

## FINITE-LAYER METHOD: BENDING AND TWISTING OF LAMINATED PLATES WITH DELAMINATIONS

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*Based on the finite-layer method, considering the multilayered package as a set of jointly deforming layers, a method to analyze laminated plates with delaminations, which enables one to determine all the functions describing the stress-strain state of the plate, including interlayer stresses, is proposed. A refined deformation model of an anisotropic plate-layer taking into account the transverse linear and shear deformations and ensuring the exact fulfillment of boundary conditions on surfaces of the plate with consideration of all derivatives of existing surface loads is presented. Calculation of the multilayer plate is reduced to the solution of a boundary-value problem for a system of ordinary differential equations including the interlaminar shear stress. The order of the system depends on the number of layers in the package. The system is stiff, and the boundary value problem is solved by the stable Godunov–Grigorenko numerical method. As examples, the calculations of bending and twisting of two-layered plates with a partial delamination are presented.*

### Introduction

In [1-3], an efficient approach, based on the finite-layer method (FLM), to analyzing the stress-strain state (SSS), interlaminar stresses, large deflections, and stability of multilayered elements of structures made of composite materials is suggested. In the same way as the finite-element method leads to a system of algebraic equations, the FLM yields a coupled system of differential equations describing the SSS of each layer and containing interlaminar stresses of contact interaction of the layers. An advantage of the method is the fact that the interlaminar stresses are resolving functions and are determined directly during solution of the problem in a strict agreement with the deformation model of a particular layer. The procedure of

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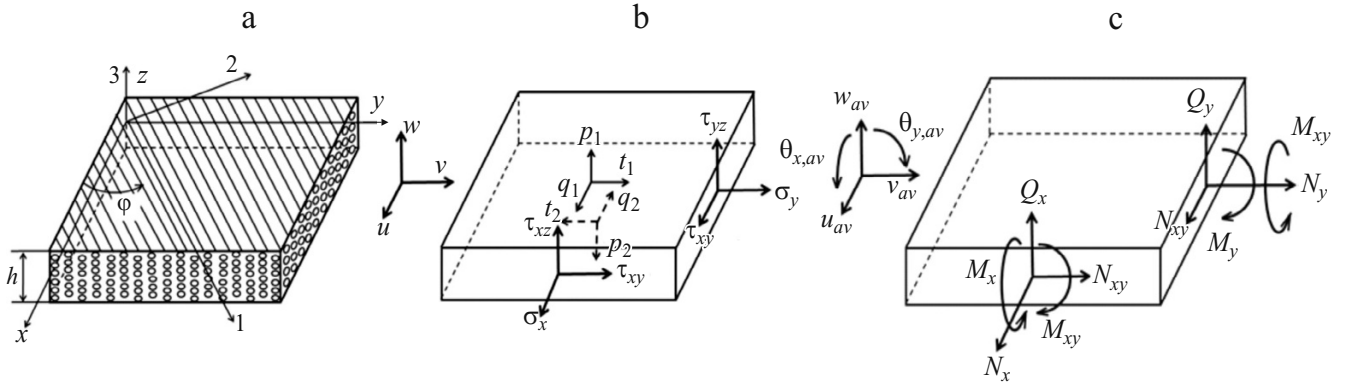


Fig. 1. Anisotropic plate: stacking scheme (a), loads and stresses (b), forces and moments per unit length (c), and displacements and rotation angles (b, c).

application of the FLM is described in detail in [1] for solving one-dimensional problems of cylindrical bending of structural elements of the type of a layered beam-strip with layers of different length. The layered elements have local delaminations and are in the conditions of plane stress or strain state.

In the present study, the method is extended to two-dimensional problems of transverse bending and torsion of orthotropic multilayered plates by edge forces in the presence of delamination defects in them.

## 1. Equations of a Refined Theory of Anisotropic Plates

Let us present a brief description of basic relations of a refined theory of anisotropic plates, which further will be used as a deformation model of a thin layer in the structure of a multilayered orthotropic package.

These relations have to obey the following principal requirements:

- the relations must maximum exactly describe the distribution of normal and tangential stresses on the surfaces of plate-layer with account of all derivatives of these stresses with respect to spatial coordinates;
- the expressions for displacements at surface points of the plate have to contain the stresses existing on these surfaces and their derivatives.

These requirements being obeyed, the contact stresses on layer interfaces are expressed in terms of resolving functions of the problem considered from the conditions of joint displacements, and additional differential equations in the interlaminar tangential stresses are formed.

Let us consider an anisotropic plate in a Cartesian system of coordinates  $x$ ,  $y$ , and  $z$  (Fig. 1a). The plate is formed by stacking an orthotropic (unidirectional) material at an angle  $\varphi$  to the  $x$  axis. The coordinate plane  $x$ ,  $y$  is the midplane dividing the thickness  $h$  in halves. The upper and lower planes of the plate are subjected to normal  $p_i(x, y)$  and tangential  $q_i(x, y)$ ,  $t_i(x, y)$  stresses ( $i = 1, 2, 3$ ), respectively.

The full system of equations includes the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0; \quad (1)$$

the geometrical relations

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \omega_x = \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \quad (2)$$

$$\begin{aligned}\varepsilon_y &= \frac{\partial v}{\partial y}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \omega_y = \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right), \\ \varepsilon_z &= \frac{\partial w}{\partial z}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right);\end{aligned}\tag{2}$$

the physical relations for an anisotropically elastic body in the orthotropy axes 1, 2, and 3 of the material

$$\begin{aligned}\bar{\varepsilon} &= \bar{S}\bar{\sigma}, \quad \bar{\sigma} = \bar{C}\bar{\varepsilon}, \\ \bar{\varepsilon} &= (\varepsilon_1 \varepsilon_2 \varepsilon_3 \gamma_{23} \gamma_{13} \gamma_{12})^T, \quad \bar{\sigma} = (\sigma_1 \sigma_2 \sigma_3 \tau_{23} \tau_{13} \tau_{12})^T, \\ \bar{S} &= \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & 0 \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & 0 \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{S}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{S}_{66} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & 0 \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{66} \end{bmatrix},\end{aligned}$$

where the elements of the compliance  $\bar{S}$  and rigidity  $\bar{C}$  matrices are

$$\begin{aligned}\bar{S}_{11} &= \frac{1}{E_1}, \quad \bar{S}_{12} = -\frac{\nu_{12}}{E_2}, \quad \bar{S}_{13} = -\frac{\nu_{13}}{E_3}, \quad \bar{S}_{22} = \frac{1}{E_2}, \quad \bar{S}_{23} = -\frac{\nu_{23}}{E_3}, \quad \bar{S}_{33} = \frac{1}{E_3}, \\ \bar{S}_{44} &= \frac{1}{G_{23}}, \quad \bar{S}_{55} = \frac{1}{G_{13}}, \quad \bar{S}_{66} = \frac{1}{G_{12}}, \\ \bar{C}_{11} &= \frac{1 - \nu_{23}\nu_{32}}{\Delta} E_1, \quad \bar{C}_{12} = \frac{\nu_{12} + \nu_{13}\nu_{32}}{\Delta} E_1, \quad \bar{C}_{13} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{\Delta} E_1, \\ \bar{C}_{22} &= \frac{1 - \nu_{13}\nu_{31}}{\Delta} E_2, \quad \bar{C}_{23} = \frac{\nu_{23} + \nu_{13}\nu_{21}}{\Delta} E_2, \quad \bar{C}_{33} = \frac{1 - \nu_{12}\nu_{21}}{\Delta} E_3, \\ \bar{C}_{44} &= G_{23}, \quad \bar{C}_{55} = G_{13}, \quad \bar{C}_{66} = G_{12}, \quad \Delta = 1 - 2\nu_{21}\nu_{32}\nu_{13} - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31}, \\ E_1\nu_{12} &= E_2\nu_{21}, \quad E_2\nu_{23} = E_3\nu_{32}, \quad E_3\nu_{31} = E_1\nu_{13};\end{aligned}$$

$E_i, G_{ij}$ , and  $\nu_{ij}$  ( $i, j = 1, 2, 3$ ) are the elastic moduli and Poisson ratios of the orthotropic material of the layer; in the coordinate axes  $x, y$ , and  $z$  of the plate,

$$\begin{aligned}\varepsilon &= S\sigma, \quad \sigma = C\varepsilon, \\ \varepsilon &= (\varepsilon_x \varepsilon_y \varepsilon_z \gamma_{yz} \gamma_{xz} \gamma_{xy})^T, \quad \sigma = (\sigma_x \sigma_y \sigma_z \tau_{yz} \tau_{xz} \tau_{xy})^T, \\ S &= \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix},\end{aligned}\tag{3}$$

the relations for elements of the  $S$  and  $C$  matrices in terms of elements of the  $\bar{S}$  and  $\bar{C}$  matrices and the stacking angle  $\varphi$  are given, for example, in [4].

The force variables — the forces and moments per unit length of the plate (Fig. 1c) are determined by the expressions

$$\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_x dz, \quad N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz, \quad N_y = \int_{-h/2}^{h/2} \sigma_y dz, \quad Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz, \\ M_x &= \int_{-h/2}^{h/2} \sigma_x z dz, \quad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz, \quad M_y = \int_{-h/2}^{h/2} \sigma_y z dz, \quad Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz. \end{aligned} \quad (4)$$

Let us present the components of stress tensor operating in cross sections of the plate in the form

$$\begin{aligned} \sigma_x &= a_0 + a_1 z, \quad \tau_{xy} = b_0 + b_1 z, \quad \tau_{xz} = c_0 + c_1 z + c_2 z^2, \\ \sigma_y &= d_0 + d_1 z, \quad \tau_{yz} = e_0 + e_1 z + e_2 z^2, \quad \sigma_z = f_0 + f_1 z + f_2 z^2 + f_3 z^3, \end{aligned}$$

where the coefficients  $a_i, b_i, d_i, c_j, e_j$ , and  $f_k$  ( $i = 0, 1; j = \overline{0, 2}; k = \overline{0, 3}$ ) depend on the coordinates  $x$  and  $y$ . Using equilibrium equations (1), expressions (4), and the boundary conditions in stresses on face planes

$$\begin{aligned} \sigma_x \left( x, y, \frac{h}{2} \right) &= p_1(x, y), \quad \tau_{xz} \left( x, y, \frac{h}{2} \right) = q_1(x, y), \quad \tau_{yz} \left( x, y, \frac{h}{2} \right) = t_1(x, y), \\ \sigma_z \left( x, y, -\frac{h}{2} \right) &= p_2(x, y), \quad \tau_{xz} \left( x, y, -\frac{h}{2} \right) = q_2(x, y), \quad \tau_{yz} \left( x, y, -\frac{h}{2} \right) = t_2(x, y), \end{aligned} \quad (5)$$

we find the functions  $a_i, b_i, d_i, c_j, e_j$ , and  $f_k$  and the following expressions for stresses in the plate:

$$\begin{aligned} \sigma_x &= \frac{N_x}{h} + \frac{12M_x}{h^3} z, \quad \sigma_y = \frac{N_y}{h} + \frac{12M_y}{h^3} z, \quad \tau_{xy} = \frac{N_{xy}}{h} + \frac{12M_{xy}}{h^3} z, \\ \tau_{xz} &= \frac{3}{2} \frac{Q_x}{h} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] - \frac{q_1 + q_2}{4} \left[ 1 - 3 \left( \frac{2z}{h} \right)^2 \right] + \frac{q_1 - q_2}{h} z, \\ \tau_{yz} &= \frac{3}{2} \frac{Q_y}{h} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] - \frac{t_1 + t_2}{4} \left[ 1 - 3 \left( \frac{2z}{h} \right)^2 \right] + \frac{t_1 - t_2}{h} z, \\ \sigma_z &= \frac{p_1 + p_2}{2} + \frac{3z}{2h} \left[ 1 - \frac{1}{3} \left( \frac{2z}{h} \right)^2 \right] (p_1 - p_2) \\ &+ \frac{h}{8} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] \left[ \left( 1 + \frac{2z}{h} \right) \left( \frac{\partial q_1}{\partial x} + \frac{\partial t_1}{\partial y} \right) - \left( 1 - \frac{2z}{h} \right) \left( \frac{\partial q_2}{\partial x} + \frac{\partial t_2}{\partial y} \right) \right], \end{aligned} \quad (6)$$

and the differential equations of equilibrium for the forces and moments per unit length of the plate are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = q_2 - q_1, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = t_2 - t_1, \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = p_2 - p_1, \quad (7)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x - \frac{q_1 + q_2}{2} h, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y - \frac{t_1 + t_2}{2} h. \quad (7)$$

Now, we pass to the expressions for kinematic variables, namely the displacements and rotation angles (Fig. 1b,c). They can be derived by using geometric relations (2), physical relations (3), and expressions (4).

The normal displacement  $w(x, y, z)$  can be found by integrating the expressions for  $\varepsilon_z$  with respect to  $z$  if  $w(x, y, 0) = w_0(x, y)$  is the displacement of midplane points.

Let us introduce the displacement averaged across the thickness of the plate

$$w_{av}(x, y) = \frac{1}{h} \int_{-h/2}^{h/2} w(x, y, z) dz.$$

Replacing  $w_0(x, y)$  with  $w_{av}(x, y)$ , we have

$$\begin{aligned} w = & w_{av} + (S_{13}N_x + S_{23}N_y + S_{36}N_{xy}) \frac{z}{h} - \frac{1}{2h} \left[ 1 - 3 \left( \frac{2z}{h} \right)^2 \right] (S_{13}M_x + S_{23}M_y + S_{36}M_{xy}) \\ & + S_{33} \left\{ \frac{p_1 + p_2}{2} z + \frac{hz}{8} \left[ 1 - \frac{1}{3} \left( \frac{2z}{h} \right)^2 \right] \left( \frac{\partial q_1}{\partial x} - \frac{\partial q_2}{\partial x} + \frac{\partial t_1}{\partial y} - \frac{\partial t_2}{\partial y} \right) - \frac{9h}{160} \left[ 1 - \frac{10}{3} \left( \frac{2z}{h} \right)^2 \right] \left[ 1 - \frac{1}{6} \left( \frac{2z}{h} \right)^2 \right] \right\} (p_1 - p_2) \\ & - \frac{7h^2}{960} \left[ 1 - \frac{30}{7} \left( \frac{2z}{h} \right)^2 \right] \left[ 1 - \frac{1}{2} \left( \frac{2z}{h} \right)^2 \right] \left( \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial x} + \frac{\partial t_1}{\partial y} + \frac{\partial t_2}{\partial y} \right) \left. \right\}. \quad (8) \end{aligned}$$

Let us present the tangential displacements  $u(x, y, z), v(x, y, z)$  in the form

$$u = u_0 + \alpha_1 z + \beta_1 z^2 + \gamma_1 z^3, \quad v = v_0 + \alpha_2 z + \beta_2 z^2 + \gamma_2 z^3,$$

where the displacements  $u_0$  and  $v_0$  of midplane points and the factors  $\alpha_i, \beta_i$ , and  $\gamma_i$  ( $i = 1, 2$ ) depend on  $x$  and  $y$ .

The thickness-averaged displacements and rotation angles are

$$\begin{aligned} u_{av}(x, y) &= \frac{1}{h} \int_{-h/2}^{h/2} u(x, y, z) dz, \quad v_{av}(x, y) = \frac{1}{h} \int_{-h/2}^{h/2} v(x, y, z) dz, \\ \theta_{x,av}(x, y) &= \frac{1}{h} \int_{-h/2}^{h/2} \omega_y(x, y, z) dz, \quad \theta_{y,av}(x, y) = -\frac{1}{h} \int_{-h/2}^{h/2} \omega_x(x, y, z) dz. \end{aligned}$$

Expressions for the tangential displacements take the form

$$\begin{aligned} u = & u_{av} + \left( \frac{\partial w_{av}}{\partial x} + 2\theta_{x,av} \right) z - \frac{h^2}{12} \left[ 1 - 3 \left( \frac{2z}{h} \right)^2 \right] \beta_1 - \frac{h^2 z}{4} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] \gamma_1, \\ v = & v_{av} + \left( \frac{\partial w_{av}}{\partial y} + 2\theta_{y,av} \right) z - \frac{h^2}{12} \left[ 1 - 3 \left( \frac{2z}{h} \right)^2 \right] \beta_2 - \frac{h^2 z}{4} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] \gamma_2. \quad (9) \end{aligned}$$

For determining the functions  $\beta_i$  and  $\gamma_i$ , we use boundary conditions (5) for the tangential stresses on face planes, expressed in terms of displacements:

$$\beta_1 = \frac{C_{44}(q_1 - q_2) - C_{45}(t_1 - t_2)}{2h\Delta} - \frac{1}{2h} \cdot \frac{\partial}{\partial x} (S_{13}N_x + S_{23}N_y + S_{36}N_{xy}) \quad (10)$$

$$\begin{aligned}
& -\frac{S_{33}}{4} \left[ \frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial x} + \frac{h}{6} \left( \frac{\partial^2 q_1}{\partial x^2} - \frac{\partial^2 q_2}{\partial x^2} + \frac{\partial^2 t_1}{\partial x \partial y} - \frac{\partial^2 t_2}{\partial x \partial y} \right) \right], \\
\beta_2 &= \frac{C_{55}(t_1 - t_2) - C_{45}(q_1 - q_2)}{2h\bar{\Delta}} - \frac{1}{2h} \cdot \frac{\partial}{\partial y} (S_{13}N_x + S_{23}N_y + S_{36}N_{xy}) \\
& -\frac{S_{33}}{4} \left[ \frac{\partial p_1}{\partial y} + \frac{\partial p_2}{\partial y} + \frac{h}{6} \left( \frac{\partial^2 q_1}{\partial x \partial y} - \frac{\partial^2 q_2}{\partial x \partial y} + \frac{\partial^2 t_1}{\partial y^2} - \frac{\partial^2 t_2}{\partial y^2} \right) \right], \\
\gamma_1 &= \frac{2}{h^3\bar{\Delta}} \left[ C_{45} \left( Q_y - \frac{t_1 + t_2}{2} h \right) - C_{44} \left( Q_x - \frac{q_1 + q_2}{2} h \right) \right] - \frac{2}{h^3} \frac{\partial}{\partial x} (S_{13}M_x + S_{23}M_y + S_{36}M_{xy}) \\
& -\frac{S_{33}}{5h} \left[ \frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{h}{12} \left( \frac{\partial^2 q_1}{\partial x^2} + \frac{\partial^2 q_2}{\partial x^2} + \frac{\partial^2 t_1}{\partial x \partial y} + \frac{\partial^2 t_2}{\partial x \partial y} \right) \right], \\
\gamma_2 &= \frac{2}{h^3\bar{\Delta}} \left[ C_{45} \left( Q_x - \frac{q_1 + q_2}{2} h \right) - C_{55} \left( Q_y - \frac{t_1 + t_2}{2} h \right) \right] - \frac{2}{h^3} \frac{\partial}{\partial y} (S_{13}M_x + S_{23}M_y + S_{36}M_{xy}) \\
& -\frac{S_{33}}{5h} \left[ \frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{h}{12} \left( \frac{\partial^2 q_1}{\partial x \partial y} + \frac{\partial^2 q_2}{\partial x \partial y} + \frac{\partial^2 t_1}{\partial y^2} + \frac{\partial^2 t_2}{\partial y^2} \right) \right].
\end{aligned} \tag{10}$$

Further, considering Eqs. (4), we come to the relations

$$\begin{aligned}
\frac{\partial w_{av}}{\partial x} + \theta_{x,av} &= \frac{C_{44}Q_x - C_{45}Q_y}{2h\bar{\Delta}}, \quad \frac{\partial w_{av}}{\partial y} + \theta_{y,av} = \frac{C_{55}Q_y - C_{45}Q_x}{2h\bar{\Delta}}, \\
\bar{\Delta} &= C_{44}C_{55} - C_{45}^2 = \bar{C}_{44}\bar{C}_{55}
\end{aligned} \tag{11}$$

and equations relating the average tangential displacements and rotation angles to the forces and moments per unit length and surface loads. These equations contain the third derivatives of the loads distributed over the face planes of the plate.

Let us introduce new variables — rotations  $\tilde{\theta}_x$  and  $\tilde{\theta}_y$  that differ from the average angles by a small value:

$$\tilde{\theta}_x = \theta_{x,av} + \frac{C_{44}Q_x - C_{45}Q_y}{2h\bar{\Delta}} - \frac{h^2}{10} \gamma_1, \quad \tilde{\theta}_y = \theta_{y,av} + \frac{C_{55}Q_y - C_{45}Q_x}{2h\bar{\Delta}} - \frac{h^2}{10} \gamma_2. \tag{12}$$

Then, the equations for displacements and rotations will contain derivatives of surface loads only to the second order inclusive and have the form

$$\begin{aligned}
& C_{11}h \frac{\partial u_{av}}{\partial x} + C_{12}h \frac{\partial v_{av}}{\partial y} + C_{16}h \left( \frac{\partial u_{av}}{\partial y} + \frac{\partial v_{av}}{\partial x} \right) \\
& = N_x - C_{13} \left\{ S_{13}N_x + S_{23}N_y + S_{36}N_{xy} + S_{33} \left[ \frac{p_1 + p_2}{2} h + \frac{h^2}{12} \left( \frac{\partial q_1}{\partial x} - \frac{\partial q_2}{\partial x} + \frac{\partial t_1}{\partial y} - \frac{\partial t_2}{\partial y} \right) \right] \right\}, \\
& C_{12}h \frac{\partial u_{av}}{\partial x} + C_{22}h \frac{\partial v_{av}}{\partial y} + C_{26}h \left( \frac{\partial u_{av}}{\partial y} + \frac{\partial v_{av}}{\partial x} \right) \\
& = N_y - C_{23} \left\{ S_{13}N_x + S_{23}N_y + S_{36}N_{xy} + S_{33} \left[ \frac{p_1 + p_2}{2} h + \frac{h^2}{12} \left( \frac{\partial q_1}{\partial x} - \frac{\partial q_2}{\partial x} + \frac{\partial t_1}{\partial y} - \frac{\partial t_2}{\partial y} \right) \right] \right\},
\end{aligned} \tag{13}$$

$$\begin{aligned}
& C_{16}h \frac{\partial u_{av}}{\partial x} + C_{26}h \frac{\partial v_{av}}{\partial y} + C_{66}h \left( \frac{\partial u_{av}}{\partial y} + \frac{\partial v_{av}}{\partial x} \right) \\
& = N_{xy} - C_{36} \left\{ S_{13}N_x + S_{23}N_y + S_{36}N_{xy} + S_{33} \left[ \frac{p_1 + p_2}{2} h + \frac{h^2}{12} \left( \frac{\partial q_1}{\partial x} - \frac{\partial q_2}{\partial x} + \frac{\partial t_1}{\partial y} - \frac{\partial t_2}{\partial y} \right) \right] \right\}, \\
\frac{\partial w_{av}}{\partial x} & = -\tilde{\theta}_x + \frac{6}{5} \frac{C_{44}Q_x - C_{45}Q_y}{h\Delta} - \frac{C_{44}(q_1 + q_2) - C_{45}(t_1 + t_2)}{10\Delta} + \frac{1}{5h} \left( S_{13} \frac{\partial M_x}{\partial x} + S_{23} \frac{\partial M_y}{\partial x} + S_{36} \frac{\partial M_{xy}}{\partial x} \right) \\
& \quad + S_{33} \frac{h}{50} \left[ \frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{h}{12} \left( \frac{\partial^2 q_1}{\partial x^2} + \frac{\partial^2 q_2}{\partial x^2} + \frac{\partial^2 t_1}{\partial x \partial y} + \frac{\partial^2 t_2}{\partial x \partial y} \right) \right], \\
\frac{\partial w_{av}}{\partial y} & = -\tilde{\theta}_y + \frac{6}{5} \frac{C_{55}Q_y - C_{45}Q_x}{h\Delta} - \frac{C_{55}(t_1 + t_2) - C_{45}(q_1 + q_2)}{10\Delta} + \frac{1}{5h} \left( S_{13} \frac{\partial M_x}{\partial y} + S_{23} \frac{\partial M_y}{\partial y} + S_{36} \frac{\partial M_{xy}}{\partial y} \right) \\
& \quad + S_{33} \frac{h}{50} \left[ \frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{h}{12} \left( \frac{\partial^2 q_1}{\partial x \partial y} + \frac{\partial^2 q_2}{\partial x \partial y} + \frac{\partial^2 t_1}{\partial y^2} + \frac{\partial^2 t_2}{\partial y^2} \right) \right], \\
& \quad C_{11} \frac{h^3}{12} \cdot \frac{\partial \tilde{\theta}_x}{\partial x} + C_{12} \frac{h^3}{12} \cdot \frac{\partial \tilde{\theta}_y}{\partial y} + C_{16} \frac{h^3}{12} \left( \frac{\partial \tilde{\theta}_y}{\partial x} + \frac{\partial \tilde{\theta}_x}{\partial y} \right) \tag{13} \\
& = M_x - C_{13} \left\{ S_{13}M_x + S_{23}M_y + S_{36}M_{xy} + S_{33} \frac{h^2}{10} \left[ p_1 - p_2 + \frac{h}{12} \left( \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial x} + \frac{\partial t_1}{\partial y} + \frac{\partial t_2}{\partial y} \right) \right] \right\}, \\
& \quad C_{12} \frac{h^3}{12} \cdot \frac{\partial \tilde{\theta}_x}{\partial x} + C_{22} \frac{h^3}{12} \cdot \frac{\partial \tilde{\theta}_y}{\partial y} + C_{26} \frac{h^3}{12} \left( \frac{\partial \tilde{\theta}_y}{\partial x} + \frac{\partial \tilde{\theta}_x}{\partial y} \right) \\
& = M_y - C_{23} \left\{ S_{13}M_x + S_{23}M_y + S_{36}M_{xy} + S_{33} \frac{h^2}{10} \left[ p_1 - p_2 + \frac{h}{12} \left( \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial x} + \frac{\partial t_1}{\partial y} + \frac{\partial t_2}{\partial y} \right) \right] \right\}, \\
& \quad C_{16} \frac{h^3}{12} \cdot \frac{\partial \tilde{\theta}_x}{\partial x} + C_{26} \frac{h^3}{12} \cdot \frac{\partial \tilde{\theta}_y}{\partial y} + C_{66} \frac{h^3}{12} \left( \frac{\partial \tilde{\theta}_y}{\partial x} + \frac{\partial \tilde{\theta}_x}{\partial y} \right) \\
& = M_{xy} - C_{36} \left\{ S_{13}M_x + S_{23}M_y + S_{36}M_{xy} + S_{33} \frac{h^2}{10} \left[ p_1 - p_2 + \frac{h}{12} \left( \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial x} + \frac{\partial t_1}{\partial y} + \frac{\partial t_2}{\partial y} \right) \right] \right\}.
\end{aligned}$$

The complete system of equations of the refined theory of anisotropic plates presented here includes differential equations (7) and (13) for the resolving functions  $N_x, N_{xy}, N_y, Q_x, Q_y, M_x, M_{xy}, M_y, u_{av}, v_{av}, w_{av}, \tilde{\theta}_x$ , and  $\tilde{\theta}_y$ , formulas for stresses (6), and expressions for displacements and rotations (8)-(12).

The boundary conditions at the edge  $x = \text{const}$  are given for five functions, one from each pair:

$$(N_x, u_{av}), (N_{xy}, v_{av}), (Q_x, w_{av}), (M_x, \theta_{x,av}), (M_{xy}, \theta_{y,av});$$

at the edge  $y = \text{const}$  —

$$(N_y, v_{av}), (N_{xy}, u_{av}), (Q_y, w_{av}), (M_y, \theta_{y,av}), (M_{xy}, \theta_{x,av}).$$

At the edges, any noncontradictory combination of the above-mentioned functions can be given too.

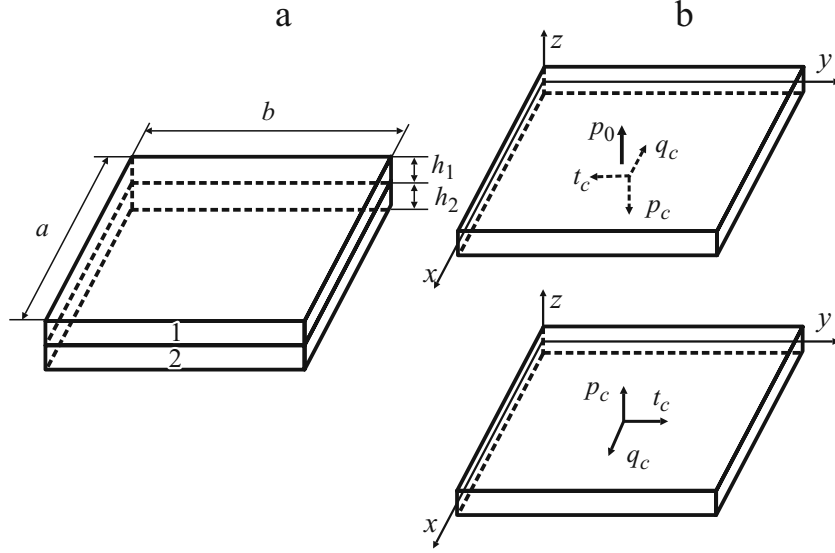


Fig. 2. A two-layer plate: dimensions (a) and the load and interlaminar stresses (b).

## 2. Two-Layer Orthotropic Plate

Let us consider a plate with dimensions indicated in Fig. 2a, which consists of two layers whose orthotropy axes coincide with the coordinate axes  $x$ ,  $y$ , and  $z$ . The upper layer of the plate is loaded with a pressure  $p_0(x, y)$ .

We divide the plate into layers (Fig. 2b) and introduce the stresses of interlaminar interaction, namely the normal stress  $p_c(x, y)$  and tangential stresses  $q_c(x, y)$  and  $t_c(x, y)$ . In what follows, the superscript in parentheses indicates the number of a layer.

The resolving system of equations for layer 1 has the form

$$\begin{aligned}
 \frac{\partial N_x^{(1)}}{\partial x} + \frac{\partial N_{xy}^{(1)}}{\partial y} &= q_c, & \frac{\partial N_{xy}^{(1)}}{\partial x} + \frac{\partial N_y^{(1)}}{\partial y} &= t_c, & \frac{\partial Q_x^{(1)}}{\partial x} + \frac{\partial Q_y^{(1)}}{\partial y} &= p_c - p_0, \\
 \frac{\partial M_x^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} &= Q_x^{(1)} - \frac{h_1}{2} q_c, & \frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_y^{(1)}}{\partial y} &= Q_y^{(1)} - \frac{h_1}{2} t_c, \\
 \frac{\partial u_{av}^{(1)}}{\partial x} &= \frac{N_x^{(1)} - \nu_{21}^{(1)} N_y^{(1)}}{E_1^{(1)} h_1} - \frac{\nu_{13}^{(1)}}{2E_3^{(1)}} \left[ p_0 + p_c - \frac{h_1}{6} \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) \right], \\
 \frac{\partial v_{av}^{(1)}}{\partial y} &= \frac{N_y^{(1)} - \nu_{12}^{(1)} N_x^{(1)}}{E_2^{(1)} h_1} - \frac{\nu_{23}^{(1)}}{2E_3^{(1)}} \left[ p_0 + p_c - \frac{h_1}{6} \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) \right], & \frac{\partial u_{av}^{(1)}}{\partial y} + \frac{\partial v_{av}^{(1)}}{\partial x} &= \frac{N_{xy}^{(1)}}{G_{12}^{(1)} h_1}, \\
 \frac{\partial w_{av}^{(1)}}{\partial x} &= -\tilde{\theta}_x^{(1)} + \frac{6}{5G_{13}^{(1)} h_1} \left( Q_x^{(1)} - \frac{h_1}{12} q_c \right) \\
 & - \frac{1}{5E_3^{(1)} h_1} \left\{ \nu_{13}^{(1)} \frac{\partial M_x^{(1)}}{\partial x} + \nu_{23}^{(1)} \frac{\partial M_y^{(1)}}{\partial x} - \frac{h_1^2}{10} \left[ \frac{\partial p_0}{\partial x} - \frac{\partial p_c}{\partial x} + \frac{h_1}{12} \left( \frac{\partial^2 q_c}{\partial x^2} + \frac{\partial^2 t_c}{\partial x \partial y} \right) \right] \right\},
 \end{aligned} \tag{14}$$



$$\begin{aligned}
\frac{\partial w_{av}^{(1)}}{\partial y} &= -\tilde{\theta}_y^{(1)} + \frac{6}{5G_{23}^{(1)}h_1} \left( Q_y^{(1)} - \frac{h_1}{12} t_c \right) - \frac{1}{5E_3^{(1)}h_1} \left\{ v_{13}^{(1)} \frac{\partial M_x^{(1)}}{\partial y} + v_{23}^{(1)} \frac{\partial M_y^{(1)}}{\partial y} - \frac{h_1^2}{10} \left[ \frac{\partial p_0}{\partial y} - \frac{\partial p_c}{\partial y} + \frac{h_1}{12} \left( \frac{\partial^2 q_c}{\partial x \partial y} + \frac{\partial^2 t_c}{\partial y^2} \right) \right] \right\}, \\
\frac{\partial \tilde{\theta}_x^{(1)}}{\partial x} &= \frac{12 \left( M_x^{(1)} - v_{21}^{(1)} M_y^{(1)} \right)}{E_1^{(1)} h_1^3} - \frac{6}{5} \frac{v_{13}^{(1)}}{E_3^{(1)} h_1} \left[ p_0 - p_c + \frac{h_1}{12} \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) \right], \\
\frac{\partial \tilde{\theta}_y^{(1)}}{\partial y} &= \frac{12 \left( M_y^{(1)} - v_{12}^{(1)} M_x^{(1)} \right)}{E_2^{(1)} h_1^3} - \frac{6}{5} \frac{v_{23}^{(1)}}{E_3^{(1)} h_1} \left[ p_0 - p_c + \frac{h_1}{12} \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) \right], \\
\frac{\partial \tilde{\theta}_x^{(1)}}{\partial y} + \frac{\partial \tilde{\theta}_y^{(1)}}{\partial x} &= \frac{12 M_{xy}^{(1)}}{G_{12}^{(1)} h_1^3};
\end{aligned} \tag{14}$$

the equations for layer 2 are

$$\begin{aligned}
\frac{\partial N_x^{(2)}}{\partial x} + \frac{\partial N_{xy}^{(2)}}{\partial y} &= -q_c, \quad \frac{\partial N_{xy}^{(2)}}{\partial x} + \frac{\partial N_y^{(2)}}{\partial y} = -t_c, \quad \frac{\partial Q_x^{(2)}}{\partial x} + \frac{\partial Q_y^{(2)}}{\partial y} = -p_c, \\
\frac{\partial M_x^{(2)}}{\partial x} + \frac{\partial M_{xy}^{(2)}}{\partial y} &= Q_x^{(2)} - \frac{h_2}{2} q_c, \quad \frac{\partial M_{xy}^{(2)}}{\partial x} + \frac{\partial M_y^{(2)}}{\partial y} = Q_y^{(2)} - \frac{h_2}{2} t_c, \\
\frac{\partial u_{av}^{(2)}}{\partial x} &= \frac{N_x^{(2)} - v_{21}^{(2)} N_y^{(2)}}{E_1^{(2)} h_2} - \frac{v_{13}^{(2)}}{2E_3^{(2)}} \left[ p_c + \frac{h_2}{6} \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) \right], \\
\frac{\partial v_{av}^{(2)}}{\partial y} &= \frac{N_y^{(2)} - v_{12}^{(2)} N_x^{(2)}}{E_2^{(2)} h_2} - \frac{v_{23}^{(2)}}{2E_3^{(2)}} \left[ p_c + \frac{h_2}{6} \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) \right], \quad \frac{\partial u_{av}^{(2)}}{\partial y} + \frac{\partial v_{av}^{(2)}}{\partial x} = \frac{N_{xy}^{(2)}}{G_{12}^{(2)} h_2}, \\
\frac{\partial w_{av}^{(2)}}{\partial x} &= -\tilde{\theta}_x^{(2)} + \frac{6}{5G_{13}^{(2)}h_2} \left( Q_x^{(2)} - \frac{h_2}{12} q_c \right) \\
&\quad - \frac{1}{5E_3^{(2)}h_2} \left\{ v_{13}^{(2)} \frac{\partial M_x^{(2)}}{\partial x} + v_{23}^{(2)} \frac{\partial M_y^{(2)}}{\partial x} - \frac{h_2^2}{10} \left[ \frac{\partial p_c}{\partial x} + \frac{h_2}{12} \left( \frac{\partial^2 q_c}{\partial x^2} + \frac{\partial^2 t_c}{\partial x \partial y} \right) \right] \right\}, \\
\frac{\partial w_{av}^{(2)}}{\partial y} &= -\tilde{\theta}_y^{(2)} + \frac{6}{5G_{23}^{(2)}h_2} \left( Q_y^{(2)} - \frac{h_2}{12} t_c \right) \\
&\quad - \frac{1}{5E_3^{(2)}h_2} \left\{ v_{13}^{(2)} \frac{\partial M_x^{(2)}}{\partial y} + v_{23}^{(2)} \frac{\partial M_y^{(2)}}{\partial y} - \frac{h_2^2}{10} \left[ \frac{\partial p_c}{\partial y} + \frac{h_2}{12} \left( \frac{\partial^2 q_c}{\partial x \partial y} + \frac{\partial^2 t_c}{\partial y^2} \right) \right] \right\}, \\
\frac{\partial \tilde{\theta}_x^{(2)}}{\partial x} &= \frac{12 \left( M_x^{(2)} - v_{21}^{(2)} M_y^{(2)} \right)}{E_1^{(2)} h_2^3} - \frac{6}{5} \frac{v_{13}^{(2)}}{E_3^{(2)} h_2} \left[ p_c + \frac{h_2}{12} \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) \right],
\end{aligned} \tag{15}$$

$$\frac{\partial \tilde{\theta}_y^{(2)}}{\partial y} = \frac{12(M_y^{(2)} - v_{12}^{(2)}M_x^{(2)})}{E_2^{(2)}h_2^3} - \frac{6}{5} \frac{v_{23}^{(2)}}{E_3^{(2)}h_2} \left[ p_c + \frac{h_2}{12} \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) \right], \quad (15)$$

$$\frac{\partial \tilde{\theta}_x^{(2)}}{\partial y} + \frac{\partial \tilde{\theta}_y^{(2)}}{\partial x} = \frac{12M_{xy}^{(2)}}{G_{12}^{(2)}h_2^3}.$$

The conditions of equality of displacements on the contact plane of layers

$$u^{(1)}\left(-\frac{h_1}{2}\right) = u^{(2)}\left(\frac{h_2}{2}\right), \quad v^{(1)}\left(-\frac{h_1}{2}\right) = v^{(2)}\left(\frac{h_2}{2}\right), \quad w^{(1)}\left(-\frac{h_1}{2}\right) = w^{(2)}\left(\frac{h_2}{2}\right)$$

yield three corresponding equations:

$$\begin{aligned} & \frac{2}{15} \left( \frac{h_1}{G_{13}^{(1)}} + \frac{h_2}{G_{13}^{(2)}} \right) q_c + \frac{31}{600} \left( \frac{h_1^2}{E_3^{(1)}} - \frac{h_2^2}{E_3^{(2)}} \right) \frac{\partial p_c}{\partial x} - \frac{7}{900} \left( \frac{h_1^3}{E_3^{(1)}} + \frac{h_2^3}{E_3^{(2)}} \right) \left( \frac{\partial^2 q_c}{\partial x^2} + \frac{\partial^2 t_c}{\partial x \partial y} \right) - u_{av}^{(1)} + \\ & + u_{av}^{(2)} + \frac{h_1}{2} \tilde{\theta}_x^{(1)} + \frac{h_2}{2} \tilde{\theta}_x^{(2)} - \frac{h_1}{12E_3^{(1)}} \left( v_{13}^{(1)} \frac{\partial N_x^{(1)}}{\partial x} + v_{23}^{(1)} \frac{\partial N_y^{(1)}}{\partial x} \right) + \frac{h_2}{12E_3^{(2)}} \left( v_{13}^{(2)} \frac{\partial N_x^{(2)}}{\partial x} + v_{23}^{(2)} \frac{\partial N_y^{(2)}}{\partial x} \right) \\ & - \frac{1}{10} \left( \frac{Q_x^{(1)}}{G_{13}^{(1)}} + \frac{Q_x^{(2)}}{G_{13}^{(2)}} \right) + \frac{1}{10E_3^{(1)}} \left( v_{13}^{(1)} \frac{\partial M_x^{(1)}}{\partial x} + v_{23}^{(1)} \frac{\partial M_y^{(1)}}{\partial x} \right) + \frac{1}{10E_3^{(2)}} \left( v_{13}^{(2)} \frac{\partial M_x^{(2)}}{\partial x} + v_{23}^{(2)} \frac{\partial M_y^{(2)}}{\partial x} \right) = -\frac{19h_1^2}{600E_3^{(1)}} \cdot \frac{\partial p_0}{\partial x}, \\ & \frac{2}{15} \left( \frac{h_1}{G_{23}^{(1)}} + \frac{h_2}{G_{23}^{(2)}} \right) t_c + \frac{31}{600} \left( \frac{h_1^2}{E_3^{(1)}} - \frac{h_2^2}{E_3^{(2)}} \right) \frac{\partial p_c}{\partial y} - \frac{7}{900} \left( \frac{h_1^3}{E_3^{(1)}} + \frac{h_2^3}{E_3^{(2)}} \right) \left( \frac{\partial^2 q_c}{\partial x \partial y} + \frac{\partial^2 t_c}{\partial y^2} \right) - v_{av}^{(1)} \\ & + v_{av}^{(2)} + \frac{h_1}{2} \tilde{\theta}_y^{(1)} + \frac{h_2}{2} \tilde{\theta}_y^{(2)} - \frac{h_1}{12E_3^{(1)}} \left( v_{13}^{(1)} \frac{\partial N_x^{(1)}}{\partial y} + v_{23}^{(1)} \frac{\partial N_y^{(1)}}{\partial y} \right) \\ & + \frac{h_2}{12E_3^{(2)}} \left( v_{13}^{(2)} \frac{\partial N_x^{(2)}}{\partial y} + v_{23}^{(2)} \frac{\partial N_y^{(2)}}{\partial y} \right) - \frac{1}{10} \left( \frac{Q_y^{(1)}}{G_{23}^{(1)}} + \frac{Q_y^{(2)}}{G_{23}^{(2)}} \right) \\ & + \frac{1}{10E_3^{(1)}} \left( v_{13}^{(1)} \frac{\partial M_x^{(1)}}{\partial y} + v_{23}^{(1)} \frac{\partial M_y^{(1)}}{\partial y} \right) + \frac{1}{10E_3^{(2)}} \left( v_{13}^{(2)} \frac{\partial M_x^{(2)}}{\partial y} + v_{23}^{(2)} \frac{\partial M_y^{(2)}}{\partial y} \right) = -\frac{19h_1^2}{600E_3^{(1)}} \cdot \frac{\partial p_0}{\partial y}, \\ & \frac{7}{20} \left( \frac{h_1}{E_3^{(1)}} + \frac{h_2}{E_3^{(2)}} \right) p_c - \frac{1}{20} \left( \frac{h_1^2}{E_3^{(1)}} - \frac{h_2^2}{E_3^{(2)}} \right) \left( \frac{\partial q_c}{\partial x} + \frac{\partial t_c}{\partial y} \right) - w_{av}^{(1)} + w_{av}^{(2)} - \frac{1}{2E_3^{(1)}} \left( v_{13}^{(1)} N_x^{(1)} + v_{23}^{(1)} N_y^{(1)} \right) \\ & - \frac{1}{2E_3^{(2)}} \left( v_{13}^{(2)} N_x^{(2)} + v_{23}^{(2)} N_y^{(2)} \right) + \frac{1}{E_3^{(1)}h_1} \left( v_{13}^{(1)} M_x^{(1)} + v_{23}^{(1)} M_y^{(1)} \right) - \frac{1}{E_3^{(2)}h_2} \left( v_{13}^{(2)} M_x^{(2)} + v_{23}^{(2)} M_y^{(2)} \right) = -\frac{3h_1}{20E_3^{(1)}} p_0. \end{aligned} \quad (16)$$

The resulting system of 29 differential equations (14)-(16) contains 26 basic resolving functions  $N_x^{(i)}, N_{xy}^{(i)}, N_y^{(i)}, Q_x^{(i)}, Q_y^{(i)}, M_x^{(i)}, M_{xy}^{(i)}, M_y^{(i)}, u_{av}^{(i)}, v_{av}^{(i)}, w_{av}^{(i)}, \tilde{\theta}_x^{(i)}$  and  $\tilde{\theta}_y^{(i)}$  ( $i=1,2$ ) and three additional resolving functions  $p_c, q_c$ , and  $t_c$ .

The coupled system of differential equations for a package consisting of  $n$  layers is derived in a similar way. Then, the number of equations in the system is  $N = 16n - 3$ ; the system contains  $13n$  basic resolving functions, namely the forces and moments per unit length and displacements, rotation angles, and  $3(n - 1)$  functions of contact stresses.

For the class of plates considered here, the coefficients of equations are constant, which makes it possible to solve the problem by using the well-known methods, for example, expansion into double trigonometrical series [5]. In this case, for each harmonic, we have a system of  $N$  linear algebraic equations in terms of amplitude values of required functions.

Further on, with particular examples, we will show a way of solving the problem of a two-layer plate with a local through-the-width delamination.

### 3. Expansion into Simple Fourier Series. Formulation and Solution of the Boundary-Value Problem for a System of Ordinary Differential Equations

Let us present the required functions, operating load, and boundary conditions in the form of expansions into trigonometric series:

$$f(x, y) = \sum_{n=1}^{NH} f_n(x) \sin\left(\frac{n\pi y}{b}\right),$$

$$g(x, y) = \sum_{n=1}^{NH} g_n(x) \cos\left(\frac{n\pi y}{b}\right) \quad (n = 1, 3, \dots, NH \text{ is odd}),$$

$$f = \left\{ N_x^{(i)}, N_y^{(i)}, Q_x^{(i)}, M_x^{(i)}, M_y^{(i)}, u_{av}^{(i)}, w_{av}^{(i)}, \tilde{\theta}_x^{(i)}, p_c, q_c, p_0 \right\}, \quad (17)$$

$$g = \left\{ N_{xy}^{(i)}, Q_y^{(i)}, M_{xy}^{(i)}, v_{av}^{(i)}, \tilde{\theta}_y^{(i)}, t_c \right\}.$$

Such an expansion allows one to satisfy the conditions of hinge support at the edges  $y = 0, b$  at the additional restrictions that these edges do not move in the direction of contour line and do not bend in the plane of edge face of the plate.

Inserting series (17) into Eqs. (14) and (15), for each number  $n$  of harmonic, we arrive at a system of 20 ordinary differential equations of the first order in the amplitude values of resolving functions  $N_{x,n}^{(i)}, N_{xy,n}^{(i)}, Q_{x,n}^{(i)}, M_{x,n}^{(i)}, M_{xy,n}^{(i)}, u_{av,n}^{(i)}, v_{av,n}^{(i)}, w_{av,n}^{(i)}, \tilde{\theta}_{x,n}^{(i)}$ , and  $\tilde{\theta}_{y,n}^{(i)}$  and the corresponding boundary conditions.

The equations contain six functions determining the contact interaction of layers:

$$p_{c,n}, q_{c,n}, t_{c,n}, p'_{c,n}, r_{c,n} = q'_{c,n} - \frac{n\pi}{b} t_{c,n}, s_{c,n} = \bar{q}_{c,n}'' - \frac{n\pi}{b} t'_{c,n}.$$

Hereinafter, the prime designates differentiation with respect to  $x$ . Two of these functions,  $q_{c,n}$  and  $r_{c,n}$ , are added to the basic resolving functions, as a result of which the system for each harmonic is supplemented by two differential equations

$$\frac{dq_{c,n}}{dx} = r_{c,n} + \frac{n\pi}{b} t_{c,n}, \quad \frac{dr_{c,n}}{dx} = s_{c,n}.$$

The corresponding boundary conditions at the edges  $x = 0, a$  are assigned for the interlaminar tangential stress  $q_{c,n}$ . In this case, the boundary conditions, namely data for distribution of the tangential stress  $\tau_{xz}$  on the end faces of the plate have to be further refined.

The remaining four functions  $p_{c,n}, t_{c,n}, p'_{c,n}$ , and  $s_{c,n}$  can be found from the conditions of equality of displacements on the contact plane of layers. For this purpose, we have three conditions of equality of displacements (16). The missing fourth

condition is obtained by differentiating the last equality in (16) with respect to the  $x$  coordinate. As a result, we come to the system of four linear algebraic equations

$$\begin{bmatrix} a_{11,n} & 0 & 0 & 0 \\ a_{21,n} & a_{22,n} & 0 & 0 \\ 0 & 0 & a_{33,n} & a_{34,n} \\ 0 & 0 & a_{43,n} & a_{44,n} \end{bmatrix} \begin{pmatrix} p_{c,n} \\ t_{c,n} \\ p'_{c,n} \\ s_{c,n} \end{pmatrix} = \begin{pmatrix} b_{1,n} \\ b_{2,n} \\ b_{3,n} \\ b_{4,n} \end{pmatrix}.$$

The matrix elements  $a_{ij,n}$  ( $i, j = \overline{1,4}$ ) depend on the physicommechanical characteristics and dimensions of layers, but the vector elements  $b_{i,n}$  depend on the resolving functions and the applied load, too.

Thus, the solution is reduced to the boundary-value problem for a system of 22 linear ordinary differential equations of the first order:

$$\frac{d\bar{Y}_n(x)}{dx} = \bar{F}_n(x, \bar{Y}_n). \quad (18)$$

Here, the vector-function of the required solution is

$$\bar{Y}_n(x) = \left( N_{x,n}^{(i)} \ N_{xy,n}^{(i)} \ Q_{x,n}^{(i)} \ M_{x,n}^{(i)} \ M_{xy,n}^{(i)} \ u_{av,n}^{(i)} \ v_{av,n}^{(i)} \ w_{av,n}^{(i)} \ \tilde{\theta}_{x,n}^{(i)} \ \tilde{\theta}_{y,n}^{(i)} \ q_{c,n} \ r_{c,n} \right)^T \quad (i = 1, 2),$$

and the right-hand sides of Eqs. (18) are given by

$$\begin{aligned} F_{1,n} &= \frac{n\pi}{b} N_{xy,n}^{(1)} + q_{c,n}, \quad F_{2,n} = -\frac{n\pi}{b} N_{y,n}^{(1)} + t_{c,n}, \quad F_{3,n} = \frac{n\pi}{b} Q_{y,n}^{(1)} + p_{c,n} - p_{0,n}, \\ F_{4,n} &= Q_{x,n}^{(1)} + \frac{n\pi}{b} M_{xy,n}^{(1)} - \frac{h_1}{2} q_{c,n}, \quad F_{5,n} = Q_{y,n}^{(1)} - \frac{n\pi}{b} M_{y,n}^{(1)} - \frac{h_1}{2} t_{c,n}, \\ F_{6,n} &= \frac{N_{x,n}^{(1)} - v_{21}^{(1)} N_{y,n}^{(1)}}{E_1^{(1)} h_1} - \frac{v_{13}^{(1)}}{2E_3^{(1)}} \left( p_{0,n} + p_{c,n} - \frac{h_1}{6} r_{c,n} \right), \quad F_{7,n} = \frac{N_{xy,n}^{(1)}}{G_{12}^{(1)} h_1} - \frac{n\pi}{b} u_{av,n}^{(1)}, \\ F_{8,n} &= -\tilde{\theta}_{x,n}^{(1)} + \frac{6}{5G_{23}^{(1)} h_1} \left( Q_{x,n}^{(1)} - \frac{h_1}{12} q_{c,n} \right) \\ &- \frac{1}{5E_3^{(1)} h_1} \left[ \left( v_{13}^{(1)} + v_{12}^{(1)} v_{23}^{(1)} \right) F_{4,n} - \frac{E_2^{(1)} v_{23}^{(1)} h_1^3}{12} \frac{n\pi}{b} F_{10,n} - \left( 1 - v_{23}^{(1)} v_{32}^{(1)} \right) \left( p'_{0,n} - p'_{c,n} + \frac{h_1}{12} s_{c,n} \right) \frac{h_1^2}{10} \right], \\ F_{9,n} &= \frac{12 \left( M_{x,n}^{(1)} - v_{21}^{(1)} M_{y,n}^{(1)} \right)}{E_1^{(1)} h_1^3} - \frac{6}{5} \frac{v_{13}^{(1)}}{E_3^{(1)} h_1} \left( p_{0,n} - p_{c,n} + \frac{h_1}{12} r_{c,n} \right), \quad F_{10,n} = \frac{12 M_{xy,n}^{(1)}}{G_{12}^{(1)} h_1^3} - \frac{n\pi}{b} \tilde{\theta}_{x,n}^{(1)}, \\ F_{11,n} &= \frac{n\pi}{b} N_{xy,n}^{(2)} - q_{c,n}, \quad F_{12,n} = -\frac{n\pi}{b} N_{y,n}^{(2)} - t_{c,n}, \quad F_{13,n} = \frac{n\pi}{b} Q_{y,n}^{(2)} - p_{c,n}, \\ F_{14,n} &= Q_{x,n}^{(2)} + \frac{n\pi}{b} M_{xy,n}^{(2)} - \frac{h_2}{2} q_{c,n}, \quad F_{15,n} = Q_{y,n}^{(2)} - \frac{n\pi}{b} M_{y,n}^{(2)} - \frac{h_2}{2} t_{c,n}, \\ F_{16,n} &= \frac{N_{x,n}^{(2)} - v_{21}^{(2)} N_{y,n}^{(2)}}{E_1^{(2)} h_2} - \frac{v_{13}^{(2)}}{2E_3^{(2)}} \left( p_{c,n} + \frac{h_2}{6} r_{c,n} \right), \quad F_{17,n} = \frac{N_{xy,n}^{(2)}}{G_{12}^{(2)} h_2} - \frac{n\pi}{b} u_{av,n}^{(2)}, \end{aligned}$$

$$\begin{aligned}
F_{18,n} &= -\tilde{\theta}_{x,n}^{(2)} + \frac{6}{5G_{23}^{(2)}h_2} \left( Q_{x,n}^{(2)} - \frac{h_2}{12} q_{c,n} \right) \\
&- \frac{1}{5E_3^{(2)}h_2} \left[ \left( v_{13}^{(2)} + v_{12}^{(2)}v_{23}^{(2)} \right) F_{14,n} - \frac{E_2^{(2)}v_{23}^{(2)}h_2^3}{12} \cdot \frac{n\pi}{b} F_{20,n} - \left( 1 - v_{23}^{(2)}v_{32}^{(2)} \right) \left( p'_{c,n} + \frac{h_2}{12} s_{c,n} \right) \frac{h_2^2}{10} \right], \\
F_{19,n} &= \frac{12 \left( M_{x,n}^{(2)} - v_{21}^{(2)} M_{y,n}^{(2)} \right)}{E_1^{(2)}h_2^3} - \frac{6}{5} \cdot \frac{v_{13}^{(2)}}{E_3^{(2)}h_2} \left( p_{c,n} + \frac{h_2}{12} r_{c,n} \right), \quad F_{20,n} = \frac{12M_{xy,n}^{(2)}}{G_{12}^{(2)}h_2^3} - \frac{n\pi}{b} \tilde{\theta}_{x,n}^{(2)}, \\
F_{21,n} &= r_{c,n} + \frac{n\pi}{b} t_{c,n}, \quad F_{22,n} = s_{c,n},
\end{aligned}$$

where

$$\begin{aligned}
N_{y,n}^{(1)} &= v_{12}^{(1)} N_{x,n}^{(1)} - E_2^{(1)} h_1 \frac{n\pi}{b} v_{av,n}^{(1)} + \frac{v_{32}^{(1)} h_1}{2} \left( p_{0,n} + p_{c,n} - \frac{h_1}{6} r_{c,n} \right), \\
M_{y,n}^{(1)} &= v_{12}^{(1)} M_{x,n}^{(1)} - \frac{E_2^{(1)} h_1^3}{12} \cdot \frac{n\pi}{b} \tilde{\theta}_{y,n}^{(1)} + v_{32}^{(1)} \left( p_{0,n} - p_{c,n} + \frac{h_1}{12} r_{c,n} \right) \frac{h_1^2}{10}, \\
Q_{y,n}^{(1)} &= \frac{5}{6} G_{23}^{(1)} h_1 \left[ \frac{n\pi}{b} w_{av,n}^{(1)} + \left( 1 - \left( \frac{n\pi}{b} \right)^2 \frac{v_{32}^{(1)} h_1^2}{60} \right) \tilde{\theta}_{y,n}^{(1)} \right] + \frac{h_1}{12} t_{c,n} \\
&+ \frac{G_{23}^{(1)}}{6E_3^{(1)}} \cdot \frac{n\pi}{b} \left[ \left( v_{13}^{(1)} + v_{12}^{(1)}v_{23}^{(1)} \right) M_{x,n}^{(1)} - \left( 1 - v_{23}^{(1)}v_{32}^{(1)} \right) \left( p_{0,n} - p_{c,n} + \frac{h_1}{12} r_{c,n} \right) \frac{h_1^2}{10} \right], \\
N_{y,n}^{(2)} &= v_{12}^{(2)} N_{x,n}^{(2)} - E_2^{(2)} h_2 \frac{n\pi}{b} v_{av,n}^{(2)} + \frac{v_{32}^{(2)} h_2}{2} \left( p_{c,n} + \frac{h_2}{6} r_{c,n} \right), \\
M_{y,n}^{(2)} &= v_{12}^{(2)} M_{x,n}^{(2)} - \frac{E_2^{(2)} h_2^3}{12} \cdot \frac{n\pi}{b} \tilde{\theta}_{y,n}^{(2)} + v_{32}^{(2)} \left( p_{c,n} + \frac{h_2}{12} r_{c,n} \right) \frac{h_2^2}{10}, \\
Q_{y,n}^{(2)} &= \frac{5}{6} G_{23}^{(2)} h_2 \left[ \frac{n\pi}{b} w_{av,n}^{(2)} + \left( 1 - \left( \frac{n\pi}{b} \right)^2 \frac{v_{32}^{(2)} h_2^2}{60} \right) \tilde{\theta}_{y,n}^{(2)} \right] + \frac{h_2}{12} t_{c,n} \\
&+ \frac{G_{23}^{(2)}}{6E_3^{(2)}} \cdot \frac{n\pi}{b} \left[ \left( v_{13}^{(2)} + v_{12}^{(2)}v_{23}^{(2)} \right) M_{x,n}^{(2)} - \left( 1 - v_{23}^{(2)}v_{32}^{(2)} \right) \left( p_{c,n} + \frac{h_2}{12} r_{c,n} \right) \frac{h_2^2}{10} \right].
\end{aligned}$$

The resolving functions and the right-hand sides of the equations include rapidly varying functions, namely the interlaminar stresses and their derivatives, which sharply change in the boundary region near the tip of the crack, therefore, the system is “stiff.” Integration of such systems requires the application of special methods taking into account the exponential character of damping of the required solution on a segment considerably smaller than the entire length of the interval of integration.

Here, the boundary-value problem for system (18) at each value of  $n$  is solved by the stable Godunov–Grigorenko numerical method [6-8], which allows one to derive an accurate numerical solution of the problem. As a result, we have a full

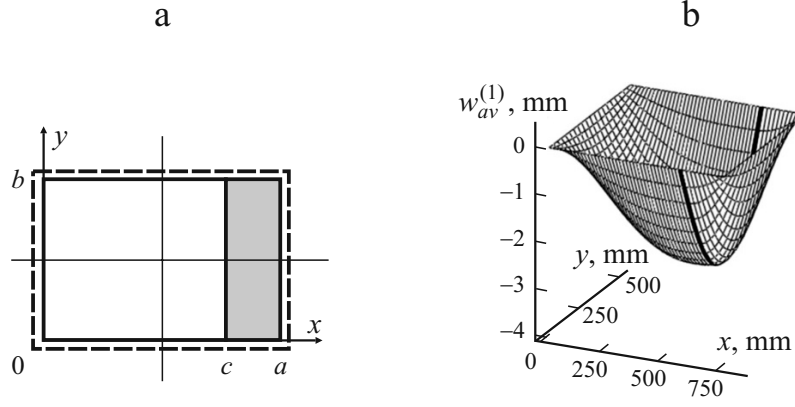


Fig. 3. Hinge-supported plate with delamination: dimensions (a) and deflections (b).

complex of parameters describing the SSS of the plate, including distributions of the stresses  $p_c, q_c$ , and  $t_c$  on the contact planes of layers.

The expansion into trigonometric series of form (17) corresponds to solutions symmetric about the midline  $y = b/2$ . To derive an antisymmetric solution, we use the expansion

$$f(x, y) = \sum_{m=1}^{NH} f_m(x) \bar{\cos}\left(\frac{m\pi y}{b}\right), \quad (19)$$

$$g(x, y) = \sum_{m=1}^{NH} g_m(x) \bar{\sin}\left(\frac{m\pi y}{b}\right) \quad (m = 1, 3, \dots, NH \text{ is odd}),$$

where sets of the functions  $f$  and  $g$ , as before, are determined by expressions (17). Such an expansion allows one to satisfy the boundary conditions  $N_{xy} = M_{xy} = Q_y = 0$  on the edges  $y = 0, b$  at the additional restrictions that these edges do not move in the plate plane and do not rotate around the contour line. In this case, all the relationships and the resolving system of equations necessary for solving the problem are obtained from the expressions given earlier if the subscript  $n$  in designations is replaced by  $m$  and  $n$  in formulas is replaced by  $-m$ .

#### 4. Two-Layer Plate with a Delamination

*Bending.* Let us consider a two-layer plate with dimensions  $a = 750$  mm,  $b = 500$  mm, and  $c = 3a/4$ , hinge-supported on the contour and having a partial delamination on the midplane (the dark zone in Fig. 3a).

The plate is loaded by a uniform pressure  $p_0 = -0.01$  MPa, which is presented in the series form

$$p(y) = \sum_{n=1}^{NH} \frac{4p_0}{n\pi} \sin\left(\frac{n\pi y}{b}\right) \quad (n = 1, 3, \dots, NH \text{ is odd}).$$

The upper layer 1:  $h_1 = 5$  mm,  $E_1^{(1)} = 20$  GPa,  $E_2^{(1)} = 30$  GPa,  $E_3^{(1)} = 7$  GPa,  $\nu_{13}^{(1)} = \nu_{23}^{(1)} = 0.07$ ,  $\nu_{12}^{(1)} = 0.24$ ,  $G_{12}^{(1)} = 10$  GPa,  $G_{13}^{(1)} = 4$  GPa, and  $G_{23}^{(1)} = 3$  GPa; the lower layer 2 represents the upper layer 1 rotated through  $90^\circ$ .

In the zone  $0 \leq x \leq c$  and  $0 \leq y \leq b$ , where the layers are fastened together, displacements on the contact plane are equal; in the delamination zone  $c \leq x \leq a$  and  $0 \leq y \leq b$ , we assume that the normal displacements of layers are equal and the layers slide without friction:  $p_c \leq 0$  and  $q_c = t_c = 0$ .

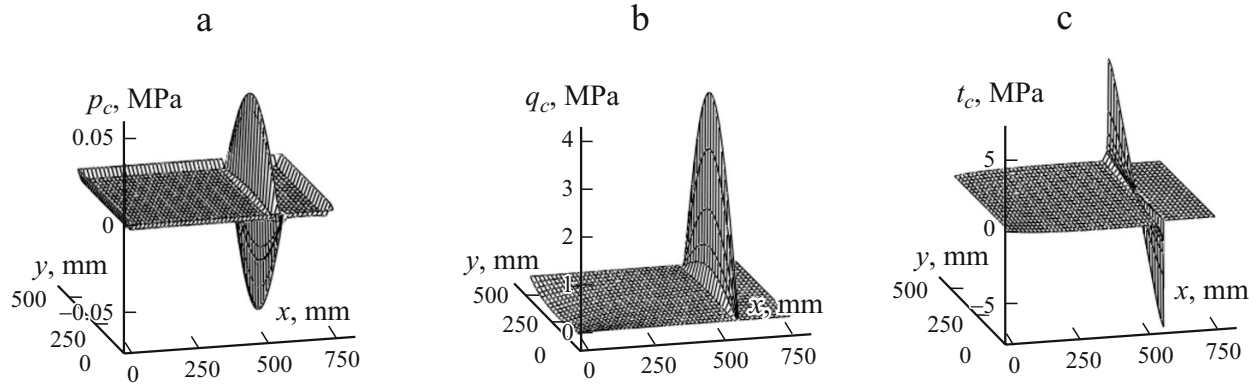


Fig. 4. Normal stress  $p_c$  (a) and tangential stresses  $q_c$  (b) and  $t_c$  (c).

The boundary conditions for the basic resolving functions are

$$x = 0, a: N_{x,n}^{(i)} = 0, M_{x,n}^{(i)} = 0, v_{av,n}^{(i)} = 0, w_{av,n}^{(i)} = 0, \tilde{\theta}_{y,n}^{(i)} = 0 \left( \theta_{y,av}^{(i)} = 0 \right) \quad (i = 1, 2).$$

We assume that, on the left support, the tangential stresses across the thickness of the plate are distributed parabolically. Then, the boundary conditions for the interlaminar tangential stress in the middle of plate thickness have the form

$$x = 0: q_{c,n} = \frac{3(Q_{x,n}^{(1)} + Q_{x,n}^{(2)})}{2(h_1 + h_2)}; \quad x = a: q_{c,n} = 0.$$

On the contact line  $x = c$  between the zones, all basic resolving functions are equal, except for the functions  $\tilde{\theta}_x$  and  $\tilde{\theta}_y$ , which are discontinuous. From the conditions of equality of the average rotation angles  $\theta_{x,av}$  and  $\theta_{y,av}$ , upon transition from the fastened section to the section with delamination, we find the jumps of the quantities  $\theta_{x,n}$  and  $\theta_{y,n}$

$$\Delta \tilde{\theta}_{x,n}^{(i)} = - \frac{1 - \nu_{23}^{(i)} \nu_{32}^{(i)}}{1 + \frac{\nu_{32}^{(i)} h_i^2}{60} \left( \frac{n\pi}{b} \right)^2} \cdot \frac{h_i^2 s_{c,n}^{(i)}}{600 E_3^{(i)}},$$

$$\Delta \tilde{\theta}_{y,n}^{(i)} = \left[ 1 - \frac{\nu_{32}^{(i)} h_i^2}{60} \left( \frac{n\pi}{b} \right)^2 \right]^{-1} \left[ \frac{t_{c,n}}{10 G_{23}^{(i)}} - \left( 1 - \nu_{23}^{(i)} \nu_{32}^{(i)} \right) \frac{n\pi}{b} \cdot \frac{h_i^2 r_{c,n}^{(i)}}{600 E_3^{(i)}} \right].$$

The solution was obtained with the use of expansions (17) with 20 harmonics ( $NH = 39$ ) retained in the series.

Figure 3b shows the average transverse displacements of layer 1 of the plate. Figure 4 depicts the three-dimensional distributions of the stresses  $p_c$ ,  $q_c$ , and  $t_c$  on the contact planes of layers.

Figure 5 demonstrates distributions of these stresses along the lines of their maximum values. As seen from the data of Figs. 4a and 5a, the assumption that  $p_c \leq 0$  in the delamination zone is fulfilled, and the layers deform without formation of a gap.

All other required functions can be presented in the same way. As an example, distributions of the running force  $N_x^{(1)}$  and moment  $M_x^{(1)}$  are shown for layer 1 in Fig. 6.

*Torsion.* The above-mentioned plate is twisted by transverse forces (Fig. 7a) linearly distributed in each layer at the edges  $x = 0, a$ :

$$Q_x^{(i)}(y) = \frac{2Q_0}{b} \left( y - \frac{b}{2} \right), \quad Q_x^{(i)}(y) = \sum_{m=1}^{NH} Q_{0,m} \cos \left( \frac{m\pi y}{b} \right), \quad Q_{0,m} = - \frac{8Q_0}{m^2 \pi^2}$$

( $m = 1, 3, \dots, NH$  — is odd).

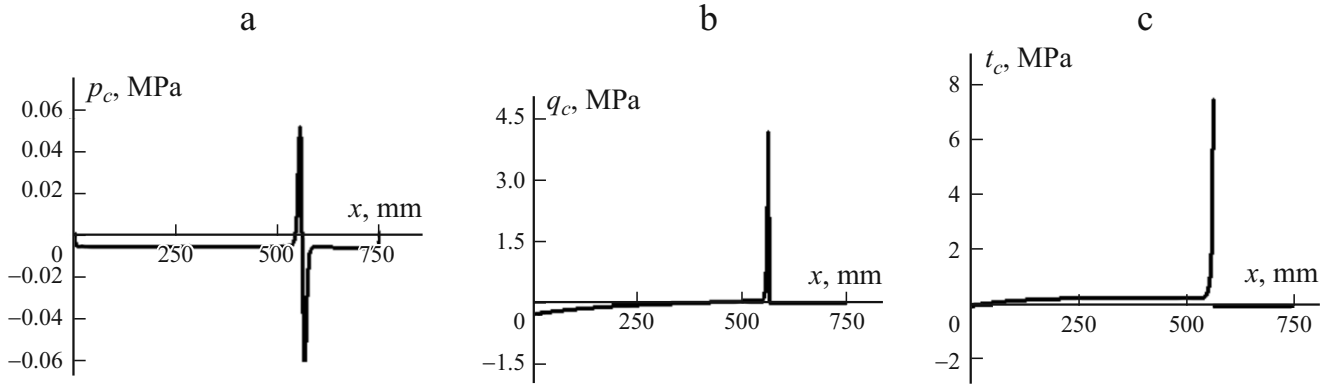


Fig. 5. Stresses:  $p_c$  — on the line  $y = b/2$  (a),  $q_c$  — on the line  $y = b/2$  (b), and  $t_c$  — on the line  $y = b$  (c).

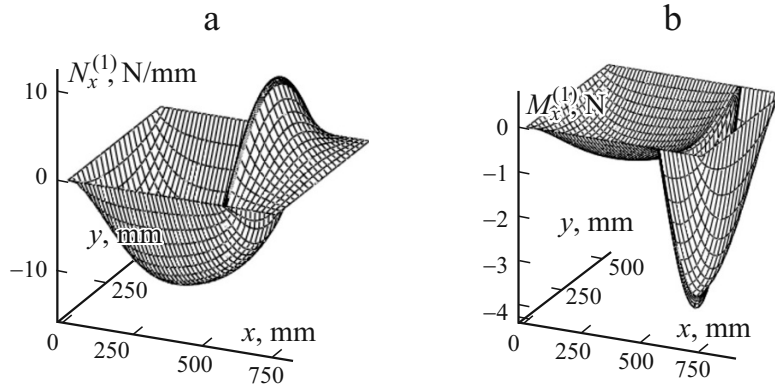


Fig. 6. The force  $N_x^{(1)}$  (a) and moment  $M_x^{(1)}$  (b) per unit length.

In the fastened zone  $0 \leq x \leq c$  and  $0 \leq y \leq b$ , the relations  $u^{(1)} = u^{(2)}$ ,  $v^{(1)} = v^{(2)}$ , and  $w^{(1)} = w^{(2)}$  are valid on the contact plane; in the delamination zone  $c \leq x \leq a$  and  $0 \leq y \leq b$ , the layers slide without friction,  $q_c = t_c = 0$ ; we assume that  $p_c = 0$ . Calculations showed that, under such loading conditions, the maximum normal stress  $p_{c,\max}$  was by an order of magnitude smaller than the tangential stress  $q_{c,\max}$  and by two orders of magnitude smaller than the stress  $t_{c,\max}$ .

The boundary conditions are

$$x = 0, a: N_{xy,m}^{(i)} = 0, Q_{x,m}^{(i)} = Q_{0,m}, M_{xy,m}^{(i)} = 0, u_{av,m}^{(i)} = 0, \tilde{\theta}_{x,m}^{(i)} = 0 \left( \theta_{x,av}^{(i)} = 0 \right) \quad (i = 1, 2);$$

$$x = 0: q_{c,m} = \frac{3(Q_{x,m}^{(1)} + Q_{x,m}^{(2)})}{2(h_1 + h_2)}; \quad x = a: q_{c,m} = 0.$$

The kinematic conditions given here at the edges  $x = 0, a$  are similar to those at the edges  $y = 0, b$  resulting from expansions (19).

The jumps of the functions  $\tilde{\theta}_x$  and  $\tilde{\theta}_y$  on the line  $x = c$  caused by the discontinuities of contact stresses and their derivatives are determined from the conditions of equality of the average rotation angles  $\theta_{x,av}$  and  $\theta_{y,av}$  on this line.



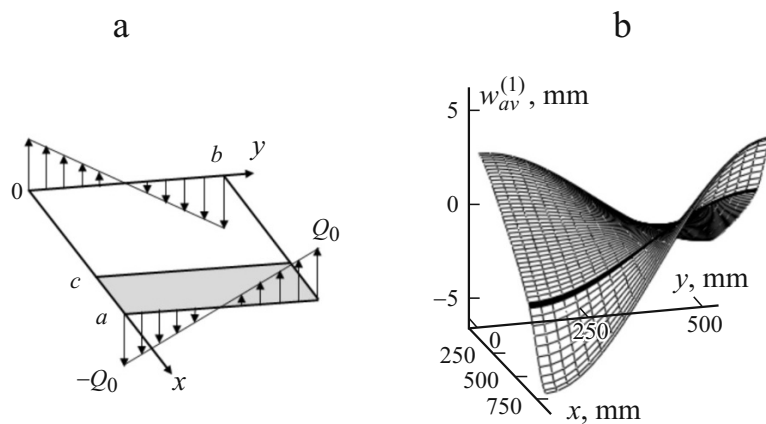


Fig. 7. Torsion of a plate with a delamination: load (a) and deflections (b).

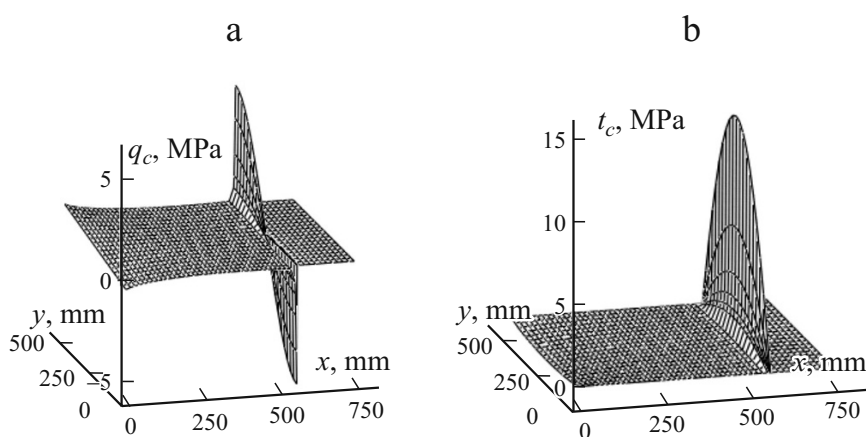


Fig. 8. Tangential stresses  $q_c$  (a) and  $t_c$  (b).

The solution is obtained with retention of 20 harmonics in expansions (19). Figure 7b shows the deflections of layer 1. Figure 8 illustrates the three-dimensional distributions of the stresses  $q_c$  and  $t_c$  on contact planes of the layers.

## Conclusions

The particular cases of boundary conditions considered on two opposite edges of the plate stem from the use of trigonometrical Fourier series for reducing the two-dimensional boundary-value problem to a one-dimensional one. Another way to decrease the dimensionality of the problem, based on the application of spline approximation and allowing one to satisfy a wider class of boundary conditions, is discussed in review [9].

The approach to solution of the problem developed here can be extended to plates with multiple delaminations and to the cases where the delamination is local and does not occupy the entire width of the plate or the plate has a local cut in one or all its layers.

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