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FREE VIBRATIONS OF A THIN ELASTIC ORTHOTROPIC CYLINDRICAL PANEL WITH FREE EDGES

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Using a system of equations corresponding to the classical theory of orthotropic cylindrical shells, the free vibrations of a thin elastic orthotropic cylindrical panel with free edges is investigated. To calculate its natural frequencies and to identify the respective vibration modes, the generalized Kantorovich–Vlasov method of reduction to ordinary differential equations is employed. To find the natural frequencies of possible types of vibrations, dispersion equations are derived. An asymptotic relation between the dispersion equations of the problem in hand and of an analogous problem for a rectangular plate with free sides is established. Determined is also a relation between the dispersion equations of the problem and of the boundary-value problem for a semi-infinite orthotropic nonclosed circular cylindrical shell with three free edges. With the example of an orthotropic cylindrical panel, the values of dimensionless characteristics of its natural frequencies are derived.

Introduction.

It is known that, at the free edge of an orthotropic plate, independently of each other, planar and flexural vibrations can occur (see [1-5] and survey work [6]). For a curved plate, both the types of vibration are coupled, giving rise to two new types of vibrations localized at the free edge (predominantly planar and predominantly flexural ones). At the free edge of an elastic cylindrical panel, they are transformated one into another. In this case, depending on the of geometrical and mechanical parameters of a shell, a complex distribution picture of natural frequencies arises in finite and infinite cylindri-

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cal shells with a free edge [7]. With growing number of free edges of a cylindrical panel, this picture becomes increasingly complex [8-14]. Therefore, investigations into the edge resonance of plates and cylindrical panels with free edges are the most difficult problems in the theory of vibrations of plates and shells [8]. These difficulties are overcome by using a combination of analytical and asymptotic theories, as well as by numerical methods.

In the present work, for the first time, investigated are free vibrations of an orthotropic cylindrical panel with free edges. Elements of such a type are important components of modern structures and constructions; therefore, the question of free vibrations of these elements is vital enough and demands attention. In this case, the orthotropy considered is such that, at each point of the panel, all three principal elasticity directions of the material coincide with directions of the corresponding coordinate lines, i.e., $B_{16} = B_{26} = 0$. It is proved, that at the boundary conditions considered, the problem does not allow the separation of variables. As was shown previously by G. R. Ghulghazaryan, such problems for cylindrical shells of orthotropic materials (for isotropic ones, was shown by V. B. Lidskii) with simple boundary conditions are self-conjugate and nonnegative definite, and therefore the generalized Kantorovich–Vlasov method can be applied to them [12, 15-19]. As the basic functions, eigenfunctions of the problem

$$w^{VIII} = \theta^8 w, \quad w\big|_{\beta=0,s} = w'\big|_{\beta=0,s} = w''\big|_{\beta=0,s} = w'''\big|_{\beta=0,s} = 0, \; 0 \le \beta \le s \,, \tag{1}$$

are used.

Problem (1) is self-conjugate and positive definite. To the eigenvalues $\theta_m^8, m = \overline{1,\infty}$, of problem (1) there correspond the eigenfunctions

$$w_m(\theta_m\beta) = \frac{\Delta_1}{\Delta} x_1(\theta_m\beta) + \frac{\Delta_2}{\Delta} x_2(\theta_m\beta) + \frac{\Delta_3}{\Delta} x_3(\theta_m\beta) + x_4(\theta_m\beta), \; 0 \le \beta \le s, \; m = \overline{1,+\infty} \,, \tag{2}$$

$$x_1(\theta_m\beta) = \cosh\theta_m\beta - \cosh\frac{\theta_m\beta}{\sqrt{2}}\cos\frac{\theta_m\beta}{\sqrt{2}} - \sinh\frac{\theta_m\beta}{\sqrt{2}}\sin\frac{\theta_m\beta}{\sqrt{2}},$$

$$x_2(\theta_m\beta) = \sinh\theta_m\beta - \sqrt{2}\cosh\frac{\theta_m\beta}{\sqrt{2}}\sin\frac{\theta_m\beta}{\sqrt{2}}, \quad x_3(\theta_m\beta) = \sin\theta_m\beta - \sqrt{2}\sinh\frac{\theta_m\beta}{\sqrt{2}}\cos\frac{\theta_m\beta}{\sqrt{2}}\,, \tag{3}$$

$$x_4(\theta_m\beta) = \cos\theta_m\beta - \cosh\frac{\theta_m\beta}{\sqrt{2}}\cos\frac{\theta_m\beta}{\sqrt{2}} + \sinh\frac{\theta_m\beta}{\sqrt{2}}\sin\frac{\theta_m\beta}{\sqrt{2}},$$

$$\Delta = \begin{vmatrix} x_1(\theta_m s) & x_2(\theta_m s) & x_3(\theta_m s) \\ x_1'(\theta_m s) & x_2'(\theta_m s) & x_3'(\theta_m s) \\ x_1''(\theta_m s) & x_2''(\theta_m s) & x_3''(\theta_m s) \end{vmatrix}, \quad \Delta_1 = -\begin{vmatrix} x_4(\theta_m s) & x_2(\theta_m s) & x_3(\theta_m s) \\ x_4'(\theta_m s) & x_2'(\theta_m s) & x_3'(\theta_m s) \\ x_4''(\theta_m s) & x_2''(\theta_m s) & x_3''(\theta_m s) \end{vmatrix},$$

$$\Delta_2 = -\begin{vmatrix} x_1(\theta_m s) & x_4(\theta_m s) & x_3(\theta_m s) \\ x_1'(\theta_m s) & x_4'(\theta_m s) & x_3'(\theta_m s) \\ x_1''(\theta_m s) & x_4''(\theta_m s) & x_3''(\theta_m s) \end{vmatrix}, \quad \Delta_3 = -\begin{vmatrix} x_1(\theta_m s) & x_2(\theta_m s) & x_4(\theta_m s) \\ x_1'(\theta_m s) & x_2'(\theta_m s) & x_4'(\theta_m s) \\ x_1''(\theta_m s) & x_2''(\theta_m s) & x_4''(\theta_m s) \end{vmatrix},$$

where $\theta_m, m = \overline{1,+\infty}$, are the positive zeros of the Wronskian of functions (3) at the point $\beta = s.$ Let us introduce the designation

$$\beta_m' = \int_0^s \left(w_m'(\theta_m\beta)\right)^2 d\beta \,/ \int_0^s \left(w_m(\theta_m\beta)\right)^2 d\beta, \;\; \beta_m'' = \int_0^s \left(w_m''(\theta_m\beta)\right)^2 d\beta \,/ \int_0^s \left(w_m'(\theta_m\beta)\right)^2 d\beta \,. \tag{4}$$

Notice that, in formulas (3) and (4), derivatives are taken with respect to $\theta_m\beta$ and $\beta_m' \to 1$, $\beta_m'' \to 1$ at $m \to +\infty$.

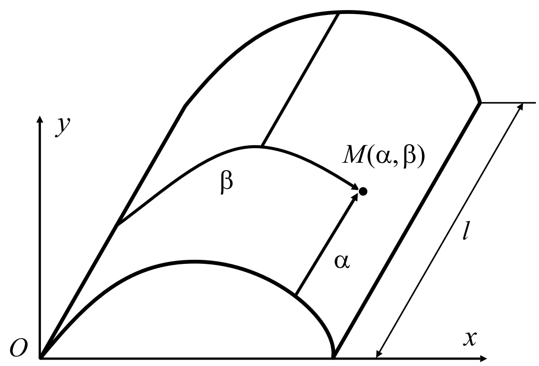

Fig. 1.

1. Statement of the Problem and the Basic Equations.

We will assume that generatrices of the cylindrical panel are orthogonal to its end faces. On the median surface of the shell, curvilinear coordinates (α, β) are introduced, where $\alpha (0 \le \alpha \le l)$ and $\beta (0 \le \beta \le s)$ are the lengths of the generatrix and directing circumference, respectively; l is length of the panel; s is length of the directing circumference.

As the initial equations describing vibrations of the panel, we will use the equations corresponding to the classical theory of orthotropic cylindrical shells written in the curvilinear coordinates α and β chosen (Fig. 1):

$$-B_{11}\frac{\partial^2 u_1}{\partial\alpha^2} - B_{66}\frac{\partial^2 u_1}{\partial\beta^2} - (B_{12}+B_{66})\frac{\partial^2 u_2}{\partial\alpha\partial\beta} + \frac{B_{12}}{R}\frac{\partial u_3}{\partial\alpha} = \lambda u_1 ,$$

$$-(B_{12}+B_{66})\frac{\partial^2 u_1}{\partial\alpha\partial\beta} - B_{66}\frac{\partial^2 u_2}{\partial\alpha^2} - B_{22}\frac{\partial^2 u_2}{\partial\beta^2} + \frac{B_{22}}{R}\frac{\partial u_3}{\partial\beta} - \frac{\mu^4}{R^2}\left(4B_{66}\frac{\partial^2 u_2}{\partial\alpha^2}\right.$$

$$\left. + B_{22}\frac{\partial^2 u_2}{\partial\beta^2}\right) - \frac{\mu^4}{R}\left(B_{22}\frac{\partial^3 u_3}{\partial\beta^3} + (B_{12}+4B_{66})\frac{\partial^3 u_3}{\partial\beta\partial\alpha^2}\right) = \lambda u_2 , \tag{1.1}$$

$$\mu^4\left(B_{11}\frac{\partial^4 u_3}{\partial\alpha^4} + 2(B_{12}+2B_{66})\frac{\partial^4 u_3}{\partial\alpha^2\partial\beta^2} + B_{22}\frac{\partial^4 u_3}{\partial\beta^4}\right) + \frac{\mu^4}{R}\left(B_{22}\frac{\partial^3 u_2}{\partial\beta^3}\right.$$

$$\left. + (B_{12}+4B_{66})\frac{\partial^3 u_2}{\partial\beta\partial\alpha^2}\right) - \frac{B_{12}}{R}\frac{\partial u_1}{\partial\alpha} - \frac{B_{22}}{R}\frac{\partial u_2}{\partial\beta} + \frac{B_{22}}{R^2}u_3 = \lambda u_3 .$$

Here, u_1, u_2, and u_3 are projections of the displacement vector on the directions α and β and on normals to the median surface of the shell, respectively; R is radius of the directing circumference of the median surface; $\mu^4 = h^2/12$ (h is shell thickness); $\lambda = \omega^2\rho$, where ω is angular frequency, ρ is the density of material; B_{ij} are elasticity coefficients. The boundary conditions of free edges are [20]

$$\left.\frac{\partial u_1}{\partial\alpha} + \frac{B_{12}}{B_{11}}\left(\frac{\partial u_2}{\partial\beta} - \frac{u_3}{R}\right)\right|_{\alpha=0,l} = 0, \qquad \left.\frac{\partial u_2}{\partial\alpha} + \frac{\partial u_1}{\partial\beta} + \frac{4\mu^4}{R}\left(\frac{\partial^2 u_3}{\partial\alpha\partial\beta} + \frac{1}{R}\frac{\partial u_2}{\partial\alpha}\right)\right|_{\alpha=0,l} = 0,$$

$$\left.\frac{\partial^2 u_3}{\partial\alpha^2} + \frac{B_{12}}{B_{11}}\left(\frac{\partial^2 u_3}{\partial\beta^2} + \frac{1}{R}\frac{\partial u_2}{\partial\beta}\right)\right|_{\alpha=0,l} = 0, \quad \left.\frac{\partial^3 u_3}{\partial\alpha^3} + \frac{B_{12}+4B_{66}}{B_{11}}\left(\frac{\partial^3 u_3}{\partial\alpha\partial\beta^2} + \frac{1}{R}\frac{\partial^2 u_2}{\partial\alpha\partial\beta}\right)\right|_{\alpha=0,l} = 0, \tag{1.2}$$

$$\left.\frac{B_{12}}{B_{22}}\frac{\partial u_1}{\partial\alpha}+\frac{\partial u_2}{\partial\beta}-\frac{u_3}{R}\right|_{\beta=0,s}=0,\quad \left.\frac{B_{12}}{B_{22}}\frac{\partial^2 u_3}{\partial\alpha^2}+\frac{\partial^2 u_3}{\partial\beta^2}+\frac{1}{R}\frac{\partial u_2}{\partial\beta}\right|_{\beta=0,s}=0, \tag{1.3}$$

$$\left.\frac{\partial^3 u_3}{\partial\beta^3}+\frac{B_{12}+4B_{66}}{B_{22}}\frac{\partial^3 u_3}{\partial\beta\partial\alpha^2}+\frac{1}{R}\frac{\partial^2 u_2}{\partial\beta^2}+\frac{4B_{66}}{B_{22}}\frac{1}{R}\frac{\partial^2 u_2}{\partial\alpha^2}\right|_{\beta=0,s}=0,\quad \left.\frac{\partial u_2}{\partial\alpha}+\frac{\partial u_1}{\partial\beta}\right|_{\alpha=0,s}=0.$$

Relations (1.2) and (1.3) are the conditions of free edges at $\alpha=0,l$ and $\beta=0,s$, respectively.

2. Derivation and an Analysis of Characteristic Equations.

In the first, second, and third equations of system (1.1), the spectral parameter λ is formally replaced by λ_1, λ_2, and λ_3 respectively. The solution of system (1.1) is sought in the form

$$(u_1,u_2,u_3)=\{u_m w_m(\theta_m\beta),\ v_m w_m'(\theta_m\beta), w_m(\theta_m\beta)\}\exp(\theta_m\chi\alpha),\quad m=\overline{1,+\infty}. \tag{2.1}$$

Here, $w_m(\theta_m\beta), m=\overline{1,\infty}$, are determined from formula (2), and u_m, v_m, and χ are indeterminate constants. In this case, conditions (1.3) are obeyed automatically. Let us insert Eq. (2.1) into Eq. (1.1). The equations found are scalarly multiplied by the vector functions $\left(w_m(\theta_m\beta), w_m'(\theta_m\beta), w_m(\theta_m\beta)\right)$ and then integrated between the limits from 0 to s. From first two equations found, we have

$$(c_m+\varepsilon_m^2a^2g_md_m)u_m=\varepsilon_m\chi\left\{a_m-a^2\frac{B_{22}(B_{12}+B_{66})}{B_{11}B_{66}}\beta_m' l_m+\varepsilon_m^2a^2\frac{B_{22}B_{12}}{B_{11}B_{66}}d_m\right\}, \tag{2.2}$$

$$(c_m+\varepsilon_m^2a^2g_md_m)v_{cm}=\varepsilon_m\left\{b_m-a^2g_ml_m\right\}, \tag{2.3}$$

but from the third equation, with account of relations (2.2) and (2.3), we obtain the characteristic equation

$$R_{mm}c_m+\varepsilon_m^2\left[c_m+b_m\beta_m'-\frac{B_{12}}{B_{22}}\chi^2a_m+a^2[R_{mm}g_md_m-2l_mb_m\beta_m']\right.$$

$$\left.+\varepsilon_m^2a^2d_m(b_m+\frac{B_{12}}{B_{11}}\chi^2)+a^4g_ml_m^2\beta_m'\right]=0, \tag{2.4}$$

$$a_m=\frac{B_{12}}{B_{11}}\chi^2+\frac{B_{22}}{B_{11}}\beta_m'+\frac{B_{12}}{B_{11}}\eta_{2m}^2+(\beta_m'-\beta_m'')\frac{B_{12}B_{22}}{B_{11}B_{66}},$$

$$b_m=B_1\chi^2-\frac{B_{22}}{B_{11}}\left(\beta_m'-\eta_{1m}^2\right),$$

$$c_m=\chi^4-B_2\chi^2+\left(\frac{B_{66}}{B_{11}}\eta_{1m}^2+\eta_{2m}^2\right)\chi^2+(\beta_m'-\eta_{1m}^2)\left(\frac{B_{22}}{B_{11}}\beta_m''-\frac{B_{66}}{B_{11}}\eta_2^2\right), \tag{2.5}$$

$$B_1=\frac{B_{11}B_{22}-B_{12}^2-B_{12}B_{66}}{B_{11}B_{66}},\ B_2=\frac{B_{11}B_{22}\beta_m''-B_{12}^2\beta_m'-2B_{12}B_{66}\beta_m'}{B_{11}B_{66}},$$

$$d_m=\frac{4B_{66}}{B_{22}}\chi^2-\beta_m'',\ g_m=\frac{B_{22}}{B_{66}}\chi^2-\frac{B_{22}}{B_{11}}\beta_m'+\frac{B_{22}}{B_{11}}\eta_{1m}^2,$$

$$l_m = \frac{B_{12}+4B_{66}}{B_{22}}\chi^2 - \beta''_m,\ \ \eta_{im}^2 = \frac{\lambda_i}{B_{66}\theta_m^2},\ i=\overline{1,3},$$

$$R_{mm} = a^2\left[\frac{B_{11}}{B_{22}}\chi^4 - \frac{2(B_{12}+2B_{66})}{B_{22}}\beta'_m\chi^2 + \beta'_m\beta''_m\right] - \frac{B_{66}}{B_{22}}\eta_{3m}^2,$$

$$a^2 = \frac{h^2}{12}\theta_m^2, \varepsilon_m = \frac{1}{R\theta_m}.$$

Let $\chi_j,\ j=\overline{1,4}$, be pairwise different roots of the Eq. (2.4) with nonpositive real parts $\chi_{4+j} = -\chi_j, j=\overline{1,4}$. Let $(u_1^{(j)}, u_2^{(j)}, u_3^{(j)}),\ j=\overline{1,8}$, be nontrivial solutions of type (2.1) of system (1.1) at $\chi=\chi_j, j=\overline{1,8}$, respectively. The solution of problem (1.1)-(1.3) is sought in the form

$$u_i = \sum\nolimits_{j=1}^{8} u_i^{(j)} w_j,\ i=\overline{1,3}. \qquad (2.6)$$

Let us insert Eq. (2.6) into boundary conditions (1.2). Each equation found, except for the second one, is multipliedy by $w(\theta_m\beta)$ and the second one — by $w'(\theta_m\beta)$ and is integrated between the limits from 0 to s. As a result, we obtain the system of equations

$$\sum_{j=1}^{8}\frac{M_{ij}^{(m)} w_j}{c_m^{(j)} + \varepsilon_m^2 a^2 g_m^{(j)} d_m^{(j)}} = 0,\quad i=\overline{1,8}, \qquad (2.7)$$

$$M_{1j}^{(m)} = \chi_j^2 a_m^{(j)} - \frac{B_{12}}{B_{11}} b_m^{(j)}\beta'_m - \frac{B_{12}}{B_{11}} c_m^{(j)} + \varepsilon_m^2 a^2\frac{B_{12}B_{22}}{B_{11}^2} d_m^{(j)}(\beta'_m - \eta_{1m}^2)$$

$$-a^2\frac{B_{22}}{B_{11}} l_m^{(j)}\beta'_m\left(\chi_j^2 + \frac{B_{12}}{B_{11}}\beta'_m - \frac{B_{12}}{B_{11}}\eta_{1m}^2\right),$$

$$M_{2j}^{(m)} = \beta'_m\chi_j\left\{a_m^{(j)} + b_m^{(j)} + a^2\left[4c_m^{(j)} - l_m^{(j)}\left(\frac{B_{22}}{B_{66}}\chi_j^2 + \frac{B_{12}B_{22}}{B_{11}B_{66}}\beta'_m + \frac{B_{22}}{B_{11}}\eta_{1m}^2\right)\right]\right.$$

$$\left. + a^2\varepsilon_m^2\left(4b_m^{(j)} + \frac{B_{12}B_{22}}{B_{11}B_{66}} d_m^{(j)} - 4a^2 g_m^{(j)}\frac{B_{12}}{B_{22}}\chi_j^2\right)\right\},$$

$$M_{3j}^{(m)} = \left(\chi_j^2 - \frac{B_{12}}{B_{11}}\beta'_m\right)c_m^{(j)}$$

$$+\varepsilon_m^2\left[a^2 g_m^{(j)}\left(\frac{4B_{66}}{B_{22}}\chi_j^4 - \frac{B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m}{B_{11}B_{22}}\chi_j^2\right) - \frac{B_{12}}{B_{11}} b_m^{(j)}\beta'_m\right], \qquad (2.8)$$

$$M_{4j}^{(m)} = \chi_j\left\{\left(\chi_j^2 - \frac{B_{12}+4B_{66}}{B_{11}}\beta'_m\right)c_m^{(j)}\right.$$

$$+\varepsilon_m^2\left[a^2 g_m^{(j)}\left(\frac{4B_{66}}{B_{22}}\chi_j^4 - \frac{B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m - 4B_{12}B_{66}\beta'_m}{B_{11}B_{22}}\chi_j^2\right)\right.$$

$$\left.\left. - \frac{B_{12}+4B_{66}}{B_{11}} b_m^{(j)}\beta'_m\right]\right\},\quad z_j = \theta_m\chi_j l,\ \ j=\overline{1,8}.$$

$$M_{5j}^{(m)} = M_{1j}^{(m)} \exp z_j \;\; M_{6j}^{(m)} = M_{2j}^{m} \exp z_j \,,$$
$$M_{7j}^{(m)} = M_{3j}^{m} \exp z_j \,, \;\; M_{8j}^{(m)} = M_{4j}^{m} \exp z_j . \tag{2.8}$$

The superscript j in parentheses means that the corresponding function is taken at $\chi = \chi_j$. In order that system (2.7) had a nontrivial solution, it is necessary and enough that

$$\det \left\| M_{ij}^{(m)} \right\|_{i,j=1}^{8} = 0 . \tag{2.9}$$

A numerical analysis shows that the left side of this equality becomes small when any two roots of Eq. (2.4) become close to each other. This highly complicates calculations and can lead to false solutions. It turns out that, from the left side of Eq. (2.9), it can be separated out a multiplier that tends to zero when two roots approach each other. For this purpose, we introduce the following designations:

$$[z_i z_j] = \theta_m l(\exp(z_i) - \exp(z_j)) / (z_i - z_j), \;\; [z_i z_j z_k] = \theta_m l\big([z_i z_j] - [z_i z_k]\big) / (z_j - z_k),$$
$$[z_1 z_2 z_3 z_4] = \theta_m l\big([z_1 z_2 z_3] - [z_1 z_2 z_4]\big) / (z_3 - z_4) \,,$$
$$\sigma_1 = \sigma_1(\chi_1, \chi_2, \chi_3, \chi_4) = \chi_1 + \chi_2 + \chi_3 + \chi_4 \,,$$
$$\sigma_2 = \sigma_2(\chi_1, \chi_2, \chi_3, \chi_4) = \chi_1\chi_2 + \chi_1\chi_3 + \chi_1\chi_4 + \chi_2\chi_3 + \chi_2\chi_4 + \chi_3\chi_4 \,,$$
$$\sigma_3 = \sigma_3(\chi_1, \chi_2, \chi_3, \chi_4) = \chi_1\chi_2\chi_3 + \chi_1\chi_2\chi_4 + \chi_1\chi_3\chi_4 + \chi_2\chi_3\chi_4 \,, \tag{2.10}$$
$$\sigma_4 = \sigma_4(\chi_1, \chi_2, \chi_3, \chi_4) = \chi_1\chi_2\chi_3\chi_4 \,,$$
$$\bar{\sigma}_k = \sigma_k(\chi_1, \chi_2, \chi_3, 0), \;\; \bar{\bar{\sigma}}_k = \sigma_k(\chi_1, \chi_2, 0, 0), \; k = \overline{1,4}.$$

In this case, $\bar{\sigma}_4 = \bar{\bar{\sigma}}_4 = \bar{\bar{\sigma}}_3 = 0$. Let f_n, $n = \overline{1,6}$, be a symmetric polynomial of *n*th degree in variables χ_1, χ_2, χ_3, and χ_4. It is known that it can be uniquely expressed in terms of elementary symmetric polynomials. Introducing the designations

$$f_n = f_n(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \;\; \bar{f}_n = f_n(\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, 0), \;\; \bar{\bar{f}}_n = f_n(\bar{\bar{\sigma}}_1, \bar{\bar{\sigma}}_2, 0, 0) \;\; n = \overline{1,6} \,, \tag{2.11}$$

$$f_1 = \sigma_1, \; f_2 = \sigma_1^2 - \sigma_2, \; f_3 = \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3, \; f_4 = \sigma_1^4 - 3\sigma_1^2\sigma_2 + \sigma_2^2 + 2\sigma_1\sigma_3 - \sigma_4,$$
$$\bar{f}_5 = \bar{\sigma}_1^5 - 4\bar{\sigma}_1^3\bar{\sigma}_2 + 3\bar{\sigma}_1\bar{\sigma}_2^2 + 3\bar{\sigma}_1^2\bar{\sigma}_3 - 2\bar{\sigma}_2\bar{\sigma}_3 \,, \;\; \bar{\bar{f}}_6 = \bar{\bar{\sigma}}_1^6 - 5\bar{\bar{\sigma}}_1^4\bar{\bar{\sigma}}_2 + 6\bar{\bar{\sigma}}_1^2\bar{\bar{\sigma}}_2^2 - \bar{\bar{\sigma}}_2^3 \,, \tag{2.12}$$

and performing elementary operations with columns of determinant (2.9), we obtain that

$$\det \left\| M_{ij}^{(m)} \right\|_{i,j=1}^{8} = K^2 \exp(-z_1 - z_2 - z_3 - z_4) \det \left\| m_{ij} \right\|_{i,j=1}^{8} \,, \tag{2.13}$$

$$K = (\chi_1 - \chi_2)(\chi_1 - \chi_3)(\chi_1 - \chi_4)(\chi_2 - \chi_3)(\chi_2 - \chi_4)(\chi_3 - \chi_4) \,. \tag{2.14}$$

Expressions for m_{ij} are given in Appendix 1. Equation (2.9) is equivalent to the equation

$$\det \left\| m_{ij} \right\|_{i,j=1}^{8} = 0 \,. \tag{2.15}$$

Taking into account the possible relations between λ_1, λ_2, and λ_3, we conclude that Eq. (2.15) determines frequencies of the corresponding types of vibrations. At $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, the equation (2.4) is the characteristic equation of system (1.1), and Eq. (2.15) — the dispersion equation of problem (1.1)-(1.3).

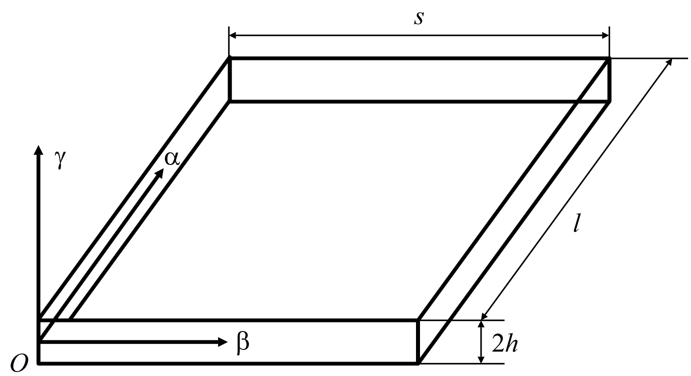

Fig. 2.

In Sect. 5, investigated is the asymptotics of dispersion equation (2.15) at $\varepsilon_m = 1/(\theta_m R) \to 0$ (transition to a plate or to vibrations localized at the free edges of the cylindrical panel) and at $\theta_m l \to \infty$ (transition to a semiinfinite cylindrical panel or to vibrations localized at the free edges of the cylindrical panel). To check the reliability of the asymptotic relations found in Sect. 5, we will investigate the free planar and flexural vibrations of a rectangular plate with free edges.

3. Planar Vibrations of an Orthotropic Rectangular Plate with the Free Sides.

Let an orthotropic rectangular plate of width s and length l be related to a three-orthogonal system of rectilinear coordinates (α, β, γ) with the origin in the end face plane in such a way, that the coordinate plane $\alpha\beta$ coincides with the median plane of the plate and the principal directions of elasticity of plate material coincide with coordinate lines (Fig. 2). Under the conditions of free vibrations, let us consider the question of existence of planar vibrations of a rectangular plate with free sides. As initial equations, we assume the equations of small planar vibrations corresponding to the classical theory of orthotropic plates [20]:

$$
\begin{gathered}
-B_{11}\frac{\partial^2 u_1}{\partial\alpha^2} - B_{66}\frac{\partial^2 u_1}{\partial\beta^2} - (B_{12}+B_{66})\frac{\partial^2 u_2}{\partial\alpha\partial\beta} = \lambda u_1, \\
-(B_{12}+B_{66})\frac{\partial^2 u_1}{\partial\alpha\partial\beta} - B_{66}\frac{\partial^2 u_2}{\partial\alpha^2} - B_{22}\frac{\partial^2 u_2}{\partial\beta^2} = \lambda u_2.
\end{gathered} \tag{3.1}
$$

Here, α $(0 \le \alpha \le l)$ and β $(0 < \beta < s)$ are the orthogonal rectilinear coordinates of a point of the median plane; u_1 and u_2 are projections of the displacement vector on the directions α and β ; B_{ik}, $i, k = 1, 2, 6$, are the elasticity constants; $\lambda = \omega^2\rho$, where ω is the angular frequency of free vibrations and ρ is the density of plate material. For the given problem ($1/R = 0$), boundary conditions (1.2) and (1.3) take the form

$$
\left.\frac{\partial u_1}{\partial\alpha} + \frac{B_{12}}{B_{11}}\frac{\partial u_2}{\partial\beta}\right|_{\alpha=0,l} = \left.\frac{\partial u_2}{\partial\alpha} + \frac{\partial u_1}{\partial\beta}\right|_{\alpha=0,l} = 0, \tag{3.2}
$$

$$
\left.\frac{B_{12}}{B_{11}}\frac{\partial u_1}{\partial\alpha} + \frac{\partial u_2}{\partial\beta}\right|_{\beta=0,s} = \left.\frac{\partial u_2}{\partial\alpha} + \frac{\partial u_1}{\partial\beta}\right|_{\beta=0,s} = 0, \tag{3.3}
$$

where relations (3.2) and (3.3) are the conditions of free edges at $\alpha = 0, l$ and $\beta = 0, s$ respectively. Problem (3.1) - (3.3) does not allow the separation of variables. The differential operator corresponding to this problem is selfconjugate and non-negative definite. Therefore, the generalized Kantorovich – Vlasov method of reduction to ordinary differential equations can be used to find vibration eigenfrequencies and eigenmodes [15-19]. The solution of system (3.1) is sought in the form

$$(u_1, u_2) = \{u_m w_m(\theta_m\beta),\ v_m w'_m(\theta_m\beta)\}\exp(\theta_m y\alpha), \quad m = \overline{1, +\infty}\,. \tag{3.4}$$

In this case, conditions (3.3) are satisfied automatically. Let us insert (3.4) into Eq. (3.2), scalarly multiply the result by the vector function $\left(w_m(\theta_m\beta),\ w'_m(\theta_m\beta)\right)$, and integrate it between limits from 0 to s. As a result, we obtain the system of equations

$$\begin{aligned}(y^2 - \frac{B_{66}}{B_{11}}(\beta'_m - \eta_m^2))u_m - \frac{B_{12}+B_{66}}{B_{11}}\, y\beta'_m\, v_m = 0,\\ \frac{B_{12}+B_{66}}{B_{66}}\, yu_m + (y^2 - \frac{B_{22}}{B_{66}}\beta''_m + \eta_m^2)v_m = 0,\end{aligned} \tag{3.5}$$

where $\eta_m^2 = \lambda / \left(\theta_m^2 B_{66}\right)$, and θ_m and β'_m, β''_m are determined in Eq. (2) and (4) respectively. Equating the determinant of system (3.5) to zero, the following characteristic equation of the system of equations (3.1) is found:

$$c_m = y^4 - B_2 y^2 + \frac{B_{11}+B_{66}}{B_{11}}\eta_m^2 y^2 + (\beta'_m - \eta_m^2)\left(\frac{B_{22}}{B_{11}}\beta''_m - \frac{B_{66}}{B_{11}}\eta_m^2\right) = 0\,. \tag{3.6}$$

Let y_1 and y_2 be various roots of the Eq. (3.6) with nonpositive real parts, $y_{2+j} = -y_j$, $j = 1, 2$. As the solution of system (3.5) at $y = y_j$, $j = \overline{1, 4}$, we take

$$u_m^{(j)} = \frac{B_{12}+B_{66}}{B_{11}}\beta' y_j,\ \ v_m^{(j)} = y_j^2 - \frac{B_{66}}{B_{11}}(\beta'_m - \eta_m^2), \quad j = \overline{1, 4}\,. \tag{3.7}$$

The solution of problem (3.1) - (3.3) can be presented in the form

$$u_1 = \sum\nolimits_{j=1}^{4} u_m^{(j)} w_m(\theta_m\beta)\exp(\theta_m y_j\alpha)w_j,\ u_2 = \sum\nolimits_{j=1}^{4} v_m^{(j)} w'_m(\theta_m\beta)\exp(\theta_m y_j\alpha)w_j. \tag{3.8}$$

Let us insert Eq. (3.8) into boundary conditions (3.2). Each of the equations obtained, except for the second one, is multiplied by $w(\theta_m, \beta)$, and the second one — by $w'(\theta_m, \beta)$, and is integrated between the limits from 0 to s. As a result, we arrive at the system of equations

$$\sum_{j=1}^{4} R_{ij}^{(m)} w_j = 0,\ i = 1, 2, \quad \sum_{j=1}^{4} R_{ij}^{(m)}\exp(z_j)\, w_j = 0,\ i = 1, 2\,, \tag{3.9}$$

where

$$R_{1j}^{(m)} = y_j^2 + \frac{B_{12}}{B_{11}}(\beta'_m - \eta_m^2),\ R_{2j}^{(m)} = y_j\left(y_j^2 + \frac{B_{12}}{B_{11}}\beta'_m + \frac{B_{66}}{B_{11}}\eta_m^2\right),\ z_j = \theta_m y_j l\,, j = \overline{1, 4}. \tag{3.10}$$

Equating the determinant Δ_e of system (3.9) to zero and performing elementary operations with columns of the determinant, we obtain the dispersion equation

$$\Delta_e = \exp(-z_1 - z_2)(y_2 - y_1)^2 \det\left\|l_{ij}\right\|_{i,j}^{4} = 0\,, \tag{3.11}$$

where

$$\begin{aligned}&l_{11} = R_{11}^{(m)},\ l_{12} = y_1 + y_2,\ l_{13} = l_{11}\exp z_1,\ l_{14} = l_{12}\exp z_2 + l_{11}[z_1 z_2];\ l_{21} = R_{21}^{(m)},\\ &l_{22} = y_1 y_2 + \left(B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m - B_{12}B_{66}\beta'\right)/(B_{11}B_{66}) - \eta_m^2,\ l_{23} = -l_{21}\exp z_1,\\ &l_{24} = -l_{22}\exp z_2 - l_{21}[z_1 z_2], \quad l_{31} = l_{13},\ l_{23} = l_{14}, l_{33} = l_{11},\ l_{34} = l_{12},\\ &l_{41} = l_{23},\ l_{42} = l_{24}, l_{43} = l_{21},\ l_{44} = l_{22},\ [z_1 z_2] = \theta_m l(\exp z_2 - \exp z_1)/(z_2 - z_1).\end{aligned} \tag{3.12}$$

Equation (3.11) is equivalent to the equation

$$\det\left|l_{ij}\right|_{i,j=1}^{4}=\left((B_{12}+B_{66})/B_{11}\right)^2 K_{2m}^2(\eta_m^2)(1+\exp 2(z_1+z_2))$$

$$-(l_{11}l_{22}+l_{21}l_{12})^2(\exp 2z_1+\exp 2z_2)+8l_{12}\,l_{11}\,l_{22}\,l_{21}\exp(z_1+z_2)$$

$$+4l_{11}l_{21}\left(l_{11}l_{22}+l_{21}l_{12}\right)(\exp z_2-\exp z_1)[z_1z_2]-4l_{11}^2l_{21}^2[z_1z_2]^2=0,$$

$$K_{2m}\left(\eta_m^{(2)}\right)=(\beta_m'-\eta_m^2)\left(\frac{B_{11}B_{22}\beta_m''-B_{12}^2\beta_m'}{B_{11}B_{66}}-\eta_m^2\right)-\eta_m^2y_1y_2\,. \tag{3.13}$$

If y_1 and y_2 are roots of Eq. (3.6) with negative real parts, then, at $\theta_m l\to\infty$, the roots of Eq. (3.13) are approximated by roots of the equation

$$K_{2m}\left(\eta_m^{(2)}\right)=(\beta_m'-\eta_m^2)\left(\frac{B_{11}B_{22}\beta_m''-B_{12}^2\beta_m'}{B_{11}B_{66}}-\eta_m^2\right)-\eta_m^2y_1y_2=0\,. \tag{3.14}$$

Equation (3.14) is an analogue of the Rayleigh equation for a long enough orthotropic a rectangular plate with free sides (compare with [11-14]). Thus, eigenfrequencies of problem (3.1) - (3.3) are found from Eqs. (3.13).

To find the corresponding eigenmodes, the coefficients w_j, $j=\overline{1,4}$ have to be determined from the system of equations (3.9) and inserted into Eqs. (3.8). As solutions of the system of equations (3.9) at a given dimensionless eigenfrequency characteristic η_m, it can be taken that

$$w_1=\frac{R_{12}^{(m)}}{R_{11}^{(m)}}\frac{R_1^{(m)}\exp(2z_1+z_2)+R_2^{(m)}\exp z_1-2R_{11}^{(m)}R_{22}^{(m)}\exp z_2}{R_1^{(m)}-R_2^{(m)}\exp 2z_1+2R_{12}^{(m)}R_{21}^{(m)}\exp(z_1+z_2)},$$

$$w_2=\frac{R_1^{(m)}\exp(2z_1+z_2)+R_2^{(m)}\exp z_2-2R_{12}^{(m)}R_{21}^{(m)}\exp z_1}{R_1^{(m)}-R_2^{(m)}\exp 2z_1+2R_{12}^{(m)}R_{21}^{(m)}\exp(z_1+z_2)}, \tag{3.15}$$

$$w_3=-\frac{R_{12}^{(m)}}{R_{11}^{(m)}}\frac{R_1^{(m)}\exp z_1+R_2^{(m)}\exp(2z_2+z_1)-2R_{11}^{(m)}R_{22}^{(m)}\exp(2z_1+z_2)}{R_1^{(m)}-R_2^{(m)}\exp 2z_1+2R_{12}^{(m)}R_{21}^{(m)}\exp(z_1+z_2)},$$

$$w_4=\exp(z_2),\; R_1^{(m)}=R_{11}^{(m)}R_{22}^{(m)}-R_{12}^{(m)}R_{21}^{(m)},\; R_2^{(m)}=R_{11}^{(m)}R_{22}^{(m)}+R_{12}^{(m)}R_{21}^{(m)}\,.$$

4. Flexural Vibrations of an Orthotropic Rectangular Plate with Free Sides

Consider an orthotropic rectangular plate (see Fig. 2). Under the condition of free vibrations, let us analyze the question of existence of flexural vibrations of a rectangular plate with free sides. As the initial equation, we take the equation of small flexural vibrations corresponding to the classical theory of orthotropic plates [20], namely,

$$\mu^4\left(B_{11}\frac{\partial^4u_3}{\partial\alpha^4}+2(B_{12}+2B_{66})\frac{\partial^4u_3}{\partial\alpha^2\partial\beta^2}+B_{22}\frac{\partial^4u_3}{\partial\beta^4}\right)=\lambda u_3\,, \tag{4.1}$$

where α $(0\le\alpha\le l)$ and β $(0\le\beta\le s)$ are the orthogonal rectilinear coordinates of a point of the median plane of the plate; u_3 is the normal component of the displacement vector of a point of the median plane; B_{ik}, $i, k=1, 2, 6$ are elasticity coefficients; $\mu^4=h^2/12$; $\lambda=\omega^2\rho$. For the given problem ($1/R=0$), boundary conditions (1.2) and (1.3) take the form

$$\left.\frac{\partial^2u_3}{\partial\alpha^2}+\frac{B_{12}}{B_{11}}\frac{\partial^2u_3}{\partial\beta^2}\right|_{\alpha=0,l}=\left.\frac{\partial^3u_3}{\partial\alpha^3}+\frac{B_{12}+4B_{66}}{B_{11}}\frac{\partial^3u_3}{\partial\alpha\partial\beta^2}\right|_{\alpha=0,l}=0\,, \tag{4.2}$$

$$\left.\frac{B_{12}}{B_{22}}\frac{\partial^2 u_3}{\partial\alpha^2}+\frac{\partial^2 u_3}{\partial\beta^2}\right|_{\beta=0,s}=\left.\frac{\partial^3 u_3}{\partial\beta^3}+\frac{B_{12}+4B_{66}}{B_{22}}\frac{\partial^3 u_3}{\partial\beta\partial\alpha^2}\right|_{\beta=0,s}=0\,. \tag{4.3}$$

Relations (4.2) and (4.3) are the conditions of free edges at $\alpha=0,l$ and $\beta=0,s$, respectively. Problem (4.1) - (4.3) does not allow the separation of variables. The differential operator corresponding to problem (4.1) - (4.3), is selfconjugate and non-negative definite. Therefore, to find vibration eigenfrequencies and eigenmodes, the Eq. Kantorovich–Vlasov method of reduction to ordinary differential equations [15-19] can be employed. The solution of Eq. (4.1) is sought in the form

$$u_3=w_m(\theta_m\beta)\exp(\theta_m y\alpha),\quad m=\overline{1,+\infty}\,, \tag{4.4}$$

where u_m, v_m, and χ are unknown constants, and $w_m(\theta_m\beta), m=\overline{1,\infty}$, are determined in (2). In this case, conditions (4.3) are fulfilled automatically. Inserting Eq. (4.4) into Eq. (4.1), multiplying the result by $w_m(\theta_m\beta)$, and integrating it between the limits from 0 to s, we arrive at the characteristic equation

$$R_{mm}=a^2\left[\frac{B_{11}}{B_{22}}y^4-\frac{2(B_{12}+2B_{66})}{B_{22}}\beta'_m y^2+\beta'_m\beta''_m\right]-\frac{B_{66}}{B_{22}}\eta_m^2=0\,, \tag{4.5}$$

$$\eta_m^2=\frac{\lambda}{\theta_m^2 B_{66}},\quad a^2=\theta_m^2 h^2/12\,, \tag{4.6}$$

where θ_m, β'_m and β''_m are determined in Eq. (2) and (4) respectively. Let y_3 and y_4 be various roots of Eq. (4.5) with non-positive real parts, $y_{2+j}=-y_j$, $j=3,4$. The solution of problem (4.1)-(4.3) is sought in the form

$$u_3=\sum\nolimits_{j=3}^{6}w_m(\theta_m\beta)\exp(\theta_m y_j\alpha)\;w_j\,. \tag{4.7}$$

Inserting Eq. (4.7) into boundary conditions (4.2), multiplying the resulting equations by $w_m(\theta_m\beta)$, and integrating them between the limits from 0 to s, we obtain the system of equations

$$\sum_{j=3}^{6}R_{ij}^{(m)}w_j=0,\ i=3,4,\quad \sum_{j=3}^{6}R_{ij}^{(m)}\exp(z_j)\,w_j=0,\ i=3,4\,, \tag{4.8}$$

$$R_{3j}^{(m)}=y_j^2-\frac{B_{12}}{B_{11}}\beta'_m,\ R_{4j}^{(m)}=y_j^3-\frac{B_{12}+4B_{66}}{B_{11}}\beta'_m y_j,\ z_j=\theta_m y_j l,\ j=\overline{3,6}\,. \tag{4.9}$$

Equating the determinant Δ_b of system (4.8) to zero and performing elementary operations with columns of the determinant, we arrive at the dispersion equation

$$\Delta_b=\exp(-z_3-z_4)(y_4-y_3)^2\det\left\|b_{ij}\right\|_{i,j}^{4}=0\,, \tag{4.10}$$

$$\begin{gathered}
b_{11}=R_{33}^{(m)},\ b_{12}=y_3+y_4,\ b_{13}=b_{11}\exp z_3,\ b_{14}=b_{12}\exp z_4+l_{11}[z_3z_4]\,,\\
b_{21}=R_{43}^{(m)},\ b_{22}=y_3y_4+\beta'_m B_{12}/B_{11},\ b_{23}=-b_{21}\exp z_3,\ b_{24}=-b_{22}\exp z_4-b_{21}[z_3z_4]\,,\\
b_{31}=b_{13},b_{32}=b_{14},b_{33}=b_{11},b_{34}=b_{12},b_{41}=b_{23},b_{42}=b_{24},b_{43}=b_{21},b_{44}=b_{22},\\
[z_3z_4]=\theta_m l(\exp(z_4)-\exp(z_3))/(z_4-z_3)\,.
\end{gathered} \tag{4.11}$$

Equation (4.10) is equivalent to the equation

$$\begin{gathered}
\det\left|b_{ij}\right|_{i,j=1}^{4}=K_{1m}^2(\eta_m)\,[1+\exp\ 2\ (z_3+z_4)]\\
-(b_{11}b_{22}+b_{21}b_{12})^2(\exp 2z_3+\exp 2z_4)+8b_{12}\,b_{11}\,b_{22}\,b_{21}\exp(z_3+z_4)
\end{gathered} \tag{4.12}$$

$$-4b_{11}b_{21}\left(b_{11}b_{22}+b_{21}b_{12}\right)(\exp z_4-\exp z_3)[z_3z_4]-4b_{11}^2b_{21}^2[z_3z_4]^2=0,$$
$$K_{1m}\left(\eta_m^2\right)=y_3^2y_4^2+4\frac{B_{66}}{B_{11}}\beta_m' y_3y_4-\left(\frac{B_{12}}{B_{11}}\right)\beta_m'^2 . \tag{4.12}$$

If y_3 and y_4 are roots of Eq. (4.5) with negative real parts, then, at $\theta_m l\to\infty$, the roots of Eq. (4.12) are approximated by roots of the equation

$$K_{1m}\left(\eta_m^2\right)=y_3^2y_4^2+4\frac{B_{66}}{B_{11}}\beta_m' y_3y_4-\left(\frac{B_{12}}{B_{11}}\right)^2\left(\beta_m'\right)^2=0 . \tag{4.13}$$

Equation (4.13) is an analogue of the Konenkov equation for a long enough orthotropic a rectangular plate with free sides (compare with [11-14]).

Thus, eigenfrequencies of problem (4.1)-(4.3) are found from Eqs. (4.12).

To find the corresponding eigenmodes, the coefficients w_j, $j=\overline{3,6}$, have to be determined from the system of Eqs. (4.8) and inserted into Eq. (4.7). As solutions of the system of Eqs. (4.8) at a given dimensionless eigenvalue characteristic η_m, it can be assumed that

$$w_3=\frac{R_{34}^{(m)}}{R_{33}^{(m)}}\frac{R_3^{(m)}\exp(2z_3+z_4)+R_4^{(m)}\exp z_3-2R_{33}^{(m)}R_{44}^{(m)}\exp z_4}{R_3^{(m)}-R_4^{(m)}\exp 2z_3+2R_{34}^{(m)}R_{43}^{(m)}\exp(z_3+z_4)},$$

$$w_4=\frac{R_3^{(m)}\exp(2z_3+z_4)+R_4^{(m)}\exp z_4-2R_{34}^{(m)}R_{43}^{(m)}\exp z_3}{R_3^{(m)}-R_4^{(m)}\exp 2z_3+2R_{34}^{(m)}R_{43}^{(m)}\exp(z_3+z_4)}, \tag{4.14}$$

$$w_5=-\frac{R_{34}^{(m)}}{R_{33}^{(m)}}\frac{R_3^{(m)}\exp z_3+R_4^{(m)}\exp(2z_3+z_4)-2R_{33}^{(m)}R_{44}^{(m)}\exp(2z_3+z_4)}{R_3^{(m)}-R_4^{(m)}\exp 2z_3+2R_{34}^{(m)}R_{43}^{(m)}\exp(z_3+z_4)},$$

$$w_6=\exp z_4,\; R_3^{(m)}=R_{33}^{(m)}R_{44}^{(m)}-R_{34}^{(m)}R_{43}^{(m)},\; R_4^{(m)}=R_{33}^{(m)}R_{44}^{(m)}+R_{34}^{(m)}R_{43}^{(m)} .$$

5. Asymptotics of Dispersion Equation (2.15)

5.1. Asymptotics of dispersion equation (2.15) at $\varepsilon_m\to 0$

Using the previous formulas, suppose that $\eta_{1m}=\eta_{2m}=\eta_{3m}=\eta_m$. Then, at $\varepsilon_m\to 0$, Eq. (2.4) will be transformed to the set of equations

$$c_m=y^4-B_2y^2+\frac{B_{11}+B_{66}}{B_{11}}\eta_m^2y^2+(\beta_m'-\eta_m^2)\left(\frac{B_{22}}{B_{11}}\beta_m''-\frac{B_{66}}{B_{11}}\eta_m^2\right)=0, \tag{5.1}$$

$$R_{mm}=a^2\left[\frac{B_{11}}{B_{22}}y^4-\frac{2(B_{12}+2B_{66})}{B_{22}}\beta_m' y^2+\beta_m'\beta_m''\right]-\frac{B_{66}}{B_{22}}\eta_m^2=0 . \tag{5.2}$$

The limiting process $\varepsilon_m\to 0$ here is understood in the sense that, at fixation of the radius R and distances b between boundary generatrices of the cylindrical panel, transition to a cylindrical panel of radius $R'=nR$ and to the limit $\varepsilon_m'=1/(n\theta_mR)=\varepsilon_m/n\to 0$ at $n\to\infty$ is performed.

Equations (5.1) and (5.2) are characteristic equations for the equations of planar and flexural, respectively, vibrations of orthotropic plates whose all sides are free. The roots of Eqs. (5.1) and (5.2) with nonpositive real parts, as in Sects. 3 and 4, are designated by y_1, y_2 and y_3, y_4, respectively. In the same way as in [21], it is proved that at

$$\varepsilon_m << 1, \quad y_i \neq y_j, \ i \neq j, \tag{5.3}$$

and the roots χ^2 of Eq. (2.4) can be presented as

$$\chi_i^2 = y_i^2 + \alpha_i^{(m)}\varepsilon_m^2 + \beta_i^{(m)}\varepsilon_m^4 + ..., \quad i = \overline{1,4}. \tag{5.4}$$

Under condition (5.3), considering relations (2.8), (2.13) and (5.4) and the equalities

$$M_{3j}^{(m)} = M_{4j}^{(m)} = M_{7j}^{(m)} = M_{8j}^{(m)} = O(\varepsilon_m^2), \ i = 1, 2, \tag{5.5}$$

Eq. (2.15) can be put into the form

$$\det\left\|m_{ij}\right\|_{i,j=1}^{8} = \left(B_{66}/B_{11}\right)^2 N^2(\eta_m^2)K_{3m}^2(\eta_m^2)\det\left|l_{ij}\right|_{i,j=1}^{4} \det\left|b_{ij}\right|_{i,j=1}^{4} + O(\varepsilon_m^2) = 0, \tag{5.6}$$

where $\det\left|l_{ij}\right|_{i,j=1}^{4}$ and $\det\left|b_{ij}\right|_{i,j=1}^{4}$ are determined by formulas (3.13) and (4.12), respectively, and

$$N(\eta_m^2) = (y_3 + y_1)(y_3 + y_2)(y_4 + y_1)(y_4 + y_2),$$

$$K_{3m}(\eta_m^2) = (\beta'_m - \eta_m^2)\left(\frac{B_{22}}{B_{11}}\beta''_m - \frac{B_{66}}{B_{11}}\eta_m^2\right)\left[\frac{B_{12}}{B_{12}+B_{66}} - a^2\frac{\left(B_{12}+4B_{66}\right)}{B_{66}}\beta'_m\right]^2$$

$$+\left(B_2 - \frac{B_{11}+B_{66}}{B_{11}}\eta_m^2\right)\left[\frac{B_{12}}{B_{12}+B_{66}} - a^2\frac{\left(B_{12}+4B_{66}\right)}{B_{66}}\beta'_m\right]\left[\frac{B_{22}\beta'_m + B_{12}\eta_m^2}{B_{12}+B_{66}} + (\beta'_m - \beta''_m)\frac{B_{12}B_{22}}{B_{66}(B_{12}+B_{66})} + a^2\frac{B_{22}}{B_{66}}\beta'_m\beta''_m\right]$$

$$+\left[\frac{B_{22}\beta'_m + B_{12}\eta_m^2}{B_{12}+B_{66}} + (\beta'_m - \beta''_m)\frac{B_{12}B_{22}}{B_{66}(B_{12}+B_{66})} + a^2\frac{B_{22}}{B_{66}}\beta'_m\beta''_m\right]^2. \tag{5.7}$$

From Eq. (5.6), it follows that, at $\varepsilon_m \to 0$, Eq. (2.15) breaks down into the equations

$$\det\left|l_{ij}\right|_{i,j=1}^{4} = 0, \ \det\left|b_{ij}\right|_{i,j=1}^{4} = 0, \ K_3(\eta_m^2) = 0. \tag{5.8}$$

The first and second equations are the dispersion equations of planar and flexural vibrations, as in the similar problems for an orthotropic rectangular plate. The roots of the third equation correspond to planar vibrations of the cylindrical panel. The third equation appears as the result of using the equation of the corresponding classical theory of orthotropic cylindrical shells.

If y_1, y_2 and y_3, y_4 are the roots of Eqs. (5.1) and (5.2), respectively, with negative real parts, then, at $\theta_m l \to \infty$, Eqs. (2.15) and (5.6) will be transformed into the equation

$$\det\left\|m_{ij}\right\|_{i,j=1}^{8} = \left(B_{66}/B_{11}\right)^2 N^2(\eta_m^2)K_{1m}^2(\eta_m^2)K_{2m}^2(\eta_m^2)K_{3m}^2(\eta_m^2) + O(\varepsilon_m^2) + \sum\nolimits_{j=1}^{4} O(\exp z_j) = 0. \tag{5.9}$$

From Eq. (5.9), it follows that, at $\varepsilon_m \to 0$ and $\theta_m l \to \infty$, the roots of dispersion equation (2.15) are approximated by roots of the equations

$$K_{1m}(\eta_m^2) = 0, \ K_{2m}(\eta_m^2) = 0, \ K_{3m}(\eta_m^2) = 0. \tag{5.10}$$

The first two equations of (5.10) are the dispersion equations of flexural and planar vibrations of long enough orthotropic rectangular plate with free sides (see Eqs. (4.13) and (3.14)). Hence, at small ε_m and great $\theta_m l$, the approximate values of roots of Eq. (2.15) are the roots of Eq. (5.8) and (5.10) (compare the data in Tabs. 1, 2 and 3).

TABLE 1. Characteristics of Eigenfrequencies of a Rectangular Plate with $s = 4$ and $l = 5$

m	θ_m	$\det\left|b_{ij}\right|_{i,j=1}^{4} = 0$	$\det\left|l_{ij}\right|_{i,j=1}^{4} = 0$	m	θ_m	$\det\left|b_{ij}\right|_{i,j=1}^{4} = 0$	$\det\left|l_{ij}\right|_{i,j=1}^{4} = 0$
1	1.95473	0.01090 0.01312	0.77652 0.92512	9	9.43718	0.06401 0.06594	0.95557 0.97863
5	2.74891	0.01917 0.02070	0.89934 0.99608	10	10.5474	0.07206 0.07566	0.95771 0.97417
3	3.52957	0.02441 0.02566	0.92803 1.01599	11	11.6577	0.07907 0.08274	0.95931 0.97117
4	4.27693	0.02859 0.02961	0.91974 0.99309	12	12.7680	0.08702 0.08991	0.96053 0.96914
5	5.04581	0.03432 0.03822	0.93579 1.00079	13	13.8782	0.09413 0.09444	0.96145 0.96770
6	6.09849	0.04133 0.04448	0.94287 1.00002	14	14.9887	0.10166 0.10195	0.96209 0.96670
7	7.21629	0.04896 0.05155	0.94870 0.99622	15	16.0962	0.10917 0.11027	0.96268 0.96604
8	8.32693	0.05718 0.05869	0.95266 0.98545	16	17.1935	0.11662 0.11686	0.96301 0.96554

5.2. Asymptotics of dispersion equation (2.15) at $\theta_m l \to \infty$.

When using the previous formulas, we will assume that the roots χ_1, χ_2, χ_3, and χ_4 of Eq. (2.4)) have negative real parts. Then, Eq. (2.15) can be put into the form

$$\det\left\|m_{ij}\right\|_{i,j=1}^{8} = \left(\det\left\|m_{ij}\right\|_{i,j=1}^{4}\right)^2 + \sum_{j=1}^{4} O(\exp z_j) = 0\,, \tag{5.11}$$

whence it follows that, at $\theta_m l \to \infty$, the roots of Eq. (2.15) are approximated by roots of the equation

$$\det\left\|m_{ij}\right\|_{i,j=1}^{4} = 0. \tag{5.12}$$

At $m \in N$, Eq. (5.12) determines all possible localized free vibrations at the free end faces of an orthotropic cylindrical panel with free edges. At $\varepsilon_m \to 0$,

$$\det\left\|m_{ij}\right\|_{i,j=1}^{4} = \left(B_{66} / B_{11}\right) N(\eta_m^2) K_{1m}(\eta_m^2) K_{2m}(\eta_m^2) K_{3m}(\eta_m^2) + O(\varepsilon_m^2)\,. \tag{5.13}$$

Hence, taking into account formulas (5.11) and (5.13), we conclude that the dispersion equation (2.15) becomes (5.9).

6. Numerical Investigations

In Tab. 1, the values of some η_m roots of the first two equations of (5.8) are given for a rectangular boron plastic plate with parameters $\rho = 2\cdot 10^3$ kg/m^3, $E_1 = 2.646\cdot 10^{11}$ N/m^2, $E_2 = 1.323\cdot 10^{10}$ N/m^2, $G = 9.604\cdot 10^9$ N/m^2, ν_1=0.2 and ν_2=0.01. In Tab. 2, some dimensionless characteristics of eigenvalues η_m for predominantly flexural and predominantly planar vibrations of an orthotropic cylindrical boron plastic panel with the same mechanical characteristics and the geometrical parameters $R = 40$, $s =$ 4.00167, and $l = 5$ are given. The results presented in Tab. 3 correspond to a cylindrical boron plastic panel with the same geometrical parameters as in Tab. 2.

TABLE 2. Characteristics Eigenfrequencies of Predominantly Flexural and Predominantly Planar Vibrations of a Cylindrical Panel with $s = 4.00167$ and $l = 5$

m	θ_m	$\eta_{1m} = \eta_{2m} = 0$, $\eta_{3m} = \eta_m$	$\eta_{1m} = \eta_{2m} = \eta_m$, $\eta_{3m} = 0$	m	θ_m	$\eta_{1m} = \eta_{2m} = 0$, $\eta_{3m} = \eta_m$	$\eta_{1m} = \eta_{2m} = \eta_m$, $\eta_{3m} = 0$
1	1.95391	0.01011 b 0.01981 b	0.56767 e 0.92511 e	9	9.43718	0.06401 b 0.06446 b	0.95545 e 0.97845 e
2	2.74776	0.02081 b 0.02099 b	0.88432 e 0.99610 e	10	10.5474	0.07154 b 0.07194 b	0.95762 e 0.97406 e
3	3.52810	0.02437 b 0.02559 b	0.91834 e 1.01584 e	11	11.6577	0.07907 b 0.07943 b	0.95924 e 0.97111 e
4	4.27542	0.02858 b 0.02960 b	0.91632 e 0.99310 e	12	12.7679	0.08660 b 0.08693 b	0.96046 e 0.96910 e
5	5.04492	0.03432 b 0.03518 b	0.93445 e 1.00046 e	13	13.8785	0.09413 b 0.09444 b	0.96138 e 0.96770 e
6	6.09841	0.04134 b 0.04204 b	0.94214 e 1.00010 e	14	14.9864	0.10165 b 0.10193 b	0.96207 e 0.96672 e
7	7.21629	0.04897 b 0.04956 b	0.94851 e 0.99562 e	15	16.1102	0.10927 b 0.10953 b	0.96261 e 0.96601 e
8	8.32693	0.05793 b 0.05869 b	0.95247 e 0.98509 e	16	17.2065	0.11671 b 0.11695 b	0.96300 e 0.96552 e

TABLE 3. Characteristics of Eigenfrequencies of a Cylindrical Panel with $s = 4.00167$ and $l = 5$

m	θ_m	$\eta_{1m} = \eta_{2m} = \eta_{3m} = \eta_m$		m	θ_m	$\eta_{1m} = \eta_{2m} = \eta_{3m} = \eta_m$	
1	1.95391	0.01218 b 0.01987 b	0.55831 e 0.92511 e	9	9.43718	0.06401 b 0.06514 b	0.95556 e 0.97864 e
2	2.74776	0.01937 b 0.02088 b	0.89931 e 0.99608 e	10	10.5474	0.07153 b 0.07566 b	0.95770 e 0.97417 e
3	3.52810	0.02438 b 0.02559 b	0.92801 e 1.01581 e	11	11.6577	0.07907 b 0.07943 b	0.95929 e 0.97118 e
4	4.27542	0.02858 b 0.02960 b	0.91972 e 0.99309 e	12	12.7679	0.08660 b 0.08693 b	0.96049 e 0.96914 e
5	5.04492	0.03243 b 0.03823 b	0.93570 e 1.00044 e	13	13.8785	0.09413 b 0.09444 b	0.96141 e 0.96773 e
6	6.09841	0.04133 b 0.04429 b	0.94286 e 1.00002 e	14	14.9864	0.10165 b 0.10193 b	0.96209 e 0.96674 e
7	7.21629	0.04896 b 0.05156 b	0.94884 e 0.99633 e	15	16.1102	0.10927 b 0.10953 b	0.96262 e 0.96603 e
8	8.32693	0.05793 b 0.05869 b	0.95265 e 0.98544 e	16	17.2065	0.11671 b 0.11695 b	0.96301 e 0.96553 e

In Tabs. 2 and 3, after the characteristics of eigenfrequencies, the type of vibrations is indicated: b — predominantly flexural, e — predominantly planar. At $1 \le m \le 16$, the third equation of (5.8) has no roots.

The elasticity moduli E_1 and E_2 correspond to the directions of generatrix and directrix, respectively. In Tabs. 2 and 3, the case with $\eta_1 = \eta_2 = \eta_3 = \eta$ corresponds to problem (1.1)-(1.3).

The case with $\eta_1 = \eta_2 = 0$ and $\eta_3 = \eta$ corresponds to problem (1.1)-(1.3), where are no tangential components of inertia force, i.e., we have predominantly flexural type of vibrations. The case with $\eta_1 = \eta_2 = \eta, \eta_3 = 0$ corresponds to predominantly planar type of vibrations.

Calculations show that the first eigenfrequencies localized at the free edges of the cylindrical panel where the normal component of inertia force operates are frequencies of the predominantly flexural type. Alongside with the first frequencies of vibrations of quasi-transverse type, there are frequencies of vibrations quasi-tangential type. With increase in m, all these vibrations become vibrations of Rayleigh type. At $\varepsilon_m \to 0$, the free vibrations in problem (1.1)-(1.3) decompose into quasi-transverse and quasi-tangential ones, and frequencies in this problem tend to the frequencies in the similar problem for a rectangular plate. Numerical results show that asymptotic formulas (5.6) and (5.9) of dispersion equation (2.15) are a good reference point for finding the eigenfrequencies of problem (1.1)-(1.3). The first eigenfrequencies depend on the chosen basic functions satisfying the same boundary conditions. At $\theta_m \to \infty$, the frequencies of vibrations at free end faces become independent of basic functions and of boundary conditions on generatrices [11, 23].

Conclusion

Using a system of equations of dynamic equilibrium for orthotropic cylindrical shells corresponding to the classical theory, for the first time, dispersion equations (2.15) for determining the eigenfrequencies of possible edge vibrations for finite and semiinfinite cylindrical panels with free edges are obtained. An asymptotic relation is established between the dispersion equation of the problem considered and the dispersion equations (5.6) of similar problems for planar and flexural vibrations of a rectangular plate. Established is also an asymptotic relation between dispersion equation (2.15) and the dispersion equation of the problem on eigenvalues (5.11) of a semiinfinite cylindrical panel with free edges. A mechanism allowing one to decompose the possible types of vibrations is presented.

Numerical results show that the asymptotic formulas (5.6) and (5.11) of dispersion equation (2.15) and the mechanism presented are a good reference point for finding the eigenfrequencies of problem (1.1)-(1.3).

Asymptotic formula (5.9) and numerical results show that the vibrations eigenfrequencies at free end faces of a finite cylindrical panel (at great values of θ_m) practically do not depend on the basic functions and the boundary conditions on generatrices.

Appendix

Here, the analytical expressions for m_{ij} are presented:

$$m_{11} = H\chi_1^4 + d_1\chi_1^2 + d_2,\ \ m_{12} = H\bar{\bar{f}}_3 + d_1\bar{\bar{f}}_1,\ \ m_{13} = H\bar{f}_2 + d_1,\ \ m_{14} = Hf,$$

$$m_{21} = T\chi_1^5 + d_3\chi_1^3 + d_4\chi_1,\ \ m_{22} = T\bar{\bar{f}}_4 + d_3\bar{\bar{f}}_2 + d_4,\ \ m_{23} = T\bar{f}_3 + d_3\bar{f}_1,\ \ m_{24} = Tf_2 + d_3,$$

$$m_{31} = \delta_m\chi_1^6 + d_5\chi_1^4 + d_6\chi_1^2 + d_7,\ \ m_{32} = \delta_m\bar{\bar{f}}_5 + d_5\bar{\bar{f}}_3 + d_6\bar{\bar{f}}_1,$$

$$m_{33} = \delta_m\bar{f}_4 + d_5\bar{f}_2 + d_6,\ \ m_{34} = \delta_m f_3 + d_5 f_1,$$

$$m_{41} = \delta_m\chi_1^7 + d_8\chi_1^5 + d_9\chi_1^3 + d_{10}\chi_1,\ \ m_{42} = \delta_m\bar{\bar{f}}_6 + d_8\bar{\bar{f}}_4 + d_9\bar{\bar{f}}_2 + d_{10},$$

$$m_{43} = \delta_m\bar{f}_5 + d_8\bar{f}_3 + d_9\bar{f}_1,\ \ m_{44} = \delta_m f_4 + d_8 f_2 + d_9,\ \delta_m = 1 + 4a^2\varepsilon_m^2,$$

$$m_{i5} = (-1)^{i+1} m_{i1}\exp z_1,\ m_{i6} = (-1)^{i+1}(m_{i2}\exp z_2 + m_{i1}[z_1 z_2]),$$

$$m_{i7} = (-1)^{i+1}(m_{i3}\exp z_3 + m_{i2}[z_2 z_3] + m_{i1}[z_1 z_2 z_3]),$$

$$m_{i8} = (-1)^{i+1}(m_{i4}\exp z_4 + m_{i3}[z_3 z_4] + m_{i2}[z_2 z_3 z_4] + m_{i1}[z_1 z_2 z_3 z_4]),\ i = \overline{1, 4},$$

where

$$H = -a^2\frac{B_{12} + 4B_{66}}{B_{11}}\beta'_m,\quad T = -\frac{B_{12}}{B_{66}}a^2\delta_m\beta'_m,\quad \delta_m = 1 + 4a^2\varepsilon_m^2,$$

$$d_1 = \frac{B_{11}B_{22} - B_{12}^2}{B_{11}^2}\beta'_m - \frac{B_{12}B_{66}}{B_{11}^2}\eta_{1m}^2 + 4a^2\varepsilon_m^2 \frac{B_{12}B_{66}}{B_{11}^2}(\beta'_m - \eta_{1m}^2)$$

$$+a^2\beta'_m\left[\frac{B_{22}}{B_{11}}\beta''_m - \frac{B_{12}(B_{12}+4B_{66})}{B_{11}^2}\left(\beta'_m - \eta_{1m}^2\right)\right],$$

$$d_2 = -\frac{B_{12}}{B_{11}^2}(\beta'_m - \eta_{1m}^2)(B_{66}\eta_{2m}^2 + B_{22}(\beta'_m - \beta''_m) + a^2\beta''_m B_{22}(\beta'_m - \varepsilon_m^2)),$$

$$d_3 = \frac{B_{11}B_{22} - B_{12}^2}{B_{11}B_{66}}\delta_m\beta'_m + a^2\beta'_m\left(4\eta_{2m}^2 - 3B_2 - 2\frac{B_{12}}{B_{11}}\beta'_m - \frac{B_{12}}{B_{11}}\eta_{1m}^2\right) + 4a^4\varepsilon_m^2\left[\frac{B_{12}}{B_{11}}(\beta'_m - \eta_{1m}^2)\right],$$

$$d_4 = \left(\frac{B_{22}}{B_{11}}\eta_{1m}^2 + \frac{B_{12}}{B_{11}}\eta_{2m}^2\right)\beta'_m + a^2\beta'_m\left[\frac{B_{22}(B_{12}+4B_{66})}{B_{11}B_{66}}\beta'_m\beta''_m - 3\frac{B_{22}}{B_{11}}\beta''_m - 4\frac{B_{66}}{B_{11}}\eta_{2m}^2\left(\beta'_m - \eta_{1m}^2\right)\right]$$

$$-a^2\varepsilon_m^2\beta'_m\frac{B_{22}}{B_{11}}\left(\frac{(B_{12}\beta'' + 4B_{66}\beta'_m)}{B_{66}} - 4\eta_{1m}^2\right) + \frac{B_{12}B_{22}}{B_{11}B_{66}}\beta'_m(\beta'_m - \beta''_m),$$

$$d_5 = \frac{B_{66}}{B_{11}}\eta_{1m}^2 + \eta_{2m}^2 - \frac{B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m - B_{12}B_{66}\beta'_m}{B_{11}B_{66}}$$

$$-a^2\varepsilon_m^2\left[\frac{B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m}{B_{11}B_{66}} + \frac{4B_{66}}{B_{11}}\left(\beta'_m - \eta_{1m}^2\right)\right],$$

$$d_6 = \frac{B_{12}}{B_{11}}B_2 + \frac{B_{22}}{B_{11}}\beta'_m\beta''_m - \left(\frac{B_{11}B_{22}\beta'' + B_{12}B_{66}\beta'_m}{B_{11}^2}\eta_{1m}^2 + \frac{B_{12}+B_{66}}{B_{11}}\beta'_m\eta_{2m}^2\right)$$

$$+\frac{B_{66}}{B_{11}}\eta_{1m}^2\eta_{2m}^2 + \varepsilon_m^2\left[a^2\frac{B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m}{B_{11}^2}(\beta'_m - \eta_{1m}^2) - \frac{B_{12}}{B_{11}}B_1\beta'_m\right],$$

$$d_7 = \frac{B_{12}}{B_{11}}\beta'_m(\beta'_m - \eta_{1m}^2)\left(\frac{B_{22}}{B_{11}}\varepsilon_m^2 - \frac{B_{22}}{B_{11}}\beta''_m + \frac{B_{66}}{B_{11}}\eta_{2m}^2\right),$$

$$d_8 = \frac{B_{66}}{B_{11}}\eta_{1m}^2 + \eta_{2m}^2 - \frac{B_{11}B_{22}\beta''_m - B_{12}^2\beta'_m - B_{12}B_{66}\beta'_m + 4B_{66}^2\beta'_m}{B_{11}B_{66}}$$

$$-a^2\varepsilon_m^2\left(B_2 + \frac{4B_{66} - 2B_{12}}{B_{11}}\beta'_m - \frac{4B_{66}}{B_{11}}\eta_{1m}^2\right),$$

$$d_9 = \frac{B_{66}}{B_{11}}\eta_{1m}^2\eta_{2m}^2 - \frac{B_{11}B_{22}\beta''_m + B_{12}B_{66}\beta'_m + 4B_{66}^2\beta'_m}{B_{11}^2}\eta_{1m}^2$$

$$+\frac{B_{12}+5B_{66}}{B_{11}}\beta'_m\eta_{2m}^2 + \frac{(B_{12}+4B_{66})}{B_{11}}B_2\beta'_m + \frac{B_{22}}{B_{11}}\beta'_m\beta''_m$$

$$+\varepsilon_m^2 \left[a^2 \frac{B_{11}B_{22}\beta_m'' - B_{12}^2\beta_m' - 4B_{12}B_{66}\beta_m'}{B_{11}^2} (\beta_m' - \eta_{1m}^2) - \frac{(B_{12} + 4B_{66})}{B_{11}} B_1 \beta_m' \right],$$

$$d_{10} = \frac{B_{12} + 4B_{66}}{B_{11}} \beta_m' (\beta_m' - \eta_{1m}^2) \left(\frac{B_{22}}{B_{11}} \varepsilon_m^2 - \frac{B_{22}}{B_{11}} \beta_m'' + \frac{B_{66}}{B_{11}} \eta_{2m}^2 \right).$$

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