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DESIGN RULES FOR THE LAMINATE STIFFNESS

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This paper aims at promoting laminate designs with specific stiffness properties: quasi-homogeneous, quasi-isotropic quasi-homogeneous, and thermally stable laminates. For this, the paper first discusses the concept of design, reviews the classical laminated plate theory and the usual design rules for laminates, and introduces the polar method for plane anisotropy. Then it develops rules and methods to design the stiffness properties that are looked for. In conclusion, it suggests that the classical design rules are too restrictive and that innovative designs should be explored for a better use of composites.

Introduction

This paper is derived from the presentation of a plenary lecture given at the MCM 2010 conference in Riga under the title "Design rules for laminates." It does not offer a theory of design of composite structures, but limits its aim to introduce and discuss practical elementary points, sometimes rules of thumb, which can aid the design. However, theoretical tools are required to derive these practical rules. Typical results are given, leaving to the reader the task to check in detail that they have the properties asserted.

Three main theoretical tools are used — the classical laminated plate theory (CLPT), some logic, and the polar description of plane anisotropy. Hereafter, the main features and the equations of CLPT are briefly reviewed, while the polar description of plane anisotropy is described with more detail. As for the logic needed, it is quite simple and well known: everybody learns from the basic mathematics that we should not confuse the necessary and sufficient conditions; however, this logical distinction should be recalled to mind again and again, as, strangely enough, it appears that this is often forgotten in the design of composites: the design rules, which are only sufficient conditions to get some properties, are often misunderstood, and even sometimes taught, as necessary ones!

The present paper is limited to the stiffness design, leaving the simultaneous strength and stiffness design to a further presentation. Moreover, it selects a limited number of typical design cases, namely quasi-homogeneous laminates, quasi-isotropic quasi-homogeneous laminates, and thermally stable laminates with a coupling.

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Design versus Analysis

In the author's opinion, analysis and design are completely opposite, as are deduction and induction. Generally speaking, a structural analysis is a deductive process. Starting from definite geometry, materials, and loading conditions, it computes, more or less easily and more or less exactly, the state of the structure. A structural design is the reverse, if any precise signification can be granted to "reverse". The aim is to find the parameters and conditions giving a required state of the structure. Mathematicians call these types of subject inverse problems, and they consider them as very difficult.

Two main concepts are encountered for design. On one side, those familiar with the modern numerical structural analysis would promote an automated design, which consists in iterating several analyses, usually with optimization techniques. This is often costly, lengthy, and heavy and may be effective or not. On the other side, the old technique was to use explicit rules derived from the state of the art, obtained by chance or experience. These rules, when existing, are very effective, fast, and easy to use, and consequently they are not obsolete. So, the research for such rules for composites, possibly with more sophisticated tools than in the past, is a promising way.

Finally, the objective of design should be questioned: is it to derive the solutions? or is it to find (or guess) some solution(s)? From a practical point of view, the time and the cost needed to obtain a result are balancing the accurateness and completeness of this result. Closed-form solutions, approximate solutions, and, more generally, closed-form or approximate information on the solutions can be valuable.

The Classical Laminated Plate Theory Reminded

The thermoelastic behaviour of laminated plates (when sufficiently thin) is correctly described by the generalized Hooke's law, which linearly relates the generalized forces (in-plane forces **N** and bending moments **M**) to the generalized strains (midplane strains $\boldsymbol{\varepsilon}^0$ and curvatures $\boldsymbol{\kappa}$), with complementary terms for the thermal effects linear with respect to the difference of temperature from a reference state, T_Δ. It is generally written in a compact form as

$$\begin{cases} \mathbf{N} = \mathbf{A}\boldsymbol{\varepsilon}^0 + \mathbf{B}\boldsymbol{\kappa} - \mathbf{r}T_\Delta , \\ \mathbf{M} = \mathbf{B}\boldsymbol{\varepsilon}^0 + \mathbf{D}\boldsymbol{\kappa} - \mathbf{s}T_\Delta . \end{cases}$$

The stiffnesses are composed of in-plane or membrane stiffnesses **A**, out-of-plane or bending stiffnesses **D**, and some membrane-bending coupling stiffnesses **B**. They are computed as weighted integrals of ply stiffnesses **Q** over the plate thickness *h*:

$$\mathbf{A}, \mathbf{B}, \mathbf{D} = \int_{-h/2}^{+h/2} \mathbf{Q}^* \, 1, z, z^2 \, dz.$$

In the same manner, the thermal effects include in-plane terms **r** and out-of-plane terms **s**, which are computed as weighted integrals of the products of ply stiffnesses **Q** and the thermal expansions of plies $\boldsymbol{\alpha}$:

$$\mathbf{r}, \mathbf{s} = \int_{-h/2}^{+h/2} \mathbf{Q}^* \boldsymbol{\alpha}^* \, 1, z, z^2 \, dz.$$

In these formulas, the superscript * applied to a quantity means that this quantity varies along the thickness *h* due to the different properties and different orientations of plies.

Although the stiffnesses are usually written as matrices, they are fourth-order tensors, while the thermal terms are second-order tensors. Consequently, a total of 18 components characterizes the elastic behaviour of the laminate, and 6 more components are used for its thermal behaviour.

The previous integrals are compact, but they do not reveal all parameters in the relationship between the laminate and lamina properties. These quantities are necessary for an analysis of a laminate; moreover, they are a part of design parameters. The formulas for the Cartesian tensor components are more explanatory. For a laminate, the integrals are reduced to sums of the material constants of plies multiplied by terms depending on the position of the plies in the thickness. Let us assume N orthotropic plies, with a ply number m in a θ_m-direction and extending from h_{m-1} to h_m across the thickness. Then, the stiffness components are

$$\begin{cases} A_{ijkl} = \sum_{m=1}^{N} Q_{ijkl}^{m}(\theta_m)(h_m - h_{m-1}), \\ B_{ijkl} = \sum_{m=1}^{N} Q_{ijkl}^{m}(\theta_m)\dfrac{h_m^2 - h_{m-1}^2}{2}, \\ D_{ijkl} = \sum_{m=1}^{N} Q_{ijkl}^{m}(\theta_m)\dfrac{h_m^3 - h_{m-1}^3}{3}, \end{cases}$$

while the thermal components are expressed as

$$\begin{cases} r_{ij} = \sum_{m=1}^{N} Q_{ijkl}^{m}(\theta_m)\alpha_{kl}^{m}(\theta_m)(h_m - h_{m-1}), \\ s_{ij} = \sum_{m=1}^{N} Q_{ijkl}^{m}(\theta_m)\alpha_{kl}^{m}(\theta_m)\dfrac{h_m^2 - h_{m-1}^2}{2}. \end{cases}$$

These equations are not completely explicit, as the terms θ_m hide cumbersome computations to rotate the stiffness and thermal components from their local axes to overall ones. With these Cartesian components, it is almost unthinkable to write closed-form formulas including all the quantities relevant to an analysis and design (on the contrary, the use of polar components, as described hereafter, allows one to completely explicate the relationship to all the parameters).

Another useful transformation is motivated by the fact that the various so-called stiffnesses do not have the dimension of stiffness, i.e., the dimension of Young's modulus, and, which is more, have different dimensions. Consequently, they cannot be compared easily with each other or with those of another material (and so are the thermal terms). By a proper transformation, they can be set to the common dimension of Young's modulus (in MPa), which is the usual dimension for the stiffness:

$$\begin{cases} \mathbf{A} = h\overline{\mathbf{Q}}, \\ \mathbf{B} = \dfrac{h^2}{\sqrt{12}}\hat{\mathbf{Q}}, \\ \mathbf{D} = \dfrac{h^3}{12}\tilde{\mathbf{Q}}. \end{cases}$$

From the presence of the factor h in the transformation for **A**, it follows that the modified in-plane stiffness $\hat{\mathbf{Q}}$ has a physical meaning: it is the apparent stiffness which will be measured using the classical tensile tests for homogeneous materials. In the same way, with the factor $h^3/12$, typical of the bending stiffness for homogeneous structures, the modified in-plane stiffness $\hat{\mathbf{Q}}$ is the apparent stiffness which will be measured using the classical bending tests for homogeneous materials. They are the tensile mean value and the bending mean value, respectively, of the stiffness of stacked plies.

Finally, a distinction should be made between two classes of laminates. One type of laminates is made of plies identical in nature (the same material and thickness), but with different orientations: these stacking sequences are completely defined by angles of the plies and are called hereafter isolaminar laminates, as opposed to hybrid laminates, which are the other type that mixes different types of plies (such as fabrics and unidirectionally reinforced plies, or laminas with different fibres). In isolaminar laminates, the position parameters h_m of plies are related to the total thickness h, the total number of plies N, and the order m of the ply as

$$h_m = \frac{h}{2}\cdot\frac{2m-N}{N}.$$

Review and Discussion of Some Classical Design Rules for Laminates

Certainly, the most famous rule is the s*ymmetric stacking sequence rule*: each ply has a counterpart with an identical nature, thickness, and orientation which is positioned symmetrically with respect to the midplane, with the possible central ply (existing when the total number of plies is odd) being resultantly symmetric itself. This suppresses the membrane-bending coupling by setting $\mathbf{B} = 0$, as the contributions of two symmetric plies to the coupling cancel each other (the possible central ply has a zero contribution). The same cancellation of contributions occurs for the thermal coupling term, so $\mathbf{s} = \mathbf{0}$ is also obtained.

Another well-known rule is the *Werren and Norris rule* [1], which was proposed in 1953 for plywood laminates. It reads: align the same numbers of plies along n directions (with n equal to 3 or more) dividing equally the circle. It gives isotropy for the membrane properties: the stiffness $\mathbf{A}$ and the thermal term $\mathbf{r}$ are isotropic, whatever the anisotropy of the constitutive plies.

Another rule is the *angle-ply or cross-ply rule* in order to get the in-plane orthotropy: for cross-ply laminates made of orthotropic plies at 0 or 90°, the orthotropy is preserved at the overall level, and, for angle-ply laminates made of the same numbers of plies at angles $+\psi$ and $-\psi$, the anisotropic contributions due to the off-axis rotation cancel each other, and the orthotropy is recovered. This also applies to the in-plane stiffness A and to the in-plane thermal term r.

First comment. These rules are widely used: almost all the composites manufactured are symmetric, while the Warren and Norris rule is at the origin of quasi-isotropic laminates or "black aluminum", i.e., any laminate with plies at 0, +45, –45, and 90° in the same proportions and, more widely, at the origin of the popular family of stacking sequences limited to these four angles.

Second comment. The Werren and Norris rule and the angle-ply rule are limited: although this is generally not emphasized, they are valid only for isolaminar laminates. The symmetric sequence rule and the cross-ply rule are larger in their scope, as they apply to hybrid laminates.

Third comment. The Werren and Norris rule is limited to the membrane thermoelastic properties: the bending properties and the strength properties (in membrane or bending) are not isotropic. Similarly, the angle-ply or cross-ply rule is limited to the membrane properties.

The major comment. These rules are sufficient conditions, not necessary ones, to get the properties requested: the rules give the properties, the properties do not require the rules. Strangely enough, this logical point is very frequently misconceived, and these sufficient conditions are erroneously presented as necessary ones. You can find this mistake in otherwise excellent books and papers. Apart from violating the logics, such a misconception greatly restricts the field of admissible laminates and restrains innovative designs.

However, examples countering this misconception have been known for long. Limiting here to discussion of the first rule, the zero coupling does not require any symmetry. Unsymmetric laminates can be uncoupled. In 1982, Caprino and Crivelli–Visconti published the uncoupled antisymmetric 8-ply laminate [2]

$$[+\theta, -\theta, -\theta, +\theta, -\theta, +\theta, +\theta, -\theta],$$

with θ an arbitrary angle.

Consider also the 7-ply laminate

$$[abbcaab]$$

with a, b, and c arbitrary angles, which is the noncoupled, nonsymmetric, isolaminar laminate with the smallest number of plies.

When the number of plies is increased, one can find more and more examples of unsymmetric uncoupled laminates. For isolaminar laminates, the number of solutions jumps from 1 for 7 or 8 plies to 40 for 13 plies and to 1638 for 18 plies (with 2 to 7 arbitrary angles).

Outlines of the Polar Method

This somewhat complex (in every sense) method is highly efficient for 2-dimensional problems, including the 2-dimensional anisotropy. Consequently, it is very useful for laminates. It finds its origin in well-known or less-known works: the complex variable and the polar coordinates for plane geometry, Mohr's circle and the Kolosov–Muskhelishvili formulas for stresses and strains [3], and the complex coordinates as interpreted by Green and Zerna [4]. For stiffnesses and compliances, Tsai and Hahn have introduced some quantities U_k prefiguring the variables used in the polar method [5].

The polar method represents 2-dimensional tensors of any order with real and complex quantities. The ones useful for the present paper are as follows:

— for a second-order tensor, such as the stress, strain, and thermal expansion tensors, the polar components are introduced in terms of Cartesian components as

$$\begin{cases} 2T = T_{xx} + T_{yy}, \\ 2Re^{2i\theta} = T_{xx} - T_{yy} + 2iT_{xy}, \end{cases}$$

from which the Cartesian components are expressed as

$$\begin{cases} T_{xx} = T + R\cos 2\theta, \\ T_{xy} = \phantom{T +{}} R\sin 2\theta, \\ T_{yy} = T - R\cos 2\theta, \end{cases}$$

where the scalars T and R are easily recognized as the position of the centre and the radius of Mohr's circle;

— for a fourth-order tensor, such as the stiffness or compliance tensors, the polar components are introduced in terms of Cartesian components in the form

$$\begin{cases} 8T_0 = T_{xxxx} - 2T_{xxyy} + 4T_{xyxy} + T_{yyyy}, \\ 8T_1 = T_{xxxx} + 2T_{xxyy} + T_{yyyy}, \\ 8Re^{4i\theta_0} = T_{xxxx} - 2T_{xxyy} - 4T_{xyxy} + T_{yyyy} + 4i(T_{xxxy} - T_{xyyy}), \\ 8Re^{2i\theta_1} = T_{xxxx} + 4T_{xyxy} + T_{yyyy} + 2i(T_{xxxy} + T_{xyyy}) \end{cases}$$

with Cartesian components expressed as

$$\begin{cases} T_{xxxx} = T_0 + 2T_1 + R_0\cos 4\theta_0 + 4R_1\cos 2\theta_1, \\ T_{xxxy} = + R_0\sin 4\theta_0 + 2R_1\sin 2\theta_1, \\ T_{xxyy} = -T_0 + 2T_1 - R_0\cos 4\theta_0, \\ T_{xyxy} = T_0 - R_0\cos 4\theta_0, \\ T_{xyyy} = - R_0\sin 4\theta_0 + 2R_1\sin 2\theta_1, \\ T_{yyyy} = T_0 + 2T_1 + R_0\cos 4\theta_0 - 4R_1\cos 2\theta_1. \end{cases}$$

The main property of the polar components is that, contrary to the heavy formulas for rotating the Cartesian components, they transform very simply when the axes are rotated: the scalars T_0 and T_1 and the moduli R_0 and R_1 remain unchanged, while the polar angles θ_0 and θ_1 are decreased by the angle of rotation of the axes. So, five invariants for the rotation exist: T_0, T_1, R_0, R_1, and $\theta_0 - \theta_1$.

Another major property is that the polar parameters are closely related to the material symmetry. For elasticity, the general anisotropy, without any symmetry, needs all the six parameters.

The orthotropy is defined by only five parameters, with the following condition on the polar angles:

$$\theta_0 - \theta_1 = K\frac{\pi}{4},$$

where $K = 0$ or $K = 1$. Consequently, the Cartesian components for orthotropy reduce to

$$\begin{cases} T_{xxxx} = T_0 + 2T_1 + (-1)^K R_0 \cos 4\theta_1 + 4R_1 \cos 2\theta_1, \\ T_{xxxy} = (-1)^K R_0 \sin 4\theta_1 + 2R_1 \sin 2\theta_1, \\ T_{xxyy} = -T_0 + 2T_1 - (-1)^K R_0 \cos 4\theta_1, \\ T_{xyxy} = T_0 - (-1)^K R_0 \cos 4\theta_1, \\ T_{xyyy} = -(-1)^K R_0 \sin 4\theta_1 + 2R_1 \sin 2\theta_1, \\ T_{yyyy} = T_0 + 2T_1 + (-1)^K R_0 \cos 4\theta_1 - 4R_1 \cos 2\theta_1. \end{cases}$$

The square symmetry is defined by four parameters, with the condition

$$R_1 = 0,$$

and the isotropy, with two parameters, is obtained at

$$R_0 = R_1 = 0.$$

Consequently, T_0 and T_1 represent the isotropic part, R_0 is the square-symmetric part with θ_0 defining the material axis, and R_1 is the orthotropic part with θ_1 defining the material axis.

The Polar Method as Applied to Laminates

The basic formulas of the classical laminated plate theory can be transcribed with polar parameters. For each type of stiffness (membrane, coupling, and bending ones), polar parameters can be introduced. As they are linear combinations of Cartesian components, each polar component of a laminate depends only on the same type of polar component of plies. Consequently, the rotation of axes has no effect on the scalar components and changes the complex components in a simple way.

Six real and six complex equations replace the 18 usual equations for **A**, **B**, and **D**. They range from

$$\overline{T}_0 = \frac{1}{h}\sum_{m=1}^{N} T_0^m (h_m - h_{m-1})$$

to

$$\widetilde{R}_1 e^{2i\widetilde{\theta}_1} = \frac{12}{h^3}\sum_{m=1}^{N} R_1^m e^{2i\theta_{1m}} \frac{h_m^3 - h_{m-1}^3}{3}.$$

These equations have a form similar to the usual ones, with the difference that all the parameters are included explicitly.

Now, restricting ourselves to isolaminar laminates, we get the following very peculiar relations:

— for the 6 isotropic parts,

$$\overline{T}_0 = T_0, \quad \overline{T}_1 = T_1,$$

$$\hat{T}_0 = 0, \quad \hat{T}_1 = 0,$$

$$\widetilde{T}_0 = T_0, \quad \widetilde{T}_1 = T_1;$$

— for the 6 complex parts,

$$\overline{R}_0 e^{4i\overline{\theta}_0} = (-1)^K R_0 \sum_{m=1}^{N} \frac{1}{N} e^{4i\theta_m},$$

$$\overline{R}_1 e^{2i\overline{\theta}_1} = R_1 \sum_{m=1}^{N} \frac{1}{N} e^{2i\theta_m},$$

$$\hat{R}_0 e^{4i\hat{\theta}_0} = (-1)^K R_0 \sum_{m=1}^{N} \sqrt{3}\, \frac{2m-N-1}{N^2} e^{4i\theta_m},$$

$$\hat{R}_1 e^{2i\hat{\theta}_1} = R_1 \sum_{m=1}^{N} \sqrt{3}\, \frac{2m-N-1}{N^2} e^{2i\theta_m},$$

$$\widetilde{R}_0 e^{4i\widetilde{\theta}_0} = (-1)^K R_0 \sum_{m=1}^{N} \frac{3(2m-N-1)^2+1}{N^3} e^{4i\theta_m},$$

$$\widetilde{R}_1 e^{2i\widetilde{\theta}_1} = R_1 \sum_{m=1}^{N} \frac{3(2m-N-1)^2+1}{N^3} e^{2i\theta_m}.$$

As in the usual expression of 18 Cartesian equations for lamination, the material properties and the stacking characteristics are separate in these 12 equations. This is typical of isolaminar stacking sequences.

Otherwise, notable differences from the usual equations appear. The most apparent point is the form of the isotropic parts: the membrane and bending isotropic parts of the laminate are equal to the isotropic parts of the constitutive ply, while the coupling isotropic parts are zero. But the main point is certainly the fact that these 12 equations are completely closed-form formulas, including all the parameters explicitly. Specially, the orientation angles θ_m of the plies also appear in them, which cannot be achieved in the usual expressions.

To summarize the advantages of the polar equations versus the Cartesian ones, instead of 18 quantities, 6 are already fixed (the isotropic ones), and the remaining 6 complex quantities explicitly present all the parameters involved in the design. From this, it is possible to look for a more complex design, beyond the range of classical ones, as shown in the following.

Design of Quasi-Homogeneous Laminates

A quasi-homogeneous laminate is defined as a laminate with the same apparent membrane and bending stiffnesses, but without the membrane-bending coupling. It satisfies the following equations, written with the transformed or the usual stiffnesses:

$$\overline{\mathbf{Q}} = \widetilde{\mathbf{Q}}, \quad \hat{\mathbf{Q}} = \mathbf{0}$$

or

$$\frac{1}{h}\mathbf{A} = \frac{12}{h^3}\mathbf{D}, \quad \mathbf{B} = \mathbf{0}.$$

The quasi-homogeneous designation for these laminates is derived from the fact that the previous relationships are satisfied by homogeneous materials.

First, such laminates do exist! At least unidirectional laminates satisfy the equations, and it will be shown that many others can be found.

Second, the quasi-homogeneous laminates possess interesting properties:

— there are less independent elastic constants, 6 instead of 18 in the case without any symmetry, and less when some symmetry exists,

— the in-plane and out-of-plane stiffnesses have the same symmetries (for instance, they are simultaneously orthotropic with the same orthotropy axes),

— when dealing with problems which involve simultaneously the in-plane and bending properties, such as buckling or transverse vibrations, they have a simpler behaviour,

— for isolaminar quasi-homogeneous laminates, not only the mechanical coupling $\mathbf{B} = \mathbf{0}$ is zero, but also the thermal coupling vanishes ($\mathbf{s} = \mathbf{0}$).

To summarize, quasi-homogeneous laminates behave like simple homogeneous materials in many respects.

It is consequently interesting and useful to try to design such laminates.

There are 12 scalar defining equations with the usual notation. Transcribed with polar parameters, there are 4 scalar and 4 complex defining equations, namely

$$\begin{cases} \overline{T}_0 = \widetilde{T}_0 , \\ \overline{T}_1 = \widetilde{T}_1 , \\ \overline{R}_0 e^{4i\overline{\theta}_0} = \widetilde{R}_0 e^{4i\widetilde{\theta}_0} , \\ \overline{R}_1 e^{2i\overline{\theta}_1} = \widetilde{R}_1 e^{2i\widetilde{\theta}_1} , \\ \hat{T}_0 = 0, \\ \hat{T}_1 = 0, \\ \hat{R}_0 e^{4i\overline{\theta}_0} = 0, \\ \hat{R}_1 e^{2i\hat{\theta}_0} = 0. \end{cases}$$

When limiting to isolaminar laminates, the material constants are factorized in the equations. Further, the scalar conditions, which represent the isotropic parts, are satisfied. After reducing the material constants, the four remaining complex conditions include only the total number N of plies and the index and angle of each ply:

$$\begin{cases} \sum_{m=1}^{N} \frac{1}{N} e^{4i\theta_m} = \sum_{m=1}^{N} \frac{3(2m-N-1)^2+1}{N^3} e^{4i\theta_m} , \\ \sum_{m=1}^{N} \frac{1}{N} e^{2i\theta_m} = \sum_{m=1}^{N} \frac{3(2m-N-1)^2+1}{N^3} e^{2i\theta_m} , \\ \sum_{m=1}^{N} \sqrt{3}\, \frac{2m-N-1}{N^2} e^{4i\theta_m} = 0, \\ \sum_{m=1}^{N} \sqrt{3}\, \frac{2m-N-1}{N^2} e^{2i\theta_m} = 0. \end{cases}$$

Consequently, for each given number of plies N, we have 4 complex equations (or 8 real equations) and N unknowns (the angles), which are the design parameters.

Unfortunately, there is no hope to find a general explicit formula solving this system of equations. However, it is possible to develop particular solutions. To this end, the sums are first split into partial sums corresponding to the contribution of

TABLE 1. Particular Solutions for Quasi-Homogeneous Laminates with 7 to 30 Plies

Number of plies	Number of particular solutions						
	Total	Symmetric	Number of orientation angles				
			2	3	4	5	6
7		1	1				
8	1		1				
9	0						
10	0						
11	3	2	3				
12	1		1				
13	4		2	2			
14	2	1		2			
15	4		2	2			
16	8	1	5	3			
17	23		15	8			
18	5			5			
19	52		30	22			
20	40		30	9	1		
21	44	2	31	13			
22	130	3	17	97	13	2	
23	594	1	95	499			
24	167		140	26	1		
25	2352		163	2132	57		
26	1495	7	54	1059	354	26	2
27	1282	1	86	918	256	21	1
28	5902	3	203	4789	871	33	6
29	45441		61	37747	7546	87	
30	6146	3	53	5552	512	29	

plies of the same orientation, with the angle terms in factors. Then, the objective is to make each partial sum equal to zero. Doing so, two scalar equations remain for each group of the same orientation:

$$\begin{cases} \sum\limits_{\text{partial}} [3(2m-N-1)^2 + 1 - N^2] = 0, \\ \sum\limits_{\text{partial}} (2m-N-1) = 0. \end{cases}$$

Consequently, there are $2P$ equations to solve, with P the (unknown) number of groups (with $P < N$). As these equations are expressed only with integers, they can be solved using a simple enumerative method. The critical task is to sort the independent solutions. All this is a tedious but very simple process, and we computed all the solutions for up to 30 plies and established a directory of these solutions, see Table 1.

The particular solutions for small numbers of plies are not numerous: only one for 7 and 8 plies and 3 for 11 plies. For more plies, there are solutions for any number of them, but the number of solutions increases dramatically. Here are the first ones:

— 7-ply symmetric solution with 2 arbitrary angles *a* and *b*,

$$[abaaaba];$$

— 8-ply antisymmetric solution with 2 arbitrary angles *a* and *b*, in fact, the sequence proposed by Caprino and Crivelli-Visconti for zero coupling, which strangely has not been recognized as quasi-homogeneous,

$$[abbabaab];$$

— 11-ply solutions with 2 arbitrary angles *a* and *b*, of which two are symmetric and one unsymmetric,

$$[abbaabaabba],\ [ababaaababa],\ [ababbaabaab].$$

Table 1 gives the number and basic properties of solutions for up to 30 plies. It can be seen that the number of solutions increases more or less regularly with the total number of plies. The number of groups increases too: starting with 13 plies, there are 3-directional solutions, and 4-directional solutions with 20 plies or more. Symmetric solutions are not common compared with unsymmetric ones, illustrating the fact that a zero coupling can be obtained with unsymmetric laminates.

It should be emphasized that these solutions are for arbitrary angles, and in fact generate families of laminates. All this means that users have at disposal a huge number of potential lay-ups, enough to satisfy most of the needs.

And finally, it should be repeated once more that the conditions used here to build the particular solutions are sufficient conditions, not necessary ones. We have some quasi-homogeneous laminates, not all the quasi-homogeneous laminates. But, hopefully, it is enough for a practical design.

Design of Quasi-Isotropic Quasi-Homogeneous Laminates

Quasi-isotropic quasi-homogeneous laminates are really similar to classical materials in their elastic properties. A sufficient method to obtain such materials is to look among the special quasi-homogeneous solutions and try to apply the Werren and Norris rule. One gets several solutions for 18, 24, 27, and 30 plies, respectively with 3, 4, 3, and 5 directions, respectively.

As examples, here are two families of solutions with 18 plies, *a*, *b*, and *c* being any different angle 0, +60, and –60°:

$$[abcabcccbbaabaaccb] \quad \text{and} \quad [abccabbcabcacababc].$$

Reminding that quasi-isotropic laminates are commonly used, although they are isotropic only for membrane properties, certain quasi-isotropic quasi-homogeneous laminates, which are isotropic for both membrane and bending properties, could be even more useful and efficient for some applications.

Design of Thermally Stable Coupled Laminates

While from the inception of composite materials, zero coupling ($\mathbf{B} = \mathbf{0}$) has been looked for, it has been recognized more recently that a nonzero coupling may be of interest for applications such as wings or blades in order to react conveniently to aerodynamic loads.

For laminates with an elastic membrane-bending coupling, a thermal coupling can also appear with the thermal term $\mathbf{s}$ in the generalized Hooke’s law presented above. This must be taken into account in their use as well as in their manufacturing, generally made by hot pressing of prepregs.

Related to these facts, the concept of thermal stability was introduced, as early as 1985, in works by Winckler [6], Chen [7], Armanios et al. [8, 9], and others. They derived stacking sequences of coupled laminates which should remain flat during the postcure cooling and, more generally, during uniform changes of temperature. All the authors considered laminates made of linear, thermoelastic unidirectional plies. The 8-ply solution family proposed by Winckler is

$$[+\theta/(\theta-\pi/2)_2/+\theta/-\theta/(\pi/2-\theta)_2/-\theta].$$

For thermally stable laminates, the generalized Hooke's law should be satisfied with zero generalized forces and no curvatures, which reduces to

$$\begin{cases} \mathbf{A}\boldsymbol{\varepsilon}^0 = \mathbf{r}T_\Delta\,, \\ \mathbf{B}\boldsymbol{\varepsilon}^0 = \mathbf{s}T_\Delta\,. \end{cases}$$

This system represents six equations for three unknown strains, which is generally impossible.

A particular case is got when the couplings **B** and **s** are zero and the system reduces to three equations: any non- coupled laminate is thermally stable.

When a coupling does exist, the system has no solutions, except when the mathematical compatibility of the equations is reached. This compatibility can be written as

$$\mathbf{s} = \mathbf{B}\mathbf{A}^{-1}\mathbf{r}.$$

Again, the general solution of this nonlinear equation is out of reach. However, when restricting to isolaminar laminates and using the polar parameters, the compatibility conditions for the above-mentioned equations can be obtained. They are:

A and **B** with the square symmetry, **r** isotropic, and **s** = **0**.

Although such conditions have not been mentioned in the previous works, all the published solutions with a unidirectional reinforcement satisfy them.

A consequence is that the reinforcement with balanced fabrics is an efficient alternative to unidirectional plies to obtain thermally stable laminates.

So, the design rule for thermally stable coupled laminates is: use any sequence of plies reinforced with the same balanced fabrics.

Final Comments and Conclusions

As most of the results in the literature, the rules and results presented in this paper are for isolaminar laminates. Some extend to hybrid ones. For instance, quasi-homogeneous hybrid laminates can be designed. The solutions depend on the thickness of plies. With the simple assumption of equal thickness for the plies of two different materials, one along the direction *a* and the other along the direction *b*, the 7-ply solution

$$[aba_3ba]$$

$$[a_5ba_{14}ba_5]\,.$$

But, generally speaking, the design of hybrid laminates is still an open subject which should be studied extensively.

It should be mentioned without further reference that the experimental evidence supports these design concepts: we manufactured a lot of laminates such as orthotropic quasi-homogeneous laminates and quasi-isotropic quasi-homogeneous ones, for which we measured properties consistent with the predicted design. Everybody — especially every sceptic — can do it and check by himself that nonclassical designs are feasible and advantageous.

This paper has shown that the classical designs of laminates are quite limiting in comparison with the large capabilities of composite materials and that other designs with promising properties can be derived using efficient methods, such as the polar method for anisotropy. Certainly, the classical designs do not pay tribute to the potentialities of composites and should be replaced by more effective rules. So, the general conclusion could be: trust and try nonclassical laminate designs, and specially trust unsymmetric laminates!

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