

## USING THE CONCEPT OF IMPOSED CONSTRAINTS IN THE PLASTICITY THEORY OF COMPOSITES

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*Results of many experimental and numerical studies indicate that, in unidirectionally reinforced composites, plastic deformations do not arise in tension along the reinforcing fibers. CFRPs and boron-aluminum composites with rectilinear fibers deform elastically up to fracture. However, unlike homogeneous materials, such composites deform plastically in hydrostatic loadings. In order to rationally reflect these features, this work proposes a plasticity model for unidirectionally reinforced composites based on the concept of materials with imposed constraints. The yield criterion and the plasticity function derived do not depend on the normal stresses in areas perpendicular to fibers. For a composite with a hexagonal packing of fibers, a micromechanical simulation of the representative element is performed, and all material parameters of the model are determined. It is known that, with respect to the elastic properties, the equivalent homogeneous material for such composites is transversely isotropic. However, a micromechanical analysis showed that, for the characteristics of plasticity, the isotropy in the plane perpendicular to fibers is violated.*

### Introduction

Fibrous composites have superior mechanical and physical characteristics, which ensure their advantage over the traditional homogeneous materials, but their drawback is the low limits of plasticity and strength in tension perpendicularly to fibers. This is especially important for unidirectionally reinforced structures, as has been established by numerous experiments and results of micromechanical analysis.

In [1], results for composites with an epoxy matrix reinforced with glass and carbon fibers are presented. It was found that, for CFRPs, the elastic modulus in the direction of fibers was 15 times higher than in the transverse direction. The ten-

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sile strength in the fiber direction 22 times exceeded that in the transverse direction. These characteristics of GFRPs differed slightly less — about 8.3 times.

Work [2] is devoted to the micromechanical modeling of the longitudinal shear of a unidirectionally reinforced carbon fiber composite with a rectangular fiber packing. The fibers are considered elastic transversely isotropic, and a diagram with a nonlinear hardening was used for the epoxy matrix. The von Mises condition was used as a criterion for the onset of plasticity. Numerical results were obtained by the finite-element method (FEM) on a 3D representative cell. For the longitudinal shear modulus, the numerical results were compared with analytical solutions based on simplified geometric models of the representative cell. A comparison showed a good agreement at low fiber volume contents. Results of an elastoplastic analysis were presented in the form of deformation diagram of a composite with 60 vol.% fibers in the longitudinal shear.

A numerical micromechanical modeling of unidirectionally reinforced composites subjected to the transverse tension was performed in [3] at a random distribution of fibers. The calculations were performed by the FEM in the ABAQUS software package in the conditions of generalized plane deformation. The random packing of fibers was generated from initial regular orthogonal and hexagonal structures. The main attention in this work was paid to considering the influence of inaccuracy in setting boundary conditions for the randomly distributed fibers. By varying the number of fibers in the representative cell, the authors achieved convergence of calculation results for the effective elastic constants of a composite with a certain volume content of fibers. An elastoplastic analysis was performed using the perfectly plastic body model and the von Mises yield criterion for the matrix. An analysis of effective stress–strain diagrams showed a violation of transverse isotropy for the composite with a hexagonal packing at the initial stage of plastic deformation. When the entire volume of matrix had passed into the plastic state, the difference between the diagrams for different directions disappeared.

The study of properties of unidirectionally reinforced composites in transverse tension was carried out in [4] at a random distribution of fibers. As the initial structure, a regular square of packing of round fibers with a thin interphase zone was taken, and then the direction and value of the allowable displacement were randomly selected for each fiber. The homogenization procedure was performed on a representative cell containing 100 fibers. A micromechanical study was carried out by the FEM, taking into account the nonlinear properties of the interphase zone. The results of numerical studies have showed that properties of the interfacial zone have a significant effect on the averaged characteristics of the equivalent homogeneous material in transverse tension.

The mechanical behavior of unidirectionally reinforced polymer composites subjected to tension and compression perpendicular to fibers was studied in [5] using the computational micromechanics. For the analysis, a representative cell of the composite with 30 randomly distributed fibers was isolated. A structural analysis of a representative cell was performed by the FEM in the ABAQUS software package. Two failure mechanisms of plastic deformation of matrix and interfacial separation were simulated. For the plasticity of matrix, a modified Drucker–Prager model considering the damage state of the material and the von Mises criterion were employed. Results of the numerical analysis were presented in the form of deformation diagrams in transverse tension and compression of the composite.

For unidirectionally reinforced composites with a random distribution of fibers, a micromechanical analysis of transverse and longitudinal shear was performed in [6]. A special algorithm was used to construct a geometric model for the representative volume with a sufficiently large number of randomly distributed fibers. The model also included coaxial interfacial zone. For the fiber material, the model of a transversely isotropic elastic body was adopted, but for the matrix - the Drucker–Prager theory of plasticity without hardening was used. For the interphase zone, a bilinear adhesion characteristic was assumed. The plasticity condition of the matrix included the hydrostatic pressure and the second and third invariants of stress deviator, which made it possible to take into account the difference between the elastic limits in tension and compression. To simulate shear loads at boundaries of the representative volume, periodicity conditions were specified. Results of numerical simulation showed that, in transverse shear, failure of the composites occurred by plastic deformation of the matrix. In longitudinal shear, damage was initiated by delamination in the interfacial zone.

In publications devoted to theoretical models of plasticity of unidirectionally reinforced composites, the identification of parameters is performed mainly for loadings in the transverse direction. Review [7] is devoted to various aspects of this problem, where work on the plastic deformation of unidirectionally reinforced composites is also analyzed. The studies addressed

to various types of regular structures and random distribution of fibers, their volume content, and interfacial separation are noted. It has been clarified that, in transverse shear, the main failure mechanism of such composites is the plastic deformation of matrix. Their transverse tensile strength is significantly lower than that in longitudinal tension. In the longitudinal shear, the weak point is the interfacial bond strength.

Constitutive relations of plasticity for unidirectionally reinforced composites were considered in [8] for the particular case of loading in the plane perpendicular to fibers. The equation of the initial yield surface was written using the system of invariants of a transversely isotropic body. When constructing physical relations for the increments of plastic strains, the linear invariant was excluded and an unassociated flow rule was used. The main content of the work is experimental studies into the plane stress state of GFRP specimens. A special arch fixing arm [9] made it possible to create a statically determinate average stress state in the sample with different ratios between normal and tangential stresses. Deformations were measured using a digital optical system, whose cameras were installed on both sides of a plane sample. By processing measurement results, parameters of the models with associated and unassociated flow rules were identified. A comparison of theoretical predictions of both the models with results of our own experiments and literature data showed that the model with an unassociated flow rule is preferable.

One of the ways to improve the mechanical characteristics of unidirectionally reinforced composites is incorporating nanoparticles into the polymer matrix. In [10], results of an experimental study into the elastoplastic properties of an epoxy matrix containing 8.7% uniformly dispersed silica nanoparticles are presented. For a theoretical description of plastic properties of matrix, a model with a nonlinear hardening with account of the damage state of the material was used. To identify material parameters of the model, the experimental deformation curves of matrix in loading and unloading were used. The effectiveness of the model proposed was tested by experimental results for unidirectionally reinforced composites with a nano-reinforced matrix.

Constitutive relations for the elastoplastic deformation of unidirectionally reinforced composites with associated and unassociated flow rules were proposed in [11]. The nonquadratic form of plasticity functions was based on the system of stress tensor invariants for a transversely anisotropic medium. When constructing plasticity functions, the stress tensor was decomposed into longitudinal and transverse shear stresses.

The numerical simulation of plastic deformation was carried out by the FEM for the representative volume of a composite with a hexagonal fiber packing scheme. The material of fibers was considered transversely isotropic, and the model of an isotropic elastoplastic body with nonlinear hardening was used for the matrix. The results found in the micro-mechanical modeling of the longitudinal shear and experimental data for the lateral shear were used to identify material parameters of the theoretical models. The analysis of advantages and disadvantages of the models were analyzed for different loading programs.

A numerical modeling of plastic deformation of unidirectionally reinforced composites was performed in [12] to analyze the development of cracks in the polymer matrix. The physical relations were based on Hill's theory of plasticity for orthotropic materials. To take into account the hydrostatic component of stress tensor, an additional factor depending on the normal stress at certain areas was introduced. The model was verified by comparing the results of numerical simulation with experimental data of [13] in the longitudinal and transverse shear.

The elastoplastic deformation of unidirectionally reinforced composites with account of geometric nonlinearity was studied in [14], where longitudinal and transverse shear and transverse compression were considered. In kinematic relations, a multiplicative decomposition of the tensor of finite strains into elastic and plastic components was used. The physical relations for elastic strains were taken in the form of a linear relationship between Green's strain tensor and the energetic stress tensor. In terms of elastic properties, the composite was transversely isotropic. The phenomenological model of plasticity was constructed by analogy with the theory of plastic deformation of crystals. It was assumed that irreversible strains develop as a result of sliding over certain planes of the matrix or interphase zone. In the case of a composite with a hexagonal packing of fibers, six slip planes were distinguished for the longitudinal and transverse shears. For the rates of shear strains in each plane, a power function of shear stresses was assumed. The model developed was implemented into the ABAQUS software package. To verify the theory, a finite-element modeling of the lateral compression of a unidirectionally reinforced composite and shear

of an orthogonally reinforced CFRP was performed. A comparison of numerical results with experimental data obtained by the authors showed a good agreement.

An analysis of publications devoted to experimental studies of unidirectionally reinforced composites indicates that, in tension along fibers, plastic deformations in the matrix practically do not develop until the fibers are destroyed. Therefore, in a theoretical modeling, too, the main attention is paid to particular models that establish rules for loading in a plane perpendicular to fibers or in longitudinal shear. To construct a more general theory, it is advisable to consider a unidirectionally reinforced composite as a material with superimposed constraints. Such an approach for nonlinear elastic materials reinforced with inextensible threads is described in monograph [15]. The theoretical relations of plasticity for composites on the assumption of their incompressibility, are presented in [16], but for the case of transverse isotropy — in [11]. In monograph [15], the internal structure of an elastic composite was not considered at all, and the material was considered as a homogeneous continuum. In [11, 16], it was assumed that transversely isotropic are also plastic properties. This assumption can be used only in the cases of random distribution of fibers. However, it has not been confirmed by the results of micromechanical analysis even for the hexagonal reinforcement structure. The hypothesis of incompressibility in hydrostatic loading is also inapplicable to composites [17].

In the present work, based on the concept of a material with imposed restrictions, constitutive relations for the plasticity of a compressible orthotropic composite are proposed. A technique for the micromechanical analysis of its representative cell and the numerical prediction of parameters for its plasticity model are developed. The initial data for the micromechanical analysis are the elastic properties of fibers and the plasticity characteristics of matrix.

## 1. Plasticity model for a composite unidirectionally reinforced with rigid fibers

The most common way for constructing a theory of plastic deformation of composite materials is to use hypotheses adopted for plasticity models of metals. Most often, it is assumed that composites are insensitive to the hydrostatic pressure and its volume does not deform plastically. However, new results of numerical simulations [6, 12, 13, 17] and experimental data indicate that plastic deformations can nevertheless develop in composite materials under a uniform all-round compression. This effect appears even if the matrix and fibers are made of materials that do not become plastic in the hydrostatic loading. In addition, characteristic of unidirectionally reinforced composites is another feature — they do not pass into the plastic state in tension along fibers up to loads that cause the break of fibers. This feature is explained by the fact that the elastic modulus of fibers significantly exceeds that of binder and fibers fail without plastic deformations.

Typical examples are CFRPs and GRPs, whose Young's modulus of fibers more than 100 times exceed that of the matrix. The same relationship holds between the tensile strength of fibers and the elastic limit of binder.

A similar difference between the mechanical characteristics of structural components is also characteristic of metal-matrix composites. For this class of composites, which are plastically compressible but not sensitive to tension along fibers, an anisotropic theory of plasticity is proposed.

Let us consider a composite material unidirectionally reinforced with rectilinear fibers parallel to a vector  $\mathbf{a}$ . The reinforcement structure is assumed, allowing one to select a periodically repeating representative element. Within the framework of the homogenization concept, the properties of the composite are replaced by properties of an equivalent homogeneous material.

The stress-strain state of the equivalent material is characterized by the stress  $\boldsymbol{\sigma}$  and strain  $\boldsymbol{\varepsilon}$  tensors obtained by averaging the Cauchy stress tensor  $\bar{\boldsymbol{\sigma}}$  and the linear strain tensor  $\bar{\boldsymbol{\varepsilon}}$  over the volume of the representative element:

$$\boldsymbol{\sigma} = \frac{1}{V} \iiint_V \bar{\boldsymbol{\sigma}} dv, \quad \boldsymbol{\varepsilon} = \frac{1}{V} \iiint_V \bar{\boldsymbol{\varepsilon}} dv,$$

The tensors  $\bar{\boldsymbol{\sigma}}$  and  $\bar{\boldsymbol{\varepsilon}}$  will be referred to as microstresses and microstrains.

When constructing a theory of plasticity for composites of this class, it is assumed that there exists a plasticity function  $f(\boldsymbol{\sigma}) = 0$  that determines a yield surface in the stress space. This function can include scalar or tensor parameters determining hardening of the material as a result of plastic deformation. In the initial, undeformed state, the function  $f(\boldsymbol{\sigma}) = 0$  determines the onset of plasticity.

In order to exclude the influence of normal stress on areas perpendicular to the fiber direction, we introduce the effective stress tensor  $\mathbf{s}$

$$\mathbf{s} = \boldsymbol{\sigma} - \sigma_n \mathbf{a} \otimes \mathbf{a}, \quad \sigma_n = \mathbf{a} \cdot \boldsymbol{\sigma} \cdot \mathbf{a}, \quad (1)$$

where  $\mathbf{a} \otimes \mathbf{a}$  is the dyadic product of unit vectors. Further the yield surface will be considered as a function  $f(\mathbf{s}) = 0$  of the equivalent stresses. When the conditions of active loading

$$f(\mathbf{s}) = 0, \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} d\boldsymbol{\sigma} > 0 \quad (2)$$

are obeyed, the tensor of plastic strain increments  $d\boldsymbol{\varepsilon}^P$  is determined by the associated flow rule

$$d\boldsymbol{\varepsilon}^P = d\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}.$$

Taking into account Eq. (1), for the effective stress, the associated flow rule assumes the form

$$d\boldsymbol{\varepsilon}^P = d\lambda \frac{\partial f}{\partial \mathbf{s}} \cdot \left( \mathbf{I} - \frac{\partial \sigma_n}{\partial \boldsymbol{\sigma}} \mathbf{a} \otimes \mathbf{a} \right),$$

where  $\mathbf{I} = \mathbf{e}_k \otimes \mathbf{e}_s \otimes \mathbf{e}_s \otimes \mathbf{e}_k$  is the fourth-rank unit tensor;  $\mathbf{e}_k$  and  $\mathbf{e}_s$  are unit vectors of the Cartesian coordinate system.

Differentiating the scalar  $\sigma_n$  with respect to the stress tensor, we obtain the final expression for the tensor of plastic strain increments, namely,

$$d\boldsymbol{\varepsilon}^P = d\lambda \left\{ \frac{\partial f}{\partial \mathbf{s}} - S_p \left[ \frac{\partial f}{\partial \mathbf{s}} \cdot (\mathbf{a} \otimes \mathbf{a}) \right] \mathbf{a} \otimes \mathbf{a} \right\}. \quad (3)$$

In accordance with Eq. (3), there are no increments of plastic strains in the direction of reinforcement:

$$\mathbf{a} \cdot d\boldsymbol{\varepsilon}^P \cdot \mathbf{a} = 0.$$

The increments of elastic strains are associated with stress increments by Hooke's law for anisotropic bodies:

$$d\boldsymbol{\varepsilon}^e = \mathbf{c} \cdot d\boldsymbol{\sigma},$$

where  $\mathbf{c}$  is the fourth rank tensor that determines the elastic properties of the equivalent homogeneous material.

## 2. Plasticity theory with an isotropic hardening

Let us consider a particular variant of constitutive relations where the plasticity function is a quadratic function of effective stresses and the strengthening is isotropic. In Cartesian coordinates, we define the plasticity function in the form

$$2f = A_{ijkl} S_{ij} S_{kl} - \Phi^2(q) = 0. \quad (4)$$

Here,  $A_{ijkl}$  are components of a fourth-rank tensor, and the hardening function  $\Phi$  depends on the specific work of stresses on the plastic deformations caused by them [18]:

$$q = \int \sigma_{ij} d\varepsilon_{ij}^P. \quad (5)$$

The associated flow rule now is presented in the form

$$d\varepsilon_{ij}^p = d\lambda \left( A_{ijkl} S_{kl} - A_{mnkl} S_{mn} a_{kl} a_{ij} \right), \quad (6)$$

where  $a_{ij}$  are components of the dyad  $\mathbf{a} \otimes \mathbf{a}$ . In active loading, when condition (2) is satisfied, the ratio between the increments of plastic strains is determined by flow rule (6), but the strains themselves are specified accurate to the factor  $d\lambda$ . To determine the factor  $d\lambda$ , the condition that the total differential of the plasticity function is equal to zero is used:

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} - \frac{\partial f}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p = 0. \quad (7)$$

Taking into account flow rule (6) and Eq. (5) for the work of plastic deformation from condition (7), we obtain that

$$d\lambda = \frac{\left( A_{ijkl} S_{kl} - A_{mnkl} S_{mn} a_{kl} a_{ij} \right) d\sigma_{ij}}{\Phi \Phi' \left( A_{ijkl} S_{kl} - A_{mnkl} S_{mn} a_{kl} a_{ij} \right) \sigma_{ij}}.$$

The constitutive relations defining ratios can be put to a more compact form by introducing the equivalent stress  $\sigma_{eq} = \sqrt{A_{ijkl} S_{ij} S_{kl}}$ . Then, the associated flow rule can be presented as

$$d\varepsilon_{ij}^p = \frac{d\sigma_{eq}}{\Phi' \sigma_{eq}^2} \left( A_{ijkl} S_{kl} - A_{mnkl} S_{mn} a_{kl} a_{ij} \right).$$

The increments of elastic strains are determined by Hooke's law:

$$d\varepsilon_{ij}^e = C_{ijkl} d\sigma_{kl}. \quad (8)$$

The tensor components  $C_{ijkl}$  satisfy the symmetry conditions  $C_{ijkl} = C_{klij} = C_{ijlk}$ . In addition, depending on the type of symmetry of the composite considered, these components will also obey additional requirements.

The variant of constitutive relations proposed for the elastoplastic deformation of unidirectionally reinforced composites contains the following parameters: tensor components  $A_{ijkl}$ ,  $C_{ijkl}$ , and the form of the hardening function.

To determining these parameters, the corresponding physical experiments are performed. This approach to modeling the plasticity of composites is extremely complex, laborious, and not always technically feasible.

The present-day capabilities of computer technology make it possible to calculate all characteristics of a composite material on the basis of a micromechanical analysis. For such an analysis, only the properties of fiber and matrix need to be determined experimentally. This approach is illustrated below by the example of a unidirectionally reinforced boron-aluminum composite with a hexagonal fiber packing.

### 3. Micromechanical analysis of elastic-plastic deformation of the composite

The investigations of microstresses and microstrains within a representative element were performed on the assumption that the contact between the fibers and matrix is ideal. The cross section of the smallest possible representative element is shown in Fig. 1.

The numerical results were obtained by the FEM in the ANSYS software package.

At the first stage of micromechanical analysis, the elastic characteristics of the equivalent transversely isotropic material were determined. A material with hexagonal symmetry has five independent components of the stiffness tensor, namely,

$$C_{1111} = C_{2222}, \quad C_{3333}, \quad C_{1122}, \quad C_{3311} = C_{3322}, \quad C_{2323} = C_{1313}.$$

The shear modulus in the isotropy plane is expressed as

$$C_{1212} = \frac{1}{2} (C_{1111} - C_{1122}).$$

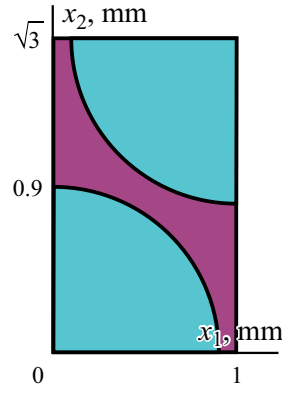


Fig. 1. Cross section of the smallest possible representative element.

TABLE 1. Mechanical Properties of Components of the Boron-Aluminum composite

Components	Elasticity modulus $E$ , GPa	Poisson ratio	Elasticity limit $\sigma_y$ , GPa	Tangential modulus $E_t$ , GPa
Boron	420	0.3	3.5	-
Aluminum matrix	71	0.34	0.262	8.75

To find the effective elastic constants of the composite, a series of independent numerical experiments were performed to simulate different types of homogeneous macrostrains: uniaxial tension in the longitudinal direction,  $\varepsilon_{33} \neq 0$ , uniaxial tension in the transverse direction,  $\varepsilon_{11} \neq 0$ , pure shear  $\varepsilon_{12} \neq 0$  in a plane perpendicular to fibers.

The numerical experiments were carried out by solving elastic problems for the generalized plane deformation. To determine the longitudinal shear moduli  $C_{5555} = C_{6666}$ , the mathematical analogy [19] between the antiplane problem of elasticity theory and the 2D problem of stationary heat conduction was employed. This allowed us to get rid of the need to build a 3D model for the representative element and to use the convenient interface of the ANSYS software package designed to solve plane problems of thermal analysis.

For each numerical experiment, special boundary conditions were set at the boundaries of cross-section of the representative element. These conditions ensured the full correspondence of microstresses and microstrains within the representative element to the state realized at a uniform average deformation of an unlimited volume of the composite. A detailed description of the procedure for forming the boundary conditions and calculating the elastic constants of an equivalent homogeneous material is presented in [20]. The calculations were performed for a boron-aluminum composite, whose mechanical properties are given in Table 1.

The finite-element modeling was performed using an 8-node PLANE 183 element. The discrete model consisted of 18,865 elements and 57,138 nodes. The average values of deformations in each experiment were given by the corresponding displacements at boundaries of the representative volume, but the average values of stresses were calculated in a postprocessor of the ANSYS software package. On the basis of these results, from Eqs. (8), components of the tensor of elastic constants were found:  $C_{1111} = C_{2222} = 272$ ,  $C_{3333} = 365$ ,  $C_{1122} = 95.7$ ,  $C_{1133} = C_{2233} = 87.1$ ,  $C_{1212} = 87.8$ , and  $C_{1313} = C_{2323} = 90.2$  GPa.

At the next stage of the micromechanical analysis, the tensor components  $A_{ijkl}$  in Eq. (4) were determined numerically. In the initial, undeformed state, then the specific work of plastic deformation is zero, and the hardening function  $\Phi(0) = 1$ . Then, Eq. (4) defines the criterion for the onset of plasticity,

$$A_{ijkl} S_{ij} S_{kl} = 1. \quad (9)$$

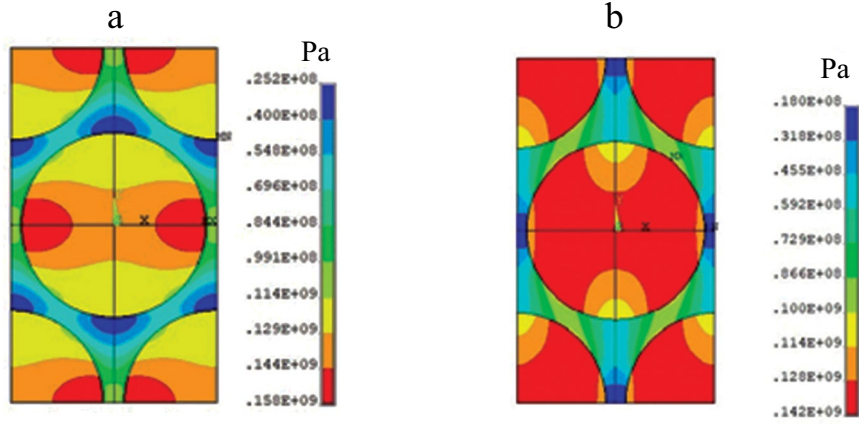


Fig. 2. Distributions of equivalent (von Mises) microstresses around fibers at the same tension along the  $x_1$  (a) and  $x_2$  (b) axes.

In the coordinate system adopted, the  $x_3$  axis is aligned with the direction of fibers. In tension along the  $x_3$  axis, the composite does not pass into the plastic state; therefore  $A_{3333} = A_{3311} = A_{3322} = 0$ . In addition, in such a coordinate system, all components of the effective stress tensor, except for  $S_{33}$ , are equal to components of the mean stress tensor.

If the  $x_1$  and  $x_2$  axes are aligned with symmetry directions of the composite, condition (9) will contain six independent tensor components  $A_{ijkl}$ :

$$A_{1111}\sigma_{11}^2 + A_{2222}\sigma_{22}^2 + 2A_{1122}\sigma_{11}\sigma_{22} + 4A_{1212}\sigma_{12}^2 + 4A_{1313}\sigma_{13}^2 + 4A_{2323}\sigma_{23}^2 = 1. \quad (10)$$

To identify the parameters of the plasticity criterion, a numerical modeling of basic experiments was carried out to determine the plasticity limits for different types of stress state: uniaxial tension in the  $x_1$  and  $x_2$  directions, simultaneous tension by equal stresses in the  $x_1$  and  $x_2$  directions, and pure shear in the three orthogonal planes  $x_1x_2$ ,  $x_2x_3$ , and  $x_1x_3$ . If the equivalent (von Mises) microstress in the matrix reaches the yield stress of aluminum, the corresponding macrostress determines the yield  $S_{ij}^*$  stress of the composite in this experiment. The yield points determined in numerical experiments were as follows:  $\sigma_{11}^* = 271$ ,  $\sigma_{22}^* = 256$ ,  $(\sigma_{11} = \sigma_{22})^* = 415$ ,  $\sigma_{12}^* = 131$ ,  $\sigma_{23}^* = 126$ , and  $\sigma_{13}^* = 110$  MPa.

A micromechanical analysis allowed us to establish the fundamental difference between the symmetries of elastic and plastic properties of the equivalent homogeneous material. In terms of elastic properties, a unidirectionally reinforced composite with a hexagonal fiber packing (on the average) is transversely isotropic. However, the yield points at tension different directions in planes perpendicular to fibers were not the same. This is explained by the fact that, at the same integral characteristics in a plane perpendicular to fibers, the local distribution of microstresses, even in the elastic state of the composite is different in different directions. To illustrate this feature, Fig. 2 shows the distribution of equivalent (von Mises) microstresses around the fiber in tension along the  $x_1$  (Fig. 2a) and the  $x_2$  (Fig. 2b) axis.

A micromechanical analysis was performed at the following integral parameters of deformation:  $\varepsilon_{11} = 0.5 \cdot 10^{-3}$ ,  $\varepsilon_{22} = 0$  and  $\varepsilon_{22} = 0.5 \cdot 10^{-3}$ ,  $\varepsilon_{11} = 0$ .

The results obtained indicated that in the elastoplastic deformation, the equivalent homogeneous material was not transversely isotropic, but orthotropic.

From the criterion of onset of plasticity (10), it was found that, in the coordinate system adopted,

$$A_{1111} = (\sigma_{11}^*)^{-2}, \quad A_{2222} = (\sigma_{22}^*)^{-2},$$

$$2A_{1122} = \left[ (\sigma_{11} = \sigma_{22})^* \right]^{-2} - A_{1111} - A_{2222},$$



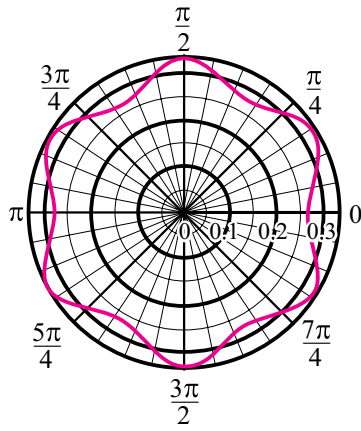


Fig. 3. Yield point vs. direction of transverse tension.

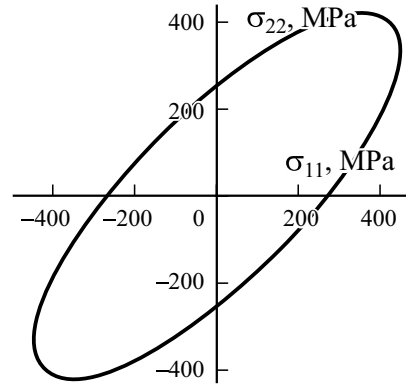


Fig. 4. Normal section of the initial yield surface.

$$A_{1212} = (2\sigma_{12}^*)^{-2}, \quad A_{1313} = (2\sigma_{13}^*)^{-2}, \quad A_{2323} = (2\sigma_{13}^*)^{-2}.$$

For the composite considered, it followed from here that  $A_{1111} = 13.6 \cdot 10^{-6}$ ,  $A_{2222} = 15.3 \cdot 10^{-6}$ ,  $A_{1122} = -11.5 \cdot 10^{-6}$ ,  $A_{1212} = 14.6 \cdot 10^{-6}$ ,  $A_{1313} = 20.7 \cdot 10^{-6}$ , and  $A_{2323} = 15.7 \cdot 10^{-6}$  MPa<sup>-2</sup>.

The violation of isotropy with respect to plastic properties in composites with a hexagonal packing of fibers is illustrated by different yield points in uniaxial transverse tension in different directions. In tension at an angle  $\theta$  to the  $x_1$  axis, the corresponding yield point  $\sigma_\theta^*$  is

$$\sigma_\theta^* = \left( A_{1111}l^4 + A_{2222}m^4 + 2A_{1122}l^2m^2 + 4A_{1212}l^2m^2 \right)^{-1/2},$$

where  $l = \cos \theta$  and  $m = \sin \theta$ . It is convenient to represent this relation by a graph (Fig. 3) in polar coordinates, where the length of radius corresponds to the yield stress (GPa) in tension at an angle  $\theta$  to the  $x_1$  axis

The parameters  $A_{ijkl}$  are found in the original coordinate system where fibers are directed along the  $x_3$  axis. Condition (10) for the onset of plasticity is invariant, which makes it possible to calculate the parameters of the plasticity criterion for an arbitrary direction of fibers. Let the direction of fibers be given by a unit vector  $\mathbf{a}$  with components  $(a_1, a_2, a_3)$  in the original coordinate system. We introduce a new coordinate system  $x'_1, x'_2, x'_3$  so that the  $x'_3$  axis coincides with the direction of fibers. The components  $A'_{ijkl}$  in plasticity criterion (10) in the new coordinate system can be calculated in terms of their values in the original system [15]

$$A'_{ijkl} = \alpha_{im}\alpha_{jn}\alpha_{kp}\alpha_{lq}A_{mnpq},$$

where  $\alpha_{ij}$  are components of the matrix degerming the position of the new coordinate system relative to the old one.

In the 3D space of normal stresses, the yield surface is an infinite cylinder whose generatrices are parallel to the direction of fibers. The trace of this cylinder on a plane perpendicular to fibers does not depend on the direction of the fibers and, based on Eq. (10), is determined by the equation

$$A_{1111}\sigma_{11}^2 + A_{2222}\sigma_{22}^2 + 2A_{1122}\sigma_{11}\sigma_{22} = 1.$$

The normal section of this surface is shown in Fig. 4.

At the final stage in identifying parameters of the plastic deformation model of the composite, the form of the hardening function  $\Phi(q)$  is predicted numerically. Since Eq. (6) is valid for any loading program, the form of the hardening function can be established from the results of numerical simulation of uniaxial tension of the composite in the direction of  $x_1$  axis.

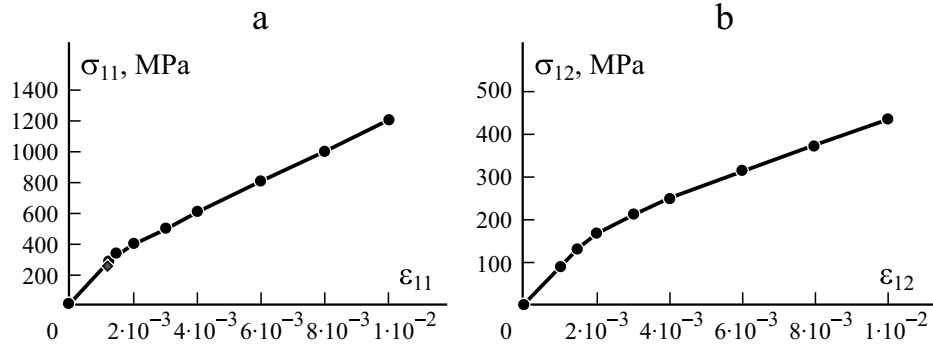


Fig. 5. Tension (a) and shear (b) diagrams of the composite.

The micromechanical modeling of the representative element was performed within the framework of a nonlinear analysis. For the matrix material, the plasticity model with bilinear hardening was used. The fibers were modeled as an isotropic linearly elastic material.

To correctly reproduce the microstressed state in the representative element in uniaxial macrotension of the composite, a special procedure for the formation of boundary conditions was used. Symmetry conditions were set on the lines  $x_1 = 0$  and  $x_2 = 0$  and boundary conditions — on the other two sides of the representative element:

$$\begin{aligned} x_1 = 1, \quad u_1 = c_1, \quad \bar{\sigma}_{12} = 0, \\ x_2 = \sqrt{3}, \quad u_2 = c_2, \quad \bar{\sigma}_{12} = 0. \end{aligned}$$

The analysis was carried out within the framework of generalized plane deformation without forces in the fiber direction.

At the first step of loading, the constants  $c_1$  and  $c_2$  were taken equal to the corresponding displacements in the ultimate elastic state with average stresses  $\sigma_{11} = \sigma_{11}^*$  and  $\sigma_{22} = \sigma_{33} = 0$ .

To simulate the successive tension of the composite, the constant  $c_1$  was increased stepwise and the value  $c_2$  was found by successive approximations. The aim of the iterative procedure was to reproduce the uniaxial stress state with a given accuracy at  $\sigma_{22} < \varepsilon \sigma_{11}$ . In calculations, it was assumed that  $\varepsilon = 10^{-3}$ , which is quite sufficient in technical applications. Upon completion of the iterative procedure, at each loading step, the average stress  $\sigma_{11}$  and average strain  $\varepsilon_{11}$  were calculated. Thus, the tensile diagram of the composite was constructed in a discrete form as  $\sigma_{11} = f(\varepsilon_{11})$ . A similar procedure was used to numerically simulate the uniaxial tension of the composite in the direction of the  $x_2$  axis.

The micromechanical modeling of pure shear in the plane normal to fibers did not require an iterative procedure, since boundary conditions were set using one constant,  $c_1$ , at each loading stage:

$$\begin{aligned} x_1 = 0, \quad u_2 = 0, \quad \bar{\sigma}_{11} = 0, \\ x_1 = 1, \quad u_2 = 0, \quad \bar{\sigma}_{11} = 0, \\ x_2 = 0, \quad u_1 = 0, \quad \bar{\sigma}_{22} = 0, \\ x_2 = \sqrt{3}, \quad u_2 = A, \quad \bar{\sigma}_{22} = 0. \end{aligned}$$

The calculated diagrams of uniaxial tension and shear are shown in Fig. 5 in the form of relations between macrostresses on macrostrains.

The violation of transverse anisotropy was also manifested in the development of plastic strains in uniaxial tension along the  $x_1$  and  $x_2$  axes.

Figure 6 shows the distribution of equivalent plastic microstrains in the matrix in tension along the  $x_1$  (Fig. 6a) and  $x_2$  (Fig. 6b) axes. These numerical experiments were carried out at the same values of macrostrains,  $\varepsilon_{11} = \varepsilon_{22} = 0.01$ .

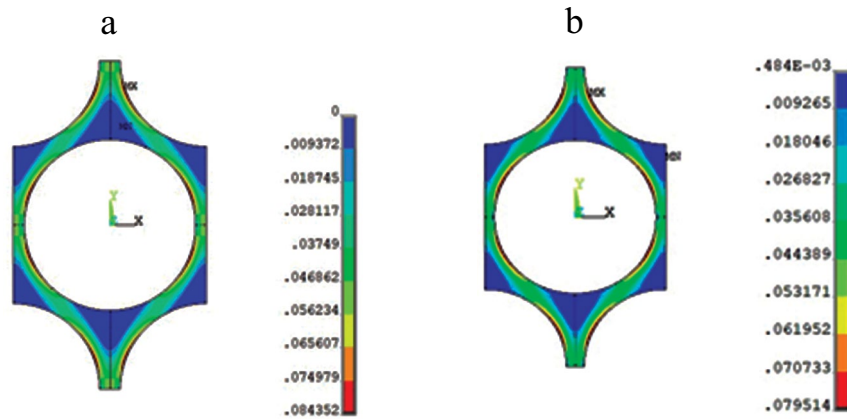


Fig. 6. Distributions of equivalent plastic strains.

In the postprocessor of the ANSYS software package, the specific work of plastic deformation was also calculated, which made it possible to establish the hardening function  $\Phi(q)$  based on the results of numerical simulation of uniaxial tension in a plane perpendicular to fibers.

## Conclusion

Constitutive relations of elastoplastic deformation of composites unidirectionally reinforced with brittle rectilinear fibers are proposed. The construction of physical relationships is based on the concept of a material with superimposed constraints. As such a constraint, we used the condition that normal stresses on areas perpendicular to the direction of fibers do not affect the development of plastic strains in the matrix. To realize this constraint, an appropriate decomposition of the stress tensor was performed and effective stresses were introduced. A quadratic yield surface for an anisotropic medium was constructed in an invariant form with respect to effective stresses by using a fourth-rank tensor. The increments of plastic strains were determined using the associated flow rule with isotropic hardening.

A micromechanical analysis and homogenization procedures were performed for a boron-aluminum composite with a hexagonal fiber packing. Numerical results were obtained by the FEM in studying the microstress state within a representative volume. The results of micromechanical modeling of basic experiments were used to identify material parameters of the theoretical model. Application of the concept of a material with imposed constraints made it possible to reduce the number of such experiments and to establish the qualitative features of plastic deformation of composites with a hexagonal fiber packing. In contrast to elastic characteristics, a violation of the transverse isotropy of macroplastic properties of composites of this type was revealed. In particular, the yield strength as a function of the direction of uniaxial tension in a plane perpendicular to the direction of the fibers is illustrated.

The invariant formulation of constitutive relations allows one to use them arbitrary fiber directions.

In a further development of the approach proposed, it is advisable to take into account the anisotropic nature of hardening. In particular, it is customary to formally associate translational hardening and the Bauschinger effect in metals with residual microstresses. For composite materials, residual stresses after a plastic deformation can be found by a micro-mechanical analysis at the level of representative cells.

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