Mechanics of Composite Materials, Vol. 50, No. 1, March, 2014 (Russian Original Vol. 50, No. 1, January-February, 2014)

AN APPROXIMATE TWO-DIMENSIONAL MODEL OF ADHESIVE JOINTS. ANALYTICAL SOLUTION

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Keywords: *adhesive joint, analytical solution, patch, stress distribution, two-dimensional model*

An analytical solution to the problem on determination of the stress state of adhesive joints is obtained by using a simplified two-dimensional Volkersen's model. The approach proposed allows one to take into consideration the transverse deformations of connected parts. The solution is constructed by the method of separation of variables.

Introduction

Historically, the first and simplest model of an adhesive joint is the Volkersen's model [1]. Further refinements of the one-dimensional models of joints were directed to consideration of the mutual influence of bending of load-carrying layers and the tangential stresses in the adhesive [2], modeling of the load-carrying layers by Timoshenko beams, studying of the effect of nonlinear behavior of the adhesive layer, and the generalization of existing models to multilayer ones. The modern development of this approach is focused on such problems as the solution of dynamic problems of adhesive joints and the problems of joints of variable thickness, with cracks in the adhesive, refinement of the stress state of adhesive layers [3, 4], an analysis of stresses near boundaries of an adhesive seam, and so on.

However, in some cases, in calculating the stress state, it is necessary to take into account the deformations in planes of a joint caused by the Poisson ratios of connected parts. Examples of such designs are presented, e.g., in [5, 6]. Construction of an analytical solution to the problem of two-dimensional stress state of adhesive joints in the general statement is an extreme challenge, because, even for one layer, the solution of the plane problem of elasticity theory is rather awkward [7]. Therefore, to solve the problem posed, several approximate approaches were suggested, which are based on the condition that the load applied to the sides of the joint is uniform. The simplest one is the quasi-two-dimensional technique [8], which

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Translated from Mekhanika Kompozitnykh Materialov, Vol. 50, No. 1, pp. 147-158, January-February, 2014. Original article submitted February 18, 2013; revision submitted May 21, 2013.

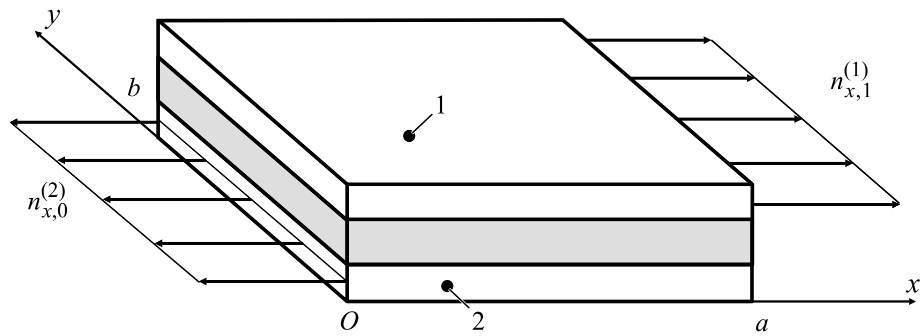

Fig. 1. Schematic sketch of a joint.

consists in successive solution of one-dimensional equilibrium equations in the longitudinal and transverse directions. At the basis of this method lies the assumption that the Poisson ratios of load-carrying layers in the transverse direction are equal to zero, which restricts the field of its application. In [9], it was offered to assume that the tangential stresses arising in the load-carrying layers upon deformation are equal to zero. This hypothesis allows one to neglect the derivatives of tangential stresses in equilibrium equations and to reduce the problem to solution of a system of two differential equations in partial derivatives with respect to the normal stresses in one layer. Physically, this situation corresponds to equating the shear moduli of materials of the load-carrying layers to zero. In that study, the general solution of the problem was not obtained; instead, a simplified approach was suggested, which, in essence, is similar to the quasi-two-dimensional method [8]. In [10], a solution to the above-mentioned system of differential equations is obtained by using a semianalytical method in the form of double Fourier series. This method is also employed in [11, 12] for solving similar problems. We should note that the technique of expansion into a double Fourier series is rather clumsy and has been introduced in the mentioned studies without any rigorous mathematical substantiation.

In the present study, an analytical solution of the problem is derived (within the framework of an approximate theory), which is constructed by the method of separation of variables and has the form of Fourier series with respect to one coordinate.

In some studies (see, for example, [13-15]), the problem in view was solved by numerical methods. A comparison of the calculation results obtained by numerical and analytical methods shows that the assumption of smallness of the derivative of tangential stresses in the load-carrying layers under a uniform load is allowable.

1. Statement of the Problem

Let us consider two glued rectangular plates (1 and 2 in Fig. 1) $D=\{(x,y)\in[0,a]\times[0,b]\}$ of thickness δ_1 and δ_2, respectively, whose lateral faces are subjected to normal forces.

The solution of the problem is based on the following hypotheses:

– the adhesive layer operates only in shear;

– stresses are uniformly distributed across the thickness of layers;

– the orthotropy directions of layers coincide with the directions of coordinate axes;

– the bending effects are not taken into account;

– the tangential stresses in the load-carrying layers are considered constant, i.e., their derivatives are equal to zero.

2. Construction of the Solution

The equilibrium equations for the differential elements of layers have the form [10]

$$\tau_x + \frac{\partial N_x^{(1)}}{\partial x} + \frac{\partial q^{(1)}}{\partial y} = 0,\ \ \tau_y + \frac{\partial N_y^{(1)}}{\partial y} + \frac{\partial q^{(1)}}{\partial x} = 0,$$

$$-\tau_x + \frac{\partial N_x^{(2)}}{\partial x} + \frac{\partial q^{(2)}}{\partial y} = 0,\ \ -\tau_y + \frac{\partial N_y^{(2)}}{\partial y} + \frac{\partial q^{(2)}}{\partial x} = 0,$$

where τ_x and τ_y are the tangential stresses in the adhesive layer in the *x* and *y* directions; $N_x^{(k)}$ and $N_y^{(k)}$ are the normal forces in a *k*th load-carrying layer ($k = 1, 2$) in the corresponding directions ($N_x^{(k)} = \delta_k \sigma_{xx}^{(k)}$ and $N_y^{(k)} = \delta_k \sigma_{yy}^{(k)}$); $q^{(k)}$ are the tangential forces in a *k*th layer ($q^{(k)} = \delta_k \tau_{xy}^{(k)}$).

According to the assumption that the tangential forces in the load-carrying layers are zero, the equilibrium equations take the form

$$\tau_x + \frac{\partial N_x^{(1)}}{\partial x} = 0\,,\ \ \tau_y + \frac{\partial N_y^{(1)}}{\partial y} = 0\,,\ \ \tau_x - \frac{\partial N_x^{(2)}}{\partial x} = 0\,,\ \ \tau_y - \frac{\partial N_y^{(2)}}{\partial y} = 0\,. \tag{1}$$

The boundary conditions

$$N_x^{(k)}\big|_{x=0} = n_{x,0}^{(k)}\ ,\ N_x^{(k)}\big|_{x=a} = n_{x,1}^{(k)}\ ,\ N_y^{(k)}\big|_{y=0} = n_{y,0}^{(k)}\ ,\ N_y^{(k)}\big|_{y=b} = n_{y,1}^{(k)} \tag{2}$$

must satisfy the equilibrium conditions of the joint

$$\begin{aligned} N_x^{(1)} + N_x^{(2)} &= n_{x,0}^{(1)} + n_{x,0}^{(2)} = n_{x,1}^{(1)} + n_{x,1}^{(2)} = F_x, \\ N_y^{(1)} + N_y^{(2)} &= n_{y,0}^{(1)} + n_{y,0}^{(2)} = n_{y,1}^{(1)} + n_{y,1}^{(2)} = F_y. \end{aligned} \tag{3}$$

Stresses in the adhesive layer are proportional to the difference of displacements of layers:

$$\tau_x = \frac{G_0}{\delta_0}\left(U_x^{(2)} - U_x^{(1)}\right), \tau_y = \frac{G_0}{\delta_0}\left(U_y^{(2)} - U_y^{(1)}\right), \tag{4}$$

where $U_x^{(k)}$ and $U_y^{(k)}$ are displacements in the corresponding directions of a *k*th layer; G_0 and δ_0 are the shear modulus and thickness of the adhesive layer.

Hooke's law for the connected parts reads as follows:

$$\varepsilon_x^{(k)} = \frac{N_x^{(k)}}{\delta_k E_x^{(k)}} - \frac{\mu_{xy}^{(k)} N_y^{(k)}}{\delta_k E_y^{(k)}} + \alpha_x^{(k)} T_k\,,\ \ \varepsilon_y^{(k)} = \frac{N_y^{(k)}}{\delta_k E_y^{(k)}} - \frac{\mu_{yx}^{(k)} N_x^{(k)}}{\delta_k E_x^{(k)}} + \alpha_y^{(k)} T_k\,,$$

where $E_x^{(k)}, E_y^{(k)}, \mu_{xy}^{(k)}, \mu_{yx}^{(k)}, \alpha_x^{(k)}$, and $\alpha_y^{(k)}$ are the elastic moduli, Poisson ratios, and the linear thermal expansion coefficients in the corresponding directions; T_k is the difference between the thickness-average temperature of formation and the operating temperature.

Differentiating the first two equations of (1) and relations (4) and then employing the Cauchy relations $\varepsilon_x^{(k)} = \frac{\partial U_x^{(k)}}{\partial x}$ and $\varepsilon_y^{(k)} = \frac{\partial U_y^{(k)}}{\partial y}$, Hooke's law, and the equilibrium conditions of joint (3) so that to exclude forces from the second layer, we arrive at a system of differential equations for forces in the first load-carrying layer:

$$\begin{cases} \dfrac{\partial^2 N_x^{(1)}}{\partial x^2} - c_{11} N_x^{(1)} + c_{12} N_y^{(1)} + c_{13} = 0, \\ \dfrac{\partial^2 N_y^{(1)}}{\partial y^2} + c_{21} N_x^{(1)} - c_{22} N_y^{(1)} + c_{23} = 0, \end{cases} \tag{5}$$

where

$$c_{11} = \frac{p_0}{\delta_1 E_x^{(1)}} + \frac{p_0}{\delta_2 E_x^{(2)}}, \; c_{12} = \frac{\mu_{xy}^{(1)} p_0}{\delta_1 E_y^{(1)}} + \frac{\mu_{xy}^{(2)} p_0}{\delta_2 E_y^{(2)}}, \; c_{21} = \frac{\mu_{yx}^{(1)} p_0}{\delta_1 E_x^{(1)}} + \frac{\mu_{yx}^{(2)} p_0}{\delta_2 E_x^{(2)}},$$

$$c_{22} = \frac{p_0}{\delta_1 E_y^{(1)}} + \frac{p_0}{\delta_2 E_y^{(2)}}, \;\; c_{13} = p_0 \left(\frac{F_x}{\delta_2 E_x^{(2)}} - \frac{\mu_{xy}^{(2)} F_y}{\delta_2 E_y^{(2)}} + \alpha_x^{(2)} T_2 - \alpha_x^{(1)} T_1 \right),$$

$$c_{23} = p_0 \left(\frac{F_y}{\delta_2 E_y^{(2)}} - \frac{\mu_{yx}^{(2)} F_x}{\delta_2 E_x^{(2)}} + \alpha_y^{(2)} T_2 - \alpha_y^{(1)} T_1 \right), \;\; p_0 = \frac{G_0}{\delta_0}.$$

In this problem, the stress state at a point is described by two normal stresses (with zero tangential ones). On the boundary, only one component of the stress tensor is assigned, while the second one is found from equilibrium equations (5). In this case, one of the equations on the boundary takes the form of an ordinary differential equation:

$$\frac{\partial^2 B_{x,j}}{\partial x^2} - c_{11} B_{x,j} + c_{13} + c_{12} n_{y,j}^{(1)} = 0, \;\; \frac{\partial^2 B_{y,j}}{\partial y^2} - c_{22} B_{y,j} + c_{23} + c_{21} n_{x,j}^{(1)} = 0, \tag{6}$$

where $j = 0, 1$; for forces along the boundaries, we introduce the designations $B_{x,0} = N_x^{(1)}\big|_{y=0}$, $B_{x,1} = N_x^{(1)}\big|_{y=b}$, $B_{y,0} = N_y^{(1)}\big|_{x=0}$, and $B_{y,1} = N_y^{(1)}\big|_{x=a}$. The solutions of Eqs. (6) have the form

$$\begin{aligned} B_{x,j}(x) &= S_{1,j} \sinh\left(\sqrt{a_{11}}\, x\right) + S_{2,j} \cosh\left(\sqrt{a_{11}}\, x\right) + \frac{c_{12} n_{y,j}^{(1)} + c_{13}}{c_{11}}, \\ B_{y,j}(y) &= S_{3,j} \sinh\left(\sqrt{a_{22}}\, y\right) + S_{4,j} \cosh\left(\sqrt{a_{22}}\, y\right) + \frac{c_{21} n_{x,j}^{(1)} + c_{23}}{c_{22}}. \end{aligned} \tag{7}$$

The constants $S_{i,j}$ are found from boundary conditions (2). Expressions (7) are the classical one-dimensional solutions to the problem of stress state of a joint [1]. Forces (7), along with conditions (2), are boundary conditions on the lateral faces of the plate, as also assumed in [10].

Excluding one unknown from system (5), we obtain the equations

$$\frac{\partial^4 N_x^{(1)}}{\partial x^2 \partial y^2} - A_1 \frac{\partial^2 N_x^{(1)}}{\partial x^2} - A_2 \frac{\partial^2 N_x^{(1)}}{\partial y^2} + A_3 N_x^{(1)} = A_4, \tag{8}$$

$$\frac{\partial^4 N_y^{(1)}}{\partial x^2 \partial y^2} - B_1 \frac{\partial^2 N_y^{(1)}}{\partial x^2} - B_2 \frac{\partial^2 N_y^{(1)}}{\partial y^2} + B_3 N_y^{(1)} = B_4, \tag{9}$$

where $A_1 = c_{22}$, $A_2 = c_{11}$, $A_3 = c_{22} c_{11} - c_{21} c_{12}$, and $A_4 = c_{22} c_{13} + c_{12} c_{23}$; $B_1 = c_{22}$, $B_2 = c_{11}$, $B_3 = c_{22} c_{11} - c_{12} c_{21}$, and $B_4 = c_{11} c_{23} + c_{21} c_{13}$.

Let us consider the construction of a solution to Eq. (8). Conditions (2) on the boundaries $x=0$ and $x=a$ and Eq. (8) are inhomogeneous. To exclude these inhomogeneities, we present the longitudinal forces in the form

$$N_x^{(1)} = R(x) + N(x,y), \tag{10}$$

where the function $R(x)$ satisfies the corresponding boundary conditions (2), but the function $N(x,y)$ turns to zero at $x=0$ and $x=a$.

Inserting Eq. (10) into Eq. (8), we come to the equations

$$-A_1 \frac{d^2 R}{dx^2} + A_3 R = A_4, \tag{11}$$

$$\frac{\partial^4 N}{\partial x^2 \partial y^2} - A_1 \frac{\partial^2 N}{\partial x^2} - A_2 \frac{\partial^2 N}{\partial y^2} + A_3 N = 0. \tag{12}$$

The general solution of Eq. (11) is

$$R(x) = V_1 \sinh\left(\sqrt{\frac{A_3}{A_1}} x\right) + V_2 \cosh\left(\sqrt{\frac{A_3}{A_1}} x\right) + \frac{A_4}{A_3}. \tag{13}$$

The constants V_1 and V_2 are determined from the corresponding boundary conditions (2). We should note that function (13) was suggested in [9] as an approximate solution to system (5).

Since the inhomogeneous boundary conditions (2) at $x=0$ and $x=a$ are satisfied by means of function (13), the corresponding boundary conditions for $N(x,y)$ will be homogeneous. This makes it possible to employ the method of separation of variables. The particular solutions of Eq. (12) are sought in the product form $N(x,y) = X(x)Y(y)$. The standard procedure of separation of variables in this equation, with the parameter of separation $-\lambda^2$ $(\lambda > 0)$, leads to two ordinary differential equations:

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0, \quad \left(A_2 + \lambda^2\right)\frac{d^2 Y}{dy^2} - \left(A_3 + \lambda^2 A_1\right) Y = 0.$$

The general solution of the first equation has the form

$$X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x.$$

The boundary conditions for $N(x,y)$ on the faces $x=0$ and $x=a$ are homogeneous, therefore, they are homogeneous for $X(x)$, too. The function $X(x)$ obeys these conditions and is not identically equal to zero only at $\lambda_n = \frac{\pi n}{a}$, $(n = 1,2,3,...)$ and $C_2 = 0$. For definiteness, we take that $C_1 = 1$. The solution of the second equation at $\lambda = \lambda_n$ is

$$Y = Y_n(y) = C_{3,n} \sinh(\mu_n y) + C_{4,n} \cosh(\mu_n y) \; \mu_n = \sqrt{\frac{A_3 a^2 + \pi^2 n^2 A_1}{A_2 a^2 + \pi^2 n^2}}.$$

The general solution of Eq. (12) can be presented in the series form

$$N(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y) = \sum_{n=1}^{\infty} \sin\frac{\pi n x}{a}\left[C_{3,n} \sinh(\mu_n y) + C_{4,n} \cosh(\mu_n y)\right].$$

Then, based on Eq. (10), we can write

$$N_x^{(1)} = R(x) + \sum_{n=1}^{\infty} \sin\frac{\pi n x}{a}\left[C_{3,n} \sinh(\mu_n y) + C_{4,n} \cosh(\mu_n y)\right]. \tag{14}$$

Satisfying boundary conditions (7) on the faces $y=0$ and $y=b$, we arrive at the expansion in Fourier series in terms of sine functions

$$\sum_{n=1}^{\infty} C_{4,n} \sin\frac{\pi nx}{a} = B_{x,0}(x) - R(x) \quad (0<x<a) \ ,$$

$$\sum_{n=1}^{\infty} \left[C_{3,n}\sinh(\mu_n b) + C_{4,n}\cosh(\mu_n b)\right]\sin\frac{\pi nx}{a} = B_{x,1}(x) - R(x) \quad (0<x<a) \ .$$

Employing the completeness and orthogonality of the system of functions $\left\{\sin\frac{\pi nx}{a}\right\}$ on the section $[0;a]$, we have

$$C_{4,n} = \frac{2}{a}\int_0^a \left(B_{x,0}-R\right)\sin\frac{\pi nx}{a}dx \ , \quad C_{3,n} = \frac{\frac{2}{a}\int_0^a \left(B_{x,1}-R\right)\sin\frac{\pi nx}{a}dx - C_{4,n}\cosh(\mu_n b)}{\sinh(\mu_n b)} \ ,$$

$$B_{y,j}(y) = S_{3,j}\sinh\left(\sqrt{a_{22}}\,y\right) + S_{4,j}\cosh\left(\sqrt{a_{22}}\,y\right) + \frac{c_{21}n_{x,j}^{(1)} + c_{23}}{c_{22}} .$$

Here, both the integrals can be calculated exactly.

Concerning the forces $N_y^{(1)}$ and Eqs. (9), a similar procedure leads to the expansion

$$N_y^{(1)} = T(y) + \sum_{n=1}^{\infty}\sin\frac{\pi ny}{b}\left[C_{5,n}\sinh(\nu_n x) + C_{6,n}\cosh(\nu_n x)\right], \tag{15}$$

where

$$T(y) = V_3\sinh\left(\sqrt{\frac{B_3}{B_2}}y\right) + V_4\cosh\left(\sqrt{\frac{B_3}{B_2}}y\right) + \frac{B_4}{B_3} ,$$

$$\nu_n = \sqrt{\frac{B_3 b^2 + \pi^2 n^2 B_2}{B_1 b^2 + \pi^2 n^2}} \ , \quad C_{5,n} = \frac{\frac{2}{b}\int_0^b \left(B_{y,1}-T\right)\sin\frac{\pi ny}{b}dy - C_{6,n}\cosh(\nu_n a)}{\sinh(\nu_n a)} \ ,$$

$$C_{6,n} = \frac{2}{b}\int_0^b \left(B_{y,0}(y) - T(y)\right)\sin\frac{\pi ny}{b}dy \ .$$

The constants V_3 and V_4 are found from the corresponding boundary conditions (2).

It is easy to verify that

$$C_{3,n}, C_{4,n}, C_{5,n}, C_{6,n} = 0\left(n^{-3}\right) \ (n\to\infty), \tag{16}$$
$$\mu_n \sim \sqrt{A_1}, \nu_n \sim \sqrt{B_2} \quad (n\to\infty) .$$

Estimates (16) ensure the convergence of series in expressions for the functions $N_x^{(1)}$ and $N_y^{(1)}$ and their particular derivatives of the first and second orders with respect to the variables x and y and uniform convergence of series for these functions and their particular derivatives of the first order. If, according to Eq. (14), the function $N_x^{(1)}$ is found, then, from the first equation of (5) for the forces $N_y^{(1)}$, apart from Eq. (15), we also obtain a different-looking representation:

$$N_y^{(1)} = W(x) + \sum_{k=1}^{\infty}\beta_k\sin\frac{\pi kx}{a}\left[C_{3,k}\sinh(\mu_k y) + C_{4,k}\cosh(\mu_k y)\right], \tag{17}$$

where $\beta_k = \frac{c_{11}}{c_{12}} + \frac{\pi^2 k^2}{a^2 c_{12}}$,

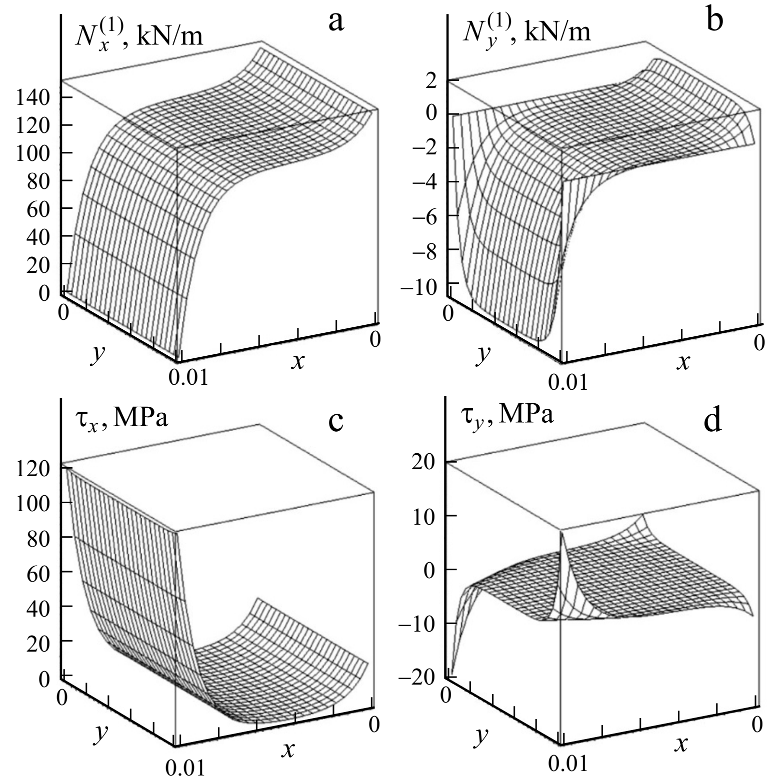

Fig. 2. Normal forces in the first layer (a, b) and tangential stresses in the adhesive (c, d).

$$W(x)=\left(\frac{c_{11}}{c_{12}}-\frac{A_3}{A_1c_{12}}\right)\left[V_1\sinh\left(\sqrt{\frac{A_3}{A_1}}x\right)+V_2\cosh\left(\sqrt{\frac{A_3}{A_1}}x\right)\right]+\frac{c_{11}A_4}{c_{12}A_3}-\frac{c_{13}}{c_{12}}.$$

Expanding the functions $W(x)$, $\sinh(\mu_k x)$ and $\cosh(\mu_k x)$ into Fourier series in terms of $\sin\frac{\pi kx}{a}$ and the functions $T(y)$ $\sinh(\nu_n y)$ and $\cosh(\nu_n y)$ in terms of $\sin\frac{\pi ny}{b}$, one can make sure that representations (15) and (17) are identical.

Based on Eqs. (1), the tangential stresses τ_x and τ_y in the adhesive layer are found by differentiation of expansions (14) and (15). Owing to the uniform convergence of the corresponding series, these stresses are continuous functions of the variables x and y.

3. Numerical Examples

Let us consider the stress state of a joint with the necessary parameters taken from [10]: $a=b=10$ mm, $\delta_1=1$ mm, $\delta_2=0.5$ mm, $\delta_0=0.15$ mm, and $G_0=4.2$ GPa. The first layer is made of a carbon-fiber-reinforced plastic (CFRP) with a reinforcement directed along the x axis ($E_x^{(1)}=181$ GPa, $E_y^{(1)}=10$ GPa, and $\mu_{yx}^{(1)}=0.28$), while the second layer is manufactured from aluminum ($E_x^{(2)}=E_y^{(2)}=72$ GPa and $\mu_{xy}^{(2)}=0.32$).

A tensile load is transferred from one load-carrying layer to another; the CFRP is loaded along fibers. The boundary conditions are

$$N_x^{(1)}\big|_{x=0}=N_x^{(2)}\big|_{x=a}=150\ \text{kN/m},\quad N_x^{(2)}\big|_{x=0}=N_x^{(1)}\big|_{x=a}=0,$$

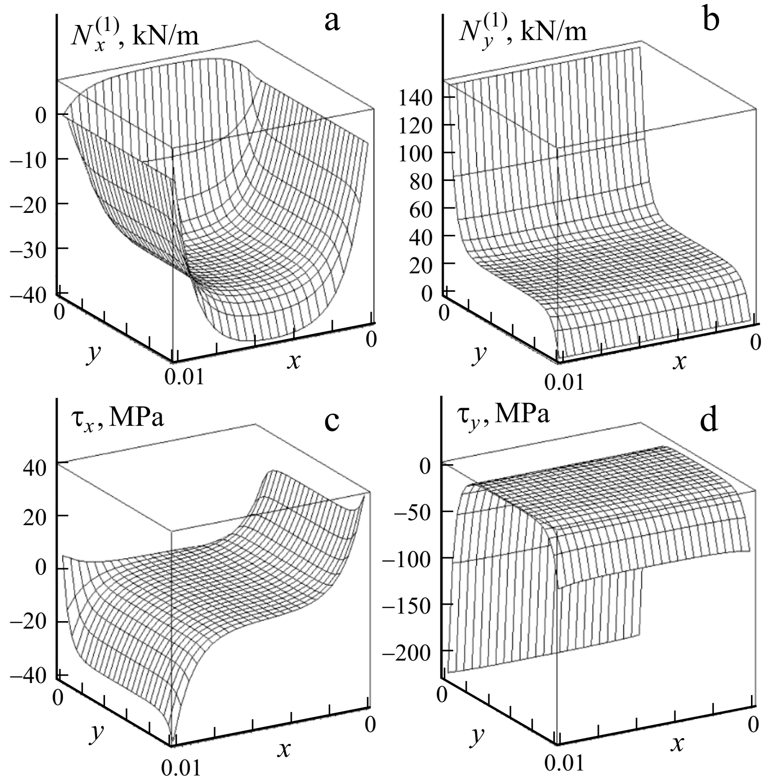

Fig. 3. Normal forces in the first layer (a, b) and tangential stresses in the adhesive (c, d).

$$N_y^{(k)}\big|_{y=0} = N_y^{(k)}\big|_{y=b} = 0\,, \quad k = 1, 2\,.$$

The graphs of forces in the first load-carrying layer and of stresses in the adhesive are shown in Fig. 2.

In the second calculated case, the load transferred from the metal layer to the CFGP is directed across the reinforcement. The boundary conditions are

$$N_y^{(1)}\big|_{y=0} = N_y^{(2)}\big|_{y=b} = 150\ \text{kN/m}, \quad N_y^{(2)}\big|_{y=0} = N_y^{(1)}\big|_{y=b} = 0\,,$$

$$N_x^{(k)}\big|_{x=0} = N_x^{(k)}\big|_{x=a} = 0\,, \quad k = 1, 2\,.$$

The forces in the first load-carrying layer (in the CFRP) and the stresses in the adhesive are illustrated in Fig. 3.

4. Analysis of Calculation Results

As seen from the graph in Fig. 2b, upon loading along the x axis, the transverse forces $N_y^{(1)}$, caused by the Poisson ratios of load-carrying layers, are relatively small compared with the forces $N_x^{(1)}$. The distribution of the latter ones is shown in Fig. 2a. The forces $N_y^{(1)}$ considerably differ at the ends $x = 0$ and $x = a$ of the joint, which is explained by the different thicknesses and elastic characteristics of the load-carrying layers. In this case, the maximum value of stresses in the transverse direction in the load-carrying layers reaches 20% of the longitudinal ones. The stresses τ_x in the adhesive joint (see Fig. 2c) are close to those calculated by means of one-dimensional techniques [1, 9]. The stresses τ_x (see Fig. 2d), similar to the forces $N_y^{(1)}$, are maximum on the side of the plate where the load is applied to the aluminum layer, but not to the CFRP, since the aluminum layer has larger transverse deformations than the CFRP layer. In the given calculation case, these stresses also reach a value about 20% of the maximum stresses τ_x in the longitudinal direction.

If the load is applied along the y axis, i.e., across the reinforcement direction of the CFRP, the stress state of the joint (see Fig. 3) will differ from that already considered in Fig. 2. The maximum transverse forces $N_x^{(1)}$ will be several times greater than those in the first calculation case. This is caused by the higher rigidity of the CFRP across the loading direction. Along the y axis, the rigidity of the aluminum layer is greater than that of the CFRP, and therefore $N_y^{(2)} > N_y^{(1)}$ on the most part of the joint, as a result of which $N_x^{(1)} < 0$ on the most part of the joint, except for a small zone in the neighborhood of $y = 0$, where the load is applied to the CFRP.

Conclusions

In the present study, an analytical solution of the problem on determination of the stress state of an adhesive joint is constructed in an approximate two-dimensional statement. A two-dimensional generalization of Volkersen's model and the condition of smallness of tangential stresses and their derivatives in the load-carrying layers [9, 10] is used. The last factor imposes restrictions on the boundary conditions on the lateral faces of the plate, namely the normal stresses must be constant along the faces, but the tangential stresses must be absent. The solution is constructed by means of the method of separation of variables in the form of Fourier series with respect to one coordinate. The method suggested has a good convergence, which allows one to develop it to solving the problems of design and optimization.

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