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Normalization of Hamiltonian in the Generalized Photogravitational Restricted Three Body Problem with Poynting–Robertson Drag

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Abstract We have performed normalization of Hamiltonian in the generalized photogravitational restricted three body problem with Poynting–Robertson drag. In this problem we have taken bigger primary as source of radiation and smaller primary as an oblate spheroid. Whittaker's method is used to transform the second order part of the Hamiltonian into the normal form.

Keywords Normalization · Generalized photogravitational · RTBP · P–R Drag

Mathematics Subject Classification (2000) 70F15

1 Introduction

The restricted three body problem (RTBP) describes the motion of an infinitesimal mass moving under the gravitational effect of the two finite masses, called primaries, which move in circular orbits around their centre of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The classical

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restricted three body problem is generalized to include the force of radiation pressure, the P–R (Poynting–Robertson) effect and oblateness effect.

Poynting (1903) considered the effect of the absorption and subsequent re-emission of sunlight by small isolated particles in the solar system. His work was later modified by Robertson (1937) who used precise relativistic treatments of the first order in the ratio of the velocity of the particle to that of light. Chernikov (1970) and Schuerman (1980) who discussed the position as well as the stability of the Lagrangian equilibrium points when radiation pressure, P–R drag force are included. Murray (1994) systematically discussed the dynamical effect of general drag in the planar circular restricted three body problem, Liou et al. (1995) examined the effect of radiation pressure, P–R drag and solar wind drag in the restricted three body problem.

Moser's conditions (1962), Arnold's theorem (1961) and Liapunov's theorem (1956) played a significant role in deciding the nonlinear stability of an equilibrium point. Moser gave some modifications in Arnold's theorem. Then Deprit and Deprit-Bartholomé (1967) investigated the nonlinear stability of triangular points by applying Moser's modified version of Arnold's theorem (1961). Maciejewski and Gozdziewski (1991) described the normalization algorithms of Hamiltonian near an equilibrium point. Niedzielska (1994) investigated the nonlinear stability of the libration points in the photogravitational restricted three body problem. Mishra and Ishwar (1995) studied second order normalization in the generalized restricted problem of three bodies, smaller primary being an oblate spheroid. Ishwar (1997) studied nonlinear stability in the generalized restricted three body problem.

In this paper normalization of Hamiltonian is performed in the generalized photogravitational restricted three body problem with Poynting–Robertson drag. Whittaker's method is used to transform the second order part of the Hamiltonian into the normal form.

2 Location of Triangular Equilibrium Points

Equations of motion are

$$\ddot{x} - 2n\dot{y} = U_x, \quad \text{where,} \quad U_x = \frac{\partial U_1}{\partial x} - \frac{W_1 N_1}{r_1^2} \tag{1}$$

$$\ddot{y} + 2n\dot{x} = U_y, \quad U_y = \frac{\partial U_1}{\partial y} - \frac{W_1 N_2}{r_1^2} \tag{2}$$

$$U_1 = \frac{n^2(x^2+y^2)}{2} + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} \tag{3}$$

$$r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x+\mu-1)^2 + y^2, \quad n^2 = 1 + \tfrac{3}{2}A_2,$$
$$N_1 = \frac{(x+\mu)[(x+\mu)\dot{x}+y\dot{y}]}{r_1^2} + \dot{x} - ny, \quad N_2 = \frac{y[(x+\mu)\dot{x}+y\dot{y}]}{r_1^2} + \dot{y} + n(x+\mu)$$

$W_1 = \frac{(1-\mu)(1-q_1)}{c_d}$, $\mu = \frac{m_2}{m_1+m_2} \le \frac{1}{2}$, m_1, m_2 be the masses of the primaries, $A_2 = \frac{r_e^2 - r_p^2}{5r^2}$ be the oblateness coefficient, r_e and r_p be the equatorial and polar radii respectively, r be the distance between primaries, $q_1 = (1 - \frac{F_p}{F_g})$ be the mass reduction factor expressed in terms of the particle's radius a, density ρ and radiation pressure efficiency factor χ (in the C.G.S. system) i.e., $q_1 = 1 - \frac{5.6 \times 10^{-5}\chi}{a\rho}$. Assumption $q_1 = constant$ is equivalent to neglecting fluctuation in the beam of solar radiation and the effect of solar radiation, the effect of the planet's shadow, obviously $q_1 \leq 1$. Triangular equilibrium points are given by $U_x = 0$, $U_y = 0$, $z = 0$, $y \neq 0$, then we have

$$x_* = x_0 \left\{ 1 - \frac{nW_1 \left[(1-\mu)\left(1+\frac{5}{2}A_2\right) + \mu\left(1-\frac{A_2}{2}\right)\frac{\delta^2}{2} \right]}{3\mu(1-\mu)y_0 x_0} - \frac{\delta^2}{2}\frac{A_2}{x_0} \right\} \tag{4}$$

$$y_* = y_0 \left\{ 1 - \frac{nW_1 \delta^2 \left[2\mu - 1 - \mu\left(1-\frac{3A_2}{2}\right)\frac{\delta^2}{2} + 7(1-\mu)\frac{A_2}{2} \right]}{3\mu(1-\mu)y_0^3} - \frac{\delta^2\left(1-\frac{\delta^2}{2}\right)A_2}{y_0^2} \right\}^{1/2} \tag{5}$$

where $x_0 = \frac{\delta^2}{2} - \mu$, $y_0 = \pm\delta\left(1-\frac{\delta^2}{4}\right)^{1/2}$ and $\delta = q_1^{1/3}$, as in Kushvah and Ishwar (2006).

3 Normalization of H_2

We used Whittaker (1965) method for the transformation of H_2 into the normal form. The Lagrangian function of the problem can be written as

$$\begin{aligned} L &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2+y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} \\ &\quad + W_1 \left\{ \frac{(x+\mu)\dot{x} + y\dot{y}}{2r_1^2} - n \arctan\frac{y}{(x+\mu)} \right\} \end{aligned} \tag{6}$$

and the Hamiltonian is $H = -L + p_x\dot{x} + p_y\dot{y}$, where p_x, p_y are the momenta coordinates given by

$$P_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - ny + \frac{W_1}{2r_1^2}(x+\mu), \quad P_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + nx + \frac{W_1}{2r_1^2}y$$

For simplicity we suppose $q_1 = 1-\varepsilon$, with $|\varepsilon| \ll 1$ then coordinates of triangular equilibrium points can be written in the form

$$x = \frac{\gamma}{2} - \frac{\epsilon}{3} - \frac{A_2}{2} + \frac{A_2\epsilon}{3} - \frac{(9+\gamma)}{6\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1 \tag{7}$$

$$y = \frac{\sqrt{3}}{2}\left\{ 1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1 \right\} \tag{8}$$

where $\gamma = 1-2\mu$. We shift the origin to L_4. For that, we change $x \to x_* + x$ and $y \to y_* + y$. Let $a = x_* + \mu$, $b = y_*$ so that

$$a = \frac{1}{2}\left\{ 1 - \frac{2\epsilon}{3} - A_2 + \frac{2A_2\epsilon}{3} - \frac{(9+\gamma)}{3\sqrt{3}}nW_1 - \frac{8\gamma\epsilon}{27\sqrt{3}}nW_1 \right\} \tag{9}$$

$$b = \frac{\sqrt{3}}{2}\left\{ 1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1 \right\} \tag{10}$$

Expanding L in power series of x and y, we get

$$L = L_0 + L_1 + L_2 + L_3 + \cdots \tag{11}$$

$$H = H_0 + H_1 + H_2 + H_3 + \cdots = -L + p_x\dot{x} + p_y\dot{y} \tag{12}$$

where L_0, L_1, L_2, L_3, ... are

$$\begin{aligned} L_0 \; &= \frac{3}{2} - \frac{2\epsilon}{3} - \frac{\gamma\epsilon}{3} + \frac{3\gamma A_2}{4} - \frac{3A_2\epsilon}{2} - \gamma A_2 \\ &\quad - \frac{\sqrt{3}nW_1}{4} + \frac{2\gamma nW_1}{3\sqrt{3}} - \frac{n\epsilon W_1}{3\sqrt{3}} - \frac{23\epsilon\gamma nW_1}{54\sqrt{3}} - nW_1 \arctan\frac{b}{a} \end{aligned} \tag{13}$$

$$\begin{aligned} L_1 = &\dot{x}\left\{-\frac{\sqrt{3}}{2} + \frac{\epsilon}{3\sqrt{3}} - \frac{5A_2}{8\sqrt{3}} + \frac{7\epsilon A_2}{12\sqrt{3}} + \frac{4nW_1}{9} - \frac{\gamma nW_1}{18}\right\} \\ &+ \dot{y}\left\{\frac{1}{2} - \frac{\epsilon}{3} - \frac{A_2}{8} + \frac{\epsilon A_2}{12} - \frac{\gamma nW_1}{6\sqrt{3}} + \frac{2\epsilon nW_1}{3\sqrt{3}}\right\} \\ &- x\left\{-\frac{1}{2} + \frac{\gamma}{2} + \frac{9A_2}{8} + \frac{15\gamma A_2}{8} - \frac{35\epsilon A_2}{12} - \frac{29\gamma\epsilon A_2}{12} + \frac{3\sqrt{3}nW_1}{8} - \frac{5\epsilon nW_1}{12\sqrt{3}} - \frac{7\gamma\epsilon nW_1}{4\sqrt{3}}\right\} \\ &- y\left\{\frac{15\sqrt{3}A_2}{2} + \frac{9\sqrt{3}\gamma A_2}{8} - 2\sqrt{3}\epsilon A_2 - 2\sqrt{3}\gamma\epsilon A_2 - \frac{nW_1}{8} + \gamma nW_1 - \frac{43\epsilon nW_1}{36} - \frac{23\gamma\epsilon nW_1}{36}\right\} \end{aligned} \tag{14}$$

$$L_2 = \frac{(\dot{x}^2 + \dot{y}^2)}{2} + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2 + y^2) - Ex^2 - Fy^2 - Gxy \tag{15}$$

$$L_3 = -\frac{1}{3!}\left\{x^3T_1 + 3x^2yT_2 + 3xy^2T_3 + y^3T_4 + 6T_5\right\} \tag{16}$$

where

$$\begin{aligned} E = \frac{1}{16}\Bigg\{&2 - 6\epsilon - 3A_2 - \frac{31A_2\epsilon}{2} - \frac{(69+\gamma)}{6\sqrt{3}}nW_1 + \frac{2(307+75\gamma)\epsilon}{27\sqrt{3}}nW_1 \\ &+ \gamma\left\{2\epsilon + 12A_2 + \frac{A_2\epsilon}{3} + \frac{(199+17\gamma)}{6\sqrt{3}}nW_1 - \frac{2(226+99\gamma)\epsilon}{27\sqrt{3}}nW_1\right\}\Bigg\} \end{aligned} \tag{17}$$

$$\begin{aligned} F = \frac{-1}{16}\Bigg\{&10 - 2\epsilon + 21A_2 - \frac{717A_2\epsilon}{18} - \frac{(67+19\gamma)}{6\sqrt{3}}nW_1 + \frac{2(413-3\gamma)\epsilon}{27\sqrt{3}}nW_1 \\ &+ \gamma\left\{6\epsilon - \frac{293A_2\epsilon}{18} + \frac{(187+27\gamma)}{6\sqrt{3}}nW_1 - \frac{4(247+3\gamma)\epsilon}{27\sqrt{3}}nW_1\right\}\Bigg\} \end{aligned} \tag{18}$$

$$\begin{aligned} G = \frac{\sqrt{3}}{8}\Bigg\{&2\epsilon + 6A_2 - \frac{37A_2\epsilon}{2} - \frac{(13+\gamma)}{2\sqrt{3}}nW_1 + \frac{2(79-7\gamma)\epsilon}{27\sqrt{3}}nW_1 \\ &- \gamma\left\{6 - \frac{\epsilon}{3} + 13A_2 - \frac{33A_2\epsilon}{2} + \frac{(11-\gamma)}{2\sqrt{3}}nW_1 - \frac{(186-\gamma)\epsilon}{9\sqrt{3}}nW_1\right\}\Bigg\} \end{aligned} \tag{19}$$

$$T_1 = \frac{3}{16}\left[\frac{16}{3}\epsilon + 6A_2 - \frac{979}{18}A_2\epsilon + \frac{(143+9\gamma)}{6\sqrt{3}}nW_1 + \frac{(459+376\gamma)}{27\sqrt{3}}nW_1\epsilon \right.$$
$$\left. + \gamma\left\{14 + \frac{4\epsilon}{3} + 25A_2 - \frac{1507}{18}A_2\epsilon - \frac{(215+29\gamma)}{6\sqrt{3}}nW_1 - \frac{2(1174+169\gamma)}{27\sqrt{3}}nW_1\epsilon\right\}\right] \tag{20}$$

$$T_2 = \frac{3\sqrt{3}}{16}\left[14 - \frac{16}{3}\epsilon + \frac{A_2}{3} - \frac{367}{18}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{(959-136\gamma)}{27\sqrt{3}}nW_1\epsilon \right.$$
$$\left. + \gamma\left\{\frac{32\epsilon}{3} + 40A_2 - \frac{382}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(2519-24\gamma)}{27\sqrt{3}}nW_1\epsilon\right\}\right] \tag{21}$$

$$T_3 = \frac{-9}{16}\left[\frac{8}{3}\epsilon + \frac{203A_2}{6} - \frac{625}{54}A_2\epsilon - \frac{(105+15\gamma)}{18\sqrt{3}}nW_1 - \frac{(403-114\gamma)}{81\sqrt{3}}nW_1\epsilon \right.$$
$$\left. + \gamma\left\{2 - \frac{4\epsilon}{9} + \frac{55A_2}{2} - \frac{797}{54}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1 - \frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon\right\}\right] \tag{22}$$

$$T_4 = \frac{-9\sqrt{3}}{16}\left[2 - \frac{8}{3}\epsilon + \frac{23A_2}{3} - 44A_2\epsilon - \frac{(37+\gamma)}{18\sqrt{3}}nW_1 - \frac{(219+253\gamma)}{81\sqrt{3}}nW_1\epsilon \right.$$
$$\left. + \gamma\left\{4\epsilon + \frac{88}{27}A_2\epsilon + \frac{(241+45\gamma)}{18\sqrt{3}}nW_1 - \frac{(1558-126\gamma)}{81\sqrt{3}}nW_1\epsilon\right\}\right] \tag{23}$$

$$T_5 = \frac{W_1}{2(a^2+b^2)^3}\left[(a\dot{x}+b\dot{y})\left\{3(ax+by)-(bx-ay)^2\right\} - 2(x\dot{x}+y\dot{y})(ax+by)(a^2+b^2)\right] \tag{24}$$

The second order part H_2 of the corresponding Hamiltonian takes the form

$$H_2 = \frac{p_x^2+p_y^2}{2} + n(yp_x - xp_y) + Ex^2 + Fy^2 + Gxy \tag{25}$$

To investigate the stability of the motion, as in Whittaker (1965), we consider the following set of linear equations in the variables x, y:

$$\begin{array}{cc} -\lambda p_x = \frac{\partial H_2}{\partial x} & \lambda x = \frac{\partial H_2}{\partial p_x} \\ -\lambda p_y = \frac{\partial H_2}{\partial y} & \lambda y = \frac{\partial H_2}{\partial p_y} \\ \text{i.e.} \quad AX = 0 & \end{array} \tag{26}$$

$$X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2E & G & \lambda & -n \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{bmatrix} \tag{27}$$

Clearly $|A| = 0$, implies that the characteristic equation corresponding to Hamiltonian H_2 is given by

$$\lambda^4 + 2(E + F + n^2)\lambda^2 + 4EF - G^2 + n^4 - 2n^2(E + F) = 0 \tag{28}$$

This is characteristic equation whose discriminant is

$$D = 4(E + F + n^2)^2 - 4\{4EF - G^2 + n^4 - 2n^2(E + F)\} \tag{29}$$

Stability is assured only when $D > 0$. i.e.,

$$\begin{aligned}\mu < \mu_{c_0} &- 0.221895916277307669\epsilon + 2.1038871010983331A_2 \\ &+ 0.493433373141671349\epsilon A_2 + 0.704139054372097028nW_1 \\ &+ 0.401154273957540929n\epsilon W_1\end{aligned}$$

where $\mu_{c_0} = 0.0385208965045513718$. When $D > 0$ the roots $\pm i\omega_1$ and $\pm i\omega_2$ (ω_1, ω_2 being the long/short-periodic frequencies) are related to each other as

$$\begin{aligned}\omega_1^2 + \omega_2^2 &= 1 - \frac{\gamma\epsilon}{2} + \frac{3\gamma A_2}{2} + \frac{83\epsilon A_2}{12} + \frac{299\gamma\epsilon A_2}{144} - \frac{nW_1}{24\sqrt{3}} + \frac{5\gamma nW_1}{8\sqrt{3}} - \frac{53\epsilon nW_1}{54\sqrt{3}} \\ &\quad - \frac{5\gamma^2 nW_1}{24\sqrt{3}} + \frac{173\gamma\epsilon nW_1}{54\sqrt{3}} - \frac{3\gamma^2\epsilon nW_1}{36\sqrt{3}}\end{aligned} \tag{30}$$

$$\begin{aligned}\omega_1^2\omega_2^2 &= \frac{27}{16} - \frac{27\gamma^2}{16} + \frac{9\epsilon}{8} + \frac{9\gamma\epsilon}{8} - \frac{3\gamma^2\epsilon}{8} + \frac{117\gamma A_2}{16} - \frac{241\epsilon A_2}{32} + \frac{2515\gamma\epsilon A_2}{192} \\ &\quad + \frac{35nW_1}{16\sqrt{3}} - \frac{55\sqrt{3}\gamma nW_1}{16} - \frac{5\sqrt{3}\gamma^2 nW_1}{4} - \frac{1277\epsilon nW_1}{288\sqrt{3}} + \frac{5021\gamma\epsilon nW_1}{288\sqrt{3}} + \frac{991\gamma^2\epsilon nW_1}{48\sqrt{3}} \\ &\quad \left(0 < \omega_2 < \frac{1}{\sqrt{2}} < \omega_1 < 1\right)\end{aligned} \tag{31}$$

From (30) and (31) it may be noted that ω_j (j = 1, 2) satisfy

$$\begin{aligned}\gamma^2 &= 1 + \frac{4\epsilon}{9} - \frac{107\epsilon A_2}{27} + \frac{2\gamma\epsilon}{3} + \frac{1579\gamma\epsilon A_2}{324} - \frac{25nW_1}{27\sqrt{3}} - \frac{55\gamma nW_1}{9\sqrt{3}} + \frac{3809\epsilon nW_1}{486\sqrt{3}} + \frac{4961\gamma\epsilon nW_1}{486\sqrt{3}} \\ &\quad + \left(-\frac{16}{27} + \frac{32\epsilon}{243} + \frac{8\gamma\epsilon}{27} + \frac{208A_2}{81} - \frac{8\gamma A_2}{27} - \frac{4868\epsilon A_2}{729} - \frac{563\gamma\epsilon A_2}{243}\right. \\ &\qquad \left. + \frac{296nW_1}{243\sqrt{3}} - \frac{10\gamma nW_1}{27\sqrt{3}} - \frac{15892\epsilon nW_1}{2187\sqrt{3}} - \frac{1864\gamma\epsilon nW_1}{729\sqrt{3}}\right)\omega_j^2 \\ &\quad + \left(\frac{16}{27} - \frac{32\epsilon}{243} - \frac{208A_2}{81} - \frac{1880\epsilon A_2}{729} - \frac{2720nW_1}{2187\sqrt{3}} + \frac{49552\epsilon nW_1}{6561\sqrt{3}} - \frac{80\gamma\epsilon nW_1}{2187\sqrt{3}}\right)\omega_j^4\end{aligned} \tag{32}$$

Alternatively, it can also be seen that if $u = \omega_1\omega_2$, then equation (31) gives

$$\gamma^2 = 1 + \frac{4\epsilon}{9} - \frac{107\epsilon A_2}{27} - \frac{25nW_1}{27\sqrt{3}} + \frac{3809\epsilon nW_1}{486\sqrt{3}} + \gamma\left(\frac{2\epsilon}{3} + \frac{1579\epsilon A_2}{324} - \frac{55\gamma nW_1}{9\sqrt{3}} + \frac{4961\gamma\epsilon nW_1}{486\sqrt{3}}\right) + \left(-\frac{16}{27} + \frac{32\epsilon}{243} + \frac{208A_2}{81} - \frac{1880\epsilon A_2}{729} + \frac{320nW_1}{243\sqrt{3}} - \frac{15856\epsilon nW_1}{2187\sqrt{3}}\right)u^2 \tag{33}$$

Following the method for reducing H_2 to the normal form, as in Whittaker (1965), use the transformation

$$X = JT \quad \text{where } X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix}, \quad J = [J_{ij}]_{1 \le i,j \le 4}, \quad T = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix} \tag{34}$$

$$P_i = (2I_i\omega_i)^{1/2}\cos\phi_i, \quad Q_i = \left(\frac{2I_i}{\omega_i}\right)^{1/2}\sin\phi_i, \quad (i = 1, 2) \tag{35}$$

The transformation changes the second order part of the Hamiltonian into the normal form

$$H_2 = \omega_1 I_1 - \omega_2 I_2 \tag{36}$$

The general solution of the corresponding equations of motion are

$$I_i = \text{const.}, \quad \phi_i = \pm\omega_i + \text{const.}, \quad (i = 1, 2) \tag{37}$$

If the oscillations about L_4 are exactly linear, the equation (37) represent the integrals of motion and the corresponding orbits will be given by

$$x = J_{13}\sqrt{2\omega_1 I_1}\cos\phi_1 + J_{14}\sqrt{2\omega_2 I_2}\cos\phi_2 \tag{38}$$

$$y = J_{21}\sqrt{\frac{2I_1}{\omega_1}}\sin\phi_1 + J_{22}\sqrt{\frac{2I_2}{\omega_2}}\sin\phi_2 + J_{23}\sqrt{2I_1}\omega_1\cos\phi_1 + J_{24}\sqrt{2I_2}\omega_2\cos\phi_2 \tag{39}$$

where

$$J_{13} = \frac{l_1}{2\omega_1 k_1}\left\{1 - \frac{1}{2l_1^2}\left[\epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}}nW_1 - \frac{(431-3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] + \frac{\gamma}{2l_1^2}\left[3\epsilon - \frac{29A_2}{36} - \frac{(187+27\gamma)}{12\sqrt{3}}nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] - \frac{1}{2k_1^2}\left[\frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}}nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon\right] - \frac{\gamma}{4k_1^2}\left[\epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}}nW_1 - \frac{(266-93\gamma)}{54\sqrt{3}}nW_1\epsilon\right] + \frac{\epsilon}{4l_1^2k_1^2}\left[\frac{3A_2}{4} + \frac{(33+14\gamma)}{12\sqrt{3}}nW_1\right] + \frac{\gamma\epsilon}{8l_1^2k_1^2}\left[\frac{347A_2}{36} - \frac{(43-8\gamma)}{4\sqrt{3}}nW_1\right]\right\} \tag{40}$$

$$
\begin{aligned}
J_{14} = \frac{l_2}{2\omega_2 k_2}\Bigg\{ & 1 - \frac{1}{2l_2^2}\left[\epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}}nW_1 - \frac{(431-3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\gamma}{2l_2^2}\left[3\epsilon - \frac{293A_2}{36} + \frac{(187+27\gamma)}{12\sqrt{3}}nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{1}{2k_2^2}\left[\frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}}nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\
& + \frac{\gamma}{2k_2^2}\left[\epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}}nW_1 - \frac{(268-9\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\epsilon}{4l_2^2k_2^2}\left[\frac{33A_2}{4} + \frac{(1643-93\gamma)}{216\sqrt{3}}nW_1\right] + \frac{\gamma\epsilon}{4l_2^2k_2^2}\left[\frac{737A_2}{72} - \frac{(13+2\gamma)}{\sqrt{3}}nW_1\right]\Bigg\}
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
J_{21} = -\frac{4n\omega_1}{l_1k_1}\Bigg\{ & 1 + \frac{1}{2l_1^2}\left[\epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}}nW_1 - \frac{(413-3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\gamma}{2l_1^2}\left[3\epsilon - \frac{293A_2}{36} + \frac{(187+27\gamma)}{12\sqrt{3}}nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{1}{2k_1^2}\left[\frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}}nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\gamma}{4k_1^2}\left[\epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}}nW_1 - \frac{(268-93\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\
& + \frac{\epsilon}{8l_1^2k_1^2}\left[\frac{33A_2}{4} + \frac{(68-10\gamma)}{24\sqrt{3}}nW_1\right] + \frac{\gamma\epsilon}{8l_1^2k_1^2}\left[\frac{242A_2}{9} + \frac{(43-8\gamma)}{4\sqrt{3}}nW_1\right]\Bigg\}
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
J_{22} = \frac{4n\omega_2}{l_2k_2}\Bigg\{ & 1 + \frac{1}{2l_2^2}\left[\epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}}nW_1 - \frac{(413-3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\gamma}{2l_2^2}\left[3\epsilon - \frac{293A_2}{36} + \frac{(187+27\gamma)}{12\sqrt{3}}nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] \\
& + \frac{1}{2k_2^2}\left[\frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}}nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\gamma}{4k_2^2}\left[\epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}}nW_1 - \frac{(268-93\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\
& + \frac{\epsilon}{4l_2^2k_2^2}\left[\frac{33A_2}{4} + \frac{(34+5\gamma)}{12\sqrt{3}}nW_1\right] + \frac{\gamma\epsilon}{8l_2^2k_2^2}\left[\frac{75A_2}{2} + \frac{(43-8\gamma)}{4\sqrt{3}}nW_1\right]\Bigg\}
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
J_{23} = \frac{\sqrt{3}}{4\omega_1 l_1 k_1}\Bigg\{ & 2\epsilon + 6A_2 + \frac{37A_2\epsilon}{2} - \frac{(13+\gamma)}{2\sqrt{3}}nW_1 + \frac{2(79-7\gamma)}{9\sqrt{3}}nW_1\epsilon \\
& - \gamma\left[6 + \frac{2\epsilon}{3} + 13A_2 - \frac{33A_2\epsilon}{2} + \frac{(11-\gamma)}{2\sqrt{3}}nW_1 - \frac{(186-\gamma)}{9\sqrt{3}}nW_1\epsilon\right] \\
& + \frac{1}{2l_1^2}\left[51A_2 + \frac{(14+8\gamma)}{3\sqrt{3}}nW_1\right] - \frac{\epsilon}{k_1^2}\left[3A_2 + \frac{(19+6\gamma)}{6\sqrt{3}}nW_1\right] \\
& - \frac{\gamma}{2l_1^2}\left[6\epsilon + 135A_2 - \frac{808A_2\epsilon}{9} - \frac{(67+19\gamma)}{2\sqrt{3}}nW_1 - \frac{(755+19\gamma)}{9\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\gamma}{2k_1^2}\left[3\epsilon - 18A_2 - \frac{55A_2\epsilon}{4} - \frac{(1-9\gamma)}{4\sqrt{3}}nW_1 + \frac{(923-60\gamma)}{12\sqrt{3}}nW_1\epsilon\right] \\
& + \frac{\gamma\epsilon}{8l_1^2k_1^2}\left[\frac{9A_2}{2} + \frac{(34-5\gamma)}{2\sqrt{3}}nW_1\right]\Bigg\}
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
J_{24} = \frac{\sqrt{3}}{4\omega_2 l_2 k_2}\Bigg\{ & 2\epsilon + 6A_2 + \frac{37A_2\epsilon}{2} - \frac{(13+\gamma)}{2\sqrt{3}}nW_1 + \frac{2(79-7\gamma)}{9\sqrt{3}}nW_1\epsilon \\
& - \gamma\left[6 + \frac{2\epsilon}{3} + 13A_2 - \frac{33A_2\epsilon}{2} + \frac{(11-\gamma)}{2\sqrt{3}}nW_1 - \frac{(186-\gamma)}{9\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{1}{2l_2^2}\left[51A_2 + \frac{(14+8\gamma)}{3\sqrt{3}}nW_1\right] - \frac{\epsilon}{k_2^2}\left[3A_2 + \frac{(19+6\gamma)}{6\sqrt{3}}nW_1\right] \\
& - \frac{\gamma}{2l_2^2}\left[6\epsilon + 135A_2 - \frac{808A_2\epsilon}{9} - \frac{(67+19\gamma)}{2\sqrt{3}}nW_1 - \frac{(755+19\gamma)}{9\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\gamma}{2k_1^2}\left[3\epsilon - 18A_2 - \frac{55A_2\epsilon}{4} - \frac{(1-9\gamma)}{4\sqrt{3}}nW_1 + \frac{(923-60\gamma)}{12\sqrt{3}}nW_1\epsilon\right] \\
& - \frac{\gamma\epsilon}{4l_1^2k_1^2}\left[\frac{99A_2}{2} + \frac{(34-5\gamma)}{2\sqrt{3}}nW_1\right]\Bigg\}
\end{aligned} \tag{45}
$$

with $l_j^2 = 4\omega_j^2 + 9, (j = 1, 2)$ and $k_1^2 = 2\omega_1^2 - 1, k_2^2 = -2\omega_2^2 + 1$.

4 Conclusion

Using Whittaker (1965) method we have found that the second order part H_2 of the Hamiltonian is transformed into the normal form $H_2 = \omega_1 I_1 – \omega_2 I_2$.

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