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A SIMPLE FINITE ELEMENT WITH FIVE DEGREES OF FREEDOM BASED ON REDDY'S THIRD-ORDER SHEAR DEFORMATION THEORY

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A simple four-node isoparametric finite element with five degrees of freedom, based on Reddy's third-order shear deformation theory, is elaborated and used in a model for analizing the bending of laminated plates. The results obtained are compared with solutions given by the three-dimensional elasticity and other theories.

1. Introduction

The use of composites materials is growing progressively compared with traditional materials, basically in the fields of application where powerful and lightweight structures are needed [1]. Laminated composite materials are extensively used in many fields, for example, in marine, civil, and mechanical engineering, and are characterized by a light weight, high strength, and high corrosion resistance. A review of recent applications and the development of composite structures for future naval ships and submarines is given in [2].

In the last decades, the finite-element method has become established as a powerful calculation tool and the most widely used method to analyze the complex behavior of composite structures [3].Review [4] reflects the recent development of the method for investigation of laminated composite plates.

Modeling of the stresses and strains of laminated composite plates is still considered as important subject of research, although many theories, such as, e.g., the classical laminated plate theory (CLPT) have been proposed for this purpose. In the early days, the CLPT was used for modeling thin laminated plates, but it neglects the transverse effect

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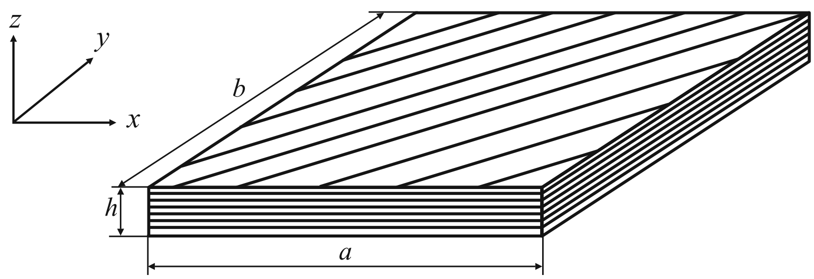

Fig. 1. Geometry of a rectangular laminated composite plate.

of shear deformation [5, 6]. This effect is taken into account in the first-order shear deformation theory (FSDT) [7-11], but it is considered constant across the thickness of the plates. Higher-order theories can describe the nonlinear shear deformations in the thickness direction without any correction factor [12-15]. Reddy proposed a third-order shear deformation theory (TSDT) [16, 17] based on a single-layer approach. It considers a parabolic variation of the transverse shear stresses across the plate thickness and satisfies zero shear stress boundary conditions on the top and bottom of the plate, but requires the C_1 continuity of the second-order derivative of transverse displacements. This theory encounters problems when the finite-element method is used, namely the requirement of C_1 continuity of displacements for common edges between two elements [18] can be satisfied only in the case of thin plate elements [19].

Many authors encountered this problem and mentioned that C_1-continuous elements are computionally inefficient and the accuracy of solution is questionable [20-24].

The objective of this paper is to employ Reddy' third-order shear deformation theory to analyze the bending of a laminated plate by using a simple isoparametric four-node finite element with five degrees of freedom at each node.

2. Kinematics

According to Reddy's third-order shear deformation theory (TSDT) [17], the displacement field can be expressed as

$$u_1 = u + z\psi_x - \frac{4z^3}{3h}\left(\psi_x + \frac{\partial w}{\partial x}\right),$$

$$u_2 = v + z\psi_y - \frac{4z^3}{3h}\left(\psi_y + \frac{\partial w}{\partial y}\right), \quad u_3 = w,$$

where u, v, w, ψ_x, and ψ_y are five unknown midplane displacement functions of the plate, and h is its thickness (see Fig. 1).

The linear strains associated with the displacement field are

$$\begin{gathered}\varepsilon_1 = \varepsilon_{xx} = \varepsilon_1^\circ + z(\kappa_1^\circ + z^2\kappa_1^2), \quad \varepsilon_2 = \varepsilon_{yy} = \varepsilon_2^\circ + z(\kappa_2^\circ + z^2\kappa_2^2), \\ \varepsilon_3 = \varepsilon_{zz} = 0, \quad \varepsilon_4 = \varepsilon_{yz} = \varepsilon_4^\circ + z^2\kappa_4^2, \\ \varepsilon_5 = \varepsilon_{xz} = \varepsilon_5^\circ + z^2\kappa_5^2, \quad \varepsilon_6 = \varepsilon_{xy} = \varepsilon_6^\circ + z(\kappa_6^\circ + z^2\kappa_6^2),\end{gathered} \tag{1}$$

where

$$\varepsilon_1^\circ = \frac{\partial u}{\partial x}, \quad \kappa_1^\circ = \frac{\partial \psi_x}{\partial x}, \quad \kappa_1^2 = -\frac{4}{3h^2}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right),$$

$$\varepsilon_2^\circ = \frac{\partial v}{\partial y}, \quad \kappa_2^\circ = \frac{\partial \psi_y}{\partial y}, \quad \kappa_2^2 = -\frac{4}{3h^2}\left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right),$$

$$\varepsilon_4^\circ = \psi_y + \frac{\partial w}{\partial y}, \quad \kappa_4^2 = -\frac{4}{h^2}\left(\psi_y + \frac{\partial w}{\partial y}\right),$$

$$\varepsilon_5^\circ = \psi_x + \frac{\partial w}{\partial x}, \quad \kappa_5^2 = -\frac{4}{h^2}\left(\psi_x + \frac{\partial w}{\partial x}\right),$$

$$\varepsilon_6^\circ = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \kappa_6^\circ = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}, \quad \kappa_6^2 = -\frac{4}{3h^2}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right).$$

3. Constitutive Equations

The constitutive equations for a single layer [11] can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{63} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{bmatrix}$$

where Q_{ij} are material constants in the material axes of the layer:

$$Q_{11} = E_1/(1-\nu_{12}\nu_{21}), \quad Q_{12} = (\nu_{12}E_2)/(1-\nu_{12}\nu_{21}),$$

$$Q_{22} = E_2/(1-\nu_{12}\nu_{21}), \quad Q_{66} = G_{12}, \quad C_{55} = G_{13}, \quad C_{44} = G_{23}.$$

The stress–strain relations in the laminate coordinates *x*, *y*, and *z* of a *k*th layer [11] are given by the relations

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_{(k)} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{23} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{63} \end{bmatrix}_{(k)} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix},$$

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix}_{(k)} = \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}_{(k)} = \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}_{(k)} \begin{bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{bmatrix}, \tag{2}$$

where the constants $\overline{Q}_{ij}$ are expresses as

$$\overline{Q}_{11} = C_{11}c^4 + 2(C_{12} + 2C_{66})c^2s^2 + C_{22}s^4,$$

$$\overline{Q}_{12} = (C_{11} + C_{22} - 4C_{66})c^2s^2 + C_{12}(c^4 + s^4),$$

$$\overline{Q}_{16} = [C_{11}c^2 + (C_{12} + 2C_{66})(s^2 - c^2) - C_{22}s^2]cs,$$

$$\overline{Q}_{22} = C_{11}s^4 + 2(C_{12} + 2C_{66})c^2s^2 + C_{22}c^4,$$

$$\overline{Q}_{26} = [C_{11}s^2 + (C_{12} + 2C_{66})(c^2 - s^2) - C_{22}c^2]cs,$$

$$\overline{Q}_{66} = (C_{11} + C_{22} - 2C_{12})c^2s^2 + C_{66}(c^2 - s^2)^2,$$

$$\overline{Q}_{44} = C_{44}c^2 + C_{55}s^2,$$

$$\overline{Q}_{45} = (C_{55} - C_{44})cs,$$

$$\overline{Q}_{55} = C_{55}c^2 + C_{44}s^2.$$

Here, $c = \cos\theta$, and $s = \sin\theta$, and θ is the angle between the global and local axes of each layer.

4. Equilibrium Equation

The static equations of the theory can be derived using the principle of virtual work, and the variation of strain energy is calculated from the equation [17]

$$\delta U - \delta K = 0.$$

In terms of stresses, strains, and external forces, this equatin can be expressed as

$$\int_{-h/2}^{h/2}\int_A (\sigma_1 \delta\varepsilon_{xx} + \sigma_2 \delta\varepsilon_{yy} + \sigma_6 \delta\varepsilon_{xy} + \sigma_4 \delta\varepsilon_{yz} + \sigma_5 \delta\varepsilon_{xz})dAdz + \int_A q\delta w dA$$

$$= \int_A \left\{ N_1 \frac{\partial \delta u}{\partial x} + M_1 \frac{\partial \delta \psi_x}{\partial x} + P_1 \left[-\frac{4}{3h^2}\left(\frac{\partial \delta \psi_x}{\partial x} + \frac{\partial^2 \delta w}{\partial x^2} \right) \right] + N_2 \frac{\partial \delta v}{\partial y} + M_2 \frac{\partial \delta \psi_y}{\partial y} \right.$$

$$+ P_2 \left[-\frac{4}{3h^2}\left(\frac{\partial \delta \psi_y}{\partial y} + \frac{\partial^2 \delta w}{\partial y^2} \right) \right] + N_6 \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) + M_6 \left(\frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} \right)$$

$$+ P_6 \left[-\frac{4}{3h^2}\left(\frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} + 2\frac{\partial^2 \delta w}{\partial x \partial y} \right) \right] + Q_2 \left(\delta \psi_y + \frac{\partial \delta w}{\partial y} \right)$$

$$+ R_2 \left[-\frac{4}{h^2}\left(\delta \psi_y + \frac{\partial \delta w}{\partial y} \right) \right] + Q_1 (\left(\delta \psi_x + \frac{\partial \delta w}{\partial x} \right)$$

$$\left. + R_2 \left[-\frac{4}{h^2} (\left(\delta \psi_x + \frac{\partial \delta w}{\partial x} \right) \right] + q\delta w \right\} dA = 0, \quad (3)$$

where

$$(N_i, M_i, P_i) = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \sigma_i (1, z, z^3) dz, \quad (i = 1, 2, 6),$$

$$(Q_2, R_2) = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \sigma_4 (1, z^2) dz, \quad (Q_1, R_1) = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \sigma_5 (1, z^2) dz.$$

Inserting Eqs. (1) into stress–strain relations (2), we have

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}_{(k)}=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & \overline{Q}_{11}z & \overline{Q}_{12}z & \overline{Q}_{13}z & \overline{Q}_{11}z^3 & \overline{Q}_{12}z^3 & \overline{Q}_{13}z^3\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & \overline{Q}_{12}z & \overline{Q}_{22}z & \overline{Q}_{23}z & \overline{Q}_{12}z^3 & \overline{Q}_{22}z^3 & \overline{Q}_{23}z^3\\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{63} & \overline{Q}_{61}z & \overline{Q}_{62}z & \overline{Q}_{61}z & \overline{Q}_{61}z^3 & \overline{Q}_{62}z^3 & \overline{Q}_{63}z^3\end{bmatrix}_{(k)}\begin{bmatrix}\varepsilon_1^0\\ \varepsilon_2^0\\ \varepsilon_6^0\\ \kappa_1^0\\ \kappa_2^0\\ \kappa_6^0\\ \kappa_1^2\\ \kappa_2^2\\ \kappa_6^2\end{bmatrix},$$

$$\begin{Bmatrix}\sigma_4\\ \sigma_5\end{Bmatrix}_{(k)}=\begin{bmatrix}\overline{Q}_{44} & \overline{Q}_{45} & \overline{Q}_{44}z^2 & \overline{Q}_{45}z^2\\ \overline{Q}_{45} & \overline{Q}_{55} & \overline{Q}_{45}z^2 & \overline{Q}_{55}z^2\end{bmatrix}_{(k)}\begin{bmatrix}\varepsilon_4^0\\ \varepsilon_5^0\\ \kappa_4^2\\ \kappa_5^2\end{bmatrix}.$$

After integrating these stresses across the thickness of each layer and summation, the generalized force–strain relations are obtained in the form [17]

$$\begin{Bmatrix}\begin{Bmatrix}N_1\\ N_2\\ N_6\end{Bmatrix}\\ \begin{Bmatrix}M_1\\ M_2\\ M_6\end{Bmatrix}\\ \begin{Bmatrix}P_1\\ P_2\\ P_6\end{Bmatrix}\end{Bmatrix}=\begin{bmatrix}\begin{bmatrix}A_{11} & A_{12} & A_{16}\\ & A_{22} & A_{26}\\ sym & & A_{66}\end{bmatrix} & \begin{bmatrix}B_{11} & B_{12} & B_{16}\\ & B_{22} & B_{26}\\ sym & & B_{66}\end{bmatrix} & \begin{bmatrix}E_{11} & E_{12} & E_{16}\\ & E_{22} & E_{26}\\ & & E_{66}\end{bmatrix}\\ & \begin{bmatrix}D_{11} & D_{12} & D_{16}\\ & D_{22} & D_{26}\\ sym & & D_{66}\end{bmatrix} & \begin{bmatrix}F_{11} & F_{12} & F_{16}\\ & F_{22} & F_{26}\\ sym & & F_{66}\end{bmatrix}\\ \text{sym} & & \begin{bmatrix}H_{11} & H_{12} & H_{16}\\ & H_{22} & H_{26}\\ sym & & H_{66}\end{bmatrix}\end{bmatrix}\begin{bmatrix}\varepsilon_1^0\\ \varepsilon_2^0\\ \varepsilon_6^0\\ \kappa_1^0\\ \kappa_2^0\\ \kappa_6^0\\ \kappa_1^2\\ \kappa_2^2\\ \kappa_6^2\end{bmatrix},$$

$$\begin{Bmatrix}Q_2\\ Q_1\\ R_2\\ R_1\end{Bmatrix}=\begin{bmatrix}A_{44} & A_{45} & D_{44} & D_{45}\\ & A_{55} & D_{45} & D_{55}\\ & & F_{44} & F_{45}\\ sym & & & F_{55}\end{bmatrix}\begin{bmatrix}\varepsilon_4^0\\ \varepsilon_5^0\\ \kappa_4^2\\ \kappa_5^2\end{bmatrix}, \tag{4}$$

where

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij})=\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\overline{Q}_{ij}(1,z,z^2,z^3,z^4,z^6)dz,\quad (i,j=1,2,6),$$

$$(A_{ij},D_{ij},F_{ij})=\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\overline{Q}_{ij}(1,z^2,z^4)dz,\quad (i,j=4,5).$$

Inserting Eqs.(4) into Eq. (3) gives

$$\int_A (\delta\varepsilon^{0T}\left[A\right]\varepsilon^0 + \delta\varepsilon^{0T}\left[B\right]\kappa^0 + \delta\varepsilon^{0T}\left[E\right]\kappa^2 + \delta\kappa^{0T}\left[B\right]\varepsilon^0 + \delta\kappa^{0T}\left[D\right]\kappa^0 + \delta\kappa^{0T}\left[F\right]\kappa^2$$

$$+\delta\kappa^{2T}\left[E\right]\varepsilon^0 + \delta\kappa^{2T}\left[F\right]\kappa^0 + \delta\kappa^{2T}\left[H\right]\kappa^2 + \delta\gamma^{sT}\left[A^s\right]\gamma^s + \delta\gamma^{sT}\left[D^s\right]\kappa^s$$

$$+\delta\kappa^{sT}\left[D^s\right]\gamma^s + \delta\kappa^{sT}\left[F^s\right]\kappa^s + q\delta w)dA = 0. \tag{5}$$

With

$$\left[\underbrace{\varepsilon_1^0 \quad \varepsilon_2^0 \quad \varepsilon_6^0} \quad \underbrace{\kappa_1^0 \quad \kappa_2^0 \quad \kappa_6^0} \quad \underbrace{\kappa_1^2 \quad \kappa_2^2 \quad \kappa_6^0}\right]^T = \left[\varepsilon^0 \quad \kappa^0 \quad \kappa^2\right]^T,$$

$$\left[\underbrace{\varepsilon_4^0 \quad \varepsilon_5^0} \quad \underbrace{\kappa_4^2 \quad \kappa_5^2}\right]^T = \left[\gamma^s \quad \kappa^s\right]^T,$$

the strain matrixes can be written as follows:

$$\varepsilon^0 = \left[B_\varepsilon^0\right]^e \{\delta\}_{(e)} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 & 0 & 0 \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \psi_x \\ \psi_y \end{Bmatrix},$$

$$\kappa^0 = \left[B_\kappa^0\right]^e \{\delta\}_{(e)} = \begin{Bmatrix} \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{\partial \psi_y}{\partial y} \\ \dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\partial}{\partial y} \\ 0 & 0 & 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \psi_x \\ \psi_y \end{Bmatrix}, \tag{6}$$

$$\kappa^2 = \left[B_\kappa^2\right]^e \{\delta\}_{(e)} = \begin{Bmatrix} -\dfrac{4}{3h^2}\left(\dfrac{\partial \psi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\right) \\ -\dfrac{4}{3h^2}\left(\dfrac{\partial \psi_y}{\partial y} + \dfrac{\partial^2 w}{\partial y^2}\right) \\ -\dfrac{4}{3h^2}\left(\dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} + 2\dfrac{\partial^2 w}{\partial x \partial y}\right) \end{Bmatrix} = c_1 \begin{bmatrix} 0 & 0 & \dfrac{\partial^2}{\partial x^2} & \dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & \dfrac{\partial^2}{\partial y^2} & 0 & \dfrac{\partial}{\partial y} \\ 0 & 0 & 2\dfrac{\partial^2}{\partial x \partial y} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \psi_x \\ \psi_y \end{Bmatrix},$$

$$\gamma^s = \left[B_\varepsilon^s\right]^e \{\delta\}_{(e)} = \begin{Bmatrix} \psi_y + \dfrac{\partial w}{\partial y} \\ \psi_x + \dfrac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \dfrac{\partial}{\partial y} & 0 & 1 \\ 0 & 0 & \dfrac{\partial}{\partial x} & 1 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \psi_x \\ \psi_y \end{Bmatrix},$$

$$\kappa^{s}=\left[B_{\kappa}^{s}\right]^{e}\{\delta\}_{(e)}=\begin{Bmatrix}-\dfrac{4}{h^2}\left(\psi_y+\dfrac{\partial w}{\partial y}\right)\\ -\dfrac{4}{h^2}\left(\psi_x+\dfrac{\partial w}{\partial y}\right)\end{Bmatrix}=c_2\begin{bmatrix}0 & 0 & \dfrac{\partial}{\partial y} & 0 & 1\\ 0 & 0 & \dfrac{\partial}{\partial x} & 1 & 0\end{bmatrix}\begin{Bmatrix}u\\ v\\ w\\ \psi_x\\ \psi_y\end{Bmatrix}, \tag{6}$$

where $c_1=-4/3h^2$ and $c_2=-4/h^2$.

5. Finite-Element Formulation

For the present study, a four-node quadrilateral C_0-continuous isoparametric finite element [25] with five degrees of freedom u, v, w, ψ_x, and ψ_y at each node is employed:

$$u(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)u_i,\quad v(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)v_i,\quad w(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)w_i,$$

$$\psi_x(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)\psi_{xi},\quad \psi_y(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)\psi_{yi},$$

$$x=\sum_{i=1}^{4}N_i(\xi,\eta)x_i,\quad y=\sum_{i=1}^{4}N_i(\xi,\eta)y_i,\qquad N_i\left(\xi,\eta\right)=\frac{1}{4}\left(1+\xi\xi_i\right)(1+\eta\eta_i).$$

The real element is determined from the reference space (ξ,η) by a geometric transformation based on the positions of nodes in the real space (x, y). Our formulation requires the first- and second-order derivatives, therefore, the following processing operations (see Appendix) [26, 27] are used:

$$\frac{\partial}{\partial\xi}=\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial\xi}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial\xi},\qquad \frac{\partial}{\partial\eta}=\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial\eta}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial\eta},$$

$$\frac{\partial^2}{\partial\xi\partial\eta}=\frac{\partial}{\partial x}\cdot\frac{\partial^2 x}{\partial\xi\partial\eta}+\frac{\partial}{\partial y}\cdot\frac{\partial^2 y}{\partial\xi\partial\eta}+\frac{\partial^2}{\partial x^2}\cdot\frac{\partial x}{\partial\xi}\cdot\frac{\partial x}{\partial\eta}+$$

$$+\frac{\partial^2}{\partial y^2}\cdot\frac{\partial y}{\partial\xi}\cdot\frac{\partial y}{\partial\eta}+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial x}{\partial\eta}\cdot\frac{\partial y}{\partial\xi}+\frac{\partial x}{\partial\xi}\cdot\frac{\partial y}{\partial\eta}\right),$$

$$\frac{\partial^2}{\partial\xi^2}=\frac{\partial}{\partial x}\cdot\frac{\partial^2 x}{\partial\xi^2}+\frac{\partial}{\partial y}\cdot\frac{\partial^2 y}{\partial\xi^2}+\frac{\partial^2}{\partial x^2}\left(\frac{\partial x}{\partial\xi}\right)^2+\frac{\partial^2}{\partial y^2}\left(\frac{\partial y}{\partial\xi}\right)^2+2\frac{\partial^2}{\partial x\partial y}\cdot\frac{\partial x}{\partial\xi}\cdot\frac{\partial y}{\partial\xi},$$

$$\frac{\partial^2}{\partial\eta^2}=\frac{\partial}{\partial x}\cdot\frac{\partial^2 x}{\partial\eta^2}+\frac{\partial}{\partial y}\cdot\frac{\partial^2 y}{\partial\eta^2}+\frac{\partial^2}{\partial x^2}\left(\frac{\partial x}{\partial\eta}\right)^2+\frac{\partial^2}{\partial y^2}\left(\frac{\partial y}{\partial\eta}\right)^2+2\frac{\partial^2}{\partial x\partial y}\cdot\frac{\partial x}{\partial\eta}\cdot\frac{\partial y}{\partial\eta}.$$

Inserting strain matrix (14) into Eq (5), the elementary stiffness matrix is written as follows:

$$\left[K\right]_e=\int_{-1}^{1}\int_{-1}^{1}\Big(\left[B_{\varepsilon}^{0}\right]^T\left[A\right]\left[B_{\varepsilon}^{0}\right]+\left[B_{\varepsilon}^{0}\right]^T\left[B\right]\left[B_{\kappa}^{0}\right]+\left[B_{\varepsilon}^{0}\right]^T\left[E\right]\left[B_{\kappa}^{2}\right]+\left[B_{\kappa}^{0}\right]^T\left[B\right]\left[B_{\varepsilon}^{0}\right]$$

$$+\left[B_{\kappa}^{0}\right]^T\left[D\right]\left[B_{\kappa}^{0}\right]+\left[B_{\kappa}^{0}\right]^T\left[F\right]\left[B_{\kappa}^{2}\right]+\left[B_{\kappa}^{2}\right]^T\left[E\right]\left[B_{\varepsilon}^{0}\right]+\left[B_{\kappa}^{2}\right]^T\left[F\right]\left[B_{\kappa}^{0}\right]$$

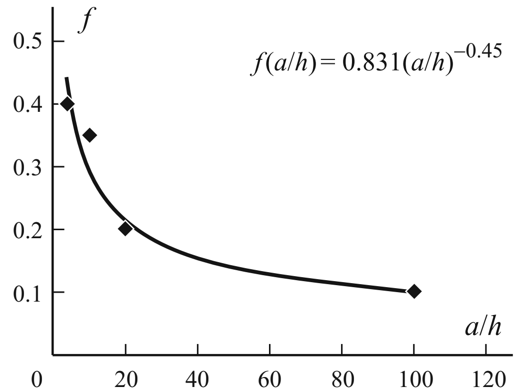

Fig. 2. Representation of the function f proposed in terms of the ratio a/h (♦) and its interpolation.

$$+\left[B_\kappa^2\right]^T[H]\left[B_\kappa^2\right]+\left[B_\varepsilon^s\right]^T\left[A^s\right]\left[B_\varepsilon^s\right]+\left[B_\varepsilon^s\right]^T\left[D^s\right]\left[B_\kappa^s\right]$$

$$+\left[B_\kappa^s\right]^T\left[D^s\right]\left[B_\varepsilon^s\right]+\left[B_\kappa^s\right]^T\left[F^s\right]\left[B_\kappa^s\right]\Big)\det[J]\,d\xi\, d\eta. \tag{7}$$

The integrals in Eq. (7) are computed numerically. The stiffness integral is obtained by considering a simple four-node finite element, 2×2 Gauss points for the bending contribution, and a 1×1 point for the shear contribution [25].

This formulation gave good results for the transverse shear stresses, but not for the normal stress and the deflection displacement w, for which an optimization procedure using a function f of the ratio a/h was performed.

Optimization of the normal stress and deflection displacement at different thicknesses required specific values of the function $f(a/h)$. These values were plotted and interpolated to obtain a proper function $f(a/h)$ (Fig. 2). The strain matrices $\left[B_\kappa^2\right]$ and $\left[B_\kappa^s\right]$ were multiplied by this function to improve results.

6. Numerical Examples

The material constants used in examples were as follows: $E_1/E_2 = 25$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, and $\nu = 0.25$.

6.1. Example 1

Deflections of a three layer (0/90/0) square laminate simply supported at all edges and subjected to a uniformly distributed load were analyzed at different mesh divisions and thickness ratios a/h. The nondimensional deflections $\bar{w}$ calculated at the centre of the plate were compared with those found in [17] and [22]. They are given in Table.1. Calculation results converged and the error decreased with increasing number of elements with different thickness ratios a/h. The deflection $\bar{w}$ was calculated by the formula

$$\bar{w} = w\left(\frac{a}{2}, \frac{b}{2}, 0\right)\left(h^3E_2/q_0a^4\right)10^2.$$

TABLE 1. Nondimensional Deflection at the Center of Simply Supported [0/90/0] Square Laminates under a Uniform Load

Reference	a/h		
	4	10	100
Present, 8×8 mesh	3.1289	1.1264	0.69285
Present, 12×12 mesh	3.0965	1.1215	0.69478
Present, 16×16 mesh	3.0857	1.120	0.69549
Present, 20×20 mesh	3.0807	1.1193	0.69582
[22], 32×32 mesh	2.9093	1.0910	0.6708
[17]	2.9091	1.0900	0.6705

TABLE 2. Nondimensional Deflection and Stresses of a Simply Supported [0/90/0] Square Laminate under a Sinusoidally Distributed Transverse Load

Reference	a/h	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
FSDT (16×16) [22]	4	1.777	0.4430	0.4843	0.0371	0.1440	0.1569
HSDT (16×16) [22]	4	1.923	0.7500	0.5080	0.0499	0.2023	0.1831
[29]	4	2.0557	0.8435	0.5610	0.0522	0.2569	0.2205
3D elasticity [28]	4	2.006	0.8010	0.5340	0.0505	0.2560	0.2172
Present (16×16)	4	2. 0491	0.42012	0.55051	0.05516	0.22878	0.24014
FSDT (16×16) [22]	10	0.6700	0.5219	0.2582	0.025	0.1623	0.0918
HSDT (16×16) [22]	10	0.7140	0.5806	0.2722	0.0279	0.2437	0.1015
[29]	10	0.7624	0.6331	0.2851	0.0291	0.3540	0.1257
3D elasticity [28]	10	0.7405	0.5900	0.2850	0.0289	0.3570	0.1228
Present (16×16)	10	0.73143	0.51394	0.27546	0.02793	0.28369	0.11146
[29]	20	0.5194	0.5826	0.2041	0.0234	0.3771	0.0944
3D elasticity [28]	20	0.5142	0.552	0.210	0.0234	0.3850	0.0938
Present (16×16)	20	0.52544	0.53324	0.21215	0.02356	0.29789	0.1130
FSDT (16×16) [22]	100	0.4350	0.5490	0.1825	0.0202	0.1568	0.0709
HSDT (16×16) [22]	100	0.4350	0.5496	0.1828	0.0215	0.2401	0.0709
[29]	100	0.4351	0.5521	0.1743	0.0213	0.3723	—
3D elasticity [28]	100	0.4368	0.5390	0.1810	0.0213	0.3950	0.0828
Present (16×16)	100	0.44734	0.53808	0.18591	0.02152	0.30384	0.07221

6.2. Example 2

A three-layer (0/90/0) square laminate simply supported at all edges and subjected to a sinusoidally distributed load was considered at different thickness ratio a/h, and its nondimensional deflection $\bar{w}$ and stresses $\bar{\sigma}_{xx}, \bar{\sigma}_{yy}, \bar{\sigma}_{xy}, \bar{\sigma}_{xz}$, and $\bar{\sigma}_{yz}$ were calculated (Table 2). It is seen that the results for the deflection and stresses are closer to those given by the exact 3D elasticity solution [28] than predictions of the HSDT and FSDT [22]. However, in the cases of $a/h = 4$ and 10, the normal stress $\bar{\sigma}_{xx}$ is comparable to that calculated by the FSDT. The normalized deflection and stresses were calculated by the formulas

$$\bar{w} = w\left(\frac{a}{2}, \frac{b}{2}, 0\right)\left(h^3 E_2 / q_0 a^4\right)10^2, \quad \bar{\sigma}_{xx} = \sigma_{xx}\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right)\left(h^2 / q_0 a^2\right), \tag{8}$$

TABLE 3. Nondimensional Deflection and Stresses of a Simply Supported [0/90/90/0] Square Laminate under a Sinusoidally Distributed Transverse Load

Reference	a/h	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
FSDT[17]	4	1.7100	0.4059	0.5765	0.0308	0.1398	0.1963
HSDT [17]	4	1.8937	0.6651	0.6322	0.0440	0.2064	0.2389
3D elasticity [30]	4	1.9540	0.7200	0.666	0.0467	0.219	0.2920
Present (16×16)	4	1.9689	0.3811	0.6509	0.0482	0.2168	0.3002
FSDT [17]	10	0.6628	0.4989	0.3615	0.0241	0.1667	0.1292
HSDT [17]	10	0.7147	0.5456	0.3888	0.0268	0.2640	0.1531
3D elasticity [30]	10	0.7430	0.5590	0.4010	0.0275	0.3010	0.1960
Present (16×16)	10	0.74602	0.49776	0.3890	0.0262	0.2967	0.1997
FSDT [17]	20	0.4912	0.5273	0.2957	0.0221	0.1749	0.1087
HSDT [17]	20	0.5060	0.5393	0.3043	0.0228	0.2825	0.1234
3D elasticity [30]	20	0.5170	0.543	0.308	0.0230	0.3280	0.156
Present (16×16)	20	0.52468	0.52902	0.3058	0.0233	0.3231	0.1621
FSDT [17]	100	0.4337	0.5382	0.2705	0.0213	0.178	0.1390
HSDT [17]	100	0.4343	0.5387	0.2708	0.0213	0.2897	0.1390
3D elasticity [30]	100	0.4347	0.5390	0.2710	0.0214	0.3390	0.1410
Present (16×16)	100	0.4455	0.5373	0.2771	0.0214	0.3346	0.1302

$$\bar{\sigma}_{yy} = \sigma_{yy}\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{6}\right)\left(h^2/q_0a^2\right), \quad \bar{\sigma}_{xy} = \sigma_{xy}\left(0, 0, \frac{h}{2}\right)\left(h^2/q_0a^2\right),$$
$$\bar{\sigma}_{xz} = \sigma_{xz}\left(0, \frac{b}{2}, 0\right)\left(h/q_0a\right), \quad \bar{\sigma}_{yz} = \sigma_{yz}\left(\frac{a}{2}, 0, 0\right)\left(h/q_0a\right). \tag{8}$$

6.3. Example 3

A four-layer (0/90/90/0) square laminate simply supported at all edges and subjected to a sinusoidally distributed load was considered at different thickness ratios a/h. Table 3 shows that the calculated nondimensional transverse displacement $\bar{w}$ and stresses $\bar{\sigma}_{xx}, \bar{\sigma}_{yy}, \bar{\sigma}_{xy}, \bar{\sigma}_{xz}$, and $\bar{\sigma}_{yz}$ are much closer to those given by the 3D elasticity solution [30] than predictions of the HSDT and FSDT [17]. It is also seen that the normal stress $\bar{\sigma}_{xx}$ at $a/h = 4$ and 10 is comparable to that given by the FSDT. The normalized deflection and stresses were calculated by the formulas

$$\bar{w} = w\left(\frac{a}{2}, \frac{b}{2}, 0\right)\left(h^3E_2/q_0a^4\right)10^2, \quad \bar{\sigma}_{xx} = \sigma_{xx}\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right)\left(h^2/q_0a^2\right),$$
$$\bar{\sigma}_{yy} = \sigma_{yy}\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{6}\right)\left(h^2/q_0a^2\right), \quad \bar{\sigma}_{xy} = \sigma_{xy}\left(0, 0, \frac{h}{2}\right)\left(h^2/q_0a^2\right), \tag{8}$$
$$\bar{\sigma}_{xz} = \sigma_{xz}\left(0, \frac{b}{2}, 0\right)\left(h/q_0a\right), \quad \bar{\sigma}_{yz} = \sigma_{yz}\left(\frac{a}{2}, 0, 0\right)\left(h/q_0a\right).$$

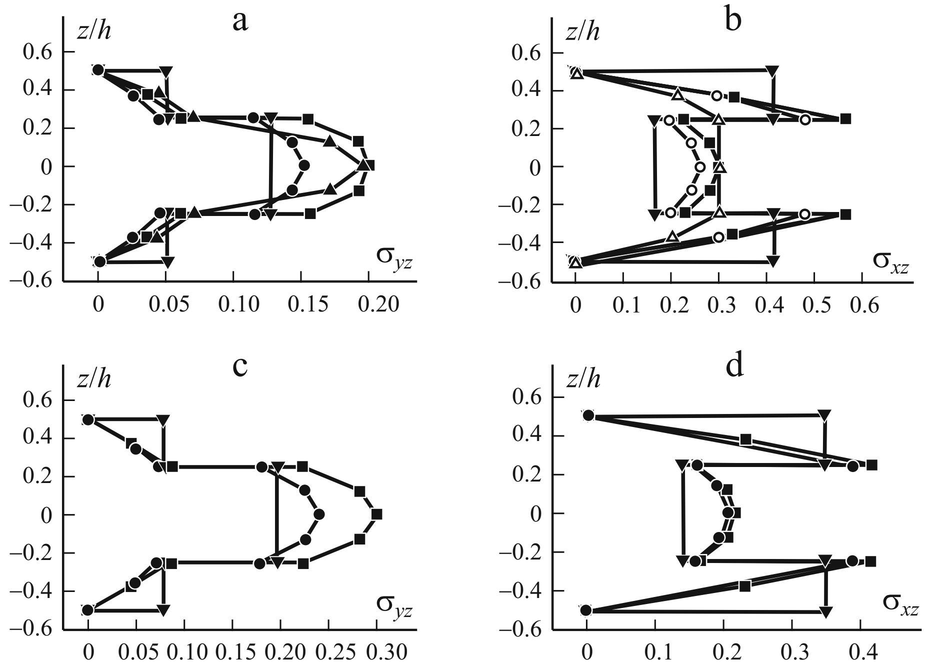

Fig. 3. Distribution of transverse shear stresses across the thickness of (0/90/90/0) plates with thickness raties a/h = 4 and 10 under the action of a sinusoidal load: ■ — present, ● — [17], ▲ — [30], ▼ — FSDT, ○ — [170, constitutive, and Δ — [17], equilibrum.

In Fig. 3, the nondimensional transverse shear stresses $\bar{\sigma}_{xz}$ and $\bar{\sigma}_{yz}$ at their maximum points are indicated. It is seen that the stresses are discontinuous, vary parabolically across the plate thickness, and satisfy zero boundary conditions on the top and bottom surfaces of the plate. In the midplane $z/h = 0$ at $a/h = 10$, the present formulation gives value of the transverse shear stress $\bar{\sigma}_{xz}$ very close to that predicted by the 3D elasticity solution. Also, it gives a good result for the transverse shear stress $\bar{\sigma}_{yz}$ at $a/h = 10$. The transverse shear stress $\bar{\sigma}_{xz}$ in the present formulation at $a/h = 4$ in the midplane $z/h = 0$ is the closest to the value of 0.219 given by the 3D elasticity theory. The same is true for the transverse shear stress $\bar{\sigma}_{yz}$ at $a/h = 4$ in the midplane $z/h = 0$.

7. Conclusion

A simple finite element with four nodes and five degrees of freedom u, v, w, ψ_x, and ψ_y at each node, based on the theory of third-order shear deformation theory which requires the second-order derivative of C1-continuous transverse displacements, was used in a model to analyze the bending of laminated plates. The results obtained for the transverse shear stresses are compared with the 3D elasticity solution, where they have a parabolic distribution across the thickness of the plates and satisfy zero boundary conditions on their top and bottom surfaces. This approach gave good results for the transverse shear stresses, but was not efficient for the normal stress and the transverse displacement, therefore, an optimization procedure for them was performed, which led to results close to the 3D elasticity solution.

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Appendix

The matrix of the first-order derivative can be presented as

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix}.$$

To obtain the second-order derivatives, the chain rule is successively applied to these relations, resulting in

$$\begin{cases} \dfrac{\partial^2}{\partial \xi^2} = \dfrac{\partial}{\partial x}\cdot\dfrac{\partial^2 x}{\partial \xi^2} + \dfrac{\partial}{\partial y}\cdot\dfrac{\partial^2 y}{\partial \xi^2} + \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial x}{\partial \xi}\right)^2 + \dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial y}{\partial \xi}\right)^2 + 2\dfrac{\partial^2}{\partial x \partial y}\cdot\dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial y}{\partial \xi}, \\ \dfrac{\partial^2}{\partial \eta^2} = \dfrac{\partial}{\partial x}\cdot\dfrac{\partial^2 x}{\partial \eta^2} + \dfrac{\partial}{\partial y}\cdot\dfrac{\partial^2 y}{\partial \eta^2} + \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial x}{\partial \eta}\right)^2 + \dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial y}{\partial \eta}\right)^2 + 2\dfrac{\partial^2}{\partial x \partial y}\cdot\dfrac{\partial x}{\partial \eta}\cdot\dfrac{\partial y}{\partial \eta}, \\ \dfrac{\partial^2}{\partial \xi \partial \eta} = \dfrac{\partial}{\partial x}\cdot\dfrac{\partial^2 x}{\partial \xi \partial \eta} + \dfrac{\partial}{\partial y}\cdot\dfrac{\partial^2 y}{\partial \xi \partial \eta} + \dfrac{\partial^2}{\partial x^2}\cdot\dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial x}{\partial \eta} + \dfrac{\partial^2}{\partial y^2}\cdot\dfrac{\partial y}{\partial \xi}\cdot\dfrac{\partial y}{\partial \eta} + \dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{\partial x}{\partial \eta}\cdot\dfrac{\partial y}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial y}{\partial \eta}\right), \end{cases}$$

$$\begin{cases} \dfrac{\partial^2}{\partial \xi^2} - \dfrac{\partial}{\partial x}\cdot\dfrac{\partial^2 x}{\partial \xi^2} - \dfrac{\partial}{\partial y}\cdot\dfrac{\partial^2 y}{\partial \xi^2} = \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial x}{\partial \xi}\right)^2 + \dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial y}{\partial \xi}\right)^2 + 2\dfrac{\partial^2}{\partial x \partial y}\cdot\dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial y}{\partial \xi}, \\ \dfrac{\partial^2}{\partial \eta^2} - \dfrac{\partial}{\partial x}\cdot\dfrac{\partial^2 x}{\partial \eta^2} - \dfrac{\partial}{\partial y}\cdot\dfrac{\partial^2 y}{\partial \eta^2} = \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial x}{\partial \eta}\right)^2 + \dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial y}{\partial \eta}\right)^2 + 2\dfrac{\partial^2}{\partial x \partial y}\cdot\dfrac{\partial x}{\partial \eta}\cdot\dfrac{\partial y}{\partial \eta}, \\ \dfrac{\partial^2}{\partial \xi \partial \eta} - \dfrac{\partial}{\partial x}\cdot\dfrac{\partial^2 x}{\partial \xi \partial \eta} - \dfrac{\partial}{\partial y}\cdot\dfrac{\partial^2 y}{\partial \xi \partial \eta} = \dfrac{\partial^2}{\partial x^2}\cdot\dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial x}{\partial \eta} + \dfrac{\partial^2}{\partial y^2}\cdot\dfrac{\partial y}{\partial \xi}\cdot\dfrac{\partial y}{\partial \eta} + \dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{\partial x}{\partial \eta}\cdot\dfrac{\partial y}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial y}{\partial \eta}\right), \end{cases}$$

$$\begin{Bmatrix} \dfrac{\partial^2}{\partial \xi^2} \\ \dfrac{\partial^2}{\partial \eta^2} \\ \dfrac{\partial^2}{\partial \xi \partial \eta} \end{Bmatrix} - \frac{1}{\det[J]} \begin{bmatrix} \dfrac{\partial^2 x}{\partial \xi^2} & \dfrac{\partial^2 y}{\partial \xi^2} \\ \dfrac{\partial^2 x}{\partial \eta^2} & \dfrac{\partial^2 y}{\partial \eta^2} \\ \dfrac{\partial^2 x}{\partial \xi \partial \eta} & \dfrac{\partial^2 y}{\partial \xi \partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = = \begin{bmatrix} \left(\dfrac{\partial x}{\partial \xi}\right)^2 & \left(\dfrac{\partial y}{\partial \xi}\right)^2 & 2\dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial y}{\partial \xi} \\ \left(\dfrac{\partial x}{\partial \eta}\right)^2 & \left(\dfrac{\partial y}{\partial \eta}\right)^2 & 2\dfrac{\partial x}{\partial \eta}\cdot\dfrac{\partial y}{\partial \eta} \\ \dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \xi}\cdot\dfrac{\partial y}{\partial \eta} & \left(\dfrac{\partial x}{\partial \xi}\cdot\dfrac{\partial y}{\partial \eta} + \dfrac{\partial x}{\partial \eta}\cdot\dfrac{\partial y}{\partial \xi}\right) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} = \left[J^B \right]^{-1} \left[\begin{Bmatrix} \dfrac{\partial^2}{\partial \xi^2} \\ \dfrac{\partial^2}{\partial \eta^2} \\ \dfrac{\partial^2}{\partial \xi \partial \eta} \end{Bmatrix} - \frac{1}{\det\left[J \right]} \begin{bmatrix} \dfrac{\partial^2 x}{\partial \xi^2} & \dfrac{\partial^2 y}{\partial \xi^2} \\ \dfrac{\partial^2 x}{\partial \eta^2} & \dfrac{\partial^2 y}{\partial \eta^2} \\ \dfrac{\partial^2 x}{\partial \xi \partial \eta} & \dfrac{\partial^2 y}{\partial \xi \partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} \right].$$