra inverse problems can be found in [3, 4, 11]. A resonance inverse problem for Jacobi matrices is considered in [7]. Recent local Borg–Marchenko results for Schrödinger operators and Jacobi matrices [13, 27] are also related to the problem we discuss here.

Jacobi matrices appear in several fields of quantum mechanics and condensed matter physics (see for example [8]).

The paper is organized as follows. In Section 2 we present some preliminary results that we need. In Section 3 we prove our results of uniqueness, reconstruction, and necessary and sufficient conditions (characterization) in the case where h_1 and h_2 are real numbers. In Section 4 we obtain similar results for the Dirichlet and Neumann boundary conditions. Finally, in the Appendix we briefly describe – for the reader's convenience – how the boundary conditions are interpreted when J is considered as a difference operator.

2 Preliminaries

Let us denote by γ the second order symmetric difference expression (see (1.1), (1.2)) such that $\gamma : f = \{f_k\}_{k\in\mathbb{N}} \mapsto \{(\gamma f)_k\}_{k\in\mathbb{N}}$, by

$$(\gamma f)_k := b_{k-1} f_{k-1} + q_k f_k + b_k f_{k+1}, \qquad k \in \mathbb{N} \setminus \{1\}, \tag{2.1}$$

$$(\gamma f)_1 := q_1 f_1 + b_1 f_2 \,. \tag{2.2}$$

Then, it is proven in Section 1.1, Chapter 4 of [1] and in Theorem 2.7 of [23] that

$$\mathrm{dom}(J^*) = \{f \in l_2(\mathbb{N}) : \gamma f \in l_2(\mathbb{N})\}, \qquad J^* f = \gamma f, \qquad f \in \mathrm{dom}(J^*).$$

The solution of the difference equation,

$$(\gamma f) = \zeta f, \qquad \zeta \in \mathbb{C}\,, \tag{2.3}$$

is uniquely determined if one gives $f_1 = 1$. For the elements of this solution the following notation is standard [1, Chap. 1, Sec. 2.1]

$$P_{n-1}(\zeta) := f_n, \qquad n \in \mathbb{N}\,,$$

where the polynomial $P_k(\zeta)$ (of degree k) is referred to as the kth orthogonal polynomial of the first kind associated with the matrix (1.3).

The sequence $\{P_k(\zeta)\}_{k=0}^{\infty}$ is not in $l_{fin}(\mathbb{N})$ but it may happen that

$$\sum_{k=0}^{\infty} |P_k(\zeta)|^2 < \infty\,, \tag{2.4}$$

in which case ζ is an eigenvalue of J^* and $f(\zeta)$ the corresponding eigenvector. Since the eigenspace is always one-dimensional, the eigenvalue of J^* is of multiplicity one . Moreover, since the (von Neumann) self-adjoint extensions of J, $J(g)$, are restrictions of J^*, it follows that the point spectrum of $J(g)$, $g \in \mathbb{R} \cup \{+\infty\}$, has multiplicity one.

The polynomials of the second kind $\{Q_k(\zeta)\}_{k=0}^{\infty}$ associated with the matrix (1.3) are defined as the solutions of

$$b_{k-1} f_{k-1} + q_k f_k + b_k f_{k+1} = \zeta f_k, \qquad k \in \mathbb{N} \setminus \{1\},$$

under the assumption that $f_1 = 0$ and $f_2 = b_1^{-1}$. Then

$$Q_{n-1}(\zeta) := f_n, \qquad n \in \mathbb{N}\,.$$

$Q_k(\zeta)$ is a polynomial of degree $k-1$.

By construction the Jacobi operator J is a closed symmetric operator. It is well known, [1, Chap. 4, Sec. 1.2] and [23, Corollary 2.9], that this operator has either deficiency indices (1, 1) or (0, 0). In terms of the polynomials of the first kind, J has deficiency indices (1, 1) when

$$\sum_{k=0}^{\infty} |P_k(\zeta)|^2 < \infty, \qquad \text{for } \zeta \in \mathbb{C} \setminus \mathbb{R}$$

(this holds for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$ if and only if it holds for one $\zeta \in \mathbb{C} \setminus \mathbb{R}$), and deficiency indices $(0, 0)$ otherwise. Since J is closed, deficiency indices $(0, 0)$ mean that $J = J^*$. The symmetric operator J with deficiency indices $(1, 1)$ has always self-adjoint extensions, which are restrictions of J^*. When studying the self-adjoint extensions of J in a more general context the self-adjoint restrictions of J^* are called von Neumann self-adjoint extensions of J [2, 23]. All self-adjoint extensions considered in this paper are von Neumann.

Let us now introduce a convenient way of parametrizing the self-adjoint extensions of J in the nonself-adjoint case. We first define the Wronskian associated with J for any pair of sequences $\varphi = \{\varphi_k\}_{k=1}^{\infty}$ and $\psi = \{\psi_k\}_{k=1}^{\infty}$ in $l_2(\mathbb{N})$ as follows

$$W_k(\varphi, \psi) := b_k(\varphi_k \psi_{k+1} - \psi_k \varphi_{k+1}), \qquad k \in \mathbb{N}\,.$$

Now, consider the sequences $v(g) = \{v_k(g)\}_{k=1}^{\infty}$ such that $\forall k \in \mathbb{N}$

$$v_k(g) := P_{k-1}(0) + g Q_{k-1}(0), \qquad g \in \mathbb{R} \tag{2.5}$$

and

$$v_k(+\infty) := Q_{k-1}(0)\,. \tag{2.6}$$

All the self-adjoint extensions $J(g)$ of the nonself-adjoint operator J are restrictions of J^* to the set [25, Lemma 2.20]

$$\begin{aligned} D(g) &:= \left\{ f = \{f_k\}_{k=1}^{\infty} \in \operatorname{dom}(J^*) : \lim_{n\to\infty} W_n\big(v(g), f\big) = 0 \right\} \\ &= \left\{ f \in l_2(\mathbb{N}) : \gamma f \in l_2(\mathbb{N}),\ \lim_{n\to\infty} W_n\big(v(g), f\big) = 0 \right\}. \end{aligned} \tag{2.7}$$

Different values of g imply different self-adjoint extensions. If J is self-adjoint, we define $J(g) := J$, for all $g \in \mathbb{R} \cup \{+\infty\}$; otherwise $J(g)$ is a self-adjoint extension of J uniquely determined by g. We have defined the domains $D(g)$ in such a way that g defines a boundary condition at infinity (see the Appendix).

It is worth mentioning that if $J \neq J^*$ then, for all $g \in \mathbb{R} \cup \{+\infty\}$, $J(g)$ has discrete spectrum. This follows from the fact that the resolvent of $J(g)$ turns out to be a Hilbert–Schmidt operator [25, Lemma 2.19].

Let us now define the self-adjoint operator $J_h(g)$ by

$$J_h(g) := J(g) - h\langle \cdot, e_1 \rangle e_1, \qquad h \in \mathbb{R}\,,$$

where $\{e_k\}_{k=1}^{\infty}$ is the canonical basis in $l_2(\mathbb{N})$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in this space. Clearly, $J_0(g) = J(g)$.

We define $J_\infty(g)$ as follows. First consider the space $l_2\big((2, \infty)\big)$ of square summable sequences $\{f_n\}_{n=2}^{\infty}$ and the sequence $v(g) = \{v_k(g)\}_{k=1}^{\infty}$ given by (2.5) and (2.6) with g fixed. Let us denote $J_\infty(g)$ the operator in $l_2\big((2, \infty)\big)$ such that

$$J_\infty(g) f = \gamma f\,,$$

where $(\gamma f)_k$ is considered for any $k \geqslant 2$ and $f_1 = 0$ in the definition of $(\gamma f)_2$, with domain given by

$$\operatorname{dom}(J_\infty(g)) := \left\{ f \in l_2\big((2, \infty)\big) : \gamma f \in l_2\big((2, \infty)\big),\ \lim_{n\to\infty} W_n\big(v(g), f\big) = 0 \right\}.$$

Clearly, the matrix

$$\begin{pmatrix} q_2 & b_2 & 0 & 0 & \cdots \\ b_2 & q_3 & b_3 & 0 & \cdots \\ 0 & b_3 & q_4 & b_4 & \\ 0 & 0 & b_4 & q_5 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix},$$

which is our original matrix (1.3) with the first column and row removed, is the matrix representation of $J_\infty(g)$ with respect to the canonical basis in $l_2\big((2,\infty)\big)$.

It follows easily from the definition of $J_h(g)$ that if $J(g)$ has discrete spectrum, the same is true for $J_h(g)$ ($h \in \mathbb{R} \cup \{+\infty\}$). Indeed, for $h \in \mathbb{R}$ this is a consequence of the invariance of the essential spectrum – that is empty in our case – under a compact perturbation [22]. We shall show in Section 4 that it is also true that $J_\infty(g)$ has discrete spectrum provided that $\sigma(J(g))$ is discrete.

For the self-adjoint operator $J_h(g)$, we can introduce the right-continuous resolution of the identity $E_{J_h(g)}(t)$, such that

$$J_h(g) = \int_{\mathbb{R}} t dE_{J_h(g)}(t)\,.$$

Let us define the function $\rho(t)$ as follows:

$$\rho(t) := \langle E_{J_h(g)}(t)e_1, e_1\rangle, \qquad t \in \mathbb{R}\,. \tag{2.8}$$

Consider the function (see [24] and [25, Chap. 2, Sec. 2.1])

$$m_h(\zeta, g) := \langle (J_h(g) - \zeta I)^{-1} e_1, e_1\rangle, \qquad \zeta \notin \sigma(J_h(g))\,. \tag{2.9}$$

$m_h(\zeta, g)$ is called the Weyl m-function of $J_h(g)$. We shall use below the simplified notation $m(\zeta, g) := m_0(\zeta, g)$. The functions $\rho(t)$ and $m_h(\zeta, g)$ are related by the Stieltjes transform (also called Borel transform):

$$m_h(\zeta, g) = \int_{\mathbb{R}} \frac{d\rho(t)}{t - \zeta}\,.$$

It follows from the definition that the Weyl m-function is a Herglotz function,i. e.,

$$\frac{\operatorname{Im} m_h(\zeta, g)}{\operatorname{Im} \zeta} > 0, \qquad \operatorname{Im} \zeta > 0\,.$$

Using the Neumann expansion for the resolvent (cf. [25, Chap. 6, Sec. 6.1])

$$(J_h(g) - \zeta I)^{-1} = -\sum_{k=0}^{N-1} \frac{(J_h(g))^k}{\zeta^{k+1}} + \frac{(J_h(g))^N}{\zeta^N}(J_h(g) - \zeta I)^{-1}\,,$$

where $\zeta \in \mathbb{C} \setminus \sigma(J(g))$, one can easily obtain the following asymptotic formula

$$m_h(\zeta, g) = -\frac{1}{\zeta} - \frac{q_1 - h}{\zeta^2} - \frac{b_1^2 + (q_1 - h)^2}{\zeta^3} + O(\zeta^{-4}), \tag{2.10}$$

as $\zeta \to \infty$ ($\operatorname{Im}\zeta \geqslant \epsilon, \epsilon > 0$).

An important result in the theory of Jacobi operators is the fact that $m(\zeta, g)$ completely determines $J(g)$ (the same is of course true for the pair $m_h(\zeta, g)$ and $J_h(g)$). There are two ways for recovering the operator from the Weyl m-function. One way consists in obtaining first $\rho(t)$ from $m(\zeta, g)$ by means of the inverse Stieltjes transform (cf. [25, Appendix B]), namely,

$$\rho(b) - \rho(a) = \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{a+\delta}^{b+\delta} (\operatorname{Im} m(x + \mathrm{i}\epsilon, g))\, dx\,.$$

The function ρ is such that all the moments of the corresponding measure are finite [1, 23]. Hence, all the elements of the sequence $\{t^k\}_{k=0}^{\infty}$ are in $L_2(\mathbb{R}, d\rho)$ and one can apply, in this Hilbert space, the Gram–Schmidt procedure of orthonormalization to the sequence $\{t^k\}_{k=0}^{\infty}$. One, thus, obtains a sequence of polynomials $\{P_k(t)\}_{k=0}^{\infty}$ normalized and orthogonal in $L_2(\mathbb{R}, d\rho)$. These polynomials satisfy a three term recurrence equation [23]

$$tP_{k-1}(t) = b_{k-1}P_{k-2}(t) + q_k P_{k-1}(t) + b_k P_k(t) \qquad k \in \mathbb{N} \setminus \{1\} \tag{2.11}$$

$$tP_0(t) = q_1 P_0(t) + b_1 P_1(t)\,, \tag{2.12}$$

where all the coefficients b_k ($k \in \mathbb{N}$) turn out to be positive and q_k ($k \in \mathbb{N}$) are real numbers. The system (2.11) and (2.12) defines a matrix which is the matrix representation of J. We shall refer to this procedure for recovering J as the method of orthogonal polynomials. The other method for determining J from $m(\zeta, g)$ was developed in [12] (see also [24]). It is based on the asymptotic behavior of $m(\zeta, g)$ and the Ricatti equation [24],

$$b_n^2 m^{(n)}(\zeta, g) = q_n - \zeta - \frac{1}{m^{(n-1)}(\zeta, g)}, \qquad n \in \mathbb{N}\,, \tag{2.13}$$

where $m^{(n)}(\zeta, g)$ is the Weyl m-function of the Jacobi operator associated with the matrix (1.3) with the first n columns and n rows removed.

After obtaining the matrix representation of J, one can easily obtain the boundary condition at infinity which defines the domain of $J(g)$ in the nonselfadjoint case. Indeed, take an eigenvalue, λ, of $J(g)$, i.e., λ is a pole of $m(\zeta, g)$. Since the corresponding eigenvector $f(\lambda) = \{f_k(\lambda)\}_{k=1}^{\infty}$ is in $\operatorname{dom}(J(g))$, it must be that

$$\lim_{n\to\infty} W_n\big(v(g), f(\lambda)\big) = 0\,.$$

This implies that either $\lim_{n\to\infty} W_n\big(\{Q_{k-1}(0)\}_{k=1}^{\infty}, f(\lambda)\big) = 0$, which means that $g = +\infty$, or

$$g = -\frac{\lim_{n\to\infty} W_n\big(\{P_{k-1}(0)\}_{k=1}^{\infty}, f(\lambda)\big)}{\lim_{n\to\infty} W_n\big(\{Q_{k-1}(0)\}_{k=1}^{\infty}, f(\lambda)\big)}.$$

If the spectrum of $J_h(g)$ is discrete, say $\sigma(J_h(g)) = \{\lambda_k\}_k$, the function $\rho(t)$ defined by (2.8) can be written as follows

$$\rho(t) = \sum_{\lambda_k \leqslant t} \frac{1}{\alpha_k},$$

where the coefficients $\{\alpha_k\}_k$ are called the normalizing constants and are given by

$$\alpha_n = \sum_{k=0}^{\infty} |P_k(\lambda_n)|^2 . \tag{2.14}$$

Thus, $\sqrt{\alpha_n}$ equals the l_2 norm of the eigenvector $f(\lambda_n) := \{P_k(\lambda_n)\}_{k=0}^{\infty}$ corresponding to λ_n. The eigenvector $f(\lambda_n)$ is normalized in such a way that $f_1(\lambda_n) = 1$.

Clearly,

$$1 = \langle e_1, e_1 \rangle = \int_{\mathbb{R}} d\rho = \sum_k \frac{1}{\alpha_k}. \tag{2.15}$$

The Weyl m-function in this case is given by

$$m_h(\zeta, g) = \sum_k \frac{1}{\alpha_k(\lambda_k - \zeta)}. \tag{2.16}$$

From this we have that

$$(\lambda_n - \zeta) m_h(\zeta, g) = (\lambda_n - \zeta) \sum_k \frac{1}{\alpha_k(\lambda_k - \zeta)} = \sum_{k \neq n} \frac{\lambda_n - \zeta}{\alpha_k(\lambda_k - \zeta)} + \frac{1}{\alpha_n}.$$

Therefore,

$$\alpha_n^{-1} = \lim_{\zeta \to \lambda_n} (\lambda_n - \zeta) m(\zeta, g) = -\operatorname*{Res}_{\zeta = \lambda_n} m(\zeta, g). \tag{2.17}$$

Let us now introduce an appropriate way for enumerating sequences that we shall use. Consider a pair of infinite real sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ that have no finite accumulation points and that interlace, i. e., between two elements of one sequence there is one and only one element of the other. We use M, a subset of $\mathbb{Z}$ to be defined below, for enumerating the sequences as follows

$$\forall k \in M \qquad \lambda_k < \mu_k < \lambda_{k+1}, \tag{2.18}$$

where

(a) If $\inf_k\{\lambda_k\}_k = -\infty$ and $\sup_k\{\lambda_k\}_k = \infty$,

$$M := \mathbb{Z} \quad \text{and we require} \quad \mu_{-1} < 0 < \lambda_1 . \tag{2.19}$$

(b) If $0 < \sup_k\{\lambda_k\}_k < \infty$,

$$M := \{k\}_{k=-\infty}^{k_{\max}}\,, \quad (k_{\max} \geqslant 1) \quad \text{and we require} \quad \mu_{-1} < 0 < \lambda_1\,. \tag{2.20}$$

(c) If $\sup_k\{\lambda_k\}_k \leqslant 0$,

$$M := \{k\}_{k=-\infty}^{0}\,. \tag{2.21}$$

(d) If $\inf_k\{\mu_k\}_k \geqslant 0$,

$$M := \{k\}_{k=0}^{\infty}\,. \tag{2.22}$$

(e) If $-\infty < \inf_k\{\mu_k\}_k < 0$,

$$M := \{k\}_{k=k_{\min}}^{\infty}\,, \qquad (k_{\min} \leqslant -1) \qquad \text{and we require} \quad \mu_{-1} < 0 < \lambda_1\,. \tag{2.23}$$

Notice that, by this convention for enumeration, the only elements of $\{\lambda_k\}_{k\in M}$ and $\{\mu_k\}_{k\in M}$ allowed to be zero are λ_0 or μ_0.

3 Rank One Perturbations with Finite Coupling Constants

In this section we consider a pair of operators $J_{h_1}(g)$ and $J_{h_2}(g)$, where $h_1, h_2 \in \mathbb{R}$, that is, rank one perturbations of the Jacobi operator $J(g)$ with finite coupling constants.

3.1 Recovering the Matrix from Two Spectra

Let $g \in \mathbb{R} \cup \{+\infty\}$ be fixed. Since $J_h(g)$ is a rank one perturbation of $J(g)$, the domain of $J(g)$ coincides with the domain of $J_h(g)$ for all $h \in \mathbb{R}$. Moreover, since the perturbation is analytic in h, the multiplicity-one eigenvalues, $\lambda_k(h)$, and the corresponding eigenvectors, are analytic functions of h [16].

Lemma 3.1 *Let $\{\lambda_k(h)\}_k$ be the set of eigenvalues of $J_h(g)$ ($h \in \mathbb{R}$). For a fixed k the following holds*

$$\frac{d}{dh}\lambda_k(h) = -\frac{1}{\alpha_k(h)}\,, \tag{3.1}$$

where $\alpha_k(h)$ is the normalizing constant corresponding to $\lambda_k(h)$.

Proof For the sake of simplifying the formulae, we write J_h and $\lambda(h)$ instead of $J_h(g)$ and $\lambda_k(h)$, respectively (k is fixed). Let us denote by $f(h)$ the eigenvector of J_h corresponding to $\lambda(h)$. Take any $\delta > 0$, taking into account that $\operatorname{dom}(J_{h+\delta}) = \operatorname{dom}(J_h)$ and that J_h is symmetric for any $h \in \mathbb{R}$, we have that

$$\begin{aligned}
&(\lambda(h+\delta) - \lambda(h))\langle f(h+\delta),\, f(h)\rangle \\
&\quad = \langle J_{h+\delta} f(h+\delta),\, f(h)\rangle - \langle f(h+\delta), J_h f(h)\rangle \\
&\quad = \langle (J_{h+\delta} - J_h + J_h) f(h+\delta),\, f(h)\rangle - \langle f(h+\delta), J_h f(h)\rangle \\
&\quad = \langle (J_{h+\delta} - J_h) f(h+\delta),\, f(h)\rangle = -\delta\,.
\end{aligned}$$

Therefore,

$$\lim_{\delta\to 0}\frac{\lambda(h+\delta)-\lambda(h)}{\delta}=-\lim_{\delta\to 0}\frac{1}{\langle f(h+\delta),\, f(h)\rangle}=-\frac{1}{\alpha_k(h)}\,. \qquad \square$$

The cornerstone of our analysis below is the Weyl m-function. Let us establish the relation between $m_h(\zeta,g)$ and $m(\zeta,g)$. Consider the second resolvent identity [26]:

$$\begin{aligned}(J_h(g)-\zeta I)^{-1}&-(J(g)-\zeta I)^{-1}\\ &=(J(g)-\zeta I)^{-1}(J(g)-J_h(g))(J_h(g)-\zeta I)^{-1}\,,\end{aligned}\tag{3.2}$$

where $\zeta\in\mathbb{C}\setminus\{\sigma(J(g))\cup\sigma(J_h(g))\}$. Then, for $h\in\mathbb{R}$,

$$\begin{aligned}m_h(\zeta,g)-m(\zeta,g)&=\left\langle\left((J_h(g)-\zeta I)^{-1}-(J(g)-\zeta I)^{-1}\right)e_1,e_1\right\rangle\\ &=\left\langle(J(g)-\zeta I)^{-1}(h\langle\cdot,e_1\rangle e_1)(J_h(g)-\zeta I)^{-1}e_1,e_1\right\rangle\\ &=\left\langle h\langle(J_h(g)-\zeta I)^{-1}e_1,e_1\rangle(J(g)-\zeta I)^{-1}e_1,e_1\right\rangle\\ &=hm_h(\zeta,g)m(\zeta,g)\,.\end{aligned}$$

Hence,

$$m_h(\zeta,g)=\frac{m(\zeta,g)}{1-hm(\zeta,g)}\,.\tag{3.3}$$

Remark 3.2 If $J(g)$ has discrete spectrum, then $m(\zeta,g)$ is meromorphic and, by (3.3), so is $m_h(\zeta,g)$. The poles of $m_h(\zeta,g)$ are the eigenvalues of $J_h(g)$. Since the poles of the denominator and numerator in (3.3) coincide, assuming that $h\neq 0$, the poles of $m_h(\zeta,g)$ are given by the zeros of $1-hm(\zeta,g)$ and the zeros of $m_h(\zeta,g)$ by the zeros of $m(\zeta,g)$. Thus, $J_{h_1}(g)$ and $J_{h_2}(g)$ have different eigenvalues, provided that $h_1\neq h_2$.

Theorem 3.3 *Consider the Jacobi operator $J(g)$ with discrete spectrum. The sequences $\{\mu_k\}_k=\sigma(J_{h_1}(g))$ and $\{\lambda_k\}_k=\sigma(J_{h_2}(g))$, $h_1\neq h_2$, together with h_1 (respectively, h_2) uniquely determine the operator J, h_2, (respectively, h_1) and, if $J\neq J^*$, the boundary condition g at infinity.*

Proof Without loss of generality we can assume that $h_1<h_2$. Consider the Weyl m-function $m(\zeta,g)$ of the operator $J(g)$. Let us define the function

$$\mathfrak{m}(\zeta,g)=\frac{m_{h_2}(\zeta,g)}{m_{h_1}(\zeta,g)}\,,\qquad \zeta\in\mathbb{C}\setminus\mathbb{R}\,.\tag{3.4}$$

Notice first that the zeros of $\mathfrak{m}(\zeta,g)$ are the eigenvalues of $J_{h_1}(g)$ while the poles of $\mathfrak{m}(\zeta,g)$ are the eigenvalues of $J_{h_2}(g)$. This follows from Remark 3.2 and (3.4). Let us now show that $\mathfrak{m}(\zeta,g)$ is a Herglotz or an anti-Herglotz function. Indeed, since $m(\zeta,g)$ is Herglotz, then

$$\mathfrak{m}(\zeta,g)=\frac{1-h_1m}{1-h_2m}=1+\frac{-1}{\frac{h_2}{h_2-h_1}+\frac{-1}{(h_2-h_1)m(\zeta,g)}}\,.\tag{3.5}$$

Therefore, $\mathfrak{m}(\zeta, g)$ is Herglotz or anti-Herglotz depending on the sign of $h_2 - h_1$. Recall that if a function f is Herglotz, then, $-\frac{1}{f}$ is also Herglotz. Since $h_2 - h_1 > 0$, $\mathfrak{m}(\zeta, g)$ is a Herglotz function.

Thus, the zeros $\{\mu_k\}_k$ of $\mathfrak{m}(\zeta, g)$ and its poles $\{\lambda_k\}_k$ interlace. Let us use the convention (2.18–2.23) for enumerating the zeros and poles of $\mathfrak{m}(\zeta, g)$. By this convention, if the sequence $\{\lambda_k\}_k$ (or $\{\mu_k\}_k$) is bounded from below, the least of all zeros is greater than the least of all poles, while, if $\{\lambda_k\}_k$ is bounded from above, the greatest of all poles is less than the greatest of all zeros. It is easy to verify, using for instance (3.1), that this is what we have for the zeros and poles of $\mathfrak{m}(\zeta, g)$ when $J(g)$ is semi-bounded.

According to [18, Chap. 7, Sec.1, Theorem 1], the meromorphic Herglotz function $\mathfrak{m}(\zeta, g)$, with its zeros and poles enumerated as convened, can be written as follows

$$\mathfrak{m}(\zeta, g) = C\frac{\zeta - \mu_0}{\zeta - \lambda_0}\prod_{k\in M}{}' \left(1 - \frac{\zeta}{\mu_k}\right)\left(1 - \frac{\zeta}{\lambda_k}\right)^{-1}, \qquad C > 0\,, \tag{3.6}$$

where the prime in the infinite product means that it does not include the factor $k = 0$.

From the asymptotic behavior of $m(\zeta, g)$, given by (2.10), one easily obtains that, as $\zeta \to \infty$ with $\operatorname{Im}\zeta \geqslant \epsilon$ ($\epsilon > 0$),

$$\mathfrak{m}(\zeta, g) = 1 + (h_1 - h_2)\zeta^{-1} + (h_1 - h_2)(q_1 - h_2)\zeta^{-2} + O(\zeta^{-3})\,. \tag{3.7}$$

Therefore,

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \mathfrak{m}(\zeta, g) = 1\,.$$

Then, using (3.6), we have

$$C^{-1} = \lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \prod_{k\in M}{}' \left(1 - \frac{\zeta}{\mu_k}\right)\left(1 - \frac{\zeta}{\lambda_k}\right)^{-1}, \qquad \epsilon > 0\,. \tag{3.8}$$

Thus, $\mathfrak{m}(\zeta, g)$ is completely determined by the spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$. Having found $\mathfrak{m}(\zeta, g)$, we can determine h_2, respectively, h_1, by means of (3.7). Hence, from (3.5) one obtains $m(\zeta, g)$ and, using the methods introduced in the preliminaries, J is uniquely determined. In the case when $J \neq J^*$, we can also find the boundary condition g at infinity as indicated in Section 2. □

In [24] (see also [9]) it is proven that the discrete spectra of $J_{h_1}(g)$ and $J_{h_2}(g)$, together with h_1 and h_2 uniquely determine J and the boundary condition g in the $(1, 1)$ case. Our result shows that it is not necessary to know both h_1 and h_2, one of them is enough.

It turns out that if one knows the spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$ together with q_1, the first element of the matrix's main diagonal, it is possible to recover uniquely the matrix, the boundary conditions h_1, h_2 and the boundary condition at infinity, g, if any. Indeed, the term of order ζ^{-1} in the asymptotic expansion of $\mathfrak{m}(\zeta, g)$ (3.7) determines $h_1 - h_2$. Since the coefficient of ζ^{-2} term is $(h_1 - h_2)(q_1 - h_2)$, if we know q_1 one finds h_2, and then h_1.

3.2 Necessary and Sufficient Conditions

Theorem 3.4 *Given $h_1 \in \mathbb{R}$ and two infinite sequences of real numbers $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ without finite points of accumulation, there is a unique real $h_2 > h_1$, a unique operator $J(g)$, and if $J \neq J^*$ also a unique $g \in \mathbb{R} \cup \{+\infty\}$, such that, $\{\mu_k\}_k = \sigma(J_{h_1}(g))$ and $\{\lambda_k\}_k = \sigma(J_{h_2}(g))$ if and only if the following conditions are satisfied.*

(a) *$\{\lambda_k\}_k$ and $\{\mu_k\}_k$ interlace and, if $\{\lambda_k\}_k$ is bounded from below, $\min_k\{\mu_k\}_k > \min_k\{\lambda_k\}_k$, while if $\{\lambda_k\}_k$ is bounded from above, $\max_k\{\lambda_k\}_k < \max_k\{\mu_k\}_k$. So we use below the convention (2.18–2.23) for enumerating the sequences.*

(b) *The following series converges*

$$\sum_{k\in M}(\mu_k - \lambda_k) = \Delta < \infty\,.$$

By condition (b) *the product* $\prod_{\substack{k\in M\\ k\neq n}} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}$ *is convergent, so define*

$$\tau_n^{-1} := \frac{\mu_n - \lambda_n}{\Delta} \prod_{\substack{k\in M\\ k\neq n}} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}, \qquad \forall n \in M\,. \tag{3.9}$$

(c) *The sequence $\{\tau_n\}_{n\in M}$ is such that, for $m = 0, 1, 2, \ldots$, the series*

$$\sum_{k\in M} \frac{\lambda_k^{2m}}{\tau_k} \quad \textit{converges.}$$

(d) *If a sequence of complex numbers $\{\beta_k\}_{k\in M}$ is such that the series*

$$\sum_{k\in M} \frac{|\beta_k|^2}{\tau_k} \quad \textit{converges}$$

and, for $m = 0, 1, 2, \ldots$,

$$\sum_{k\in M} \frac{\beta_k \lambda_k^m}{\tau_k} = 0\,,$$

then $\beta_k = 0$ for all $k \in M$.

Proof We first prove that if $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are the spectra of $J_{h_2}(g)$ and $J_{h_1}(g)$, with $h_2 > h_1$, then (a), (b), (c), and (d) hold true. The condition (a) follows directly from the proof of the previous theorem. To prove that (b) holds, observe that (3.1) implies

$$\mu_k - \lambda_k = \int_{h_1}^{h_2} \frac{dh}{\alpha_k(h)}\,.$$

Consider a sequence $\{M_n\}_{n=1}^{\infty}$ of subsets of M, such that $M_n \subset M_{n+1}$ and $\cup_n M_n = M$, then, using (2.15), we have

$$s_n := \sum_{k\in M_n} (\mu_k - \lambda_k) = \sum_{k\in M_n} \int_{h_1}^{h_2} \frac{dh}{\alpha_k(h)} = \int_{h_1}^{h_2} \sum_{k\in M_n} \frac{dh}{\alpha_k(h)} \leqslant h_2 - h_1 \,.$$

The sequence $\{s_n\}_{n=1}^{\infty}$ is then convergent and clearly

$$\sum_{k\in M} (\mu_k - \lambda_k) = \lim_{n\to\infty} s_n = h_2 - h_1 \,.$$

Thus, $\Delta = h_2 - h_1$.

The convergence of the series in (b) allows us to write (3.8) as follows

$$C^{-1} = \prod_{k\in M}{}' \frac{\lambda_k}{\mu_k} \lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \prod_{k\in M}{}' \frac{\mu_k - \zeta}{\lambda_k - \zeta}, \qquad \epsilon > 0\,.$$

Now, using again (b), it easily follows that for any $\epsilon > 0$

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \prod_{k\in M}{}' \frac{\mu_k - \zeta}{\lambda_k - \zeta} = \lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \prod_{k\in M}{}' \left(1 + \frac{\mu_k - \lambda_k}{\lambda_k - \zeta}\right) = 1\,.$$

Thus, $C = \prod'_{k\in M} \mu_k/\lambda_k$ and by (3.6),

$$\mathfrak{m}(\zeta, g) = \prod_{k\in M} \frac{\mu_k - \zeta}{\lambda_k - \zeta}\,. \tag{3.10}$$

Let us now find formulae for the normalizing constants in terms of the sets of eigenvalues for different boundary conditions. By (2.17),

$$\alpha_n^{-1}(h_2, g) = \lim_{\zeta\to\lambda_n} (\lambda_n - \zeta) m_{h_2}(\zeta, g)\,.$$

Using the second resolvent identity, as we did to obtain (3.3), we have that

$$m_{h_1}(\zeta, g) = \frac{m_{h_2}(\zeta, g)}{1 - (h_1 - h_2) m_{h_2}(\zeta, g)}\,.$$

Therefore,

$$\mathfrak{m}(\zeta, g) = \frac{m_{h_2}(\zeta, g)}{m_{h_1}(\zeta, g)} = 1 - (h_1 - h_2) m_{h_2}, \qquad \zeta \in \mathbb{C}\setminus\mathbb{R}\,. \tag{3.11}$$

Then, the normalizing constants are given by

$$\alpha_n^{-1}(h_2, g) = \lim_{\zeta\to\lambda_n} (\lambda_n - \zeta) \frac{\mathfrak{m}(\zeta, g) - 1}{h_2 - h_1} = \frac{1}{h_2 - h_1} \lim_{\zeta\to\lambda_n} (\lambda_n - \zeta) \mathfrak{m}(\zeta, g)\,.$$

Now,

$$\lim_{\zeta\to\lambda_n}(\lambda_n-\zeta)\mathfrak{m}(\zeta,g)=\lim_{\zeta\to\lambda_n}(\lambda_n-\zeta)\prod_{k\in M}\frac{\mu_k-\zeta}{\lambda_k-\zeta}$$
$$=(\mu_n-\lambda_n)\prod_{\substack{k\in M\\ k\neq n}}\frac{\mu_k-\lambda_n}{\lambda_k-\lambda_n}\,. \tag{3.12}$$

Hence,

$$\alpha_n^{-1}(h_2,g)=\frac{\mu_n-\lambda_n}{h_2-h_1}\prod_{\substack{k\in M\\ k\neq n}}\frac{\mu_k-\lambda_n}{\lambda_k-\lambda_n}\,. \tag{3.13}$$

Notice that, since $\Delta=h_2-h_1$, it follows from (3.13) that $\tau_n=\alpha_n$ for all $n\in M$. Hence the spectral function ρ of the self-adjoint extension $J_{h_2}(g)$ is given by the expression $\rho(t)=\sum_{\lambda_k\leqslant t}\tau_k^{-1}$. Thus (c) follows from the fact that all the moments of ρ are finite [1, 23]. Similarly, (d) stems from the density of polynomials in $L_2(\mathbb{R},d\rho)$, which takes place since ρ is N-extremal [1], [23, Proposition 4.15].

We now prove that conditions (a), (b), (c), and (d) are sufficient. Let $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ be sequences as in (a) and (b). Then,

$$0<\prod_{\substack{k\in M\\ k\neq n}}\frac{\mu_k-\lambda_n}{\lambda_k-\lambda_n}<\infty\,. \tag{3.14}$$

The convergence of this product allows us to define the sequence of numbers $\{\tau_n\}_{n\in M}$. Observe that for all $n\in M$, $\tau_n>0$. Indeed, $\Delta>0$ and (2.18–2.23) yield $\mu_n-\lambda_n>0$ for all $n\in M$. Thus, taking into account (3.14), we obtain

$$\tau_n>0,\qquad\forall\, n\in M\,. \tag{3.15}$$

Let us now define the function

$$\rho(t):=\sum_{\lambda_k\leqslant t}\frac{1}{\tau_k},\qquad t\in\mathbb{R}\,. \tag{3.16}$$

Since (3.15) holds, ρ is a monotone non-decreasing function and has an infinite number of points of growth. Notice also that ρ is right continuous. Now, we want to show that for the measure corresponding to ρ all the moments are finite and

$$\int_{\mathbb{R}}d\rho(t)=1\,. \tag{3.17}$$

The fact that the moments are finite follows directly from condition (c). Indeed,

$$\int_{\mathbb{R}}t^m d\rho(t)=\sum_{k\in M}\frac{\lambda_k^m}{\tau_k}\,.$$

We show next that (3.17) holds true. Given the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ satisfying (a) and (b) we can define the function

$$\widetilde{\mathfrak{m}}(\zeta) := \prod_{k\in M} \frac{\mu_k - \zeta}{\lambda_k - \zeta}\,. \tag{3.18}$$

Taking into account (3.9), one obtains that

$$\operatorname*{Res}_{\zeta=\lambda_n}(\widetilde{\mathfrak{m}}(\zeta) - 1) = -\frac{\Delta}{\tau_n}\,.$$

In view of (b), we easily find that

$$\begin{aligned}\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}} (\widetilde{\mathfrak{m}}(\zeta) - 1) &= \lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}} \prod_{k\in M} \frac{\mu_k - \zeta}{\lambda_k - \zeta} - 1\\ &= \lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}} \prod_{k\in M} \left(1 + \frac{\mu_k - \lambda_k}{\lambda_k - \zeta}\right) - 1 = 0\,. \end{aligned} \tag{3.19}$$

Thus, on the basis of Čebotarev's theorem on the representation of meromorphic Herglotz functions [18, Chap. VII, Section 1 Theorem 2], one obtains

$$\widetilde{\mathfrak{m}}(\zeta) - 1 = \sum_{k\in M} \frac{\Delta}{(\lambda_k - \zeta)\tau_k}\,. \tag{3.20}$$

We now define the function $\widetilde{m}(\zeta) := \frac{\widetilde{\mathfrak{m}}(\zeta)-1}{\Delta}$. Then, (3.20) yields

$$\widetilde{m}(\zeta) = \sum_{k\in M} \frac{1}{\tau_k(\lambda_k - \zeta)}\,. \tag{3.21}$$

We next show that

$$\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}} \zeta\widetilde{m}(\zeta) = -1\,.$$

Indeed,

$$\begin{aligned}\frac{\widetilde{\mathfrak{m}}(\zeta)}{\Delta} &= \frac{1}{\Delta}\prod_{k\in M}\frac{\mu_k - \zeta}{\lambda_k - \zeta}\\ &= \frac{1}{\Delta}\exp\left\{\sum_{k\in M}\ln\left(\frac{\mu_k - \zeta}{\lambda_k - \zeta}\right)\right\}\\ &= \frac{1}{\Delta}\exp\left\{\sum_{k\in M}\ln\left(1 + \frac{\mu_k - \lambda_k}{\lambda_k - \zeta}\right)\right\}\\ &= \frac{1}{\Delta}\exp\left\{\sum_{k\in M}\sum_{p=1}^{\infty}(-1)^{p-1}\left(\frac{\mu_k - \lambda_k}{\lambda_k - \zeta}\right)^p\right\}\,.\end{aligned}$$

Thus, as $\zeta \to \infty$ with $\operatorname{Im}\zeta \geqslant \epsilon$ ($\epsilon > 0$),

$$\frac{\widetilde{\mathfrak{m}}(\zeta)}{\Delta} = \frac{1}{\Delta} + \frac{1}{\Delta}\sum_{k\in M}\frac{\mu_k - \lambda_k}{\lambda_k - \zeta} + O(\zeta^{-2})\,.$$

Then,

$$\begin{aligned}\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}} \zeta\widetilde{m}(\zeta) &= \lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}} \zeta\frac{1}{\Delta}\sum_{k\in M}\frac{\mu_k-\lambda_k}{\lambda_k-\zeta}\\ &= -\frac{1}{\Delta}\sum_{k\in M}(\mu_k-\lambda_k) = -1\,.\end{aligned}$$

Also, from (3.21) one has

$$\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}} \zeta\widetilde{m}(\zeta) = -\sum_{k\in M}\frac{1}{\tau_k}\,.$$

Therefore,

$$1 = \sum_{k\in M}\frac{1}{\tau_k} = \int_{\mathbb{R}} d\rho(t)\,.$$

Having found a function ρ with infinitely many growing points and such that (3.17) is satisfied and all the moments exist, one can obtain, applying the method of orthogonal polynomials (see Section 2), a tridiagonal semi-infinite matrix. Let us denote by $\widehat{J}$ the operator whose matrix representation is the obtained matrix. By what has been explained before, this operator is closed and symmetric. Now, define $h_2 := \Delta + h_1$ and $J := \widehat{J} + h_2\langle\cdot, e_1\rangle\, e_1$.

If $\widehat{J} = \widehat{J}^*$, we know that $\rho(t) = \langle E(t)_{\widehat{J}} e_1, e_1\rangle$, where $E_{\widehat{J}}(t)$ is the spectral decomposition of the self-adjoint Jacobi operator $\widehat{J}$. Then, obviously, $\widehat{J} = J_{h_2}$.

If $\widehat{J} \neq \widehat{J}^*$, the Stieltjes transform of ρ is the Weyl m-function, we denote it by $w(\zeta)$, of some self-adjoint extension of $\widehat{J}$ that we denote by $\widetilde{J}$. This is true because of the density of polynomials in $L_2(\mathbb{R}, d\rho)$. Indeed, (d) means that the polynomials are dense in $L_2(\mathbb{R}, d\rho)$. Thus, $w(\zeta)$ lies on the Weyl circle, and then, it is the Weyl m-function of some self-adjoint extension of $\widehat{J}$ [1], [23, Proposition 4.15]. Therefore, $\widetilde{J} + h_2\langle\cdot, e_1\rangle\, e_1$ is a self-adjoint extension of J and hence, $\widetilde{J} + h_2\langle\cdot, e_1\rangle\, e_1 = J(g)$ for some unique $g \in \mathbb{R}\cup\{\infty\}$. Furthermore, we obviously have that, $\widetilde{J} = J_{h_2}(g)$ and $w(\zeta) = m_{h_2}(\zeta, g)$. We uniquely reconstruct $m(\zeta, g)$ from $m_{h_2}(\zeta, g)$ using (3.3) and then, we uniquely reconstruct g as explained in Section 2.

Notice that we have

$$m_{h_2}(\zeta, g) = \int_{\mathbb{R}}\frac{d\rho(t)}{t-\zeta} = \widetilde{m}(\zeta)\,.$$

It remains to show that $\sigma(J_{h_2}(g)) = \{\lambda_k\}_k$ and $\sigma(J_{h_1}(g)) = \{\mu_k\}_k$. To this end consider the function $\mathfrak{m}(\zeta, g)$ for the pair J_{h_2} and J_{h_1}:

$$\mathfrak{m}(\zeta, g) = \frac{m_{h_2}(\zeta, g)}{m_{h_1}(\zeta, g)}, \qquad \zeta \in \mathbb{C}\setminus\mathbb{R}\,.$$

Let the sequence $\{\gamma_k\}_k$ denote the spectrum of J_{h_1}. Then, arguing as in the proof of (3.10) we obtain that

$$\mathfrak{m}(\zeta, g) = \prod_{k\in M} \frac{\gamma_k - \zeta}{\lambda_k - \zeta}\,.$$

Since we have already proven that (a) and (b) are necessary conditions, we have that

$$\sum_{k\in M}(\gamma_k - \lambda_k) = \Delta < \infty\,.$$

Then, as in the proof of (3.19), it follows that

$$\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}} (\mathfrak{m}(\zeta) - 1) = 0\,.$$

Hence by Čebotarev's theorem [18, Chap. VII, Section 1 Theorem 2],

$$\mathfrak{m}(\zeta, g) = 1 + \sum_{k\in M} \frac{h_2 - h_1}{(\lambda_k - \zeta)\alpha_k(h_2, g)}\,,$$

where we compute the residues of $\mathfrak{m}(\zeta)$ as in (3.12). Thus, since $\alpha_k(h_2, g) = \tau_k$, $\forall k \in M$,

$$\mathfrak{m}(\zeta, g) = 1 + \sum_{k\in M} \frac{\Delta}{(\lambda_k - \zeta)\tau_k} = \widetilde{\mathfrak{m}}(\zeta, g)\,.$$

But $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are the poles and zeros of $\widetilde{\mathfrak{m}}(\zeta, g)$ and then, the eigenvalues of $J_{h_2}(g)$ and $J_{h_1}(g)$, respectively. □

Remark 3.5 We draw the reader's attention to the fact that the matrix associated with the function ρ, constructed in the proof of the previous theorem, may have deficiency indices $(1, 1)$ [1, 23, 25].

If we drop the condition of the density of polynomials in $L_2(\mathbb{R}, d\rho)$ and our reconstruction method yields a nonself-adjoint operator J, then the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ correspond to the spectra of some generalized self-adjoint extensions of J_{h_2} and J_{h_1}, respectively (see [23]). The generalized extensions of symmetric operators, which are not von Neumann extensions, were first introduced by Naimark (see Appendix I in [2] on Naimark's theory).

In [15] the case of Jacobi operators bounded from below is considered. A uniqueness result is proven, and some sufficient conditions for a pair of sequences to be the spectra of a Jacobi operator with different boundary conditions are given.

4 Dirichlet–Neumann Conditions

4.1 Recovering the Matrix from Two Spectra

In this section we shall consider the pair of Jacobi operators $J_0(g) = J(g)$ and $J_\infty(g)$. Here, as before, we keep the convention of writing $J(g)$ even if $J = J^*$. The matrix representation of $J_\infty(g)$ corresponds to the matrix representation of $J(g)$ with the first column and row removed. From the Ricatti equation (2.13), taking into account that $m^{(0)}(\zeta, g) = m(\zeta, g)$ and $m^{(1)}(\zeta, g) = m_\infty(\zeta, g)$, we have

$$m_\infty(\zeta, g) = -\frac{1}{b_1^2}\left((\zeta - q_1) + \frac{1}{m(\zeta, g)}\right). \tag{4.1}$$

As before, we assume that the spectrum of $J = J(g)$ is discrete.

If $m(\zeta, g)$ is a meromorphic function, then, by (4.1), $m_\infty(\zeta, g)$ is also meromorphic and the spectrum of $J_\infty(g)$ is discrete. The poles of $m(\zeta, g)$ are the eigenvalues of $J(g)$, while the zeros of $m(\zeta, g)$ are the eigenvalues of $J_\infty(g)$. Since $m(\zeta, g)$ is always a Herglotz function, under our assumption on the discreteness of $\sigma(J(g))$, $m(\zeta, g)$ is a meromorphic Herglotz function. This implies that $\sigma(J(g))$ and $\sigma(J_\infty(g))$ are interlaced, that is, between two successive eigenvalues of one operator there is exactly one eigenvalue of the other.

Let the sequence $\{\lambda_k\}_k$ denote the eigenvalues of $J(g)$ (the poles of $m(\zeta, g)$). Furthermore, $\{\mu_k\}_k$ will stand for the eigenvalues of $J_\infty(g)$ (the zeros of $m(\zeta, g)$). It is worth remarking that, in contrast to the case of boundary conditions being rank one perturbations with finite coupling constant, here our convention for enumerating the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ does not work in the case when $J(g)$ is semi-bounded from above. Indeed, it follows from the mini-max principle [21] that if $J(g)$ is bounded from below, the smallest of all poles is less than the smallest of all zeros of $m(\zeta, g)$, and if $J(g)$ is bounded from above, the min–max principle applied to $-J(g)$ implies that the greatest of all zeros is less than the greatest of all poles of $m(\zeta, g)$.

So let us consider first the case when $J(g)$ is not semi-bounded or semi-bounded from below and enumerate the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ by (2.18), (2.19), (2.22), and (2.23). Then, by the same theorem we used to obtain (3.6) [18], $m(\zeta, g)$ can be written as follows

$$m(\zeta, g) = C\frac{\zeta - \mu_0}{\zeta - \lambda_0}{\prod_{k\in M}}'\left(1 - \frac{\zeta}{\mu_k}\right)\left(1 - \frac{\zeta}{\lambda_k}\right)^{-1}, \qquad C > 0\,, \tag{4.2}$$

where, as before, the prime in the infinite product means that it does not include the factor $k = 0$.

If $J(g)$ is bounded from above, then we are still able to use (2.18), (2.20) and (2.21) for enumerating the zeros and poles of the meromorphic Herglotz function $-\frac{1}{m(\zeta,g)}$. Thus,

$$-\frac{1}{m(\zeta,g)}=\widetilde{C}\frac{\zeta-\lambda_0}{\zeta-\mu_0}\prod_{k\in M}{}'\left(1-\frac{\zeta}{\lambda_k}\right)\left(1-\frac{\zeta}{\mu_k}\right)^{-1}, \qquad \widetilde{C}>0\,. \tag{4.3}$$

Notice that, since we have enumerated zeros and poles of $-\frac{1}{m(\zeta,g)}$ by our convention, we have now

$$\forall k\in M, \qquad \mu_k<\lambda_k<\mu_{k+1}\,, \tag{4.4}$$

and

(a) If $0<\sup_k\{\mu_k\}_k<\infty$,

$$M:=\{k\}_{k=-\infty}^{k_{\max}}\,, \qquad (k_{\max}\geqslant 1) \quad \text{requiring} \quad \lambda_{-1}<0<\mu_1\,, \tag{4.5}$$

(b) If $\sup_k\{\mu_k\}_k\leqslant 0$,

$$M:=\{k\}_{k=-\infty}^{0}\,. \tag{4.6}$$

Here again λ_0 or μ_0 are the only ones allowed to be zero.

Equations (4.2) and (4.3) can be written in one formula

$$m(\zeta,g)=K\frac{\zeta-\mu_0}{\zeta-\lambda_0}\prod_{k\in M}{}'\left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}, \tag{4.7}$$

where, if $J(g)$ is not semi-bounded from above, $K=C$ and $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are enumerated by (2.18), (2.19), (2.22), and (2.23), while $K=-\widetilde{C}^{-1}$ and $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are enumerated by (4.4–4.6) if $J(g)$ is semi-bounded from above.

We give now, for the reader's convenience, a simple proof of a theorem that was proven by Fu and Hochstadt [10] for regular Jacobi operators (a regular Jacobi matrix is defined in [10]), and by Teschl [24] in the general case.

Theorem 4.1 (Fu and Hochstadt, Teschl) *Consider the Jacobi operator $J(g)$ with discrete spectrum. The sequences $\{\lambda_k\}_k=\sigma(J(g))$ and $\{\mu_k\}_k=\sigma(J_\infty(g))$ uniquely determine the operator J and, if $J\neq J^*$, the boundary condition, g, at infinity.*

Proof From (2.10) we know that

$$\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}}\zeta m(\zeta,g)=-1, \qquad \epsilon>0\,.$$

Then, if $J(g)$ is not semi-bounded from above, (4.2) yields

$$C^{-1}=-\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geqslant\epsilon}}\zeta\prod_{k\in M}{}'\left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}, \qquad \epsilon>0\,, \tag{4.8}$$

where $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are enumerated by (2.18), (2.19), (2.22), and (2.23). On the other hand, in the semi-bounded from above case (4.3) implies

$$\widetilde{C}^{-1} = \lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \frac{1}{\zeta} \prod_{k\in M}{}' \left(1-\frac{\zeta}{\lambda_k}\right)\left(1-\frac{\zeta}{\mu_k}\right)^{-1}, \qquad \epsilon > 0\,. \tag{4.9}$$

where $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are enumerated by (4.4–4.6). Thus, in any case, one can find K, the constant in (4.7), from the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$. Therefore, the spectra $\sigma(J(g))$ and $\sigma(J_\infty(g))$ uniquely determine $m(\zeta, g)$. Having found $m(\zeta, g)$ we can, using the methods introduced in Section 2, determine J and, in the case when $J \neq J^*$, also find uniquely the boundary condition at infinity, g. □

Remark 4.2 It turns out that, by (4.8) and (4.9), K can be written as

$$K^{-1} = -\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \zeta \prod_{k\in M}{}' \left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}, \qquad \epsilon > 0\,, \tag{4.10}$$

where the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ have been enumerated by (2.18), (2.19), (2.22), and (2.23), when $J(g)$ is not semi-bounded from above and by (4.4–4.6), otherwise.

In what follows the Weyl m-function will be written through (4.7) with K given by (4.10). From (4.7) one can obtain straightforward formulae for the normalizing constants (2.14) in terms of the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$. Indeed, when $n \neq 0$

$$\begin{aligned}\lim_{\zeta\to\lambda_n} (\lambda_n - \zeta) m(\zeta, g) &= \lim_{\zeta\to\lambda_n} (\lambda_n - \zeta) K \frac{\zeta-\mu_0}{\zeta-\lambda_0} \prod_{k\in M}{}' \frac{1-\frac{\zeta}{\mu_k}}{1-\frac{\zeta}{\lambda_k}} \\ &= K\frac{\lambda_n}{\mu_n}(\mu_n-\lambda_n)\frac{\lambda_n-\mu_0}{\lambda_n-\lambda_0} \prod_{\substack{k\in M \\ k\neq n}}{}' \frac{1-\frac{\lambda_n}{\mu_k}}{1-\frac{\lambda_n}{\lambda_k}}\,.\end{aligned}$$

Formulae (2.17) and (4.10) then give , for $n \neq 0$,

$$\alpha_n^{-1} = -\frac{\frac{\lambda_n}{\mu_n}(\mu_n-\lambda_n)\frac{\lambda_n-\mu_0}{\lambda_n-\lambda_0} \prod_{\substack{k\in M \\ k\neq n}}{}' \left(1-\frac{\lambda_n}{\mu_k}\right)\left(1-\frac{\lambda_n}{\lambda_k}\right)^{-1}}{\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \zeta \prod_{k\in M}{}' \left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}}\,. \tag{4.11}$$

Analogously,

$$\alpha_0^{-1} = -\frac{(\mu_0 - \lambda_0) \prod'_{k\in M} \left(1 - \frac{\lambda_0}{\mu_k}\right)\left(1 - \frac{\lambda_0}{\lambda_k}\right)^{-1}}{\lim\limits_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \zeta \prod'_{k\in M} \left(1 - \frac{\zeta}{\mu_k}\right)\left(1 - \frac{\zeta}{\lambda_k}\right)^{-1}} . \tag{4.12}$$

4.2 Necessary and Sufficient Conditions

The following result establishes necessary and sufficient conditions for two given sequences of real numbers to be the spectra of $J(g)$ and $J_\infty(g)$.

Theorem 4.3 *Given two infinite sequences of real numbers $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ without finite points of accumulation, there is a unique operator $J(g)$, and if $J \neq J^*$ also a unique $g \in \mathbb{R} \cup \{+\infty\}$, such that $\{\lambda_k\}_k = \sigma(J(g))$ and $\{\mu_k\}_k = \sigma(J_\infty(g))$ if and only if the following conditions are satisfied.*

(a) *$\{\lambda_k\}_k$ and $\{\mu_k\}_k$ interlace and, if $\{\lambda_k\}_k$ is bounded from below,* $\min_k\{\mu_k\}_k > \min_k\{\lambda_k\}_k$, *if $\{\lambda_k\}_k$ is bounded from above,* $\max_k\{\lambda_k\}_k > \max_k\{\mu_k\}_k$. *So we use below the convention* (2.18), (2.19), (2.22), *and* (2.23) *for enumerating the sequences when $J(g)$ is not semi-bounded from above, and* (4.4–4.6) *otherwise.*
By condition (a) *the product*

$$\prod'_{k\in M} \left(1 - \frac{\zeta}{\mu_k}\right)\left(1 - \frac{\zeta}{\lambda_k}\right)^{-1}$$

converges uniformly on compact subsets of $\mathbb{C}$ (see the proof below and [18, *Chap.* 7, *Section* 1].

(b) *The limit*

$$\lim_{\substack{\xi\to\infty \\ \xi\in\mathbb{R}}} \mathrm{i}\xi \prod'_{k\in M} \left(1 - \frac{\mathrm{i}\xi}{\mu_k}\right)\left(1 - \frac{\mathrm{i}\xi}{\lambda_k}\right)^{-1} \tag{4.13}$$

is finite and negative when the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are not bounded from above, and it is finite and positive otherwise.

(c) *Let $\{\tau_n\}_{n\in M}$ be defined by*

$$\tau_n^{-1} = -\frac{\frac{\lambda_n}{\mu_n}(\mu_n - \lambda_n)\frac{\lambda_n - \mu_0}{\lambda_n - \lambda_0} \prod'_{\substack{k\in M \\ k\neq n}} \left(1 - \frac{\lambda_n}{\mu_k}\right)\left(1 - \frac{\lambda_n}{\lambda_k}\right)^{-1}}{\lim\limits_{\substack{\xi\to\infty \\ \xi\in\mathbb{R}}} \mathrm{i}\xi \prod'_{k\in M} \left(1 - \frac{\mathrm{i}\xi}{\mu_k}\right)\left(1 - \frac{\mathrm{i}\xi}{\lambda_k}\right)^{-1}}$$

for $n \in M$, $n \neq 0$, *and*

$$\tau_0^{-1} = -\frac{(\mu_0 - \lambda_0) \prod_{k\in M}' \left(1 - \frac{\lambda_0}{\mu_k}\right)\left(1 - \frac{\lambda_0}{\lambda_k}\right)^{-1}}{\lim\limits_{\substack{\xi\to\infty \\ \xi\in\mathbb{R}}} \mathrm{i}\xi \prod_{k\in M}' \left(1 - \frac{\mathrm{i}\xi}{\mu_k}\right)\left(1 - \frac{\mathrm{i}\xi}{\lambda_k}\right)^{-1}}.$$

The sequence $\{\tau_n\}_{n\in M}$ *is such that, for* $m = 0, 1, 2\ldots$, *the series*

$$\sum_{k\in M} \frac{\lambda_k^{2m}}{\tau_k} \qquad \textit{converges}.$$

(d) *If a sequence of complex numbers* $\{\beta_k\}_{k\in M}$, *is such that the series*

$$\sum_{k\in M} \frac{|\beta_k|^2}{\tau_k} \quad \textit{converges}$$

and, for $m = 0, 1, 2, \ldots$,

$$\sum_{k\in M} \frac{\beta_k \lambda_k^m}{\tau_k} = 0\,,$$

then $\beta_k = 0$ *for all* $k \in M$.

Proof We begin the proof by showing that the sequences $\sigma(J(g)) = \{\lambda_k\}_k$ and $\sigma(J_\infty(g)) = \{\mu_k\}_k$ satisfy (a), (b), (c), and (d). Since the Weyl m function is Herglotz, the eigenvalues of $J(g)$ and $J_\infty(g)$ interlace as indicated in (a). To prove that (b) holds, consider first the case when $J(g)$ is not semi-bounded or only bounded from below, then (4.8) yields (b). If $J(g)$ is semi-bounded from above, (4.9) implies (b).

On the basis of (4.11) and (4.12), τ_n coincides with the normalizing constant α_n for all $n \in M$. Hence the spectral function ρ of the self-adjoint extension $J(g)$ is given by the expression $\rho(t) = \sum_{\lambda_k \leqslant t} \tau_k^{-1}$. Thus (c) follows from the fact that all the moments of ρ are finite [1, 23]. Similarly, (d) stems from the density of polynomials in $L_2(\mathbb{R}, d\rho)$, which takes place since ρ is N-extremal [1], [23, Proposition 4.15].

Let us now suppose that we are given two real sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ that satisfy (a). It can be shown that

$$0 < \prod_{\substack{k\in M \\ k\neq n}}' \left(1 - \frac{\lambda_n}{\mu_k}\right)\left(1 - \frac{\lambda_n}{\lambda_k}\right)^{-1} < \infty\,. \tag{4.14}$$

Indeed, the convergence of the infinite product follows from (a) and is part of the Theorem 1 in [18, Chap. 7, Sec.1] used to obtain (3.6). We give here, for the reader's convenience, some details. The product in (4.14) converges if and only if

$$\sum_{\substack{k\in M\\k\neq n}}{}'\left\{\left(1-\frac{\lambda_n}{\mu_k}\right)\left(1-\frac{\lambda_n}{\lambda_k}\right)^{-1}-1\right\}=\lambda_n\sum_{\substack{k\in M\\k\neq n}}{}'\left(\frac{1}{\lambda_k}-\frac{1}{\mu_k}\right)\left(1-\frac{\lambda_n}{\lambda_k}\right)^{-1}<\infty\,,$$

where prime means that the summand $k=0$ is excluded. Thus, we have to prove that

$$\sum_{k\in M}{}'\left(\frac{1}{\lambda_k}-\frac{1}{\mu_k}\right)<\infty\,.$$

It will suffice to consider that in (a) the sequences are ordered by (2.18) with M given by (2.19). For any $k\in\mathbb{N}$, (2.18) implies

$$0<\left(\frac{1}{\lambda_k}-\frac{1}{\mu_k}\right)<\left(\frac{1}{\lambda_k}-\frac{1}{\lambda_{k+1}}\right),\qquad\forall k\in\mathbb{N}\,.$$

Clearly, $\sum_{k\in\mathbb{N}}\left(\frac{1}{\lambda_k}-\frac{1}{\lambda_{k+1}}\right)$ is convergent. Analogously, it can be proven that

$$\sum_{k\in\mathbb{N}}\left(\frac{1}{\lambda_{-k}}-\frac{1}{\mu_{-k}}\right)<\infty\,.$$

Having established the convergence of the the product in (4.14), its positivity follows easily.

We have, therefore, a sequence of real numbers $\{\tau_k\}_{k\in M}$ and let us now show that $\tau_n>0$, $\forall n\in M$. First notice that (2.18), (2.19), (2.22), and (2.23), yield

$$\frac{\lambda_n}{\mu_n}(\mu_n-\lambda_n)\frac{\lambda_n-\mu_0}{\lambda_n-\lambda_0}>0\quad(n\neq 0)\quad\text{and}\quad\mu_0-\lambda_0>0\,.$$

On the other hand (4.4–4.6) imply

$$\frac{\lambda_n}{\mu_n}(\mu_n-\lambda_n)\frac{\lambda_n-\mu_0}{\lambda_n-\lambda_0}<0\quad(n\neq 0)\quad\text{and}\quad\mu_0-\lambda_0<0\,.$$

From these last inequalities, taking into account (4.14) and condition (b) we obtain

$$\tau_n>0,\qquad\forall\, n\in M\,. \tag{4.15}$$

Let us now define the function

$$\rho(t):=\sum_{\lambda_k\leqslant t}\frac{1}{\tau_k},\qquad\forall t\in\mathbb{R}\,. \tag{4.16}$$

In view of (4.15), ρ is a monotone non-decreasing function and has an infinite number of points of growth. Now, we want to show that, for the measure corresponding to ρ, all the moments are finite and

$$\int_{\mathbb{R}}d\rho(t)=1\,. \tag{4.17}$$

The fact that the moments are finite follows directly from condition (c). Indeed,

$$\int_{\mathbb{R}} t^m d\rho(t) = \sum_{k\in M} \frac{\lambda_k^m}{\tau_k}\,.$$

We show next that (4.17) is true. Given the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ satisfying (a) and (b), we can define the function

$$\widetilde{m}(\zeta) := -\frac{\frac{\zeta-\mu_0}{\zeta-\lambda_0}\prod'_{k\in M}\left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}}{\lim_{\substack{\xi\to\infty\\ \xi\in\mathbb{R}}} \mathrm{i}\xi \prod'_{k\in M}\left(1-\frac{\mathrm{i}\xi}{\mu_k}\right)\left(1-\frac{\mathrm{i}\xi}{\lambda_k}\right)^{-1}}\,. \tag{4.18}$$

Now, arguing as in the proof of (4.11) and (4.12), we obtain

$$\operatorname*{Res}_{\zeta=\lambda_n} \widetilde{m}(\zeta) = -\tau_n^{-1}\,.$$

On the other hand,

$$\lim_{\substack{\xi\to\infty\\ \xi\in\mathbb{R}}} \widetilde{m}(\mathrm{i}\xi) = -\lim_{\substack{\xi\to\infty\\ \xi\in\mathbb{R}}} \frac{\prod'_{k\in M}\left(1-\frac{\mathrm{i}\xi}{\mu_k}\right)\left(1-\frac{\mathrm{i}\xi}{\lambda_k}\right)^{-1}}{\mathrm{i}\xi\prod'_{k\in M}\left(1-\frac{\mathrm{i}\xi}{\mu_k}\right)\left(1-\frac{\mathrm{i}\xi}{\lambda_k}\right)^{-1}} = 0\,.$$

Thus, using again Čebotarev's theorem [18] we find that

$$\widetilde{m}(\zeta) = \sum_{k\in M} \frac{1}{\tau_k(\lambda_k-\zeta)}\,. \tag{4.19}$$

It follows from (4.18) that

$$\lim_{\substack{\xi\to\infty\\ \xi\in\mathbb{R}}} \mathrm{i}\xi\widetilde{m}(\mathrm{i}\xi) = -\lim_{\substack{\xi\to\infty\\ \xi\in\mathbb{R}}} \frac{\mathrm{i}\xi\prod'_{k\in M}\left(1-\frac{\mathrm{i}\xi}{\mu_k}\right)\left(1-\frac{\mathrm{i}\xi}{\lambda_k}\right)^{-1}}{\mathrm{i}\xi\prod'_{k\in M}\left(1-\frac{\mathrm{i}\xi}{\mu_k}\right)\left(1-\frac{\mathrm{i}\xi}{\lambda_k}\right)^{-1}} = -1\,.$$

Also from (4.19) one has

$$\lim_{\substack{\xi\to\infty\\ \xi\in\mathbb{R}}} \mathrm{i}\xi\widetilde{m}(\mathrm{i}\xi) = -\sum_{k\in M}\frac{1}{\tau_k}\,.$$

Therefore,

$$1 = \sum_{k\in M}\frac{1}{\tau_k} = \int_{\mathbb{R}} d\rho(t)\,.$$

We have found a function $\rho(t)$ with infinitely many growing points, such that all the moments exist for the corresponding measure and (4.17) holds. Therefore one can obtain, applying the method of orthogonal polynomials (see Section 2), a tridiagonal semi-infinite matrix. Let us denote by J the operator whose matrix representation is the obtained matrix. As was mentioned before, J is symmetric and closed. Now, if $J = J^*$, we know that $\rho(t) = \langle E_J(t)e_1, e_1\rangle$, where $E_J(t)$ is the spectral decomposition of the self-adjoint Jacobi operator J. If $J \neq J^*$, then the Stieltjes transform of $\rho(t)$ is the Weyl m-function $m(\zeta, g)$

of some self-adjoint extension of J with boundary conditions at infinity given by g, that is,

$$m(\zeta, g) = \int_{\mathbb{R}} \frac{d\rho(t)}{t - \zeta}\,.$$

This last assertion is true because of the density of polynomials in $L_2(\mathbb{R}, d\rho)$, which follows from (d). Hence ρ is N-extremal [1]. This implies that $m(\zeta, g)$ lies on the Weyl circle, and then it is the Weyl m-function of some self-adjoint extension $J(g)$ [1], [23].

It remains to show that $\sigma(J(g)) = \{\lambda_k\}_k$ and $\sigma(J_\infty(g)) = \{\mu_k\}_k$.

So we start from (4.16) and find the Weyl m-function of $J(g)$ using (4.19)

$$m(\zeta, g) = \int_{\mathbb{R}} \frac{d\rho(t)}{t - \zeta} = \sum_{k \in M} \frac{1}{\tau_k(\lambda_k - \zeta)} = \widetilde{m}(\zeta)\,.$$

But $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are the poles and zeros of $\widetilde{m}$ and then the eigenvalues of $J(g)$ and $J_\infty(g)$, respectively. □

For Jacobi operators semi-bounded from below, necessary and sufficient conditions are given in [14]. Note that Remark 3.5 can also be made here.

It is worth mentioning that, from (4.8) and (4.9), it follows that, when (b) is seen as a necessary condition, one could write

$$\lim_{\substack{\zeta \to \infty \\ \operatorname{Im}\zeta \geqslant \epsilon}} \zeta \prod_{k \in M}{}' \left(1 - \frac{\zeta}{\mu_k}\right)\left(1 - \frac{\zeta}{\lambda_k}\right)^{-1}, \qquad \epsilon > 0\,,$$

instead of (4.13).

Appendix

Boundary Conditions for Jacobi Operators

The difference expression γ defined by (2.1) and (2.2) can be written together in one equation with the help of some conditions. Indeed, consider the difference expression $\tilde{\gamma}$ given by

$$(\tilde{\gamma} f)_k = b_{k-1} f_{k-1} + q_k f_k + b_k f_{k+1}, \qquad k \in \mathbb{N} \quad (b_0 = 1)\,. \tag{A.1}$$

Clearly, γf is equal to $\tilde{\gamma} f$ provided that

$$f_0 = 0\,. \tag{A.2}$$

This requirement can be considered as a boundary condition for the difference equation (A.1). Notice that, although f_0 is not an element of the sequence $\{f_k\}_{k=1}^{\infty}$, it can be used to introduce boundary conditions for (A.1) which turn out to be completely analogous to the boundary conditions at the origin for the Sturm–Liouville operator on the semi-axis. We shall refer to (A.2) as the

Dirichlet boundary condition. Thus, J is the closure of the operator which acts on sequences of $l_{fin}(\mathbb{N})$ by (A.1) with the Dirichlet boundary condition (A.2).

Suppose that the deficiency indices of J are $(1,1)$ and consider now the following solution of (2.1)

$$\tilde{v}_k(\beta) := Q_{k-1}(0)\cos\beta + P_{k-1}(0)\sin\beta, \qquad \beta \in [0,\pi)\,.$$

Let us define the set

$$\left\{ f = \{f_k\}_{k=1}^{\infty} \in l_2(\mathbb{N}) : \tilde{\gamma} f \in l_2(\mathbb{N}),\ \lim_{n\to\infty} W_n(\tilde{v}(g),\, f) = 0 \right\}. \tag{A.3}$$

Notice that $D(g)$, defined by (2.7), coincides with (A.3) as long as $g = \cot\beta$. As pointed out in Section 2, the domain of every self-adjoint extension of J is given by (A.3) for some β, and different β's define different self-adjoint extensions [25]. Let us denote these self-adjoint extensions by $J(g)$, as we did in Section 2, bearing in mind that $g = \cot\beta$. The condition

$$\lim_{n\to\infty} W_n(\tilde{v}(g),\, f) = 0, \qquad f \in \operatorname{dom}(J^*) \tag{A.4}$$

determining the restriction of J^* is considered to be a boundary condition at infinity.

In analogy with the case of Sturm–Liouville operators one can define general boundary conditions at zero for the difference expression (A.1). To this end, consider the operator $J(\alpha, g)$ defined by the difference expression (A.1) with boundary condition at infinity (A.4) if necessary, and boundary condition 'at the origin'

$$f_1\cos\alpha + f_0\sin\alpha = 0, \qquad \alpha \in [0,\pi)\,. \tag{A.5}$$

Thus, if $\alpha \in (0,\pi)$,

$$J(\alpha, g) = J(g) - \cot\alpha\langle\cdot, e_1\rangle e_1\,.$$

Therefore $J(\alpha, g) = J_h(g)$, provided that $h = \cot\alpha$.

When $\alpha = 0$, from (A.5), one has $f_1 = 0$ and (A.1) is used to define the action of the operator for $k \geqslant 2$. $J(0, g)$ is said to be operator $J(g)$ with Neumann boundary condition. For this case we have that $J(0, g)$ is equal to $J_\infty(g)$.

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