Self-Adaptation in Collective Adaptive Systems

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Abstract An adaptive system is currently on spot: collective adaptive system (CAS), which is inspired by the socio-technical systems. CASs are characterized by a high degree of adaptation, giving them resilience in the face of perturbations. In CASs, highest degree of adaptation is *self-adaptation*. The overarching goal of CAS is to realize systems that are tightly entangled with humans and social structures. Meeting this grand challenge of CASs requires a fundamental approach to the notion of self-adaptation. To this end, taking advantage of the categorical approach we establish, in this paper, a firm formal basis for modeling self-adaptation in CASs.

1 Introduction

The socio-technical structure of our community increasingly depends on systems, which are built as a collection of varied agents and are tightly coupled with humans and social interrelations. Their agents more and more need to be able to develop, cooperate and work all by themselves as a part of an artificial community. Hence, for such collective adaptive systems (CASs), one of major challenges is how to support self-adaptation in the face of changing interactions [7, 8]. In other words, how does a CAS understand relevant interrelations and then self-adapt to become better able to live in its interactions?

Dealing with this grand challenge of CASs requires a well-founded modeling and in-depth analysis on the notion of self-adaptation. With this aim, we develop a firm formal approach in which a collective of entities is able to self-adapt its configuration and self-optimize its performance in the face of changing interactions [9].

CASs focus on self-adaptation in preference to build systems. In this view, we see that rigorously approaching to self-adaptation requests fundamental research in all aspects of the self-adaptation. As a novel development for the self-adaptation, we consider to formalize aspects of the self-adaptation taking advantage of categorical language, whose content is presented in this paper.

2 Outline

The paper is a reference material for readers who already have a basic understanding of CAS and are now ready to know the novel approach for formalizing self-adaptation in CAS using categorical language.

Formalization is presented in a straightforward fashion by discussing in detail the necessary components and briefly touching on the more advanced components. Several notes explaining how to use the formal aspects, including justifications needed in order to achieve the particular results, are presented.

We attempt to make the presentation as self-contained as possible, although familiarity with the notion of self-adaptation in CAS is assumed. Acquaintance with the algebra and the associated notion of categorical language is useful for recognizing the results, but is almost everywhere not strictly necessary.

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The rest of this paper is organized as follows: Sections 3 and 4 present the notions of collective adaptive systems (CASs) and self-adaptation, respectively. In Section 5, formal aspects of self-adaptation in CASs are developed in detail. In Section 6, we briefly discuss our development. Finally, a short summary is given in Section 7.

3 Collective adaptive systems (CASs)

We define collective adaptive systems (CASs) as the following among various definitions that have been offered by different researchers:

Definition 1 CASs are systems that consist of a collective of heterogeneous components, often called agents, that interact and adapt or learn.

Hence, CASs are characterized by a high degree of adaptation, giving them resilience in the face of perturbations. We see that, in CASs, highest degree of adaptation is *self-adaptation* and we are interested in approaches to this characteristic of CASs.

This definition is concerned with three major factors of CAS:

- *A collective of heterogeneous agents* is large enough to build up systems that are tightly entangled with humans and social structures. Their agents increasingly need to be able to evolve, collaborate and function as a part of an artificial society. More importantly, the agents interact dynamically, and their interactions are either physical or involving the exchange of information.
- *Interactions* are rich, non-linear and primarily, but not exclusively, with immediate neighbors. They can be recurrent, i.e. any interaction can feed back onto itself directly or after a number of intervening stages. CASs are dynamic networks of interactions
- *Self-adaptation* is the self-evolutionary process whereby a CAS becomes better able to live in its interactions.

4 Self-adaptation

An interesting aspect of CASs is that it makes distinction between self-adaptation (i.e. system-driven personalization and modifications) and self-adaptability (i.e. user-driven personalization and modifications). *Self-adaptedness* is the state of being self-adapted, i.e. the degree to which a CAS is able to live and reproduce in a given set of interactions. A *self-adaptive trait* is an aspect of the developmental pattern of the CAS which enables or enhances the probability of that CAS surviving and reproducing.

Formally, let self-* be the set of self-_'s. Each self-_ to be an element in self-* is called a *self-* facet* [6]. That is,

$$\text{self} - * = \{\text{self} - \_ \mid \text{self} - \_ \text{ is a self} - * \text{ facet}\} \quad (1)$$

Thus, self-adaptation is a facet of self-*, that is, self-adaptation is a member of self-*. In other words, using categorical language, this is written as $1 \xrightarrow{\text{self-adaptation}} \text{self-*}$. CASs are self-adaptive in that the individual and collective behavior mutate and self-organize corresponding to interactions. Self-adaptation indicates that CAS is a mimicry of socio-technical systems.

5 Self-adaptation in CASs

Self-adaptation is achieved when CASs are constructed. In this way, we start with considering CASs by categorical approach in this paper. CASs we want to abstract are intuitionally multiple partial morphism applications, such as

$$s_0 \xrightarrow{\sigma_0} s_1 \xrightarrow{\sigma_1} s_2 \xrightarrow{\sigma_2} s_3 \cdots \quad (2)$$

where

- All indexes $i \in T \ (= \mathbb{N} \cup \{0\})$ refer to times,
- s is a self-adaptedness representing a state of CAS in the set, denoted by CAS, of states. s_i is the state s at the time i,
- σ is an interaction in the set, denoted by $Inter$, of interactions. σ_i is interaction σ at the time i, which makes change of the state s_i to become s_{i+1}.

The representation of Eq. 2 can be understood as

$$\ldots s_2(s_1(s_0())) = \ldots s_2(s_1(\sigma_0)) = \ldots s_2(\sigma_1) = \ldots \sigma_2 \quad (3)$$

The self-adaptation in Eq. 2 can also be descriptively drawn as

$$s_0() \ \sigma_0 \ \sigma_1 \ \sigma_2 \ \cdots \longmapsto s_1(\sigma_0) \ \sigma_1 \ \sigma_2 \ \cdots \longmapsto \sigma_0 \ s_2(\sigma_1) \ \sigma_2 \ \cdots \quad (4)$$

or, in another representation

$$\xrightarrow{s_0} \sigma_0 \ \sigma_1 \ \sigma_2 \ \cdots \longmapsto \sigma_0 \xrightarrow{s_1} \sigma_1 \ \sigma_2 \ \cdots \longmapsto \sigma_0 \ \sigma_1 \xrightarrow{s_2} \sigma_2 \ \cdots \quad (5)$$

Note that in Eqs. 4 and 5, we want to represent the above-mentioned self-adaptation of CAS based on interaction where each step of the self-adaptation is an application of

unary partial morphism $1 \xrightarrow{s_i} CAS$ on $1 \xrightarrow{\sigma_{i-1}} Inter$ for all i in T.

The self-adaptation, in Eqs. 4 and 5, describes self-evolutionary process of CASs including the self-adaptation steps to change *configurations* of the system.

Definition 2 (**Configuration of CAS**) We define a configuration of CAS at a self-adaptation step to be a member of the set $CAS \times Inter^{i \in T}$, where $Inter^{i \in T}$ stands for

$$Inter^{i \in T} = \underbrace{Inter \times Inter \times \ldots \times Inter}_{i \text{ times}} \qquad (6)$$

As we know, when we combine sets by multiplication, each set is a *factor* and the resulting set is the *product*. Hence, each set $Inter$ is a factor of the resulting set $Inter^{i \in T}$, CAS and $Inter^{i \in T}$ are two factors of the set $CAS \times Inter^{i \in T}$. The definition of multiplication of sets is very natural. Just remember that a product is not just a set, but a set with two morphisms as in

- When i=2 then $Inter^2 = \{<\sigma_1, \sigma_2 > | \sigma_1, \sigma_2 \in Inter\}$ is obtained by
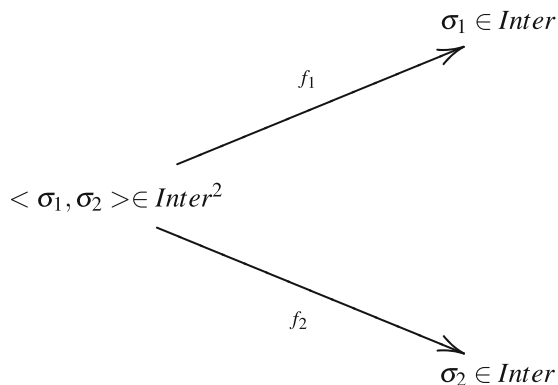
- When i=3 then $Inter^3 = \{<< \sigma_1, \sigma_2 >, \sigma_3 > | \sigma_1, \sigma_2, \sigma_3 \in Inter\}$ is obtained by
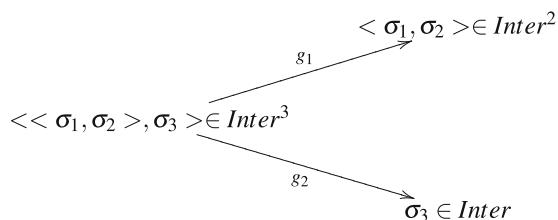
Specially, we have

- If i=0 then $Inter^0 = \{\}$
- If i=1 then $Inter^1 = Inter = \{\sigma_1 | \sigma_1 \in Inter\}$

We hope that these diagrams seem suggestive to readers. Our aim is to learn to use them as precise tools of understanding and reasoning, not merely as intuitive guides.

The self-adaptation paradigm, which we want to approach to, is based on mapping a configuration to another. Let us see the following examples

Example 1: A specific self-adaptation can be specified by the following morphism:

$$Self - A : (CAS \times Inter) \longrightarrow CAS \qquad (7)$$

(i.e., $Self - A : (CAS \times Inter^1) \longrightarrow (CAS \times Inter^0)$ or denoted by $Self - A(CAS \times Inter, CAS)$)

Example 2: Another specific self-adaptation can be specified by

$$Self - A : (CAS \times Inter) \longrightarrow (CAS \times Inter) \qquad (8)$$

(i.e., $Self - A : (CAS \times Inter^1) \longrightarrow (CAS \times Inter^1)$ or denoted by $Self - A(CAS \times Inter, CAS \times Inter)$)

Example 3: Again, we can also specify another specific self-adaptation as

$$Self - A : (CAS \times Inter^n) \longrightarrow (CAS \times Inter) \qquad (9)$$

(i.e., $Self - A : (CAS \times Inter^n) \longrightarrow (CAS \times Inter^1)$ or denoted by $Self - A(CAS \times Inter^n, CAS \times Inter)$)

and we can, in the completely same way, do for any other specific self-adaptation.

Definition 3 (**Self-adaptation**) Generally, an arbitrary self-adaptation is specified by

$$Self - A : (CAS \times Inter^{i \in T}) \longrightarrow (CAS \times Inter^{j \in T}) \qquad (10)$$

Now, let us investigate the following corollaries.

Corollary 1 (**Self-adaptation in CASs**) *Morphism Self-A in* Eq. 10 *defines self-adaptation in* CASs

Proof This stems from Eq. 10 and the fact that self-adaptation emerges through interactions in CASs. □

Morphism *Self-A* is called a self-adaptation. Morphism *Self-A* in Eq. 10 defines a set $\{Self - A_{k \in \mathbb{N}}\}$ of mappings such that

$$Self - A_{k \in \mathbb{N}} : (CAS \times Inter^{i \in T}) \longrightarrow (CAS \times Inter^{j \in T}) \qquad (11)$$

Corollary 2 (**Self-adaptive traits in CASs**) *Set* $\{Self - A_{k \in \mathbb{N}}\}$ *in* Eq. 11 *defines self-adaptive traits in* CASs. *Each mapping Self-A$_{k \in \mathbb{N}}$ is called a self-adaptive trait.*

Proof This originates as the result of the truth that self-adaptation is the set of self-adaptive traits. □

For further well-founded investigation, we can construct a category of the sets of CAS configurations and establish *Self-A*-algebras as described in the following corollaries.

Corollary 3 (**Category of the sets of CAS configurations**) *Sets of* CAS *configurations as in definition 2 define a category.*

Proof In fact, let **Cat(CAS)** be such a category of the sets of CAS configurations, whose structure is constructed as follows:

- Each set of configurations $CAS \times Inter^{i \in T}$ defines an object.
  That is, $Obj(\mathbf{Cat(CAS)}) = \{CAS \times Inter^{i \in T}\}$.
- Each $Self - A$ defines a morphism.
  That is, $Arc(\mathbf{Cat(CAS)}) = \{Self - A : (CAS \times Inter^{i \in T})(CAS \times Inter^{j \in T})\}$.

It is easy to check that identity in Eq. 30 and associativity in Eq. 31 on all $Self - A$s are satisfied. □

Corollary 4 (**Self-A-algebra(CAS)**) *Each morphism Self-A in the category* **Cat(CAS)** *defines an algebra, so-called Self-A-algebra(CAS).*

Proof This stems from definition of T-algebra, see Appendix, where functor T is defined such that $T = \biguplus \{Self - A\}$. Note that the notation $\biguplus$ stands for *disjoint union* or *coproduct*. □

With the result of corollary 4, we obtain a compact formal definition of CAS as in

Definition 4 (**CAS**) Each $Self-A$-algebra(CAS) defines a CAS

Both CAS and $Inter$ may be infinite. If both CAS and $Inter$ are finite, then we have a finite CAS, otherwise we have an infinite CAS.

At this point we can extend the representation of CASs in Eq. 2 to become *extended* CASs (ECASs) as follows

$$s_0 \xrightarrow{\sigma_0 \mid x_0} s_1 \xrightarrow{\sigma_1 \mid x_1} s_2 \xrightarrow{\sigma_2 \mid x_2} s_3 \cdots \quad (12)$$

where

- All indexes i in T, s_i and σ_i are similar in meaning to the ones mentioned in Eq. 2
- x_i is a real number that can be thought of as the *multiplicity* (or *weight*) with which the self-adaptation from s_i to s_{i+1} occurs.

Self-adaptation of ECASs in diagram (12) can be separated into two complementary parts as follows:

$$s_0 \xrightarrow{\sigma_0} s_1 \xrightarrow{\sigma_1} s_2 \xrightarrow{\sigma_2} s_3 \cdots \quad (13)$$

and

$$s_0 \xrightarrow{x_0} s_1 \xrightarrow{x_1} s_2 \xrightarrow{x_2} s_3 \cdots \quad (14)$$

On the one hand, diagram (13) emphasizes $1 \xrightarrow{\sigma_i} Inter$ for all i in T, in the self-adaptation. This allows us to discover conveniently sequence of σ_i as series of interactions. On the other hand, diagram (14) gives rise to $1 \xrightarrow{x_i} \mathbb{R}$, for all i in T, as weights of the series of interactions in the self-adaptation to support an evaluation of weight-based quantitative behaviors of the series of interactions.

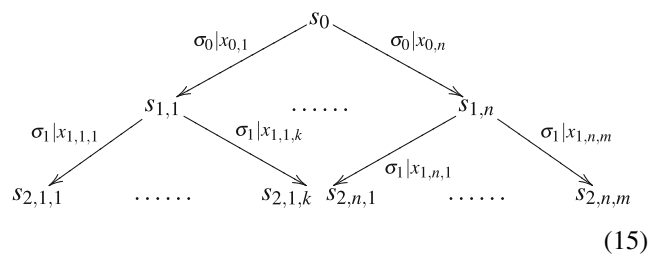Some first steps of the self-adaptation in Eq. 12 can also be descriptively drawn as

$$
\begin{array}{c}
s_0 \\
\sigma_0 \mid x_{0,1} \swarrow \qquad \searrow \sigma_0 \mid x_{0,n} \\
s_{1,1} \quad \cdots\cdots \quad s_{1,n} \\
\sigma_1 \mid x_{1,1,1} \downarrow \quad \sigma_1 \mid x_{1,1,k} \searrow \quad \sigma_1 \mid x_{1,n,1} \swarrow \quad \searrow \sigma_1 \mid x_{1,n,m} \\
s_{2,1,1} \quad \cdots\cdots \quad s_{2,1,k} \quad s_{2,n,1} \quad \cdots\cdots \quad s_{2,n,m}
\end{array}
$$
$$(15)$$

Diagram (15) is thought of as

- For the first step,
  $$s_1 \in \{s_{1,1}, \ldots, s_{1,n}\} \subset CAS$$
  and
  $$x_0 \in \{x_{0,1}, \ldots, x_{0,n}\} \subset \mathbb{R}$$
- For the second step,
  $$s_2 \in \{s_{2,1,1}, \ldots, s_{2,1,k}\} \cup \ldots \cup \{s_{2,n,1}, \ldots, s_{2,n,m}\} \subset CAS$$
  and
  $$x_1 \in \{x_{1,1,1}, \ldots, x_{1,1,k}\} \cup \ldots \cup \{x_{1,n,1}, \ldots, x_{1,n,m}\} \subset \mathbb{R}$$

and the meaning of Eq. 12 is viewed as the following morphism.

$$Self - A : (CAS \times Inter) \longrightarrow (CAS \longrightarrow \mathbb{R}) \quad (16)$$

The self-adaptation morphism *Self-A* in Eq. 16 is nondeterministic and this can be explained as follows: *Self-A* assigns to each configuration in $CAS \times Inter$ a morphism $CAS \longrightarrow \mathbb{R}$ that can be seen as a kind of *nondeterministic configuration* (or so-called *distributed configuration*) and specifies for every state s' in CAS a multiplicity (or weight) $Self-A(<s, \sigma>)(s')$ in $\mathbb{R}$.

This nondeterminism of self-adaptation makes extension in representation of the categorical models mentioned above. Let us see the following examples

Example 4: A specific self-adaptation, which is specified by the following morphism, is an extension of Eq. 7:

$$Self - A : (CAS \times Inter) \longrightarrow (CAS \longrightarrow \mathbb{R}) \quad (17)$$

(i.e., $Self\text{-}A:(CAS \times Inter^1) \longrightarrow ((CAS \times Inter^0) \longrightarrow \mathbb{R})$)

or denoted by $Self - A ((CAS \times Inter), (CAS \longrightarrow \mathbb{R})))$

Example 5: The model in Eq. 8 extended for self-adaptation is specified by

$$Self - A : (CAS \times Inter) \longrightarrow ((CAS \times Inter) \longrightarrow \mathbb{R}) \tag{18}$$

(i.e., $Self\text{-}A:(CAS \times Inter^1) \longrightarrow ((CAS \times Inter^1) \longrightarrow \mathbb{R})$)

or denoted by $Self - A((CAS \times Inter), ((CAS \times Inter) \longrightarrow \mathbb{R})))$

Example 6: Again, we specify another specific self-adaptation as an extension of Eq. 9 in

$$Self - A : (CAS \times Inter^n) \longrightarrow ((CAS \times Inter) \longrightarrow \mathbb{R}) \tag{19}$$

(i.e., $Self - A : (CAS \times Inter^n) \longrightarrow ((CAS \times Inter^1) \longrightarrow \mathbb{R})$)

or denoted by $Self - A(\longrightarrow (CAS \times Inter^n), ((CAS \times Inter) \longrightarrow \mathbb{R})))$

and, in the completely same way, we do for an arbitrary self-adaptation as in

Definition 5 (**Self-adaptation**) Generally, an arbitrary self-adaptation is specified by

$$Self - A : (CAS \times Inter^{i \in T}) \longrightarrow ((CAS \times Inter^{j \in T}) \longrightarrow \mathbb{R}) \tag{20}$$

We have the following corollaries.

Corollary 5 (**Self-adaptation in ECASs**) *Morphism Self-A in* Eq. 20 *defines self-adaptation in* ECASs

Proof This stems from Eq. 20 and the fact that self-adaptation emerges through interactions in ECASs. □

Morphism *Self-A* in Eq. 20 defines a set $\{Self - A_{k \in \mathbb{N}}\}$ of mappings such that

$$\{Self - A_{k \in \mathbb{N}}\} : (CAS \times Inter^{i \in T}) \longrightarrow ((CAS \times Inter^{j \in T}) \longrightarrow \mathbb{R}) \tag{21}$$

Corollary 6 (**Self-adaptive traits in ECASs**) *Set* $\{Self - A_{k \in \mathbb{N}}\}$ *in* Eq. 21 *defines self-adaptive traits in* ECASs. *Each mapping Self-$A_{k \in \mathbb{N}}$ is called a self-adaptive trait.*

Proof This originates as the result of the truth that self-adaptation is the set of self-adaptive traits. □

Corollary 7 (**Category of the sets of ECAS configurations**) *Category* **Cat(CAS)** *equipped with structure* $(CAS \times Inter^{i \in T}) \longrightarrow ((CAS \times Inter^{j \in T}) \longrightarrow \mathbb{R})$ *defines a category* **Cat(ECAS)** *of the sets of* ECAS *configurations.*

Proof This result comes immediately from corollary 3. □

Corollary 8 (***Self-A*-algebra(ECAS)**) *Structure* $(CAS \times Inter^{i \in T}) \longrightarrow ((CAS \times Inter^{j \in T}) \longrightarrow \mathbb{R})$ *in the category* **Cat(ECAS)** *defines an algebra, so-called Self-A-algebra(ECAS).*

Proof This originates from definition on T-algebra in Appendix, where functor T is defined such that $\mathsf{T} = \biguplus\{Self - A\}$ (similar to corollary 4) with *Self-A* defined in Eq. 20. □

With this result of corollary 8, we obtain a compact formal definition of ECAS as in

Definition 6 (**ECAS**) Each *Self-A*-algebra(ECAS) defines an ECAS

Moreover, we have the following corollary to obtain a significant relationship between CASs and ECASs.

Corollary 9 (**Relationship between CASs and ECASs**) CASs *are just of specific* ECASs. *In other words, using categorical language,* CASs $\overset{\subset}{\longrightarrow}$ ECASs

Proof In fact, by the self-adaptation morphism in Eq. 20 of ECASs, let f be the morphism $f : (CAS \times Inter^{j \in T}) \longrightarrow \mathbb{R}$, *Conf* be $CAS \times Inter^{j \in T}$ and the finite set $\mathbb{R}(Conf) = \{1 \xrightarrow{c} Conf | f(c) \neq 0\} \xrightarrow{\subseteq} Conf$. Hence it follows that when $\exists! 1 \xrightarrow{c} Conf : f(c) = 1$ but $\forall c' \neq c : f(c') = 0$ (i.e., the set $\mathbb{R}(Conf)$ is a singleton set of configuration with weight of 1. Note that the notation $\exists!$ is read as "exist only") then Eq. 20 becomes the self-adaptation morphism of CASs as in Eq. 10. In other words, in the case, ECASs will become CASs. □

6 Discussions

The aim of this paper has been both to give an in-depth analysis as well as to present the new material on the notion of self-adaptation in both CASs and ECASs.

In the considered context, we apply the category theory, which deals in an abstract way with algebraic objects and relationships between them for specifying interaction behaviors in both CASs and ECASs. For modeling, analyzing and verifying the interaction behaviors, category theory

is much better-approaching than other ones such as *process algebras* (or *process calculi*), FSM (Finite State Machine) or UML (Unified Modeling Language). In fact, the categorical approach becomes more powerful since process algebras and FSM are just of algebraic objects of category and UML is really a semi-formal approach. Categories were first described by Samuel Eilenberg and Saunders Mac Lane in 1945 [4], but have since grown substantially to become a branch of modern mathematics. Category theory spreads its influence over the development of both mathematics and theoretical computer science. The categorical structures themselves are still the subject of active research, including work to increase their range of practical applicability.

7 Conclusions

In this paper, we have rigorously approached to the notion of self-adaptation in both CASs and ECASs from which formal aspects of the self-adaptation emerge.

We have investigated configuration of both CASs and ECASs at every self-adaptation step as a member in the set $CAS \times Inter^{i \in T}$, then self-adaptation as a morphism from a configuration to another and, moreover, self-adaptation as a set of self-adaptive traits. By that way we have discovered some significant properties of the self-adaptation.

Appendix

We recall some concepts from the category theory [1–5] used in this paper.

What is a category?

■ A category **C** can be viewed as a graph $(Obj(\mathbf{C}), Arc(\mathbf{C}), s, t)$, where

- $Obj(\mathbf{C})$ is the set of nodes we call *objects*,
- $Arc(\mathbf{C})$ is the set of edges we call *morphisms* and
- $s, t : Arc(\mathbf{C}) \longrightarrow Obj(\mathbf{C})$ are two maps called *source* (or *domain*) and *target* (or *codomain*), respectively.

We write $f : \mathcal{X} \longrightarrow \mathcal{Y}$ when f is in $Arc(\mathbf{C})$ and $s(f) = \mathcal{X}$ and $t(f) = \mathcal{Y}$.

Explanation on terminology An object in the category is an algebraic structure such as a set. We are probably familiar with some notations for finite sets: {*Student A, Student B, Student C*} is a name for the set whose three elements are *Student A, Student B, Student C*. Note that the order in which the elements are listed is irrelevant.

A morphism f in the category consists of three things: a set $\mathcal{X}$, called the source of the morphism; a set $\mathcal{Y}$, called the target of the morphism and a rule assigning to each element x in the source an element y in the target. This y is denoted by $f(x)$, read "f of x". Note that the morphism is also called the *map*, *function*, *transformation*, *operator* or *arrow*. For example, let $\mathcal{X} = \{$*Student A, Student B, Student C*$\}$, $\mathcal{Y} = \{$*Math, Physics, Chemistry, History*$\}$ and let f assign each student his or her favorite subject. The following internal diagram is an illustration.

$$\{Student\ A \qquad Student\ B \qquad Student\ C\}$$
$$f = \text{favorite subject}$$
$$\{Math \qquad Chemistry \qquad History \qquad Physics\} \tag{22}$$
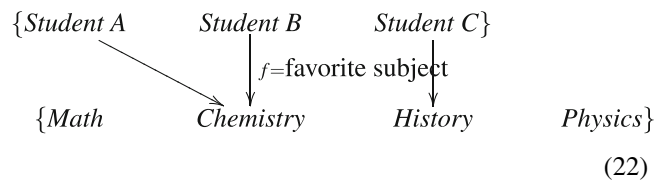
This states that the favorite subject of the *Student C* is *History*, written by $f(Student\ C) = History$, while *Student A* and *Student B* prefer *Chemistry*. There are some important properties of any morphism

- From each element in the source {*Student A, Student B, Student C*}, there is exactly one arrow leaving.
- To an element in the target {*Math, Physics, Chemistry, History*}, there may be zero, one or more arrows arriving.

It is possible that the source and target of the morphism could be the same set. The following internal diagram is an example.

$$\{Student\ A \qquad Student\ B \qquad Student\ C\}$$
$$e = \text{favorite classmate}$$
$$\{Student\ A \qquad Student\ B \qquad Student\ C\} \tag{23}$$
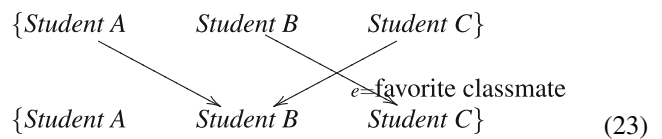
and, in the case, the morphism is called an *endomorphism* whose representation is available as in

$$\{Student\ A \longrightarrow Student\ B \longleftarrow Student\ C\} \tag{24}$$
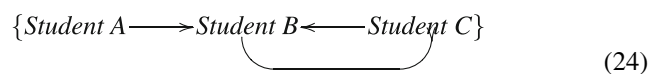
■ Associated with each object $\mathcal{X}$ in $Obj(\mathbf{C})$, there is a morphism $1_{\mathcal{X}} = \mathcal{X} \longrightarrow \mathcal{X}$, called the *identity* morphism on $\mathcal{X}$, and to each pair of morphisms $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and $g : \mathcal{Y} \longrightarrow \mathcal{Z}$, there is an associated morphism $f; g : \mathcal{X} \longrightarrow \mathcal{Z}$, called the *composition* of f with g. The representations in

Eq. 25 include the external diagrams of identity morphism and composition of morphisms.

$$1_{\mathcal{X}} \qquad \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \atop f;g \qquad (25)$$

Explanation on terminology Here are the corresponding internal diagrams of the identity morphism.

$$\{Student\ A \qquad Student\ B \qquad Student\ C\} \atop \downarrow \qquad \downarrow 1_{\mathcal{X}} \qquad \downarrow \atop \{Student\ A \qquad Student\ B \qquad Student\ C\} \qquad (26)$$

Or

$$\{Student\ A \qquad Student\ B \qquad Student\ C\} \qquad (27)$$

And here, the composition of morphisms is described in the internal diagram

$$\{Student\ A \qquad Student\ B \qquad Student\ C\}$$
$$e = \text{favorite classmate}$$
$$\{Student\ A \qquad Student\ B \qquad Student\ C\}$$
$$f = \text{favorite subject}$$
$$\{Math \qquad Chemistry \qquad History \qquad Physics\} \qquad (28)$$

Or, in the external diagram $\mathcal{X} \xrightarrow{e} \mathcal{X} \xrightarrow{f} \mathcal{Y}$ By the diagram (28), we can obtain answers for the question "What should each student support to his or her favorite classmate for subject?". In fact, the answers are such as " *Student A* likes *Student B*, *Student B* likes *Chemistry*, so *Student A* should support *Chemistry*", "*Student B* likes *Student C*, *Student C* likes *History*, so *Student B* should support *History*" and "*Student C* likes *Student B*, *Student B* likes *Chemistry*, so *Student C* should support *Chemistry*".

The composition of two morphisms e and f means that e and f are combined to obtain a third morphism $\mathcal{X} \xrightarrow{e;f} \mathcal{Y}$ This is represented in the following internal diagram.

$$\{Student\ A \qquad Student\ B \qquad Student\ C\}$$
$$e;f$$
$$\{Math \qquad Chemistry \qquad History \qquad Physics\} \qquad (29)$$

where, for example, $e; f(Student\ B) = History$ is read as "the favorite subject of the favorite classmate of *Student B* is *History*".

■ The following equation must hold for all objects $\mathcal{X}$, $\mathcal{Y}$ in $Obj(\mathbf{C})$ and morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ in $Arc(\mathbf{C})$:

$$Identity: \qquad 1_{\mathcal{X}}; f = f = f; 1_{\mathcal{Y}} \qquad (30)$$

$$1_{\mathcal{X}} \mathcal{X} \xrightarrow{f} \mathcal{Y} \quad = \quad \mathcal{X} \xrightarrow{f} \mathcal{Y} \quad = \quad \mathcal{X} \xrightarrow{f} \mathcal{Y} 1_{\mathcal{Y}}$$

The following equation must hold for all objects $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ in $Obj(\mathbf{C})$ and morphisms $f : \mathcal{X} \longrightarrow \mathcal{Y}$, $g : \mathcal{Y} \longrightarrow \mathcal{Z}$ and $h : \mathcal{Z} \longrightarrow \mathcal{T}$ in $Arc(\mathbf{C})$:

$$Associativity: \qquad (f; g); h = f; (g; h) \qquad (31)$$

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \xrightarrow{h} \mathcal{T} \atop f;g \quad = \quad \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \xrightarrow{h} \mathcal{T} \atop g;h$$

Isomorphism

A morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ in the category $\mathbf{C}$ is an *isomorphism* if there exists a morphism $g : \mathcal{Y} \longrightarrow \mathcal{X}$ in that category such that $f; g = 1_{\mathcal{X}}$ and $g; f = 1_{\mathcal{Y}}$.

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{X} \atop f;g=1_{\mathcal{X}} \qquad \text{and} \qquad \mathcal{Y} \xrightarrow{g} \mathcal{X} \xrightarrow{f} \mathcal{Y} \atop g;f=1_{\mathcal{Y}} \qquad (32)$$

That is, if the following diagram commutes.

$$1_{\mathcal{X}} \mathcal{X} \overset{g}{\underset{f}{\rightleftarrows}} \mathcal{Y} 1_{\mathcal{Y}} \qquad (33)$$

Element of a set

For any set A, $x \in A$ iff $1 \xrightarrow{x} A$ (or $x : 1 \longrightarrow A$) where 1 denotes a singleton set. Focus on one element of {*Math, Physics, Chemistry, History*}, say {*subject*}, and call this set "1". Let us see what the morphisms from 1 to {*Math, Physics, Chemistry, History*} are. There are exactly four of them.

$$\{subject\} \qquad \{Math \qquad Chemistry \qquad History \qquad Physics\}$$
$$Math$$
$$(34)$$

$$\{subject\} \qquad \{Math \qquad Chemistry \qquad History \qquad Physics\}$$
$$Chemistry$$
$$(35)$$

$$\{subject\} \qquad \{Math \qquad Chemistry \qquad History \qquad Physics\}$$
$$\underbrace{\hspace{5cm}}_{History}$$

$$(36)$$

$$\{subject\} \qquad \{Math \qquad Chemistry \qquad History \qquad Physics\}$$
$$\underbrace{\hspace{5cm}}_{Physics}$$

$$(37)$$

By this way, we can write $1 \xrightarrow{2} \mathbb{N}$ (or $2 : 1 \longrightarrow \mathbb{N}$) for $2 \in \mathbb{N}$, $1 \xrightarrow{i} \mathbb{N}$ (or $i : 1 \longrightarrow \mathbb{N}$) for $i \in \mathbb{N}$ and so on.

Functor

Functor is a special type of mapping between categories. Functor from a category to itself is called an *endofunctor*. Note that the functors are also viewed as morphisms in a category, whose objects are smaller categories.

T-algebra

Let **C** be a category, $\mathcal{A}$ an object in $Obj(\mathbf{C})$, $\mathsf{T} : \mathbf{C} \longrightarrow \mathbf{C}$ an endofunctor and f a morphism $\mathsf{T}(\mathcal{A}) \xrightarrow{f} \mathcal{A}$; then T-*algebra* is a pair $\langle \mathcal{A}, f \rangle$. $Obj(\mathbf{C})$ is called a *carrier* of the algebra and T a *signature* of the algebra.

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