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PREDICTION OF THE ELASTIC MODULI OF COMPOSITES WITH ISOLATED INCLUSIONS BY THE METHOD OF EFFECTIVE VOLUMES OF AVERAGING

A. F. Fedotov*

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A method for calculating the macroscopic elastic moduli of isotropic composites with isolated inclusions is proposed. The distinctive feature of the method is calculation of the concentration factors of average strains and stresses by using the effective volumes of averaging of phases. The effective volumes are found by solving the boundary-value problem of elastic deformation of the representative cell of a two-phase composite. In this case, the limiting version of a conventionally porous composite with zero material constants of inclusions is taken into account. A good agreement between calculation results and experimental data is obtained for various combinations of the elastic moduli and volume fraction of inclusions.

Introduction

Different methods have been suggested for calculating the elastic moduli of composites (see, for example, [1-5]). At the same time, the questions on prediction of the effective properties of composites with arbitrary volume content and greatly differing elastic moduli of their components remain unresolved in many respects. The asymptotic averaging method [3] makes it possible to correctly describe the effective properties of composites at any distinction between the properties of its components and any geometry of inclusions. However, a certain "payment for accuracy" is the complexity of the mathematical technique and constructions to be performed [3], as well as the fact that the method is accessible only to a small number of experts. The numerical method of finite elements allows one to create a three-dimensional representative cell reflecting the actual heterogeneous structure and accurately describing the mechanical properties of a composite. However, in creation of a

Samara State Technical University, Russia
*Corresponding author; e-mail: a.fedotov50@mail.ru

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three-dimensional structural model and its discretization, a complex software has to be developed, or a commercial software, for example, ANSYS, should be used. In this connection, more simple and accessible methods of continuum mechanics of composite materials remain to be in demand.

The problem of predicting the elastic properties is also important for porous materials, which are biphase composites with zero material constants of one of their phases. Porous materials have the highest possible distinction between phase characteristics, and it is the pores that affect properties of the materials to the greatest extent. In this connection, of attention is the approach where, in constructing the model of elastic deformation of a composite, the case of zero material constants of one of its phases is taken into account beforehand. In [6], based on the model of elastic deformation of porous materials, a method for calculating the effective elastic moduli of granular composites is developed. A particular feature of this method is calculation of the concentration factors of average strains in terms of effective volumes of averaging of material phases. In approximation of a plane interface between phases, analytical relationships for calculating the macroscopic shear and bulk moduli are obtained. Calculation results agree well with experimental data. The variant of composites with rigid inclusions is opposite to that of porous materials. However, the model proposed in [6] does not allow one to describe the properties of composites with undeformable inclusions. In the present study, the method of effective volumes of averaging is generalized for the case of a curvilinear interface and an arbitrary combination of elastic moduli of the phases of composites with isolated inclusions.

1. Method of Effective Volumes of Averaging of Strains and Stresses

Following the technique described in [7], let us derive the basic relations for calculating the effective elastic moduli of a biphase composite. In this case, we will assume that each phase and the composite as a whole are homogeneous.

The tensor of effective elastic moduli C_{ijmn} is determined according to the generalized Hooke's law [1, 7]

$$\left\langle \sigma_{ij} \right\rangle_V = C_{ijmn} \left\langle \varepsilon_{ij} \right\rangle_V ,$$

where $\left\langle \sigma_{ij} \right\rangle_V$ and $\left\langle \varepsilon_{ij} \right\rangle_V$ are the macroscopic average stresses and strains calculated by averaging the microscopic stresses σ'_{ij} and strains ε'_{ij} over the volume V of the representative cell of the composite:

$$\left\langle \sigma_{ij} \right\rangle_V = \frac{1}{V} \int_V \sigma'_{ij} dV, \quad \left\langle \varepsilon_{ij} \right\rangle_V = \frac{1}{V} \int_V \varepsilon'_{ij} dV. \tag{1}$$

The tensor C_{ijmn} of a biphase composite is determined through the concentration factors $K^k_{\varepsilon\, ijmn}$ of the average strain of a kth phase [7] as

$$C_{ijmn} = c_1 C^1_{ijkl}\, K^1_{\varepsilon\, klmn} + c_2 C^2_{ijkl}\, K^2_{\varepsilon\, klmn},$$

where C^k_{ijmn} is the tensor of elastic moduli of this phase. Hereinafter, the super- or subscript k indicates different phases (k = 1, 2). The concentration factors $K^k_{\varepsilon\, ijmn}$ are connected with the average strains $\left\langle \varepsilon_{ij} \right\rangle_{V_k}$ in the phases and the macroscopic strains $\left\langle \varepsilon_{ij} \right\rangle_V$ of the composite by the relation

$$\left\langle \varepsilon_{ij} \right\rangle_{V_k} = K^k_{\varepsilon\, ijmn} \left\langle \varepsilon_{mn} \right\rangle_V . \tag{2}$$

Similar to the tensor of elastic moduli C_{ijmn} (3), the tensor of elastic compliances S_{ijmn} of the composite is expressed as

$$S_{ijmn} = c_1 S^1_{ijkl}\, K^1_{\sigma\, klmn} + c_2 S^2_{ijkl}\, K^2_{\sigma\, klmn} .$$

Here, $K^k_{\sigma\, ijmn}$ are the concentration factors of average stresses in phases:

$$\left\langle \sigma_{ij} \right\rangle_{V_k} = K^k_{\sigma\, ijmn} \left\langle \sigma_{mn} \right\rangle_V . \tag{3}$$

The elastic properties of isotropic materials are characterized by two independent constants. As the base ones, we assume Young's modulus E and the Poisson ratio. Accordingly, the uniaxial tension of the composite will be considered.

The effective Young's modulus E of an isotropic composite consisting of isotropic components is found in the form

$$E = c_1 E_1 K_{\varepsilon 1} + c_2 E_2 K_{\varepsilon 2} , \tag{4a}$$

$$\frac{1}{E} = \frac{c_1 K_{\sigma 1}}{E_1} + \frac{c_2 K_{\sigma 2}}{E_2} , \tag{4b}$$

where E_1 and E_2 are the Young's moduli of phases; $K_{\varepsilon 1}$ and $K_{\varepsilon 2}$ ($K_{\sigma 1}$ and $K_{\sigma 2}$) are concentration factors of the average strains (stresses) of uniaxial tension ε_{11} (σ_{11}) , which, according to Eqs. (2) and (3), are

$$K_{\varepsilon k} = \frac{\left\langle \varepsilon_{11} \right\rangle_{V_k}}{\left\langle \varepsilon_{11} \right\rangle_V} , \quad K_{\sigma k} = \frac{\left\langle \sigma_{11} \right\rangle_{V_k}}{\left\langle \sigma_{11} \right\rangle_V} . \tag{5}$$

Here, $\left\langle \varepsilon_{11} \right\rangle_{V_k}$ and $\left\langle \sigma_{11} \right\rangle_{V_k}$ are the average tensile strains and stresses in the phase volumes V_k; $\left\langle \varepsilon_{11} \right\rangle_V$ and $\left\langle \sigma_{11} \right\rangle_V$ are the average tensile strains and stresses in the composite volume V:

$$\left\langle \varepsilon_{11} \right\rangle_{V_k} = \frac{1}{V_k} \int_{V_k} \varepsilon'_{11} \, dV , \; \left\langle \varepsilon_{11} \right\rangle_V = \frac{1}{V} \int_V \varepsilon'_{11} \, dV , \; \left\langle \sigma_{11} \right\rangle_{V_k} = \frac{1}{V_k} \int_{V_k} \sigma'_{11} \, dV , \; \left\langle \sigma_{11} \right\rangle_V = \frac{1}{V} \int_V \sigma'_{11} \, dV . \tag{6}$$

The concentration factors satisfy the relations [7]

$$c_1 K_{\varepsilon 1} + c_2 K_{\varepsilon 2} = 1, \quad c_1 K_{\sigma 1} + c_2 K_{\sigma 2} = 1. \tag{7}$$

The concentration factors of average tensile strains $K_{\varepsilon k}$ are found as follows. Each phase in the average strain of a composite has its own effective fraction and a corresponding effective volume. As follows from the condition of uniqueness of the total strain in phase volume, the total average tensile strain $\left\langle \varepsilon_{11} \right\rangle_V$ of the composite in terms of the effective volumes of averaging of phases, $V_{\alpha k}$, will be equal to the sum of the average tensile strains $\left\langle \varepsilon_{11}^k \right\rangle_k$ in the phase volumes V_k:

$$\left\langle \varepsilon_{11} \right\rangle_V V_{\alpha k} = \left\langle \varepsilon_{11}^k \right\rangle_{V_k} V_k . \tag{8}$$

From relation (8), we have

$$\left\langle \varepsilon_{11}^k \right\rangle_{V_k} = \frac{V_{\alpha k}}{V_k} \left\langle \varepsilon_{11} \right\rangle_V = \frac{\alpha_{\varepsilon k}}{c_k} \left\langle \varepsilon_{11} \right\rangle_V , \tag{9}$$

where $\alpha_{\varepsilon k} = V_{\alpha k} / V$ is the volume fraction of the effective volume of averaging of strains of a kth component. From comparison of relations (5) and (9), it follows that the concentration factors $K_{\varepsilon k}$ are

$$K_{\varepsilon k} = \frac{\alpha_{\varepsilon k}}{c_k} . \tag{10}$$

From Eq. (9), with account of Eqs. (1) and (6), we obtain

$$\alpha_{\varepsilon k} = \frac{V_{\alpha k}}{V} = \int_{V_k} \varepsilon'_{11} \, dV \Big/ \int_V \varepsilon'_{11} \, dV . \tag{11}$$

Thus, according to Eq. (11), the fractions of effective volumes of averaging of strains quantitatively present the ratio of total tensile strain in the volume of a respective component to that in the volume of a composite.

Inserting Eq. (10) into Eq. (4a) yields

$$E = \alpha_{\varepsilon 1} E_1 + \alpha_{\varepsilon 2} E_2 . \tag{12}$$

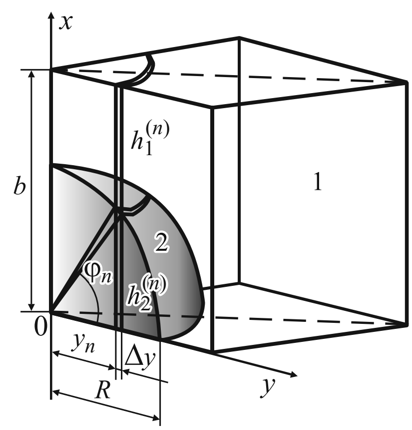


Fig. 1. Calculated model of the representative cell of a composite with isolated inclusions.

With account of Eq. (10), the first relation of (7) takes the form

$$\alpha_{\varepsilon 1} + \alpha_{\varepsilon 2} = 1 . \tag{13}$$

In estimating the effective Young's modulus using the concentration factors $K_{\sigma k}$ of average tensile stresses, the volume fractions $\alpha_{\sigma k}$ of effective volumes of averaging of the stresses are determined as follows:

$$\alpha_{\sigma k} = \int_{V_k} \sigma'_{11}\, dV \Big/ \int_{V} \sigma'_{11}\, dV .$$

For the concentration factors of average tensile stresses $K_{\sigma k}$, we have

$$K_{\sigma k} = \frac{\alpha_{\sigma k}}{c_k} . \tag{14}$$

With account of Eq. (14), expression (4b) for the effective Young's modulus can be written as

$$\frac{1}{E} = \frac{\alpha_{\sigma 1}}{E_1} + \frac{\alpha_{\sigma 2}}{E_2} , \tag{15}$$

and the second relation of (7) takes the form

$$\alpha_{\sigma 1} + \alpha_{\sigma 2} = 1 .$$

Relation (12), in its structure, corresponds to the known Voigt relation, while relations (15) — to the Reuss equation. Contrary to the Voigt and Reuss models, in the model suggested here, instead of the volume fractions of phases, the fractions of effective volumes of averaging are employed.

2. Elastic Characteristics of Biphase Composites

2.1. Effective volumes of averaging. The functional relationships for the effective volumes of averaging $\alpha_{\varepsilon k}$ and $\alpha_{\sigma k}$ as functions of the elastic moduli of phases and their content can be found from the solution of the boundary-value problem of elastic deformation of the representative cell of the biphase composite. The representative cell of an isotropic composite with isolated inclusions is taken in the form of a cube with a side b and a spherical inclusion of radius R. Due to symmetry, we consider only 1/8 of the cell (Fig. 1).

On the upper face of the cell, a tensile strain $\varepsilon_x = \varepsilon_{11}$ is assigned. According to [1], we assume that the strain state on cell faces is homogeneous, but the stress-strain state (SSS) inside the cell is inhomogeneous. In the case of a homogeneous strain state, faces of the cell and its structural elements remain mutually parallel. Then, with the assumptions accepted, it is reasonable to use the hypothesis of plane sections.

Under the condition of homogeneity, the total tensile macrostrain $\varepsilon_{x\Sigma}$ in the representative cell of volume $V = b^3$ is

$$\varepsilon_{x\Sigma} = \int_V \varepsilon_x \, dV = \varepsilon_x b^3 . \tag{16}$$

Within the framework of the hypothesis of plane sections, the volume V_{01} of the matrix around an inclusion will deform homogeneously, and the tensile strain in this volume will be $\varepsilon_{x1}^0 = \varepsilon_x$. The total tensile strain in the volume V_{01} is

$$\varepsilon_{x1\Sigma}^0 = \varepsilon_x V_{01} = \varepsilon_x \left(b^3 - \frac{\pi}{4} R^2 b \right). \tag{17}$$

Deformation of the central region of the cell, including the interface, is homogeneous. The summation of strains in the interface region is reduced to calculation of the integrals [6, 8]

$$I = \int \frac{dx}{a + f(x)}, \quad I = \int \frac{f(x)\, dx}{a + f(x)}, \tag{18}$$

where $f(x)$ is a function depending on the form of the interface. In the case of a plane interface, integrals (18) are calculated in quadratures [6]. If the interface is curvilinear, the relations for the strains are nonlinear, which makes it impossible to calculate integrals (18). Therefore, instead of the integral sums, we will find arithmetic ones.

The central region of the representative cell with an interphase is divided into N parallel-connected cylindrical cells of thickness Δy. In turn, each cylindrical cell consists of serially connected biphase elements with a variable concentration of phases. Independently of each other, the cells are subjected to an assigned macroscopic tensile strain ε_x. In view of the digitization assumed, we come to the following dependences:

— for the angle φ_n determining the position of an nth cell,

$$\varphi_n = \frac{\pi n}{2N} \quad n = 1, 2, \ldots, N ;$$

— for the thickness of a cylindrical cell,

$$\Delta y = \frac{\pi R}{2N} \sin \varphi_n ;$$

— for the concentration of phases in an nth cell,

$$c_1^{(n)} = \frac{h_1^{(n)}}{b} = 1 - \frac{R}{b} \sin \varphi_n , \; c_2^{(n)} = \frac{h_2^{(n)}}{b} = \frac{R}{b} \sin \varphi_n ; \tag{19}$$

— for the volumes of phases in this cell,

$$V_1^{(n)} = \frac{\pi^2}{8N} R^2 b\, c_1^{(n)} \sin 2\varphi_n \; V_2^{(n)} = \frac{\pi^2}{8N} R^2 b\, c_2^{(n)} \sin 2\varphi_n .$$

The total strains in the biphase region of the representative cell for the matrix are

$$\varepsilon_{x1\Sigma} = \sum_{n=1}^{N} \varepsilon_{x1}^{(n)} V_1^{(n)} = \frac{\pi^2 R^2 b}{8N} \sum_{n=1}^{N} \varepsilon_{x1}^{(n)} c_1^{(n)} \sin 2\varphi_n \tag{20}$$

and for the inclusion —

$$\varepsilon_{x2\Sigma} = \sum_{n=1}^{N} \varepsilon_{x2}^{(n)} V_2^{(n)} = \frac{\pi^2 R^2 b}{8N} \sum_{n=1}^{N} \varepsilon_{x2}^{(n)} c_2^{(n)} \sin 2\varphi_n . \tag{21}$$

The fractions $\alpha_{\varepsilon k}$ of the effective volumes of averaging of strains, determined by expression (11) with account of Eqs. (16), (17), (20), and (21), are given by

$$\alpha_{\varepsilon 1} = \frac{\varepsilon_{1\Sigma}}{\varepsilon_{x\Sigma}} = \left(1 - \frac{\pi R^2}{4b^2}\right) + \frac{\pi^2}{8N\varepsilon_x}\frac{R^2}{b^2}\sum_{n=1}^{N}\varepsilon_{x1}^{(n)}c_1^{(n)}\sin 2\varphi_n ,$$
$$\alpha_{\varepsilon 2} = \frac{\varepsilon_{2\Sigma}}{\varepsilon_{x\Sigma}} = \frac{\pi^2}{8N\varepsilon_x}\frac{R^2}{b^2}\sum_{n=1}^{N}\varepsilon_{x2}^{(n)}c_2^{(n)}\sin 2\varphi_n . \tag{22}$$

The ratio of characteristic dimensions R/b of the cell can be found by using the model of elastic deformation of porous materials [9]. Let us assume that the inclusion is a pore. Then, Young's modulus $E_2 = 0$, and we have from Eq. (12) that

$$E = \alpha_0 E_0 , \tag{23}$$

where α_0 is the fraction of the effective volume of averaging of the solid phase with Young's modulus E_0. If the inclusion is a pore, then, in the case of series connection of cylindrical cells, the strain of matrix in the central region is equal to zero ($\varepsilon_{x1}^{(n)} = 0$), and the effective volume of averaging $\alpha_{\varepsilon 1}$ of the matrix will be equal to that of the solid phase, α_{01}, of a conditionally porous composite:

$$\alpha_{\varepsilon 1} = \alpha_{01} = 1 - \frac{\pi R^2}{4b^2} . \tag{24}$$

Using Eq. (24), from Eqs. (22), we obtain expressions for the effective volumes of averaging of strains of composite phases:

$$\alpha_{\varepsilon 1} = \alpha_{01} + A\sum_{n=1}^{N}\varepsilon_{x1}^{(n)}c_1^{(n)}\sin 2\varphi_n , \quad \alpha_{\varepsilon 2} = A\sum_{n=1}^{N}\varepsilon_{x2}^{(n)}c_2^{(n)}\sin 2\varphi_n ,$$

where $A = \dfrac{\pi(1-\alpha_{01})}{2\varepsilon_E N}$ is a coefficient depending on the number N of cylindrical cells accepted.

The ratio of characteristic dimensions R/b in relations (22) for phase concentration in an nth cell can be expressed in terms of the volume fraction of inclusions c_2. The volume of the body is proportional to the cube of its linear dimension, and we have for c_2

$$c_2 = \frac{V_2}{V} \sim \left(\frac{R}{b}\right)^3 .$$

Now, the ratio R/b takes the form

$$\frac{R}{b} = \sqrt[3]{k\, c_2} , \tag{25}$$

where k is a constant depending on the type of packing of spherical inclusions.

The structural model of the matrix composite considered is correct if inclusions are isolated and do not make contact with each other. The limit volume fraction of isolated inclusions c_2^* corresponds to the rise of contacts and the formation of a bonded packing. Upon contact of spheres, the ratio of inclusion radius R to the side b of the cubic cell becomes equal to unity: $R/b = 1$. Then, the volume fraction of inclusions c_2 is equal to the limit one, $c_2 = c_2^*$, and, from Eq. (25), we find the constant k

$$k = 1/c_2^* . \tag{26}$$

With account of this equality, we have

$$\frac{R}{b} = \sqrt[3]{\frac{c_2}{c_2^*}} .$$

The limit volume fraction of inclusions c_2^* depends on the character of packing of spheres. Ordered and disordered (statistical) packings are distinguished [10]. If composites are manufactured by mixing their components, the formation of statistically loose packings is most probable. Preliminary calculations have also shown that the best agreement with experiment gives the model of statistically loose packings. In this case, the limit volume fraction of inclusions $c_2^* = 0.601$ [10]. For the structure of a granular composite assumed here, we come to the following relations for calculating the concentration of components in an nth cell:

$$c_1^{(n)} = 1 - \sqrt[3]{1.66\,c_2}\,\sin\varphi_n\,, \quad c_2^{(n)} = \sqrt[3]{1.66\,c_2}\,\sin\varphi_n\,.$$

The effective volumes $\alpha_{\sigma k}$ of averaging of tensile stresses in phases are expressed as ratios of the sum of tensile stresses $\sigma_{xk\Sigma}$ in the volume of a kth phase to the sum of tensile stresses $\sigma_{x\Sigma}$ in the volume of the representative cell of the composite:

$$\alpha_{\sigma k} = \frac{\sigma_{xk\Sigma}}{\sigma_{x\Sigma}}\,.$$

Upon determination of the stresses $\sigma_{xk}^{(n)} = E_k\,\varepsilon_{xk}^{(n)}$ in the corresponding volumes, we find for the effective volumes of averaging of stresses

$$\alpha_{\sigma 1} = \frac{\alpha_{01}\,E_1 + A\sum_{n=1}^{N}\sigma_E^{(n)} c_1^{(n)}\sin 2\varphi_n}{\alpha_{01}\,E_1 + A\sum_{n=1}^{N}\sigma_E^{(n)}\sin 2\varphi_n}\,, \qquad \alpha_{\sigma 2} = 1 - \alpha_{\sigma 1}\,. \tag{27}$$

Relationships (27) were deduced using the condition of equality of stresses in phases for an nth extended cell

$$\sigma_{x1}^{(n)} = \sigma_{x2}^{(n)} = \sigma_x^{(n)}\,. \tag{28}$$

The effective Poisson ratio is equal to the ratio between the average transverse strain $\left\langle \varepsilon_y \right\rangle_V$ and the average longitudinal strain $\left\langle \varepsilon_x \right\rangle_V$ of the composite

$$\nu = -\frac{\left\langle \varepsilon_y \right\rangle_V}{\left\langle \varepsilon_x \right\rangle_V}\,.$$

After averaging, the longitudinal $\left\langle \varepsilon_x \right\rangle_V$ and transverse $\left\langle \varepsilon_y \right\rangle_V$ strains are distributed homogeneously in the volume of the composite. By definition, strains in the effective volumes of averaging of strains are equal to the corresponding average strains of the composite,

$$\varepsilon_{x1} = \varepsilon_{x2} = \left\langle \varepsilon_x \right\rangle_V\ \varepsilon_{y1} = \varepsilon_{y2} = \left\langle \varepsilon_y \right\rangle_V\,. \tag{29}$$

Taking into account Eqs. (29) and (13), the average strains can be expressed in terms of fractions of the effective volumes of averaging:

$$\left\langle \varepsilon_x \right\rangle_V = \varepsilon_{x1}\,\alpha_{\varepsilon 1} + \varepsilon_{x2}\alpha_{\varepsilon 2}\ \left\langle \varepsilon_y \right\rangle_V = \varepsilon_{y1}\,\alpha_{\varepsilon y1} + \varepsilon_{y2}\alpha_{\varepsilon y2}\,,$$

and, for the Poisson ratio, we have

$$\nu = -\frac{\varepsilon_{y1}\,\alpha_{\varepsilon y1} + \varepsilon_{y2}\,\alpha_{\varepsilon y2}}{\varepsilon_{x1}\,\alpha_{\varepsilon 1} + \varepsilon_{x2}\,\alpha_{\varepsilon 2}}\,, \tag{30}$$

where $\alpha_{\varepsilon y1}$ and $\alpha_{\varepsilon y2}$ are fractions of the effective volumes of averaging of transverse strains. In Eq. (30), the transverse strain ε_{ky} is related to the longitudinal strain ε_{kx} of a kth phase by the Poisson law: $\varepsilon_{ky} = -\nu_k\,\varepsilon_{kx}$. Expressing the quantity ε_{kx} in terms of ε_{ky} and performing some transformations with account of Eqs. (13) and (29), we come to the relationship for calculating the Poisson ratio of the composite with isolated inclusions

$$\nu = \frac{\nu_1\,\nu_2}{\nu_1\,\alpha_{\varepsilon 2} + \nu_2\,\alpha_{\varepsilon 1}}\,.$$

The strains of phases $\varepsilon_{xk}^{(n)}$ can be found from the solution of the elastic problem of uniaxial tension for each cylindrical cell at a given tensile strain ε_x. From the condition of equal stresses in components of an extended nth cell, $\sigma_{x1}^{(n)} = \sigma_{x2}^{(n)}$, and from the equation of relation between phase strains

$$\varepsilon_x = \varepsilon_{x1}^{(n)} c_1^{(n)} + \varepsilon_{x2}^{(n)} c_2^{(n)} ,$$

we find the required strains $\varepsilon_{xk}^{(n)}$

$$\varepsilon_{x1}^{(n)} = \frac{\varepsilon_x E_2}{c_1^{(n)} E_2 + c_2^{(n)} E_1} , \quad \varepsilon_{x2}^{(n)} = \frac{\varepsilon_x E_1}{c_1^{(n)} E_2 + c_2^{(n)} E_1} .$$

After calculation of the effective volumes of averaging of strains $\alpha_{\varepsilon k}$ or stresses $\alpha_{\sigma k}$ by using Eq. (12) or (15), the effective Young's modulus E of the composite is found. The accuracy of calculation according to the method suggested depends on the number of elementary cells N adopted. Investigations of convergence of the numerical solution showed that an increase in the discretization parameter N by a factor exceeding 100 practically did not affect the calculation results, therefore, we can assume that $N = 100$.

2.2. Effective volumes of averaging of porous materials. In the scientific literature, various relationships for calculating the elastic moduli of porous materials are presented. The shear μ and bulk K moduli are determined most frequently. By analogy with relation (23), for the shear modulus of a porous material, we have

$$\mu = \alpha_{s0} \mu_0 ,$$

where μ_0 is the shear modulus of the solid phase, and α_{s0} is the effective volume of averaging of the solid phase in shear. To a high accuracy, the elastic properties of powders and sintered porous materials are described by the modified Bal'shin relation [9]

$$\alpha_{s0} = \rho^n \frac{\rho - \rho_0}{1 - \rho_0} , \quad n = \frac{2 - \rho - \rho_0}{1 - \rho_0} , \tag{31}$$

where ρ is the relative density and ρ_0 is the initial (bulk) relative density of the powder.

Let us express the fraction of the effective volume of averaging in tension α_0 in terms of that in shear α_{s0}. For this purpose, we use the relationship for the macroscopic bulk modulus K of a porous material [9]

$$K = \frac{4}{3} \mu_0 \frac{(1+\nu_0)\alpha}{2(1-2\nu_0) + (1+\nu_0)(1-\alpha)}$$

and the relation among Young's, shear, and bulk moduli

$$E_0 = \frac{9K_0 \mu_0}{3K_0 + \mu_0} , \quad E = \frac{9K\mu}{3K + \mu} .$$

After simple transformations, we have

$$\alpha_0 = \frac{6\alpha_{s0}}{6 + (1+\nu_0)(1-\alpha_{s0})} ,$$

where ν_0 is the Poisson ratio of the solid phase of the porous body. For a composite, the role of the solid phase is played by the matrix.

In calculating the effective volumes of averaging $\alpha_{\varepsilon k}$ and $\alpha_{\sigma k}$, the function α_{01} was obtained by replacing the relative density ρ in Eq. (31) with the volume fraction c_1 of composite matrix. For a composite with isolated inclusions, we should take that $c_{10} = 0$ and write (31) in the form

$$\alpha_{s0} = c_1^{\,3-c_1} . \tag{32}$$

A good agreement with experimental data for the shear viscosity η of suspensions of spherical particles is shown by the semi-phenomenological Dougherty–Krieger formula [11]

$$\eta = \eta_0 \left(1 - \frac{c_2}{c_2^*} \right)^{-2.5 c_2^*} , \tag{33}$$

where η_0 is viscosity of the fluid. The shear viscosity of a porous liquid, by analogy with Eq. (23), will be $\eta = \alpha_{s0} \eta_0$.

TABLE 1. Elastic Properties of Components

Composite	Component	Young's modulus, GPa	Poisson ratio
$NiAl–Al_2O_3$	NiAl	$E_1 = 186$	$\nu_1 = 0.31$
	Al_2O_3	$E_2 = 401$	$\nu_2 = 0.24$
Al–SiC	Al	$E_1 = 70$	$\nu_1 = 0.34$
	SiC	$E_2 = 450$	$\nu_2 = 0.22$
Co–WC	Co	$E_1 = 207$	$\nu_1 = 0.31$
	WC	$E_2 = 700$	$\nu_2 = 0.19$
W–glass	W	$E_1 = 355$	$\nu_1 = 0.2$
	Glass	$E_2 = 81$	$\nu_2 = 0.24$

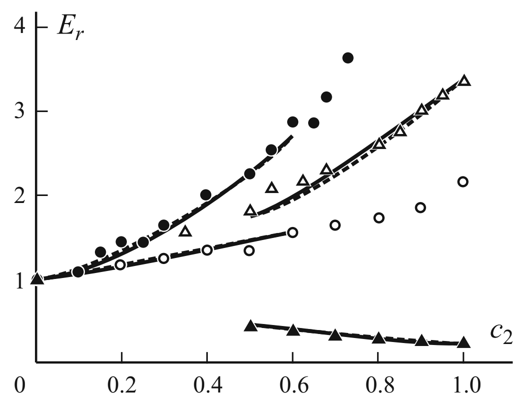


Fig. 2. Reduced Young's modulus of composites as a function of volume fraction of inclusions: (——) — calculation by Eq. (32) and (– – –) — calculation by Eq. (35); experimental points: ○ — $NiAl–Al_2O_3$ [12], ● — Al–SiC [12], Δ — Co–WC [12], and ▲ — W–glass [12].

Replacing the volume fraction of particles c_2 in Eq. (33) with the porosity $\theta = 1 - \rho$, we arrive at the relation for the fraction of effective volume of averaging in shear of a porous body

$$\alpha_{s0} = \left(\frac{\rho - \rho_0}{1 - \rho_0} \right)^{2.5(1-\rho_0)} . \tag{34}$$

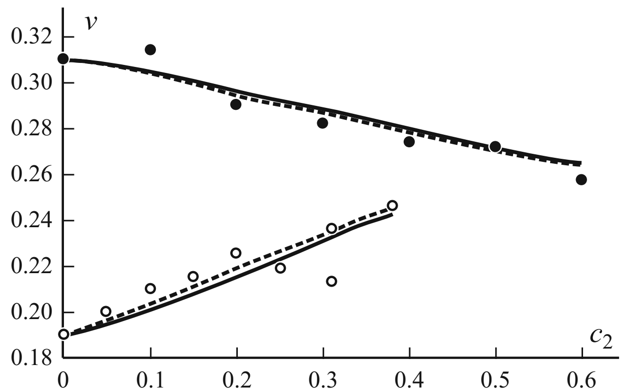
For a composite with isolated inclusions, in view of the relations $\rho = c_1$ and $\rho_0 = c_{10} = 0$, we have

$$\alpha_{s0} = c_1^{2.5} . \tag{35}$$

3. Experimental Verification

The verification of adequacy of the method suggested was carried out according to experimental data on the elastic properties of biphase composites, porous materials, and suspensions with undeformable inclusions. The compositions of biphase composites and the elastic properties of their components are presented in Table 1. The first component of the composition serves as a matrix and the second one as an inclusion. The elastic properties of the components are taken from [12].

The calculated Young's moduli of the composites reduced to that of the matrix, $E_r = E_2/E_1$, are shown in Fig. 2. The results obtained by the method suggested agree well with experimental data for all the composites considered. In this case, the calculation of the effective volume of averaging α_{01} by Eqs. (32) or (35) yielded practically identical results.



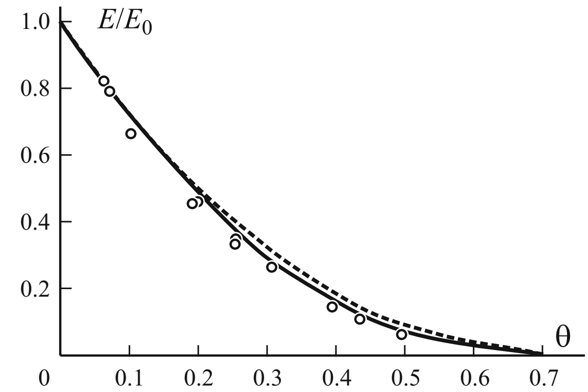


Fig. 3. Poisson ratio ν of composites as a function of volume fraction of inclusions c_2: (——) — calculation by Eq. (32) and (– – –) — by Eq. (35); ○ — WC–Co [13] and ● — NiAl–Al_2O_3 [14].

Fig. 4. Relative Young's modulus of powdered copper as a function of θ: (——) — calculation by Eq. (32), (– – –) — calculation by Eq. (35), and ○ — experiment [15].

At a content of inclusions exceeding the limit volume fraction, $c_2 > c_2^*$, the properties of composites should be described within the framework of the model with interpenetrating components.

The macroscopic Young's moduli of a composite can be determined in terms of the effective volumes of averaging of either strains $\alpha_{\varepsilon k}$ from Eq. (12) or stresses $\alpha_{\sigma k}$ from Eq. (15). Calculations showed that both variants give similar results. Since in uniaxial tension of cylindrical cells, along with the phase strains $\varepsilon_{xk}^{(n)}$, the phase stresses $\sigma_{xk}^{(n)}$ are also determined, the use of the effective volumes $\alpha_{\varepsilon k}$ or $\alpha_{\sigma k}$ is equivalent from the viewpoint of the algorithm and time of solution of the problem. However, it is preferable to use the effective volumes of averaging of strains, because in this case, the calculation relations are slightly simpler than in the case of stresses.

Figure 3 shows calculation and experimental data for the Poisson ratio of WC–Co and NiAl–Al_2O_3 composites at a different volume content of cobalt and alumina inclusions. As seen, the calculation results agree rather well with experimental data.

Thus, despite the approximate method of solution of the problem of elastic deformation of the representative cell, the calculation relations obtained make it possible to quite accurately describe the elastic properties of composites with isolated inclusions at different combinations of elastic moduli and arbitrary content of phases.

The maximum distinction between the elastic moduli of phases is realized in the case of porous materials, where one of phases (pores) has zero material constants. The Young's modulus of a porous material is determined by the elastic modulus E_0 of the solid phase and by the effective volume of averaging α_0 of the phase according to Eq. (23). Figure 4 presents calculation and experimental relations between the relative Young's modulus E/E_0 and the porosity $\theta = 1 - \rho$ for a pressed copper powder. The relative bulk density ρ_0 of the powdered copper $\rho_0 = 0.3$ [15]. The best agreement with experimental data was found in the case of calculation of the effective volume α_{s0} by using Eq. (31). The calculation of α_{s0} by relation (34) only slightly overestimated the value of Young's modulus.

Along with porous materials, the maximum distinction between the elastic moduli of components is typical of granular composites with undeformable rigid inclusions, for example, suspensions. Let component 2 be a rigid inclusion. Then, $E_2 = \infty$, and the macroscopic Young's modulus of the composite is calculated by using relation (15) and the effective volumes of averaging of stresses $\alpha_{\sigma k}$. At $E_2 = \infty$, the Young's modulus of the composite with rigid inclusions is

$$E = E_{\mathrm{m}} / \alpha_{\sigma 0} ,$$

where E_{m} is the Young's modulus of matrix and $\alpha_{\sigma 0}$ is the effective volume of averaging of stresses in the matrix of the composite.

In a porous material, the pore is deformed ($\varepsilon_{11}' \neq 0$), but does not resist the strain ($\sigma_{11}' = 0$), and the effective volume of averaging of stresses of the pore is $\alpha_{\sigma 0} = 0$. Since the elastic modulus of the pore $E_2 = 0$, the field of microstrains in the

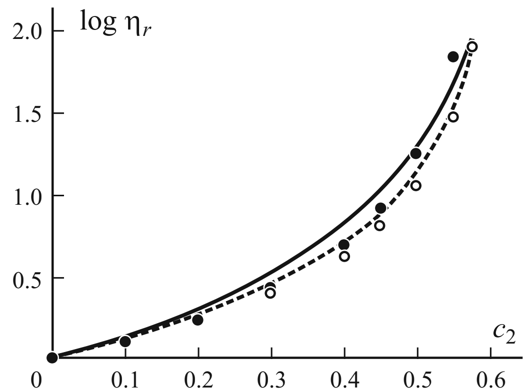


Fig. 5. Relative viscosity of suspensions $\log \eta_r$ as a function of volume fraction of inclusions c_2: (——) — calculation by Eq. (36) and (– – –) — calculation by Eq. (37); ○ and ● — experiment [11].

pore is not determined, and the macroscopic properties of the porous material are calculated by using the effective volume of averaging of deformations α_{s0} of the solid phase. A composite with rigid inclusions is deformed in a different way. The rigid inclusion is loaded ($\sigma'_{11} \neq 0$), but not deformed ($\varepsilon'_{11} = 0$), and the effective volume of averaging of strains of the inclusion is $\alpha_{s0} = 0$. Since the elastic modulus of inclusion $E_2 = \infty$, the field of microscopic stresses in the inclusion is not determined, and the macroscopic properties of the composite are found by using the effective volume of averaging of matrix stresses $\alpha_{\sigma 0}$. Let us assume that the effective volume of averaging of stresses of a composite with rigid inclusions $\alpha_{\sigma 0}$ is equal to that of strains of a porous material α_{s0}. Then, relations (31) and (34) for the volume α_{s0} with reference to the volume $\alpha_{\sigma 0}$ in shear can be written in terms of the volume fraction of inclusions c_2

$$\alpha_{\sigma 0} = (1 - c_2)^n \frac{c_2^* - c_2}{c_2^*}, \, n = \frac{c_2 + c_2^*}{c_2^*}, \tag{36}$$

$$\alpha_{\sigma 0} = \left(1 - \frac{c_2}{c_2^*}\right)^{2.5 c_2^*}. \tag{37}$$

Within the framework of the principle of elastic-viscous analogy, the solution of elastic problems can be applied to a linearly viscous fluid by replacing strains with strain rates and the shear modulus — with viscosity [2]. Therefore, for the viscosity η of a suspension, we have

$$\eta = \eta_0 / \alpha_{\sigma 0},$$

where η_0 is the viscosity of the liquid phase of the suspension.

For suspensions, the limiting volume concentration of inclusions at which the suspension loses its fluidity is $c_2^* = 0.61 \pm 0.02$ [11]. Figure 5 shows calculated relations between the relative viscosity of suspension $\eta_r = \eta / \eta_0$ and the volume fraction of particles at $c_2^* = 0.61$. The best agreement with experimental data is found in the case of the effective volume $\alpha_{\sigma 0}$ calculated according to Eq. (37). The calculation of $\alpha_{\sigma 0}$ by using Eq. (36) gave overestimated values of viscosity.

Thus, in predicting the elastic properties of porous materials with absolutely compliant inclusions, most accurate is the calculation with the modified Bal'shin relation. The Dougherty–Krieger relationship more accurately describes the viscosity of suspensions with undeformable rigid inclusions. For other variants of deformational properties of inclusions, the relationships of the effective volumes of averaging of conventionally porous composites considered here give practically identical results.

Conclusions

Invoking the model of elastic deformation of porous materials, a method for calculating the effective Young's modulus and Poisson ratio of granular composites with isolated inclusions has been developed. A particular feature of the method consists in calculating the concentration factors of average strains and stresses by using the effective volumes of averaging of phases. The effective volumes of averaging are calculated upon solution of the boundary-value problem of elastic deformation of the representative cell of biphase composites with the use of a simple scheme of digitization and a computational algorithm. The calculated effective elastic Young's moduli, Poisson ratio, and viscosity of suspensions are found to be in good agreement with experimental data at different combinations of material constants and arbitrary volume concentrations of isolated inclusions.

REFERENCES

1. T. D. Shermergor, Elasticity Theory of Microinhomogeneous Media [in Russian], Nauka, Moscow (1977).
2. R. M. Christensen, Mechanics of Composite Materials, John Wiley & Sons, New York, (1979).
3. N. S. Bakhvalov and G. P. Panasenko, Averaging of Processes in Periodic Media [in Russian], Nauka, Moscow (1984).
4. G. A. Vanin, Micromechanics of Composite Materials [in Russian], Naukova Dumka, Kiev (1985).
5. G. N. Dul'nev and V. V. Novikov, Transport Processes in Heterogeneous Media [in Russian], Energoatomizdat, Leningrad (1991).
6. A. F. Fedotov, "Application of a deformation model of porous materials to calculating the effective elastic moduli of granular composites," Mekh. Kompozits. Mater. Konstr., **17**, No. 1, 3-18 (2011).
7. J. Sendetski (ed.), Composite Materials. Vol. 2. Mechanics of Composite Materials [in Russian], Mir, Moscow (1978), pp. 61-101.
8. T. Fudzii and M. Dzako, Fracture Mechanics of Composite Materials [Russian translation], Mir, Moscow (1982).
9. A. F. Fedotov, "Model of averaging of local stresses and strains and the effective elastic moduli of powdery and porous sintered materials," Izv. Vuzov. Poroshk. Metallurg. Funkts. Pokr., No. 4, 19-26 (2010).
10. L. E. Nielsen, Mechanical Properties of Polymers and Composites, Marcel Dekker, Inc., New York (1974).
11. N. B. Ur'ev and A. A. Potanin, Fluidity of Suspensions and Powders [in Russian], Khimiya, Moscow (1992).
12. C. L. Hsieh and W. H. Tuan, "Elastic properties of ceramic-metal particulate composites," Mater. Sci. Eng. A, **393**, 133-139 (2005).
13. C. L. Hsieh and W. H. Tuan, "Elastic and thermal expansion behavior of two-phase composites," Mater. Sci. Eng. A, **425**, 349-360 (2006).
14. C. L. Hsieh, W. H. Tuan, and T. T. Wu, "Elastic behaviour of a model two-phase material," J. Europ. Ceramic Soc., **24**, 3789-3793 (2004).
15. M. Yu. Bal'shin, Scientific Foundations of Powder Metallurgy and Fiber Metallurgy [in Russian], Metallurgiya, Moscow (1972).