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FREE VIBRATION ANALYSIS OF LAMINATED COMPOSITE PLATES RESTING ON ELASTIC FOUNDATIONS BY USING A REFINED HYPERBOLIC SHEAR DEFORMATION THEORY

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The free vibration of laminated composite plates on elastic foundations is examined by using a refined hyperbolic shear deformation theory. This theory is based on the assumption that the transverse displacements consist of bending and shear components where the bending components do not contribute to shear forces, and likewise, the shear components do not contribute to bending moments. The most interesting feature of this theory is that it allows for parabolic distributions of transverse shear stresses across the plate thickness and satisfies the conditions of zero shear stresses at the top and bottom surfaces of the plate without using shear correction factors. The number of independent unknowns in the present theory is four, as against five in other shear deformation theories. In the analysis, the foundation is modeled as a two-parameter Pasternak-type foundation, or as a Winkler-type one if the second foundation parameter is zero. The equation of motion for simply supported thick laminated rectangular plates resting on an elastic foundation is obtained through the use of Hamilton's principle. The numerical results found in the present analysis for free the vibration of cross-ply laminated plates on elastic foundations are presented and compared with those available in the literature. The theory proposed is not only accurate, but also efficient in predicting the natural frequencies of laminated composite plates.

1. Introduction

Laminated composite plates are widely used in industry and new fields of technology. Due to the high degrees of anisotropy and the low rigidity in transverse shear of the plates, the Kirchhoff hypothesis as a classical theory is no longer adequate. The hypothesis states that the normal to the midplane of a plate remains straight and normal after deformation because

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of the negligible transverse shear effects. Refined theories without this assumption have been used recently. The free vibration frequencies calculated by using the classical theory of thin plates are higher than those obtained by the Mindlin theory of plates [1], in which the transverse shear and rotary inertia effects are included.

A number of shear deformation theories have been proposed to date. The first such theory for laminated isotropic plates was apparently [2]. This theory was generalized to laminated anisotropic plates in [3]. It was shown in [4-6] that the Yang–Norris–Stavski (YNS) theory [3] is adequate for predicting the flexural vibration response of laminated anisotropic plates in the first few modes. In [7], the YNS theory was employed to study the cylindrical bending of antisymmetric cross-ply and angle-ply plate-strips under sinusoidal loading and the free vibration of antisymmetric angle-ply plate-strips (see also [8, 9]. Using the YNS theory, a closed-form solution for the free vibration of simply supported rectangular plates of antisymmetric angle-ply laminates was obtained in [10]. In [11] were also presented exact three-dimensional elasticity solutions for the free vibration of isotropic and anisotropic composite laminated plates, which serve as benchmark solutions for comparison by many researchers. The free vibration of antisymmetric angle-ply laminated plates, with account of transverse shear deformations, was investigated in [12] by using the finite-element method The author also derived a set of variationally consistent equilibrium equations for the kinematic models originally proposed by Levinson and Murthy [13]. In [14], analytical and finite- element solutions for the vibration and buckling of laminated composite plates were found by using various theories of plates to prove the necessity for shear deformation theories to predict the behavior of composite laminates. Using a higher-order shear deformation theory, finite-element solutions for free vibration analysis of laminated composite plates were also obtained in [15]. The complete set of linear equations of a second-order theory was derived in [16] to analyze the free vibration behavior of cross-ply and antisymmetric angle-ply laminated plates. In [17], the natural frequencies of composite plates with random material properties were determined by using a higher-order shear deformation theory (including the rotatory inertia effect). The natural frequencies of laminated composite plates were also found in [18] by employing a third-order shear deformation theory. In [19], the dynamic deflections and the stresses of a functionally graded simply supported beam subjected to a moving mass were investigated by using the Euler–Bernoulli, Timoshenko, and the parabolic shear deformation theory of beams. In [20], the free vibration of functionally graded beams with different boundary conditions was examined by using the classical, first-order, and different higher-order shear deformation theories of beams. A stress analysis of a functionally graded plate subjected to thermal and mechanical loads was performed in [21] by using a two-dimensional higher-order theory. A new trigonometric shear deformation theory for isotropic and composite laminated and sandwich plates was developed recently in [22], where displacements of the middle surface were expanded in terms of tangential trigonometric functions of the thickness coordinate, and the transverse displacements were assumed to be constant across the thickness.

In this paper, a refined and simple theory of plates is presented and applied to the investigation of free vibration behavior of laminated composite plates on elastic foundations. This theory is based on the assumption that the in-plane and transverse displacements consist of bending and shear components where the bending components do not contribute to shear forces, and likewise, the shear components do not contribute to bending moments. The most interesting feature of this theory is that it allows for parabolic distributions of transverse shear stresses across the plate thickness and satisfies zero shear stress conditions at the top and bottom surfaces of the plate without using shear correction factors. In addition, it contains four independent variables, as against five in other shear deformation theories. The elastic foundation is modeled as a two-parameter Pasternak foundation. The equations of motion are derived using Hamilton's principle. The fundamental frequencies are found by solving an eigenvalue equation. The results obtained by the present method are compared with solutions derived from other models known from the literature and are found to be in good agreement with them.

2. Theoretical Formulations

2.1. Basic assumptions

The assumptions of the present theory are as follows.

(i) Displacements are small in comparison with plate thickness, and therefore the strains involved are infinitesimal.

(ii) The transverse displacement w includes two components — the bending w_b and shear w_s ones, which are functions only of x and y coordinates,

$$w(x, y, z) = w_b(x, y) + w_s(x, y). \tag{1}$$

(iii) The transverse normal stress σ_z is negligible in comparison with the in-plane stresses σ_x and σ_y.

(iv) The displacements u in the x-direction and v in the y-direction consist of extension, bending, and shear components,

$$u = u_0 + u_b + u_s, \quad v = v_0 + v_b + v_s. \tag{2}$$

The bending components u_b and v_b are assumed to be similar to the displacements given by the classical theory of plates, namely

$$u_b = -z\frac{\partial w_b}{\partial x}, \quad v_b = -z\frac{\partial w_b}{\partial y}. \tag{3}$$

The shear components u_s and v_s, in conjunction with w_s, give rise to parabolic variations in the shear strains γ_{xz} and γ_{yz} and hence in the shear stresses τ_{xz} and τ_{yz} across the thickness of the plate in such a way that the stresses τ_{xz} and τ_{yz} are zero at the top and bottom faces of the plate. Consequently, the expression for u_s and v_s can be given as

$$u_s = -f(z)\frac{\partial w_s}{\partial x}, \quad v_s = -f(z)\frac{\partial w_s}{\partial y}. \tag{4}$$

2.2. Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (1)-(4):

$$\begin{aligned} u(x, y, z) &= u_0(x, y) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x}, \\ v(x, y, z) &= v_0(x, y) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y}, \\ w(x, y, z) &= w_b(x, y) + w_s(x, y), \end{aligned} \tag{5}$$

where the shape function $f(z)$ is given as

$$f(z) = \frac{(h/\pi)\sinh\left(\dfrac{\pi}{h}z\right) - z}{\cosh(\pi/2) - 1}.$$

This function ensures zero transverse shear stresses at the top and bottom surfaces of the plate. The parabolic distributions of transverse shear stresses across the plate thickness are taken into account in the analysis by means of a hyperbolic function of the displacement field assumed.

The strains associated with the displacements in Eq. (5) are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} + f(z) \begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g(z) \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix},$$

where

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_s}{\partial x^2} \\ -\dfrac{\partial^2 w_s}{\partial y^2} \\ -2\dfrac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w_s}{\partial y} \\ \dfrac{\partial w_s}{\partial x} \end{Bmatrix} \tag{6}$$

and

$$g(z) = 1 - f'(z) \text{ with } f'(z) = \frac{df(z)}{dz}.$$

2.3. Constitutive equations

The stress state in each layer is given by Hooke's law

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix},$$

where Q_{ij} are the stiffnesses, which are defined in terms of engineering constants in the material axes of the layer:

$$Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13}.$$

Since the laminate is made of several orthotropic layers with their material axes oriented arbitrarily with respect to laminate coordinates, the constitutive equations of each layer must be transformed to the laminate coordinates x, y, and z. The stress–strain relations in the laminate coordinates of a kth layer are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)}, \tag{7}$$

where $\overline{Q}_{ij}$ are the transformed material constants, which an given in [23].

2.4. Governing equations

Using Hamilton's energy principle, we derive the equation of motion of the laminated composite plate

$$\delta\int_{t_1}^{t_2}\left(U+U_F-V-T\right)dt=0,$$

where U is the strain energy, T is the kinetic energy of the plate, U_F is the strain energy of foundation, and V is the work of external forces. Employing the principle of minimum total energy leads to the general equation of motion and boundary conditions. Taking the variation of the above equation and integrating by parts, we obtain

$$\int_{t_1}^{t_2}\Bigg[\int_V\Big(\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{xy}\delta\gamma_{xy}+\tau_{yz}\delta\gamma_{yz}+\tau_{zx}\delta\gamma_{zx}\Big)$$

$$-\rho\left(\ddot{u}_0\delta u_0+\ddot{v}_0\delta v_0+(\ddot{w}_b+\ddot{w}_s)\delta\;(w_b+w_s)\right)dv-\int_A f_e\delta(w_b+w_s)dA\Bigg]dt=0, \tag{8}$$

where two points above a variable means the second derivative with respect to time, and f_e is the density of the reaction force of foundation. For the Pasternak foundation,

$$f_e=k_0w-k_1\nabla^2 w. \tag{9}$$

If the foundation is modeled as a linear Winkler foundation, the coefficient k_1 in Eq. (9) is zero. With account of Eqs. (6), Eq. (8) takes the form

$$\int_{t_1}^{t_2}\Bigg[\int_A\Big(-\delta u_0N_{x,x}-\delta v_0N_{y,y}-\delta u_0N_{xy,y}-\delta v_0N_{xy,x}-\delta w_bM^b_{x,xx}-\delta w_bM^b_{y,yy}-2\delta w_bM^b_{xy,xy}$$

$$-\delta w_sM^s_{x,xx}-\delta w_sM^s_{y,yy}-2\delta w_sM^s_{xy,xy}-\delta w_sS^s_{xz,x}-\delta w_sS^s_{yz,y}\Big)dA$$

$$+\int_A f_e(\delta\;w_b+\delta\;w_s)dA-\int_A\Big\{\delta u_0\left(I_1\ddot{u}_0-I_2\ddot{w}_{b,x}-I_4\ddot{w}_{s,x}\right)+\delta v_0\left(I_1\ddot{v}_0-I_2\ddot{w}_{b,y}-I_4\ddot{w}_{s,y}\right)$$

$$+\delta w_b\left[I_1(\ddot{w}_b+\ddot{w}_s)+I_2(\ddot{u}_{0x}+\ddot{v}_{0y})-I_3(\ddot{w}_{b,xx}+\ddot{w}_{b,yy})-I_5(\ddot{w}_{s,xx}+\ddot{w}_{s,yy})\right]$$

$$+\delta w_s\left[I_1(\ddot{w}_b+\ddot{w}_s)+I_4(\ddot{u}_{0x}+\ddot{v}_{0y})-I_5(\ddot{w}_{b,xx}+\ddot{w}_{b,yy})-I_6(\ddot{w}_{s,xx}+\ddot{w}_{s,yy})\right]\Big\}dA\Bigg]dt=0. \tag{10}$$

The stress resultants N, M, and S are defined as

$$\begin{Bmatrix} N_x, & N_y, & N_{xy} \\ M^b_x, & M^b_y, & M^b_{xy} \\ M^s_x, & M^s_y, & M^s_{xy}\end{Bmatrix}=\int_{-h/2}^{h/2}\left(\sigma_x,\sigma_y,\tau_{xy}\right)\begin{Bmatrix}1\\ z\\ f(z)\end{Bmatrix}dz,$$

$$\left(S^s_{xz},S^s_{yz}\right)=\int_{-h/2}^{h/2}\left(\tau_{xz},\tau_{yz}\right)g(z)dz. \tag{11}$$

Inserting Eq. (7) into Eqs. (11) and integrating across the thickness of the plate, the stress resultants are obtained

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A & B & B^s \\ A & D & D^s \\ B^s & D^s & H^s \end{bmatrix} \begin{Bmatrix} \varepsilon \\ k^b \\ k^s \end{Bmatrix}, \quad S = A^s \gamma ,$$

where

$$N = \left\{N_x, N_y, N_{xy}\right\}^t, \quad M^b = \left\{M_x^b, M_y^b, M_{xy}^b\right\}^t, \quad M^s = \left\{M_x^s, M_y^s, M_{xy}^s\right\}^t,$$

$$\varepsilon = \left\{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0\right\}^t, \quad k^b = \left\{k_x^b, k_y^b, k_{xy}^b\right\}^t, \quad k^s = \left\{k_x^s, k_y^s, k_{xy}^s\right\}^t,$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}, \tag{12}$$

$$B^s = \begin{bmatrix} B_{11}^s & B_{12}^s & B_{16}^s \\ B_{12}^s & B_{22}^s & B_{26}^s \\ B_{16}^s & B_{26}^s & B_{66}^s \end{bmatrix}, \quad D^s = \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix}, \quad H^s = \begin{bmatrix} H_{11}^s & H_{12}^s & H_{16}^s \\ H_{12}^s & H_{22}^s & H_{26}^s \\ H_{16}^s & H_{26}^s & H_{66}^s \end{bmatrix},$$

$$S = \left\{S_{xz}^s, S_{yz}^s\right\}^t, \quad \gamma = \left\{\gamma_{xz}, \gamma_{yz}\right\}^t, \quad A^s = \begin{bmatrix} A_{44}^s & A_{45}^s \\ A_{45}^s & A_{55}^s \end{bmatrix},$$

and the stiffness components and inertias are given as

$$\left(A_{ij}, B_{ij}, D_{ij}, B_{ij}^s, D_{ij}^s, H_{ij}^s\right) = \int_{-h/2}^{h/2} \overline{Q}_{ij} \left(1, z, z^2, f(z), z\, f(z), f^2(z)\right) dz , \quad (i,j) = (1,2,6),$$

$$A_{ij}^s = \int_{-h/2}^{h/2} \overline{Q}_{ij} \left[g(z)\right]^2 dz, \quad (i,j) = (4,5), \tag{13}$$

$$(I_1, I_2, I_3, I_4, I_5, I_6) = \int_{-h/2}^{h/2} \rho \ (1, z, z^2, f(z), z\, f(z), [f(z)]^2) dz.$$

Collecting the coefficients of δu_0, δv_0, δw_b, and δw_s in Eq. (10), the equations of motion are obtained as

$$\delta u_0 : N_{x,x} + N_{xy,y} = I_1 \ddot{u}_0 - I_2 \ddot{w}_{b,x} - I_4 \ddot{w}_{s,s},$$

$$\delta v_0 : N_{xy,x} + N_{y,y} = I_1 \ddot{v}_0 - I_2 \ddot{w}_{b,y} - I_4 \ddot{w}_{s,y},$$

$$\delta w_b : M_{x,xx}^b + 2\ M_{xy,xy}^b + M_{y,yy}^b - k_0 (w_b + w_s) + k_1 \nabla^2 (w_b + w_s)$$

$$= I_1 (\ddot{w}_b + \ddot{w}_s) + I_2 \left(\ddot{u}_{0,x} + \ddot{v}_{0,y}\right) - I_3 \left(\ddot{w}_{b,xx} + \ddot{w}_{b,yy}\right) - I_5 \left(\ddot{w}_{s,xx} + \ddot{w}_{s,yy}\right), \tag{14}$$

$$\delta w_s : M_{x,xx}^s + 2\ M_{xy,xy}^s + M_{y,yy}^s + S_{xz,x}^s + S_{yz,y}^s - k_0 (w_b + w_s) + k_1 \nabla^2 (w_b + w_s)$$

$$= I_1 (\ddot{w}_b + \ddot{w}_s) + I_4 \left(\ddot{u}_{0,x} + \ddot{v}_{0,y}\right) - I_5 \left(\ddot{w}_{b,xx} + \ddot{w}_{b,yy}\right) - I_6 \left(\ddot{w}_{s,xx} + \ddot{w}_{s,yy}\right).$$

Clearly, when the effect of transverse shear deformation is neglected ($w_s = 0$), Eqs. (14) yield the equations of motion of a composite plate based on the classical theory of plates.

2.5. Analytical solutions for simply supported rectangular laminates

Rectangular laminated composites plates are generally classified in accordance with the type of support used. We are concerned here with analytical solutions of Eqs. (14) for simply supported composite plate. The following boundary conditions are imposed at the side edges:

$$v_0(0,y)=w_b(0,y)=w_s(0,y)=\frac{\partial w_b}{\partial y}(0,y)=\frac{\partial w_s}{\partial y}(0,y)=0\,,$$

$$v_0(a,y)=w_b(a,y)=w_s(a,y)=\frac{\partial w_b}{\partial y}(a,y)=\frac{\partial w_s}{\partial y}(a,y)=0\,,$$

$$N_x(0,y)=M_x^b(0,y)=M_x^s(0,y)=N_x(a,y)=M_x^b(a,y)=M_x^s(a,y)=0\,, \tag{15}$$

$$u_0(x,0)=w_b(x,0)=w_s(x,0)=\frac{\partial w_b}{\partial x}(x,0)=\frac{\partial w_s}{\partial x}(x,0)=0\,,$$

$$u_0(x,b)=w_b(x,b)=w_s(x,b)=\frac{\partial w_b}{\partial x}(x,b)=\frac{\partial w_s}{\partial x}(x,b)=0\,,$$

$$N_y(x,0)=M_y^b(x,0)=M_y^s(x,0)=N_y(x,b)=M_y^b(x,b)=M_y^s(x,b)=0\,.$$

The displacement functions that satisfy boundary conditions (15) are taken in the form Fourier series

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \end{Bmatrix}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\begin{Bmatrix} U_{mn}\cos(\lambda x)\sin(\mu y)e^{i\omega t} \\ V_{mn}\sin(\lambda x)\cos(\mu y)e^{i\omega t} \\ W_{bmn}\sin(\lambda x)\sin(\mu y)e^{i\omega t} \\ W_{smn}\sin(\lambda x)\sin(\mu y)e^{i\omega t} \end{Bmatrix},$$

where U_{mn}, V_{mn}, W_{bmn}, and W_{smn} are arbitrary parameters to be determined, ω is the eigenfrequency associated with an (m, n)th eigenmode; $\lambda = m\pi/a$ and $\mu = n\pi/b$.

Inserting Eqs. (15), (12), and (13) into the equations of motion (14) we get the following eigenvalue equations for the free vibration problem at any fixed values of m and n:

$$\left(\left[K\right]-\omega^2\left[M\right]\right)\left\{\Delta\right\}=\left\{0\right\}, \tag{16}$$

where $\{\Delta\}$ denotes the column

$$\left\{\Delta\right\}^T=\left\{U_{mn},V_{mn},W_{bmn},W_{smn}\right\},$$

and

$$\left[K\right]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix},\quad \left[M\right]=\begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{34} & m_{44} \end{bmatrix}.$$

Here,

$$a_{11}=A_{11}\lambda^2+A_{66}\mu^2\,,\ a_{12}=\lambda\mu\left(A_{12}+A_{66}\right),\ a_{13}=-B_{11}\lambda^3\,,\ a_{14}=-B_{11}^s\lambda^3\,,$$

TABLE 1. Displacement Models

Theory	Unknown functions
Classical theory of laminated plates (CLPT)	3
First-order shear deformation theory (FSDT) [7]	5
Parabolic shear deformation theory (PSDT) [13]	5
Exponential shear deformation theory [24] (ESDT) [24]	5
Sinusoidal shear deformation theory [25] (SSDT) [25]	5
Present refined shear deformation theory of plates	4

$$a_{22} = A_{66}\lambda^2 + A_{22}\mu^2\,,\ a_{23} = B_{11}\mu^3\,,\ a_{24} = B_{11}^s\mu^3\,,$$

$$a_{33} = D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4 + k_0 + k_1(\lambda^2 + \mu^2)\,,$$

$$a_{34} = D_{11}^s\lambda^4 + 2(D_{12}^s + 2D_{66}^s)\lambda^2\ \mu^2 + D_{22}^s\ \mu^4 + k_0 + k_1(\lambda^2 + \mu^2)\,,$$

$$a_{44} = H_{11}^s\lambda^4 + 2(H_{12}^s + 2H_{66}^s)\lambda^2\mu^2 + H_{22}^s\mu^4 + A_{55}^s\lambda^2 + A_{44}^s\mu^2 + k_0 + k_1(\lambda^2 + \mu^2)\,,$$

$$m_{11} = m_{22} = I_1\,,\ m_{33} = I_1 + I_3\left(\lambda^2 + \mu^2\right),\ m_{34} = I_1 + I_5\left(\lambda^2 + \mu^2\right),\ m_{44} = I_1 + I_6\left(\lambda^2 + \mu^2\right).$$

The natural frequencies of the laminates can be obtained by setting the determinant of the coefficient matrix in Eq. (16) to zero.

3. Numerical Results and Discussion

In this study, a free vibration analysis of symmetrically and antisymmetrically laminated composite plates resting on an elastic foundation by using the present shear deformation theory for laminated plates is suggested. The Navier solutions for free vibrations of laminated composite plates are found by solving eigenvalue equations. Comparisons are made with various theories of plates and with exact solutions of three-dimensional elasticity theory. The description of various displacement models is given in Table 1.

In order to verify the accuracy of the present analysis, some numerical examples were solved. It was assumed that the thickness and the material properties for all laminas were the same. In the analysis, the elastic properties of a lamina were taken to be as follows:

$$G_{12} = G_{13} = 0.6E_2\,,\quad G_{23} = 0.5E_2\,,\quad \nu_{12} = \nu_{13} = 0.25\,.$$

The following nondimensional fundamental frequency, nondimensional linear Winkler foundation parameter, and nondimensional Pasternak foundation parameter were used:

$$\bar{\varpi} = \omega\frac{a^2}{h}\sqrt{\frac{\rho}{E_2}},\quad K_0 = \frac{k_0 L^4}{E_2 h^3}\,,\quad K_1 = \frac{k_1 L^4}{E_2 h^3}\,.$$

The fundamental frequencies of the systems were calculated by Eq. (16) as an eigenvalue problem.

In Tables 2 and 3, the nondimensional fundamental frequencies of antisymmetrically laminated cross-ply plates obtained by using different shear deformation theories are shown for various values of a/h and moduli ratios. It can be seen that, in general, the present theory gives more accurate results in predicting the natural frequencies than the PSDT and the three-dimensional elasticity solution given in [11]. It should be noted that unknown functions in present theory are four, while

TABLE 2. Nondimensional Fundamental Frequencies of Antisymmetric Square Plates at Various Values of Orthotropy Ratio with $a/h = 5$

No. of layers	Theory	E_1/E_2				
		3	10	20	30	40
$(0/90)_1$	Exact [11]	6.2578	6.9845	7.6745	8.1763	8.5625
	FSDT	6.2085	6.9392	7.7060	8.3211	8.8333
	PSDT	6.2169	6.9887	7.8210	8.5050	9.0871
	ESDT	6.2224	7.0066	7.8584	8.5630	9.1661
	SSDT	6.2188	6.9964	7.8379	8.5316	9.1236
	Present	6.2164	6.9839	7.8095	8.4863	9.0610
$(0/90)_2$	Exact [11]	6.5455	8.1445	9.4055	10.1650	10.6790
	FSDT	6.5043	8.2246	9.6885	10.6198	11.2708
	PSDT	6.5008	8.1954	9.6265	10.5348	11.1716
	ESDT	6.5034	8.1939	9.6201	10.5261	11.1628
	SSDT	6.5012	8.1929	9.6205	10.5268	11.1628
	Present	6.5017	8.1999	9.6353	10.5467	11.1853
$(0/90)_3$	Exact [11]	6.6100	8.4143	9.8398	10.6950	11.2720
	FSDT	6.5569	8.4183	9.9427	10.8828	11.5264
	PSDT	6.5558	8.4052	9.9181	10.8547	11.5012
	ESDT	6.5595	6.5595	9.9313	10.8756	11.5314
	SSDT	6.5567	8.4066	9.9211	10.8603	11.5102
	Present	6.5563	8.4069	9.9205	10.8568	11.5019
$(0/90)_5$	Exact [11]	6.6458	8.5625	10.0843	11.0027	11.6245
	FSDT	6.5837	8.5132	10.0638	11.0058	11.6444
	PSDT	6.5842	8.5126	10.0674	11.0197	11.6730
	ESDT	6.5885	8.5229	10.0881	11.0522	11.7180
	SSDT	6.5854	8.5156	10.0740	11.0309	11.6894
	Present	6.5846	8.5131	10.0670	11.0175	11.6682

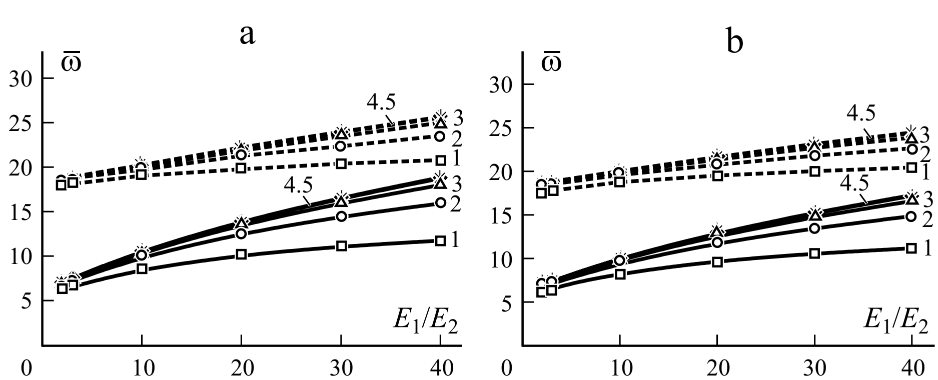

Fig. 1. Effect of the orthotropy ratio E_1/E_2 on the nondimensional fundamental frequencies $\bar{\omega}$ of laminated cross-ply plates with layups (0/90/90/0) (a) and (0/90/0/90) (b) on an elastic foundation. $(K_0, K_1) = (0, 0)$ (——) and (100, 10) (– – –); a/h = 5 (1), 10 (2), 20 (3), 50 (4), and 100 (5).

the unknown functions in the FSDT and higher-order shear deformation theories (PSDT, ESDT, and SSDT) are five. It can be concluded that the present theory is not only accurate, but also simple in predicting the natural frequencies of laminated plates.

TABLE 3. Nondimensional Fundamental Frequencies of Antisymmetric Square Plates at Various Values of a/h with $E_1/E_2 = 40$

No. of layers	Theory	a/h					
		2	4	10	20	50	100
$(0/90)_1$	CLPT	8.6067	10.4244	11.1537	11.2693	11.3023	11.3070
	FSDT	5.2104	8.0349	10.4731	11.0779	11.2705	11.2990
	PSDT	5.7170	8.3546	10.5680	11.1052	11.2751	11.3002
	ESDT	5.8948	8.4561	10.5964	11.1132	11.2764	11.3005
	SSDT	5.8000	8.4017	10.5811	11.1089	11.2757	11.3003
	Present	5.6568	8.3208	10.5587	11.1025	11.2746	11.3000
$(0/90)_2$	CLPT	14.1036	16.3395	17.1448	17.2682	17.3032	17.3083
	FSDT	5.6656	9.8148	14.9214	16.6008	17.1899	17.2796
	PSDT	5.7546	9.7357	14.8463	16.5733	17.1849	17.2784
	ESDT	5.8129	9.7362	14.8338	16.5683	17.1840	17.2781
	SSDT	5.7794	9.7314	14.8376	16.5700	17.1843	17.2782
	Present	5.7413	9.7464	14.8571	16.5773	17.1857	17.2786
$(0/90)_3$	CLPT	15.0895	17.2676	18.0461	18.1652	18.1990	18.2038
	FSDT	5.6992	9.9852	15.5010	17.3926	18.0673	18.1706
	PSDT	5.8741	9.9878	15.4632	17.3772	18.0644	18.1698
	ESDT	5.9888	10.0323	15.4702	17.3787	18.0646	18.1699
	SSDT	5.9243	10.0036	15.4634	17.3768	18.0643	18.1698
	Present	5.8428	9.9838	15.4676	17.3792	18.0648	18.1699
$(0/90)_5$	CLPT	15.6064	17.7314	18.4916	18.6080	18.6410	18.6457
	FSDT	5.7140	10.0628	15.7790	17.7800	18.4995	18.6100
	PSDT	5.9524	10.1241	15.7700	17.7743	18.4984	18.6097
	ESDT	6.0889	10.1854	15.7847	17.7784	18.4991	18.6099
	SSDT	6.0133	10.1481	15.7739	17.7751	18.4985	18.6097
	Present	5.9129	10.1137	15.7716	17.7753	18.4986	18.6098

TABLE 4. Comparison of the Fundamental Frequency Parameter $\Omega = \omega a^2 \sqrt{\rho h / D}$ of Isotropic Square Plates

Thickness-to-length ratio	K_0, K_1	Theory		
		[27]	[26]	Present method
$h/a = 0.001$	0, 0	19.7391	19.7392	19.7322
	10^2, 10	26.2112	26.2112	26.2049
	10^3, 10^2	57.9961	57.9962	57.9894
$h/a = 0.1$	0, 0	19.0840	19.0658	19.0657
	10^2, 10	25.6368	25.6236	25.6235
	10^3, 10^2	57.3969	57.3923	57.3922
$h/a = 0.2$	0, 0	17.5055	17.4531	17.4531
	10^2, 10	24.3074	24.2728	24.2727
	10^3, 10^2	56.0359	56.0311	56.0310

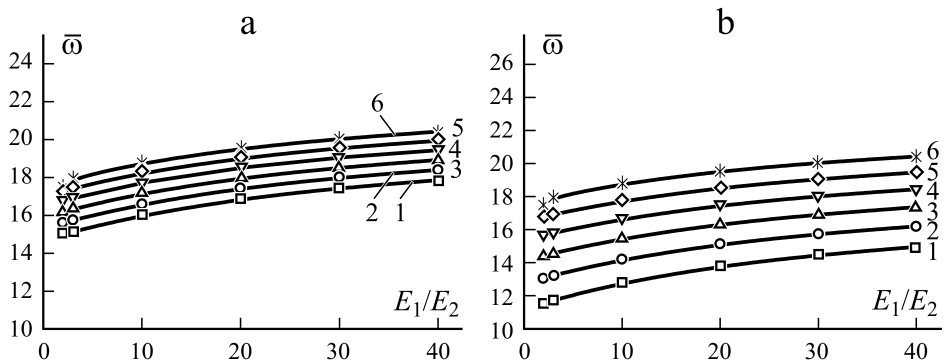

Fig. 2. Effect of the orthotropy ratio E_1/E_2 on the nondimensional fundamental frequency $\bar{\omega}$ of antisymmetrically laminated (0/90/0/90) cross-ply plate on an the elastic foundation at (K_0, K_1) = (0, 10) (1), (20, 10) (2), (40, 10) (3), (60, 10) (4), (80, 10) (5), and (100, 10) (6) (a) and (K_0, K_1) = (100, 0) (1), (100, 2) (2), (100, 4) (3), (100, 6) (4), (100, 8) (5), and (100, 10) (6) (b).

In order to validate the present theory in the case of plates resting on an elastic foundation, the results for the fundamental natural frequency parameter of an isotropic thick plate with three different values of thickness-to-length ratios and three different values of Winkler elastic coefficients are compared in Table 4 with those obtained in [26, 27]. A excellent agreement of the three methods can be seen. We should note here that, in Table 4, $D = Eh^3/12(1-\nu^2)$, as defined in [26].

In Fig. 1, variations in the nondimensional fundamental frequencies of symmetrically and antisymmetrically laminated orthotropic cross-ply plates on an elastic foundation are given. It is seen from the figures that an increase in the degree of orthotropy produces an increase in the fundamental frequency. The effect of foundation stiffness on the vibration of thick laminated plates is illustrated in Fig. 2. The figure shows that the frequencies of laminates increase when foundation parameters increase.

4. Conclusions

A refined hyperbolic shear deformation theory of plates has been successfully developed for the free vibration of simply supported laminated plates on an elastic foundation. The theory allows for a square-law variation in the transverse shear strains across the plate thickness and satisfies the zero-traction boundary conditions on the top and bottom surfaces of the plate without using shear correction factors. The equations of motion were derived from Hamilton's principle. All comparison studies show that the natural frequencies obtained by the proposed theory with four unknowns are almost identical to those predicted by the shear deformation theories containing five unknowns. It can be concluded that the theory proposed is accurate and efficient in predicting the vibration responses of composite plates.

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