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TWO NEW HYPERBOLIC SHEAR DISPLACEMENT MODELS FOR ORTHOTROPIC LAMINATED COMPOSITE PLATES

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Two hyperbolic displacement models, HPSDT1 and HPSDT2, are developed for a bending analysis of orthotropic laminated composite plates. These models take into account the parabolic distribution of transverse shear stresses and satisfy the condition of zero shear stresses on the top and bottom surfaces of the plates. The accuracy of the analysis presented is demonstrated by comparing the results with solutions derived from other higher-order models and with data found in the literature. It is established that the HPSDT1 model is more accurate than some theories of laminates developed previously, and therefore the analysis can be expanded to laminated composite shells.

Introduction

In the classical plate theory (CPT), which is the simplest theory of plates, the normals to their midplane remain straight and perpendicual to the plane after deformation. This is the result of neglecting the transverse shear strains. The displacement field in the CPT is

$$u(x, y, z) = u_0 - z\frac{\partial w_0}{\partial x}, \quad v(x, y, z) = v_0 - z\frac{\partial w_0}{\partial y}, \quad w(x, y, z) = -w_0(x, y),$$

where u, v, and w are displacements along the x, y, and z coordinate directions, respectively, and u_0, v_0, and w_0 are the midplane displacements.

However, in thick and moderately thick plates, the transverse shear strains have to be taken into account. There are numerous plate theories that include these strains. The first-order shear deformation theory (FSDT), which is known as the Mindlin plate theory [1-3], considers the displacement field as linear variations of midplane displacements:

$$u(x, y, z) = u_0 + z\theta_x, \quad v(x, y, z) = v_0 + z\theta_y, \quad w(x, y, z) = w_0(x, y).$$

In this theory, the relation between the resultant shear forces and the shear strains depends on shear correction factors [4-6].

Some other plate theories, e.g., higher-order shear deformation theories (HSDT), which include the effect of transverse shear strains, are reported in the literature.

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Higher-order theories based on series expansions were developed by Donnel [7], Reissner [8], and Lo et al. [9, 10] and were modified by Levinson [11], Murthy [12], and Reddy [13]. The displacement field in these theories is

$$u(x, y, z) = u_0 + z\theta_x + z^2\psi_x + z^3\xi_x, \;\; v(x, y, z) = v_0 + z\theta_y + z^2\psi_y + z^3\xi_y,$$

$$w(x, y, z) = w_0 + z\theta_z + z^2\psi_z.$$

Reddy [13, 14] put forward a modified third-order theory which considers not only the transverse shear strains, but also their parabolic variation across the plate thickness. As a result, there is no need to use shear correction coefficients in computing the shear stresses. In this theory,

$$u(x, y, z) = u_0 + z\left[\theta_x - \frac{4}{3h^2} z^2\left(\theta_x + \frac{\partial w_0}{\partial x}\right)\right],$$

$$v(x, y, z) = v_0 + z\left[\theta_y - \frac{4}{3h^2} z^2\left(\theta_y + \frac{\partial w_0}{\partial y}\right)\right], \;\; w(x, y, z) = w_0(x, y).$$

Touratier [15] used trigonometric functions for describing the parabolic distribution of transverse shear strains across the plate thickness and took the displacement field in the form

$$u(x, y, z) = u_0 - z\frac{\partial w_0}{\partial x} + \left(\frac{h}{\pi}\right)\sin\left(\frac{\pi z}{h}\right)\theta_x,$$

$$v(x, y, z) = v_0 - z\frac{\partial w_0}{\partial y} + \left(\frac{h}{\pi}\right)\sin\left(\frac{\pi z}{h}\right)\theta_y, \;\; w(x, y, z) = w_0.$$

Soldatos [16] employed hyperbolic functions for this purpose:

$$u(x, y, z) = u_0 - z\frac{\partial w_0}{\partial x} + \left[h\sinh\left(\frac{z}{h}\right) - z\cosh\left(\frac{1}{2}\right)\right]\theta_x,$$

$$v(x, y, z) = v_0 - z\frac{\partial w_0}{\partial y} + \left[h\sinh\left(\frac{z}{h}\right) - z\cosh\left(\frac{1}{2}\right)\right]\theta_y, \;\; w(x, y, z) = w_0(x, y),$$

Karama et al. [17] used an exponential model:

$$u(x, y, z) = u_0 - z\frac{\partial w_0}{\partial x} + ze^{-2(z/h)^2}\theta_x,$$

$$v(x, y, z) = v_0 - z\frac{\partial w_0}{\partial y} + ze^{-2(z/h)^2}\theta_y, \;\; w(x, y, z) = w_0.$$

Ferreira et al. [18] assumed the displacement field in the form

$$u(x, y, z) = u_0 - z\frac{\partial w_0}{\partial x} + \sin\left(\frac{\pi z}{h}\right)\theta_x,$$

$$v(x, y, z) = v_0 - z\frac{\partial w_0}{\partial y} + \sin\left(\frac{\pi z}{h}\right)\theta_y, \;\; w(x, y, z) = w_0(x, y).$$

There are also many comparison studies on the behavior of transverse shear stresses in laminated composite plates [19-21].

In this study, two new hyperbolic displacement models for an analysis of simply supported laminated composite plates are proposed. Analytical solutions for bending deflections of symmetric cross-ply laminates are obtained. The governing equations are derived from the principle of minimum total potential energy. It is found that the HPSDT1 model provides more accurate results than some other higher-order shear deformation theories and can easily be used for computations.

2. Displacement Field and Constitutive Equations

In the present analysis, displacement field models satisfying the condition of zero transverse shear stresses on the top and bottom surface of the plate are considered. The displacement field is written in the unified form

$$u = u_0(x, y) - z\frac{\partial w}{\partial x} + f(z)\theta_x, \quad v = v_0(x, y) - z\frac{\partial w}{\partial y} + f(z)\theta_y, \quad w = w_0(x, y), \tag{1}$$

where u, v, and w are displacements in the x, y, and z directions; u_0, v_0, and w_0 are midplane displacements; θ_x and θ_y are the rotations of normals to the midplane about the y and x axes, respectively; $f(z)$ is a hyperbolic shape function.

The function $f(z)$ is chosen in the form

$$f(z) = \frac{3\pi}{2} h \tanh\left(\frac{z}{h}\right) - \frac{3\pi}{2} z \operatorname{sech}^2\left(\frac{1}{2}\right) \text{ for the HPSDT1 model and}$$

$$f(z) = z \operatorname{sech}\left(\frac{\pi z^2}{h^2}\right) - z \operatorname{sech}\left(\frac{\pi}{4}\right)\left[1 - \frac{\pi}{2}\tanh\left(\frac{\pi}{4}\right)\right] \text{ for the HPSDT2 model.}$$

Assumption (1) yields the following relations for the normal and shear strains:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} + f(z)\frac{\partial \theta_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} + f(z)\frac{\partial \theta_y}{\partial y},$$

$$\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x \partial y} + f(z)\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right), \tag{2}$$

$$\gamma_{yz} = f'(z)\theta_y, \quad \gamma_{xz} = f'(z)\theta_x,$$

where

$$f'(z) = \frac{df(z)}{dz}. \tag{3}$$

The constitutive equations for a rectangular plate consisting of an orthotropic material can be expressed in its symmetry axes as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}, \quad \begin{Bmatrix} \tau_{23} \\ \tau_{31} \end{Bmatrix} = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{23} \\ \gamma_{31} \end{Bmatrix}.$$

If fiber orientations do not coincide with the principal directions of the plate, the plate stresses can be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix},$$

where Q_{ij} and $\overline{Q}_{ij}$ are material constants, which are defined in [14].

3. Equilibrium Equations and Boundary Conditions

The equilibrium equations are derived by using the virtual work principle, which can be written for the plate as

$$\int_V \{\delta\varepsilon\}^T \{\sigma\} dV + \int_V \{\delta\gamma\}^T \{\tau\} dV - \int_V \{q\} \{\delta u\} dV = 0.$$

Using Eqs. (2) and (3), the previous equation can be put in the form

$$\int_0^t \Bigg[\int_A \Bigg\{ \delta u \frac{\partial N_x}{\partial x} + \delta v \frac{\partial N_y}{\partial y} + \delta u \frac{\partial N_{xy}}{\partial y} + \delta v \frac{\partial N_{xy}}{\partial x} - \delta w \frac{\partial^2 M_x}{\partial x^2} - \delta w \frac{\partial^2 M_y}{\partial y^2}$$

$$- \delta w 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \delta\theta_x \frac{\partial P_x}{\partial x} + \delta\theta_y \frac{\partial P_y}{\partial y} + \delta\theta_x \frac{\partial P_{xy}}{\partial y} + \delta\theta_y \frac{\partial P_{xy}}{\partial x}$$

$$+ \delta\theta_y (-R_y) + \delta\theta_x (-R_x) \Bigg\} dA + \int_A q \delta w \, dx \, dy \Bigg] dt = 0, \tag{4}$$

where the stress and moment resultants are

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & C_{11} & C_{12} & C_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & C_{12} & C_{22} & C_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & C_{16} & C_{26} & C_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & E_{11} & E_{12} & E_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & E_{12} & E_{22} & E_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & E_{16} & E_{26} & E_{66} \\ C_{11} & C_{12} & C_{16} & E_{11} & E_{12} & E_{16} & G_{11} & G_{12} & G_{16} \\ C_{12} & C_{22} & C_{26} & E_{12} & E_{22} & E_{26} & G_{12} & G_{22} & G_{26} \\ C_{16} & C_{26} & C_{66} & E_{16} & E_{26} & E_{66} & G_{16} & G_{26} & G_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\ -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \\ \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{Bmatrix}, \begin{Bmatrix} R_y \\ R_x \end{Bmatrix} = \begin{bmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{bmatrix} \begin{Bmatrix} \theta_y \\ \theta_x \end{Bmatrix}$$

The stiffness components are expressed as follows:

$$\{A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, G_{ij}\}$$

$$= \int_{-h/2}^{h/2} \{1, z, f(z), z^2, zf(z), [f(z)]^2\} \overline{Q}_{ij}^k \, dz, \quad i, j = 1, 2, 6;$$

$$\{F_{ij}\} = \int_{-h/2}^{h/2} [f'(z)]^2 \overline{Q}_{ij}^k \, dz, \quad i, j = 4, 5. \tag{5}$$

For symmetric cross-ply plates,

$$A_{16} = A_{26} = F_{45} = 0, \quad C_{16} = C_{26} = 0, \quad D_{16} = D_{26} = 0, \quad E_{16} = E_{26} = 0,$$

$$G_{16} = G_{26} = 0, \quad B_{ij} = 0, \quad i, j = 1, 2, 6.$$

Integrating the expressions in Eq. (4) by parts and collecting the coefficients of δu, δv, δw, $\delta\theta_x$, and $\delta\theta_y$, we arrive at the equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \quad \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - q = 0,$$

$$\frac{\partial P_x}{\partial x} + \frac{\partial P_{xy}}{\partial y} - R_x = 0, \quad \frac{\partial P_{xy}}{\partial x} + \frac{\partial P_y}{\partial y} - R_y = 0. \tag{6}$$

Substituting Eq. (5) into Eqs. (6), these equations are obtained in terms of displacements:

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{12}\frac{\partial^2 v_0}{\partial x \partial y} + A_{66}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y}\right)$$

$$+ C_{11}\frac{\partial^2 \theta_x}{\partial x^2} + C_{12}\frac{\partial^2 \theta_y}{\partial x \partial y} + C_{66}\left(\frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial^2 \theta_y}{\partial x \partial y}\right) = 0, \tag{7}$$

$$A_{12}\frac{\partial^2 u_0}{\partial x \partial y} + A_{22}\frac{\partial^2 v_0}{\partial y^2} + A_{66}\left(\frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial^2 v_0}{\partial x^2}\right)$$

$$+ C_{12}\frac{\partial^2 \theta_x}{\partial x \partial y} + C_{22}\frac{\partial^2 \theta_y}{\partial y^2} + C_{66}\left(\frac{\partial^2 \theta_x}{\partial x \partial y} + \frac{\partial^2 \theta_y}{\partial x^2}\right) = 0, \tag{8}$$

$$-D_{11}\frac{\partial^4 w_0}{\partial x^4} - 2D_{12}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} - D_{22}\frac{\partial^4 w_0}{\partial y^4} - 4D_{66}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + E_{11}\frac{\partial^3 \theta_x}{\partial x^3}$$

$$+ E_{12}\left(\frac{\partial^3 \theta_y}{\partial x^2 \partial y} + \frac{\partial^3 \theta_x}{\partial x \partial y^2}\right) + E_{22}\frac{\partial^3 \theta_y}{\partial y^3} + 2E_{66}\left(\frac{\partial^3 \theta_x}{\partial x \partial y^2} + \frac{\partial^3 \theta_y}{\partial x^2 \partial y}\right) = q, \tag{9}$$

$$C_{11}\frac{\partial^2 u_0}{\partial x^2} + C_{12}\frac{\partial^2 v_0}{\partial x \partial y} + C_{66}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y}\right) - E_{11}\frac{\partial^3 w_0}{\partial x^3} - E_{12}\frac{\partial^3 w_0}{\partial x \partial y^2}$$

$$-2E_{66}\frac{\partial^3 w_0}{\partial x \partial y^2} + G_{11}\frac{\partial^2 \theta_x}{\partial x^2} + G_{12}\frac{\partial^2 \theta_y}{\partial x \partial y} + G_{66}\left(\frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial^2 \theta_y}{\partial x \partial y}\right) - F_{55}\theta_x = 0, \tag{10}$$

$$C_{12}\frac{\partial^2 u_0}{\partial x\partial y}+C_{22}\frac{\partial^2 v_0}{\partial y^2}+C_{66}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}\right)-E_{12}\frac{\partial^3 w_0}{\partial x^2\partial y}-E_{22}\frac{\partial^3 w_0}{\partial y^3}$$

$$-2E_{66}\frac{\partial^3 w_0}{\partial x^2\partial y}+G_{12}\frac{\partial^2\theta_x}{\partial x\partial y}+G_{22}\frac{\partial^2\theta_y}{\partial y^2}+G_{66}\left(\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{\partial^2\theta_y}{\partial x^2}\right)-F_{44}\theta_y=0. \tag{11}$$

The boundary conditions which should be prescribed at plate edges are as follows:

$$\begin{array}{ll}
\text{at } x=\text{const}, & \text{at } y=\text{const},\\
u \text{ or } N_x, & v \text{ or } N_y,\\
v \text{ or } N_{xy}, & u \text{ or } N_{xy},\\
w \text{ or } \dfrac{\partial M_x}{\partial x}+2\dfrac{\partial M_{xy}}{\partial y}, & w \text{ or } \dfrac{\partial M_y}{\partial y}+2\dfrac{\partial M_{xy}}{\partial x},\\
\dfrac{\partial w}{\partial x} \text{ or } M_x, & \dfrac{\partial w}{\partial y} \text{ or } M_y,\\
\theta_x \text{ or } P_x, & \theta_x \text{ or } P_{xy},\\
\theta_y \text{ or } P_{xy}, & \theta_y \text{ or } P_y.
\end{array} \tag{12}$$

4. Numerical Procedure

From Eqs. (12), we have the boundary conditions

$$N_x=v=w=M_x=P_x=\theta_y=0 \quad \text{at} \quad x=0, a,$$

$$N_y=u=w=M_y=P_y=\theta_x=0 \quad \text{at} \quad y=0, b.$$

Equations (7-11) are solved by the Navier method. The displacement and load functions are expanded in the Fourier series

$$u(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\quad v(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b},$$

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\quad \theta_x=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}T_{xmn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{13}$$

$$\theta_y=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}T_{ymn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b},\quad q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

where $U_{mn}, V_{mn}, W_{mn}, T_{xmn}$, and T_{ymn} are arbitrary parameters to be determined, and

$$q_{mn}=\begin{cases} q_0 & \text{for a sinusoidal load, } m=n=1,\\ \dfrac{16q_0}{mn\pi^2} & \text{for a uniform load, } m,n=1,3,5,\ldots. \end{cases}$$

Substituting Eqs. (13) into Eqs. (7)-(11) leads to the equation

TABLE 1. Comparison of Normalized Displacements and Stresses for a $[0/90]_s$ Laminated Square Plate

a/h	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{yz}$	$\bar{\tau}_{xz}$
4	HPSDT1	1.9073	0.6779	0.6349	0.2446	0.2137
	HPSDT2	1.7845	0.6271	0.6073	0.2058	0.1704
	HPSDT [13]	1.8937	0.6651	0.6322	0.2389	0.2064
	[18]	1.8987	0.6856	0.6316		0.2093
	[22]	1.954	0.720	0.666	0.292	0.219
10	HPSDT1	0.7184	0.5479	0.3904	0.1567	0.2745
	HPSDT2	0.6856	0.5398	0.3753	0.1315	0.2109
	HPSDT [13]	0.7147	0.5456	0.3888	0.1531	0.2640
	[18]	0.7194	0.5491	0.6909		0.2999
	[22]	0.743	0.559	0.401	0.196	0.301
100	HPSDT1	0.4343	0.5387	0.2708	0.1139	0.3017
	HPSDT2	0.4339	0.5386	0.2706	0.0989	0.2283
	HPSDT [13]	0.4343	0.5387	0.2708	0.1117	0.2897
	[18]	0.4339	0.5384	0.2707		0.3360
	[22]	0.438	0.539	0.276	0.141	0.337

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ T_{xmn} \\ T_{ymn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_{mn} \\ 0 \\ 0 \end{Bmatrix},$$

where

$$a_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad a_{12} = \alpha\beta(A_{12} + A_{66}), \quad a_{13} = 0,$$

$$a_{14} = C_{11}\alpha^2 + C_{66}\beta^2, \quad a_{15} = \alpha\beta(C_{12} + C_{66}), \quad a_{22} = A_{66}\alpha^2 + A_{22}\beta^2,$$

$$a_{23} = 0, \quad a_{24} = \alpha\beta(C_{12} + C_{66}), \quad a_{25} = C_{66}\alpha^2 + C_{22}\beta^2,$$

$$a_{33} = D_{11}\alpha^4 + 2D_{12}\alpha^2\beta^2 + 4D_{66}\alpha^2\beta^2 + D_{22}\beta^4,$$

$$a_{34} = -E_{11}\alpha^3 - E_{12}\alpha\beta^2 - 2E_{66}\alpha\beta^2, \; a_{35} = -E_{12}\alpha^2\beta - 2E_{66}\alpha^2\beta - E_{22}\beta^3,$$

$$a_{44} = F_{55} + G_{11}\alpha^2 + G_{66}\beta^2, \quad a_{45} = \alpha\beta(G_{12} + G_{66}),$$

$$a_{55} = F_{44} + G_{66}\alpha^2 + G_{22}\beta^2,$$

with $\alpha = \frac{m\pi}{a}$ and $\beta = \frac{n\pi}{b}$.

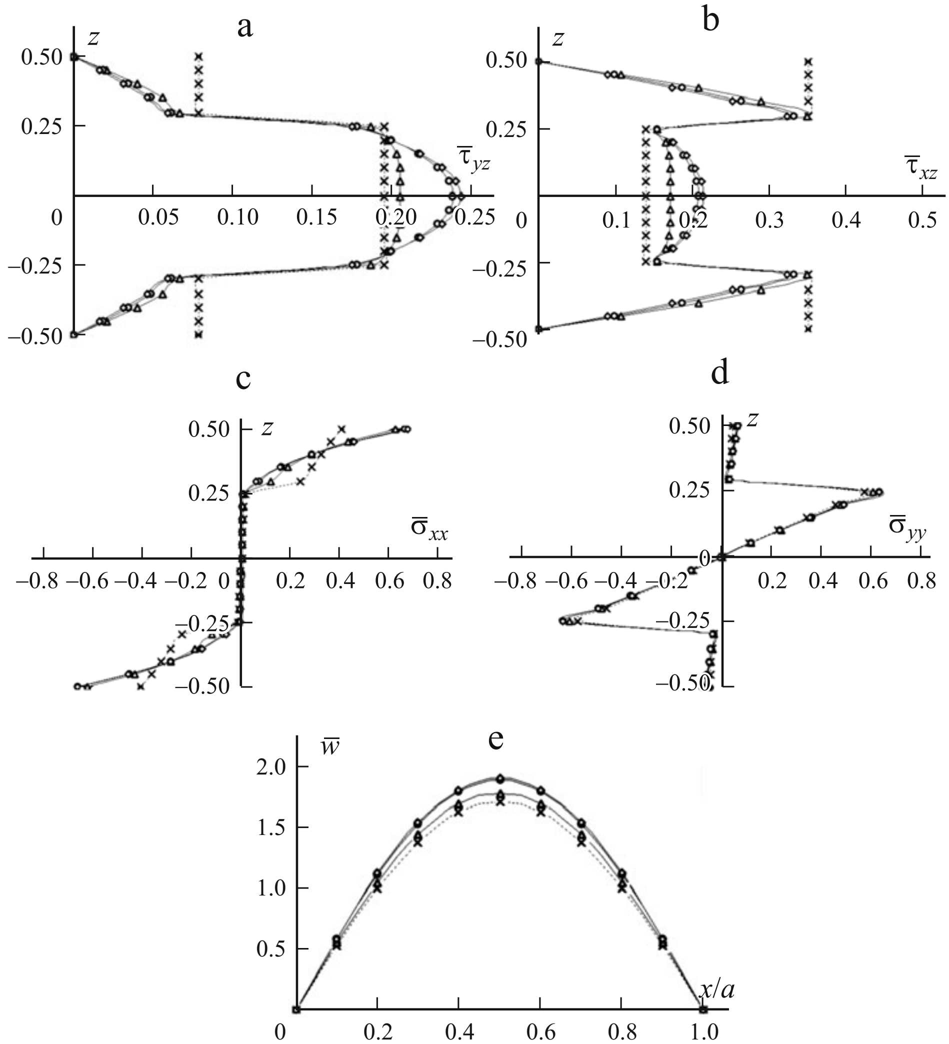

Fig. 1. Distributions of stresses $\bar{\tau}_{yz}$ (a), $\bar{\tau}_{xz}$ (b), $\bar{\sigma}_{xx}$ (c), $\bar{\sigma}_{yy}$ (d), and $\bar{w}$ (e) in a symmetric $[0/90]_s$ laminated square plate with a/h = 2.5 under a sinusoidal transverse load according to various theories: FSDT (x), HSDT (○), HPSDT1 (◇), and HPSDT2 (△).

5. Numerical Results

In this study, two new shear deformation theories for laminated plates are considered, Navier solutions for symmetrically laminated composite plates are presented, and comparisons are made with solutions available in the literature.

Three numerical examples are presented here to estimate the accuracy of the shear theories for laminated plates with simply supported boundary conditions. In all the examples, the lamina properties are assumed to be as follows: E_1 = 172 GPa, E_2 = 6.89 GPa, $G_{12} = G_{13}$ = 3.45 GPa, G_{23} = 1.38 GPa, and $\nu_{12} = \nu_{13}$ = 0.25.

The normalized transverse displacement $\bar{w}$ and the normalized stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{yz}$, and $\bar{\tau}_{xz}$ were calculated by the formulas

$$\bar{w} = \frac{E_2 h^3 w\left(\frac{a}{2}, \frac{b}{2}, 0\right)}{q_0 a^4} \cdot 100,$$

TABLE 2. Normalized Displacements and Stresses in Simply Supported Symmetrical Cross-Ply (0/90/0) Rectangular ($b = 3a$) Laminates

a/h	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{yz}$	$\bar{\tau}_{xz}$
4	HPSDT1	2.6625	1.0762	0.1034	0.0353	0.2821
	HPSDT2	2.4546	0.9526	0.0968	0.0305	0.2230
	HPSDT [13]	2.6411	1.0356	0.1028	0.0348	0.2724
	[19]	2.666	1.067	0.103	0.0355	0.285
	[23]	2.82	1.10	0.119	0.0334	0.387
10	HPSDT1	0.8679	0.6994	0.040	0.0171	0.2974
	HPSDT2	0.8220	0.6769	0.0382	0.0153	0.2280
	HPSDT [13]	0.8622	0.6924	0.0398	0.0170	0.2859
	[19]	0.870	0.698	0.040	0.017	0.302
	[23]	0.919	0.725	0.0435	0.0152	0.420
20	HPSDT1	0.5951	0.6424	0.0289	0.0140	0.2998
	HPSDT2	0.5834	0.6367	0.0284	0.0128	0.2288
	HPSDT [13]	0.5937	0.6407	0.0289	0.0139	0.2880
	[19]	0.5960	0.6420	0.0290	0.0141	0.3040
	[23]	0.610	0.650	0.0299	0.0119	0.434
100	HPSDT1	0.5070	0.6240	0.0252	0.0130	0.3006
	HPSDT2	0.5065	0.6238	0.0252	0.0120	0.2290
	HPSDT [13]	0.5070	0.6240	0.0253	0.0129	0.2886
	[23]	0.508	0.624	0.0253	0.0108	0.439
	CPT	0.5030	0.6230	0.0252	–	–

$$\bar{\sigma}_x = \frac{h^2}{q_0 a^2} \sigma_x\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right), \quad \bar{\sigma}_y = \frac{h^2}{q_0 a^2} \sigma_y\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{6}\right),$$

$$\bar{\tau}_{yz} = \frac{h}{q_0 a} \tau_{yz}\left(\frac{a}{2}, 0, 0\right), \quad \bar{\tau}_{xz} = \frac{h}{q_0 a} \tau_{xz}\left(0, \frac{b}{2}, 0\right).$$

Example 1. First, a symmetric $[0/90]_s$ cross-ply square laminate subjected to a sinusoidal transverse load was considered.

The normalized displacements and stresses obtained for the plate by using the plate theories mentioned are presented in Table 1 as compared with those given by the Reddy higher-order shear deformation theory [13], the Ferreira trigonometric shear deformation theory [18], and the exact elastic solution [22]. It is seen that the HPSDT1 model provides better results than HPSDT2 and they are very close to those following from the Ferreira theory even for great values of a/h.

In Fig. 1, the normalized stresses and displacements in a laminated plate with $a/h = 2.5$ calculated according to various theories are illustrated (HSDT: the Reddy higher-order shear deformation theory [13]; FSDT: the first-order shear deformation theory with a correction factor $k = 5/6$). As can be seen, the results of HSDT1 are in good agreement with the Reddy theory [13].

Example 2. Here, a symmetric $[0/90/0]_s$ cross-ply rectangular ($b = 3a$) simply supported laminate under a sinusoidal load is considered.

In Table 2, the results of present methods are compared with solutions given by the three-dimensional elasticity theory [23], the Touratier trigonometric shear deformation theory [19], and the Reddy higher-order theory [13].

As seen, the results obtained by using the present formulations agree with the Touratier [19] and Reddy [13] theories, but HPSDT1 is more accurate than HPSDT2.

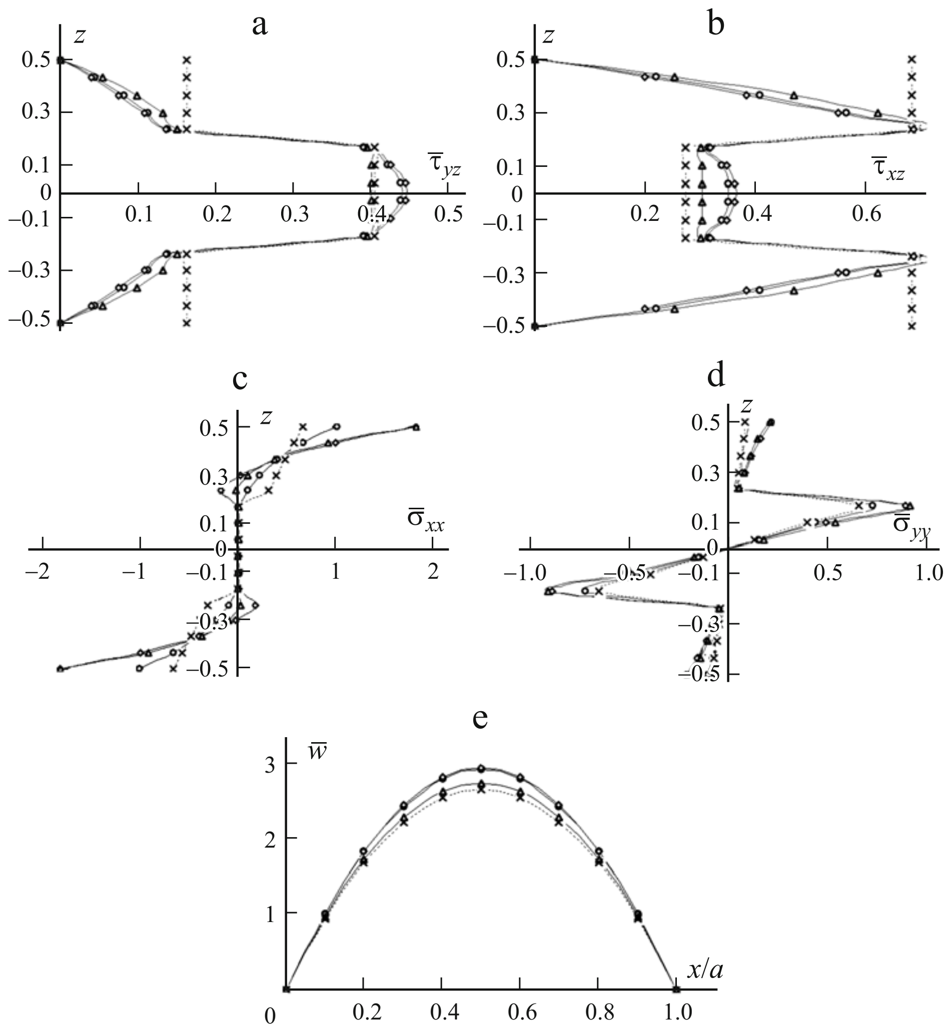

Fig. 2. The same for a $[0/90/0]_s$ composite plate.

Example 3. This example considers a symmetric $(0/90/0)_s$ cross-ply square laminate with a $a/h = 2.5$ under a uniform load.

Figure 2 shows the variations of normalized in-plane stresses, transverse shear stresses, and transverse displacements across the plate thickness. It is seen that the results of HPSDT1 are very close to those of the Reddy theory [13]. In addition, the HPSDT2 model provides better results than the first-order shear deformation theory for the transverse displacement and shear stresses.

6. Conclusions

In this study, two theories of orthotropic laminated plates have been developed on the assumption that the transverse shear displacements vary as a hyperbolic function across the plate thickness. For a symmetric rectangular cross-ply plate under static loading, the equilibrium equations and associated boundary conditions were derived from the virtual work principle. The Navier method was used for the analytical solutions of a laminated plate with simply supported boundary conditions. In order to investigate the accuracy of the present theories, the numerical results obtained were compared with those of the third-order shear deformation theory, the trigonometric shear deformation theory, and the three-dimensional elasticity theory.

The comparison shows that both the present theories are in good agreement with the other theories of laminated plates, but HPSDT1 is slightly better than the third-order shear deformation theory for all a/h ratios when compared with exact solutions.

HPSDT2 gives adequate results when compared with the first-order shear deformation theory; for in-plane stresses, it is even better than the first-order theory. It can be seen from Tables 1 and 2 that the results of HPSDT1 are very close to those of trigonometric shear deformation theories [18, 19].

The numerical results indicate that the hyperbolic models presented, especially HPSDT1, predict the deflections and stresses more accurately than some other displacement theories. As a result, we can assert that the HPSDT1 model can be used for a static bending analysis of symmetrical cross-ply laminated orthotropic plates. The analysis presented can be expanded to laminated composite shells.

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