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Range Assignment for Biconnectivity and *k*-Edge Connectivity in Wireless Ad Hoc Networks*

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Abstract. Depending on whether bidirectional links or unidirectional links are used for communications, the network topology under a given range assignment is either an undirected graph referred to as the bidirectional topology, or a directed graph referred to as the unidirectional topology. The Min-Power Bidirectional (resp., Unidirectional) k-Node Connectivity problem seeks a range assignment of minimum total power subject to the constraint that the produced bidirectional (resp. unidirectional) topology is k-vertex connected. Similarly, the Min-Power Bidirectional (resp., Unidirectional) k-Edge Connectivity problem seeks a range assignment of minimum total power subject to the constraint the produced bidirectional (resp., unidirectional) topology is k-edge connected.

The Min-Power Bidirectional Biconnectivity problem and the Min-Power Bidirectional Edge-Biconnectivity problem have been studied by Lloyd et al. [23]. They show that range assignment based the approximation algorithm of Khuller and Raghavachari [18], which we refer to as *Algorithm KR*, has an approximation ratio of at most $2(2 - 2/n)(2 + 1/n)$ for Min-Power Bidirectional Biconnectivity, and range assignment based on the approximation algorithm of Khuller and Vishkin [19], which we refer to as *Algorithm KV*, has an approximation ratio of at most $8(1 - 1/n)$ for Min-Power Bidirectional Edge-Biconnectivity.

In this paper, we first establish the NP-hardness of Min-Power Bidirectional (Edge-) Biconnectivity. Then we show that *Algorithm KR* has an approximation ratio of at most 4 for both Min-Power Bidirectional Biconnectivity and Min-Power Unidirectional Biconnectivity, and *Algorithm KV* has an approximation ratio of at most $2k$ for both Min-Power Bidirectional k-Edge Connectivity and Min-Power Unidirectional k-Edge Connectivity. We also propose a new simple constant-approximation algorithm for both Min-Power Bidirectional Biconnectivity and Min-Power Unidirectional Biconnectivity. This new algorithm applies only to Euclidean instances, but is best suited for distributed implementation.

Keywords: topology control, approximation algorithms, NP-hardness, power assignment, distributed algorithm

1. Introduction

Recently, range assignment problems for wireless ad hoc networks have been studied extensively. In wireless ad hoc networks no wired backbone infrastructure is installed and communication sessions are achieved either through a single-hop transmission if the communication parties are close enough, or through relaying by intermediate nodes otherwise. Omnidirectional antennas are used by all nodes to transmit and receive signals. Such antennas are attractive due to their broadcast nature. A single transmission by a node can be received by all nodes within its vicinity. We assume that every node can dynamically adjust its transmitting power based on the distance to the receiving node and the background noise. In the most common power-attenuation model [24], the signal power falls as $\frac{1}{d^k}$ where d is the distance from the transmitter antenna and k is a real *constant* between 2 and 5 dependent on the wireless environment. We assume that all receivers have the same threshold for signal detection, and normalize this threshold to one. With these assumptions, the power required to support a link between two nodes separated by a distance d is d^k.

The network topology of a wireless ad hoc network, which consists of all possible one-hop communication links among the nodes, is determined by the transmission ranges of the nodes. Depending on whether *unidirectional* links or *bidirectional* links are used for communications, the network topology is represented by either a directed graph referred to as the *unidirectional topology*, or an undirected graph referred to as the *bidirectional topology*. In the unidirectional topology, there is an arc from a node u to another node v if and only if v is within the transmission range of u. In the bidirectional topology, there is an edge between two nodes u and v if and only if they are within the transmission ranges of each other.

Connectivity is one of the most important properties of a wireless ad hoc network. By unidirectional k-node (resp., k-edge) connectivity we mean that the unidirectional topology is (strongly) k-node (resp., k-edge) connected, and by bidirectional k-node (resp., k-edge) connectivity we mean that the bidirectional topology is k-node (resp., k-edge) con-

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nected. Recall that a graph or digraph is k-node (resp., k-edge) connected if there are k internally node-disjoint (resp., k edge-disjoint) paths from any node to any other node. For $k = 1$, edge connectivity and node connectivity are identical, and thus are simply referred to as connectivity. For $k = 2$, 2-node connectivity is simply referred to as biconnectivity, and 2-edge connectivity is simply referred to as edge-biconnectivity. For a given transmission range, the unidirectional connectivity is always at least the bidirectional connectivity. However, if the transmission ranges are not identical, the unidirectional connectivity may be higher than the bidirectional connectivity. On the other hand, if all nodes have the same transmission range, the unidirectional topology and the bidirectional topology always have the same connectivity.

The requirement on the network connectivity (either unidirectional or unidirectional) imposes a constraint on the transmission ranges of all nodes. A crucial issue is how to find a range assignment of the smallest total power to meet a specified connectivity requirement. The Min-Power Bidirectional (resp., Unidirectional) k-Node Connectivity problem seeks a range assignment of minimum total power subject to the constraint that the produced bidirectional (resp., unidirectional) topology is k-connected. Similarly, the Min-Power Bidirectional (resp., Unidirectional) k-Edge Connectivity problem seeks a range assignment of minimum total power subject to the constraint the produced bidirectional (resp., unidirectional) topology is k-edge connected. Clearly, the smallest total power for unidirectional k-node (resp., edge) connectivity is no more than the smallest total power for bidirectional k-node (resp., edge) connectivity.

The study of the Min-Power Unidirectional Connectivity problem was started by Chen and Huang [5], who gave a 2-approximation algorithm based on a minimum spanning tree. Kirousis et al. [20], among other results, rediscover the 2-approximation algorithm and show the problem is NP-hard in three dimensions, and Clementi et al. [7] show the problem is NP-hard in two dimensions. The related broadcast problem was studied in [27,29] and [6]. The recent survey [8] presents the state of the art for these "unidirectional" problems. The Min-Power Bidirectional Connectivity problem was proposed in [2] and [4]. Both papers claim that Min-Power Bidirectional Connectivity is NP-hard, and [4] presents a $(1 + \ln 2)$-approximation algorithm. In [1], this approximation ratio is improved to $5/3 + \epsilon$, for any $\epsilon > 0$.

The Min-Power Bidirectional Biconnectivity problem has been first studied by Ramanathan and Rosales-Hain [25], who proposed one reasonable heuristic but without a proven approximation ratio. Lloyd et al. [23] studied both Min-Power Bidirectional Biconnectivity and Min-Power Bidirectional Edge-Biconnectivity. Among other results, they show that the range assignment based on the approximation algorithm of Khuller and Raghavachari [18], which we refer to as *Algorithm KR*, has an approximation ratio of at most $2(2 - 2/n)(2 + 1/n)$ for Min-Power Bidirectional Biconnectivity, and the range assignment based on the approximation algorithm of Khuller and Vishkin [19], which we refer to as *Algorithm KV*, has an approximation ratio of at most $8(1 - 1/n)$ for Min-Power Bidirectional Edge-Biconnectivity.

In this paper, we present a reduction that establishes the NP-hardness of both Min-Power Bidirectional Biconnectivity and Min-Power Bidirectional Edge-Biconnectivity. The NP-hardness holds for plane instances, not only for arbitrary graph weights. We show that the range assignment based on the *Algorithm KR* has an approximation ratio of at most 4 for both Min-Power Bidirectional Biconnectivity and Min-Power Unidirectional Biconnectivity. Specifically, we prove that the total power of this range assignment is less than four times the smallest power for unidirectional biconnectivity. We also show that the range assignment based on *Algorithm KV* has an approximation ratio of at most $2k$ for both Min-Power Bidirectional k-Edge Connectivity and Min-Power Unidirectional k-Edge Connectivity. Specifically, we prove that the total power of this range assignment is less than $2k$ times the smallest power for unidirectional k-edge connectivity. As both algorithms are graph algorithms, the approximation ratios hold also if the nodes are in three dimensional space, if the possible ranges come from a discrete set of values, if obstacles completely block the communication in between certain pairs of nodes, and if there is a maximum value on the ranges. The previous result of Lloyd et al. [23] also has this desirable property.

Although the range assignments based *Algorithm KR* and *Algorithm KV* have constant approximation ratios, they have very complicated implementations and are not practical for wireless ad hoc networks. This motivates us to seek a tradeoff between the approximation ratio and the implementation complexity. We propose a very simple range assignment, called MST-Augmentation, which achieves both bidirectional and unidirectional biconnectivity. The total power of this range assignment is less than 8 times the smallest power for unidirectional connectivity for plane instances with $k = 2$, while for $k > 2$ we prove a $3.2 \cdot 2^k$-approximation.

In parallel with us (our conference version is one month later than theirs), Hajiaghayi et al. [17] published results which overlap or complement ours. They obtain a $O(k)$-approximation for Min-Power Bidirectional k Connectivity in graphs. They also propose a MST augmentation algorithm similar to ours, and include a general version for k-connectivity for which they prove a $O(k^{2k+2})$ approximation ratio for plane instances. For biconnectivity they prove an approximation ratio of $2(4 \cdot 2^{k-1} + 1)$, which is weaker than ours.

The remainder of this paper is organized as follows. In Section 2, we present the NP-hardness of Min-Power Bidirectional (Edge-) Biconnectivity. In Section 3, we describe an alternative problem formulation and some basic properties of the power costs. In Sections 4 and 5, we derive tighter upper bounds on the approximation ratios of the range assignments based *Algorithm KR* and *Algorithm KV* respectively. In Section 6, we present the new algorithm, MST-Augmentation, and analyze its approximation ratio. Finally, in Section 7, we conclude the paper.

2. NP-hardness

In this section we describe the reduction proving the NP-hardness of both Min-Power Bidirectional Biconnectivity and Min-Power Bidirectional Edge-Biconnectivity. NP-hardness holds for plane instances, not only for arbitrary graph weights.

Theorem 1. *Min-Power Bidirectional Biconnectivity and Min-Power Bidirectional Edge-Biconnectivity are NP-hard.*

Proof: The reduction is from Hamiltonian Circuit in Planar Cubic Graphs, proved to be NP-Complete in [16]. The intuition comes from the following simple reduction showing that finding a biconnected spanning subgraph with minimum number of edges is NP-Hard. The simple reduction is also from Hamiltonian Circuit in Planar Cubic Graphs, keeps the graph and lets n be the desired number of edges of the biconnected spanning subgraph. If the graph has a Hamiltonian circuit, then this circuit is a biconnected spanning subgraph with n edges, and a biconnected spanning subgraph with n edges must be a Hamiltonian circuit.

Let $G = (V, E)$ be a planar cubic (all vertices having degree three) graph with n vertices. We construct an instance U of Min-Power Bidirectional Two-(Edge)-Connectivity as follows. We first apply the polynomial time algorithm in [3, 23] to obtain a planar orthogonal grid drawing of G in which each vertex u has integer coordinates, each edge uv has at most one bend, and each horizontal or vertical line segment has length between 6 and a polynomial function of n. Note that the bends also occur only at integer coordinates, since an edge connects vertices with integer coordinates and has at most one bend. Scale the construction up by n, so that a point x on the embedding of edge uv with $\|xu\| > n$ and $\|xw\| > n$ is at distance at least n to any point on some embedded edge other than uv.

Let L be the total length of the edges. Then L is bounded by a polynomial in n. Next, subdivide every edge of length l into lL^2 equidistant points but remove in the middle of the edge, in a place not containing a bend, L^2 of these new points, leaving a *gap* of length 1. For an illustration of the result, please refer to figure 1.

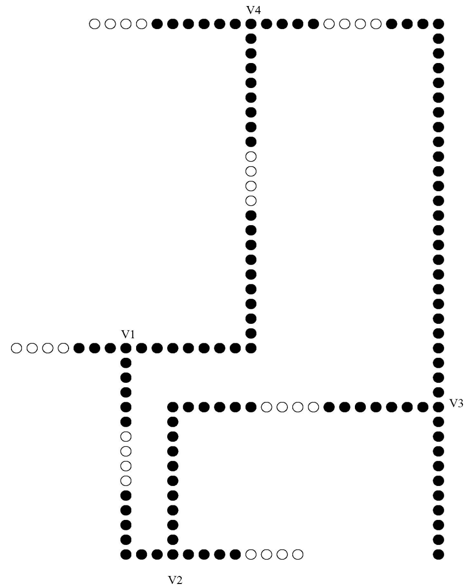

Figure 1. A portion of a planar cubic graph containing the circuit v_1, v_2, v_3, v_4. The circles denote the points obtained by subdivision, with the removed points being empty circles. The picture suggests that many points are removed—in fact on an edge at most a fraction of $\frac{1}{6n}$ is removed, and the "gaps" are very small.

Place a node in each of the points mentioned above, except the removed ones. Finally, for every already placed node in the plane, place arbitrarily at distance $1/L^2$ to it another new node; two such nodes are called *twins*. The total number of nodes introduced is $O(L^3)$, and therefore the construction is polynomial.

If we consider the graph induced only by nodes at most $3/L^2$ apart, it has n components, each corresponding to a vertex of the original graph G. We call such a component *the cluster* of the original vertex v. Moreover, each component is two connected, as we prove below. The nodes obtained from the subdivision are at distance $1/L^2$ apart and form a connected graph. Each node added as a twin is adjacent to its twin. Removing a newly added twin cannot destroy connectivity, since it is preserved by the nodes obtained from the subdivision. Removing a node obtained from the subdivision also does not break a cluster into connected components since the twin of the removed node is adjacent to the nodes "close" (at distance $1/L^2$) of the node removed. For an illustration of one cluster (including the twins) (see figure 2).

Let n' denote the number of nodes in the resulting instance. Recall that the power of a node is at least the square of its assigned range. If the original graph is Hamiltonian, we obtain a range assignment of total power not exceeding $2n + 9n'/L^4$

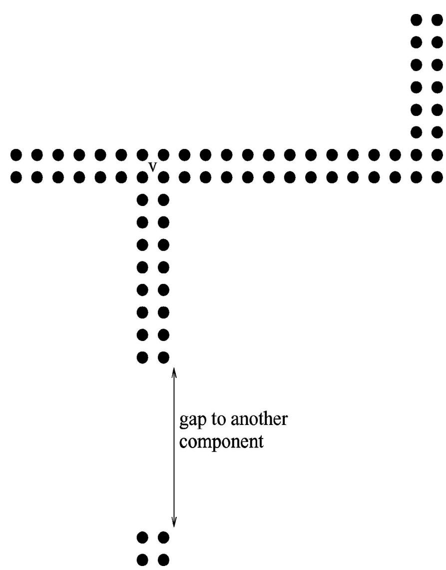

Figure 2. The cluster of v, with a gap of length 1 to another cluster.

by assigning to every node a range of $3/L^2$ and, for every edge uv of the Hamiltonian path, we pick the two nodes next to the uv-gap, one in the cluster of u and one in the cluster of v, and assign them range 1. Note that $n' \leq 2L^3$ and therefore $2n + 9n'/L^4 < 2n + 1$ (where we use $L \geq n > 18$; if $n \leq 18$ there is no need for a reduction as we could solve Hamiltonian Circuit in Planar Cubic Graphs). We proved that if the original graph is Hamiltonian, we obtain in the constructed graph a range assignment ensuring biconnectivity of total power strictly less than $2n + 1$.

Next we show that any range assignment ensuring edge biconnectivity of total power less than $2n + 1$ implies that the original graph G is Hamiltonian. Let H' be the two-(edge-) connected graph established by the range assignment, and H be the multigraph obtained from H' by contracting every cluster to a single vertex. Every cluster must be incident to at least two edges of H. Recall that a point x on the embedding of edge uv with $\|xu\| > n$ and $\|xv\| > n$ is at distance at least n to any point on some embedded edge other than uv. Thus for a node x in a cluster to have edges of H' incident to nodes in two other clusters, it must have a range of at least n, contributing at least n^2 to the total power. So we may assume that any node is, in H', incident only with nodes in its own cluster, or only one extra cluster. A range of at least 1 is needed to establish links to another cluster.

For $U \subseteq V$, let $P(U)$ be the minimum total power required to establish the edges of H' with both endpoints in the clusters of $H[U]$, the subgraph of H induced by U. We claim that if $U \subseteq V$, $|U| \geq 3$, and $H[U]$ is edge-biconnected, then $P(U) \geq 2|U|$. Indeed, if every cluster corresponding to U has two vertices with range 1, then the claim holds. If the cluster corresponding to a vertex $v \in U$ has only one node x with range at least 1, then v is adjacent in $H[U]$ to only one other vertex, which we call u, by at least two parallel edges. Then, in the cluster of u, two nodes must have range at least 1 and be adjacent to x in H'. Also, $H[U - v]$ must be two-edge connected. If $|U - v| = 2$, the same reasoning as above implies that $P(U - v) \geq 3$ and therefore $P(U) \geq 6$: the two nodes of the cluster of u and the one node in the cluster of v each contribute another 1 to the power of U. If $|U| \geq 4$, the claim follows by induction, as in this case $P(U - v) \geq 2(|U| - 1)$.

The previous claim and its proof imply that if $H[U]$ is two-edge connected, $|U| \geq 4$, and $P(U) < 2|U| + 1$, then every cluster corresponding to U has exactly two nodes with range at least 1, establishing links to two other clusters. For $U = V$, this implies that $H[V]$ is Hamiltonian, and therefore G is Hamiltonian. Thus, the theorem follows. □

3. Problem reformulation

A wireless ad hoc network can be represented by a weighted complete graph $G = (V, E, c)$ with $c(e) = ||e||^k$ where $||e||$ is the length of the edge e. For any spanning subgraph H of G, define $p_H(v) = \max_{uv \in E(H)} c(uv)$ for each $v \in V$ and $p(H) = \sum_{v \in V} p_{\mathrm{H}}(v)$; we call $p(H)$ the *power* of H (note that we re-define the notion of "power of a graph" and we never use the classical graph-theoretic definition in this paper). Since assigning $p(v) \geq p_H(v)$ is necessary to produce the subgraph H and $p(v) > p_H(v)$ just wastes power, the Min-Power Bidirectional k-Node (resp., k-Edge) Connectivity problem is equivalent to finding a k-vertex (resp., k-edge) connected spanning subgraph H of G with minimum $p(H)$.

For any subgraph H of G, we use $\vec{H}$ to represent the weighted graph obtained from H by replacing every edge uv of H with two oppositely oriented arcs uv and vu with the same weight as the edge uv in H. For any spanning subdigraph D of $\vec{G}$, we define $p_D(u) = \max_{uv \in E(D)} c(uv)$ for each $u \in V$ and $p(D) = \sum_{u \in V} p_D(v)$; we call $p(D)$ the *power* of D. Similarly, the Min-Power Unidirectional k-Node (resp., k-edge) Connectivity problem is equivalent to finding a (strongly) k-vertex (resp., k-edge) spanning subgraph D of $\vec{G}$ with minimum $p(D)$.

Next, we discuss some basic properties of powers of graphs and digraphs. As usual, the *weight* of a subgraph H of G is defined as $c(H) = \sum_{e \in E(H)} c(e)$, and the *weight* of a subdigraph D of $\vec{G}$ is defined as $c(D) = \sum_{e \in E(D)} c(e)$. Clearly, for any subgraph H of G, $p(\vec{H}) = p(H)$ and $c(\vec{H}) = 2c(H)$.

Lemma 2. *For any subdigraph D of $\vec{G}$, $p(D) \leq c(D)$. For any subgraph H of G, $p(H) \leq 2c(H)$.*

Proof: Since

$$p(D) = \sum_{u \in V} p_D(u) = \sum_{u \in V} \max_{uv \in E(D)} c(uv)$$
$$\leq \sum_{u \in V} \sum_{uv \in E(D)} c(uv) = \sum_{e \in E(D)} c(e) = c(D),$$

the first inequality holds. Since

$$p(H) = p(\vec{H}) \leq c(\vec{H}) = 2c(H),$$

the second inequality holds. □

A spanning subdigraph D of $\vec{G}$ is said to be a *branching* rooted at some vertex $s \in V$ if D contains exactly $|V| - 1$ arcs and there is a path in D to s from any other vertex. In other words, branchings in directed graphs are directed analogs of spanning trees in undirected graphs. It is easy to verify that if D is a branching, then $p(D) = c(D)$.

For any subdigraph D of $\vec{G}$, we use $\bar{D}$ to represent the undirected graph obtained from D by ignoring the orientations of the arcs and then removing multiple edges between any pair of nodes. Then,

$$c(D) \geq c(\bar{D}), \quad p(D) \leq p(\bar{D})$$

4. Algorithm KR for k-edge connectivity

A digraph is said to be *k-edge inconnected* to a vertex s if it contains k arc-disjoint paths to s from any other vertex. The min-weight spanning subdigraph of a given weighted digraph which is k-edge-inconnected to a specified vertex, if there is any, can be found in polynomial time by the weighted matroid intersection algorithm due to Lawler [21] and Edmonds [11]. The fastest implementation of a weighted matroid intersection algorithm is given by Gabow [13]. If a digraph D is k-edge-inconnected, then $\bar{D}$ is k-edge connected [18]. *Algorithm KR* [18] constructs a k-edge-connected spanning subgraph of a given weighted graph G as follows. For some node s, find the minimum-weight subdigraph D of $\bar{G}$ which is k-edge inconnected to s, and then output the graph $\bar{D}$.

Let *opt* be the power cost of an optimum range assignment for unidirectional k-edge connectivity. Lloyd et al. [23] proved that for $k = 2$, $p(\bar{D}) \leq 8(1 - 1/n) \cdot opt$, where n is the number of nodes. We prove the following stronger bound, which also applies to larger values of k.

Theorem 3. $p(\bar{D}) < 2k \cdot opt$

Proof: Let D^* be the digraph produced by the optimum range assignment for unidirectional k-edge connectivity. Then D^* is strongly k-edge connected. By a theorem due to Edmonds [10], D^* contains k arc-disjoint branchings B_1, B_2, ..., B_k rooted at s. As $\cup_{i=1}^{k} B_i$ is k-edge inconnected to s,

$$c(D) \leq c\left(\bigcup_{i=1}^{k} B_i\right) = \sum_{i=1}^{k} c(B_i)$$
$$= \sum_{i=1}^{k} p(B_i) < kp(D^*) = k \cdot opt.$$

Using Lemma 2, we conclude:

$$p(\bar{D}) \leq 2c(\bar{D}) \leq 2c(D) < 2k \cdot opt.$$

Theorem 3 implies that the approximation ratio of *Algorithm KR* is at most $2k$. □

5. Algorithm KV for biconnectivity

A digraph is said to be *k-vertex inconnected* to a vertex s if it contains k internally vertex-disjoint paths to s from any other vertex. The min-weight spanning subdigraph of a given weighted digraph which is k-vertex inconnected to a specified vertex, if there is any, can be found in polynomial time by an algorithm of Frank and Tardos [12]. Gabow [14] has given a faster implementation of the Frank-Tardos algorithm. Suppose that D is a 2-vertex inconnected digraph to a vertex s in which s has exactly two incoming neighbors x and y. Then the graph $\left(\bar{D} - s\right) \cup \{xy\}$ is biconnected [19]. *Algorithm KV* [19] constructs a biconnected spanning subgraph of a given weighted graph G as follows.

1. Let xy be the edge of G of minimum weight and s be a vertex not in V. Add two edges xs and ys of weight 0 to G. The resulting graph is denoted by G^+.
2. Find the minimum-weighted spanning subgraph D of $\vec{G}^+$ which is 2-vertex-inconnected to s.
3. Output the graph $\left(\bar{D} - s\right) \cup \{xy\}$.

The result of this section (Theorem 6 below) makes use of the following two previously-known graph-theoretic results. The first is a corollary of Menger's Theorem:

Theorem 4 (Fan Lemma) (see, for example, [9]). *Suppose that D is a k-vertex connected directed graph and U is a proper subset of its vertices with $|U| = k$. Then for any vertex v not in U, there are k internally vertex-disjoint paths that link v to distinct vertices of U.*

Theorem 5 (Whitty) [28]. *Suppose that, given a directed graph $D = (V, A)$ and a specified vertex $s \in V$, there are two internally vertex-disjoint paths to s from any other vertex of D. Then D has two arc-disjoint branchings rooted at s such that for any vertex $v \in V - s$ the two paths to s from v uniquely determined by the branchings are internally vertex-disjoint.*

Let *opt* be the power cost of an optimum range assignment for unidirectional biconnectivity. Lloyd et al. [23] proved that $p((\bar{D} - s) \cup \{xy\}) \leq 2(2 - 2/n)(2 + 1/n) \cdot opt$. The next theorem gives a tighter bound.

Theorem 6. $p((\bar{D} - s) \cup \{xy\}) \leq 4 \cdot opt$.

Proof: Let D^* be the digraph produced by the optimum range assignment for unidirectional biconnectivity. Then by the Fan lemma (Theorem 4), $D^* \cup \{xs, ys\}$ is 2-vertex inconnected to s. By Theorem 5, $D^* \cup \{xs, ys\}$ contains two arc-disjoint branchings B_1 and B_2 rooted at s such that, for every vertex $v \in V$, the two paths in B_1 and B_2 from v to s are internally vertex-disjoint. So $B_1 \cup B_2$ is 2-vertex inconnected to s. Hence,

$$c(D) \leq c(B_1 \cup B_2) = c(B_1) + c(B_2) = p(B_1) + p(B_2)$$
$$< 2p(D^* \cup \{xs, ys\}) = 2p(D^*) = 2 \cdot opt.$$

By Lemma 2 and the selection of the edge xy,

$$p((\bar{D} - s) \cup \{xy\}) = p(\bar{D} - s) \leq 2c(\bar{D} - s)$$
$$= 2c(\bar{D}) \leq 2c(D) < 4 \cdot opt. \qquad \square$$

Theorem 6 implies that the approximation ratio of *Algorithm KR* is at most 4.

6. Algorithm MST-augmentation for biconnectivity

In this section, we present a simple algorithm which produces a biconnected spanning graph H by augmenting an MST. The algorithm first finds a Euclidean MST T and initializes H to T. At any non-leaf node v of T, a local Euclidean MST T_v over all the neighbors of V in T is constructed and added to H. Thus H is a union of a big MST T and many small MSTs.

Hajiaghayi et al. [17] devised a similar MST augmentation algorithm, but they use paths instead of trees to connect the neighbors of V in the minimum spanning tree T.

H is 2-connected, as it follows from the following argument. Only internal nodes of T can be articulation points of H; let u be such a node. Removing u from T creates a number of connected components of T, each having one vertex neighbor with u in T. But the neighbors of u in T remain connected by T_u, the local MST which does not include u.

We refer to this algorithm as *MST-Augmentation*. Besides being simple and very fast (as every vertex has constant degree in T, the total running time is dominated by constructing T and is $O(n)\log n$), this algorithm is best suited to efficient distributed implementation. Indeed, after the computation of the minimum spanning tree, each node can compute its power with a constant number of messages to other nodes (since T has degree bounded by six, see the next paragraph). The minimum spanning tree can be computed by the algorithm of Gallager et al. [15] in $5n \log n + 2m$ messages and $O(n)$ time, where m is the number of valid communication links. Another advantage of this algorithm is the independence of the path-loss exponent k, since only the Euclidean distances between the nodes are used (only the approximation ratio depends on k, not the algorithm itself).

To bound the approximation ratio of *MST-Augmentation*, we introduce a geometric constant α defined below. Let o be the origin of the Euclidean plane. A set U of at least two points is called a *star-set* if its Euclidean MST for $\{o\} \cup U$ is a star centered at o. The star is denoted by S_U. Note that each star-set contains at least two but at most six points, as the maximum degree of the Euclidean minimum spanning tree is six. Indeed, if uw and uv are two edges of the Euclidean minimum spanning tree, the angle in between these edges cannot be smaller than $\Pi/3$ since otherwise the triangle uvw has a bigger angle, and therefore at least one of uw and uv is longer than vw and can be replaced by vw in the tree. Seven edges incident to a vertex imply an angle of less than $\Pi/3$. Also, we use below the fact that having six edges incident to a vertex in the Euclidean minimum spanning tree implies that the six angles in between consecutive (in clockwise order) edges are equal and, by the replacement argument above, all the six edges are equal. For any star-set U, let T_U be the minimum spanning tree of U. Then α is defined as the supreme of the ratio $c(T_U)/c(S_U)$ over all star-sets.

Lemma 7. *For any $k \geq 2$, $2^{k-1} \leq \alpha \leq 1.6 \cdot 2^{k-1}$. If $k = 2$, then α=2.*

Proof: The lower bound 2^{k-1} is achieved by U consisting of two points u_1 and u_2 such that o is the midpoint of the line segment u_1u_2. Next, we prove the upper bound $1.6 \cdot 2^{k-1}$. Consider any star-set U. If U has exactly six points, then these points form a regular hexagon centered at o, and hence

$$c(T_U) = \frac{5}{6} c(S_U) < 1.6 \cdot 2^{\kappa-1} c(S_U).$$

So we assume U has $m \leq 5$ points. For any two points u and w in U,

$$\begin{aligned} c(uw) &= \|uw\|^k \leq (\|ou\| + \|ow\|)^k \\ &= 2^k \left(\frac{\|ou\| + \|ow\|}{2} \right)^k \\ &\leq 2^k \frac{\|ou\|^k + \|ow\|^k}{2} \\ &= 2^{k-1} (c(ou) + c(ow)). \end{aligned}$$

Thus, the total weight of the convex polygon formed by the points of U is at most $2^k c(S_U)$. On the other hand, as removing the largest edge of the polygon creates a tree on U, $c(T_U)$ is at most $(1 - \frac{1}{m})$ times the total weight of this polygon. Thus,

$$\begin{aligned} c(T_U) &\leq \left(1 - \frac{1}{m}\right) \cdot 2^k c(S_U) \\ &\leq \left(1 - \frac{1}{5}\right) \cdot 2^k c(S_U) \\ &= 1.6 \cdot 2^{k-1} c(S_U). \end{aligned}$$

The lemma thereby follows.

Now we assume $k = 2$ and show that $\alpha = 2$. Since $\alpha \geq 2$, we only have to show that $\alpha \leq 2$. Consider a star-set (each point given by its coordinates):

$$U = \{(a_i, b_i) : 1 \leq i \leq m\}.$$

Let K_U denote the complete graph over U. We first claim that

$$c(S_U) \geq \frac{1}{m} c(K_U).$$

To see this, we make use of the following inequality:

$$\begin{aligned} \sum_{i=1}^{m} a_i^2 &= \frac{\left(\sum_{i=1}^{m} a_i\right)^2 + \sum_{1 \leq i < j \leq m} (a_i - a_j)^2}{m} \\ &\geq \frac{\sum_{1 \leq i < j \leq m} (a_i - a_j)^2}{m}. \end{aligned}$$

Thus,

$$\begin{aligned} c(S_U) &= \sum_{i=1}^{m} \left(a_i^2 + b_i^2\right) \\ &\geq \frac{\sum_{1 \leq i < j \leq m} [(a_i - a_j)^2 + (b_i - b_j)^2]}{m} \\ &= \frac{1}{m} c(K_U). \end{aligned}$$

Next, we claim that

$$c(T_U) \leq \frac{2}{m} c(K_U).$$

This claim can be proved by a simple counting argument. Note that a complete graph of order m has m^{m-2} spanning trees, and each edge appears in

$$\frac{m^{m-2}(m-1)}{\frac{m(m-1)}{2}} = 2m^{m-3}$$

spanning trees (see, for example, Chap. 2 of [26]). The total weight of all spanning trees of K_U is thus $2m^{m-3}c(K_U)$. Hence,

$$c(T_U) \leq \frac{2m^{m-3}c(K_U)}{m^{m-2}} = \frac{2}{m}c(K_U).$$

From the two previous claims, we have

$$\frac{c(T_U)}{c(S_U)} \leq \frac{\frac{2}{m}c(K_U)}{\frac{1}{m}c(K_U)} = 2.$$

So the lemma follows for $k = 2$. □

Now we are ready to present the upper bound on $p(H)$ in terms of α and the power cost of an optimum range assignment for unidirectional connectivity which is denoted by *opt*.

Theorem 8. $p(H) < 4\,\alpha \cdot opt$.

The proof of this theorem consists of the following several lemmas. The first of these lemmas is implicit in [20] and it follows immediately from the fact that T is a minimum spanning tree and one argument used in the proof of Theorem 3.

Lemma 9. $c(T) < opt$.

Let E_1 be the set of all edges of T incident to leaves. Let E_2 be the set of all edges of the trees T_v for all non-leaf nodes v. Let H' be the graph $(V, E_1 \cup E_2)$. Then H' is a subgraph of H, and thus $p(H) \geq p(H')$. The next lemma states that the equality actually holds.

Lemma 10. *For every node* v, $p_H(v) = p_H{}'(v)$, *and consequently* $p(H) = p(H')$.

Proof: We prove the lemma by contradiction. Assume that $p_H(v) > p_H{}'(v)$ for some node v. Let $p_H(v) = c(uv)$. Then uv must be an edge of T and neither of u and v is a leaf. Since u is not a leaf, u has a neighbor w in T other than v such that vw is an edge in T_u. So vw is an edge of E_2. Since both uv and uw are edges of MST T, $|uv| \leq |vw|$, and thus $c(uv) \leq c(vw)$. Therefore,

$$p_H(v) = c(uv) \leq c(vw) \leq p_{H'}(v),$$

which is a contradiction. □

The next lemma provides an upper bound on the total weight of H'.

Lemma 11. $c(H') \leq 2\alpha \cdot c(T)$.

Proof: From Lemma 7, we have

$$c(T_u) \leq \alpha \sum_{uv \in E(T)} c(uv).$$

Then

$$\begin{aligned} c(H') &= c(E_1) + c(E_2) \\ &= \sum_{u \text{ leaf}} \sum_{vu \in E(T)} c(uv) + \sum_{u \text{ internal}} c(T_u) \\ &\leq \alpha \sum_{u \text{ leaf}} \sum_{vu \in E(T)} c(uv) + \alpha \sum_{u \text{ internal}} \sum_{vu \in E(T)} c(uv) \\ &= 2\alpha c(T), \end{aligned}$$

as every edge of T appears exactly twice in the summation. □

Now Theorem 8 follows immediately from Lemmas 2, 9, 10, and 11:

$$p(H) = p(H') \leq 2c(H') < 4\alpha \cdot c(T) < 4\alpha \cdot opt.$$

Theorem 8 and Lemma 7 imply that the approximation ratio of *MST-Augmentation* is at most 8 for $k = 2$ and at most $3.2 \cdot 2^k$ for general k.

7. Summary

We presented improved analyses of existing algorithms for Min-Power Bidirectional Biconnectivity and Min-Power Bidirectional k-Edge Connectivity, and showed the bidirectional output of these algorithms is also a good approximation for Min-Power Unidirectional Biconnectivity and Min-Power Unidirectional k-Edge Connectivity, respectively. We showed that Min-Power Bidirectional Biconnectivity and Min-Power Bidirectional Edge-Biconnectivity are NP-hard. We introduced the new algorithm *MST-Augmentation* and showed it also has constant approximation ratio.

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