Mechanics of Composite Materials, Vol. 41, No. 3, 2005

FORCED VIBRATION OF A PRESTRETCHED TWO-LAYER SLAB ON A RIGID FOUNDATION

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***Keywords**: forced vibration, two-layer slab, prestretching, stress distribution, incompressible material*

Within the framework of a piecewise homogeneous body model, with the use of the three-dimensional linearized theory of elastic waves in initially stressed bodies, the forced vibration of a prestretched two-layer slab resting on a rigid foundation is studied. To the upper plane of the slab, a harmonic point force is applied. It is assumed that the layer materials are incompressible, and their elastic properties are characterized by Treloar's potential. Numerical results are presented for the case where the material stiffness of the lower layer is greater than that of the upper one. The influence of prestretching the layers on the frequency dependences of the normal stresses operating on the interface between the layers and between the slab and the rigid foundation are analyzed.

1. Introduction

An interesting and urgent problem, which cannot be solved within the framework of the classical linear theory of elastic waves, is the elastodynamics problem for initially stressed bodies. This problem is encountered in many applications. It is well known that initial stresses arise in composites during their production, in structural elements during their manufacture and assembly, in the Earth crust under the action of geostatic and geodynamic forces, etc. Therefore, up to now, a large number of theoretical and experimental investigations have been carried out in this field. A systematic survey and analysis of these results were made in [1]. The reviews of recent researches can be found in [2-6] and other papers. It follows from the overviews that almost all these investigations have been performed within the framework of the three-dimensional linearized theory of elastic waves in initially stressed bodies (TLTEWISB), and a considerable part of them deals with wave propagation in layered composite materials with homogeneous initial stresses.

It is evident that studying the influence of initial stresses on a dynamical stress field in homogeneous and layered materials has also a great theoretical and practical importance. Thus far, investigations in this field are few in number; however, they deal with prestressed half-planes [7] and half-planes covered with a prestretched layer [8-10]. Moreover, in these studies, it is assumed that a normal linearly distributed force changing harmonically with time is applied to the upper plane.

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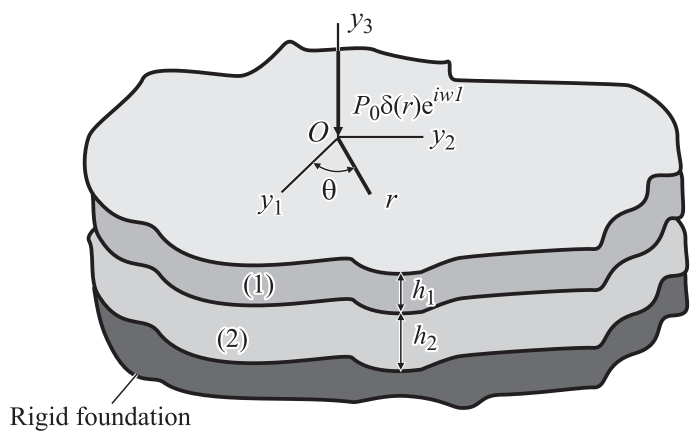

Fig. 1. Geometry of a two-layer slab resting on a rigid foundation.

The domains considered in the above-mentioned investigations are semi-infinite. Therefore, the results obtained in [8-10] cannot be used, for example, in the cases where a layered material rests on a rigid foundation. Note that such situations are characteristic of structural members whose basic material is covered with a layered one. If the stiffness of the basic material (elastic modulus) is significantly greater than that of the cover layers, the basic material can be modeled as a rigid foundation. It is well known that, as a result of the covering procedure, residual (initial) stresses inevitably arise in the cover layers. Therefore, in studying the dynamical stress field under forced vibrations of such structural members, these stresses must be taken into account.

In the present paper, the investigations carried out in [8-10] are extended to an infinite two-layer slab on a rigid foundation. To the upper plane of the slab, a normal point force changing harmonically with time is applied, and the axisymmetric stress state in this slab is studied. It is assumed that the layers of the slab are radially prestretched. The layer materials are incompressible, and the stress–strain relation for them is defined through the use of Treloar's potential. The investigation is carried out within the framework of a piecewise homogeneous body model with the use of the TLTEWISB. The numerical results are presented for the case where the material stiffness of the lower layer is greater than that of the upper one.

2. Problem Formulation

We consider a two-layer slab resting on a rigid foundation (Fig. 1). In the natural state, the thickness of the upper and lower layers of the slab are h_1 and h_2, respectively, and the slab is related to Cartesian $Oy_1y_2y_3$ and cylindrical $Or\theta y_3$ systems of coordinates. The layers are infinite in the radial direction. Before their bonding with each other and with the rigid foundation, the layers are stretched separately in the radial directions, and, as a result, a homogeneous axisymmetric initial strain state appears in them. With the initial state of the layers, we associate Lagrangian cylindrical $O'r'\theta'y'_3$ and Cartesian $O'y'_1y'_2y'_3$ systems of coordinates. We assume that the materials of the layers are incompressible. The quantities related to the upper and lower layers of the slab will be denoted by the superscripts (1) and (2), respectively. In addition, the quantities referring to the initial state will be labeled with the superscript 0. Thus, according to the above-stated, the initial state in the layers can be described as follows:

$$u_m^{(k),0} = (\lambda_m^{(k)} - 1)y_m, \quad \lambda_1^{(k)} = \lambda_2^{(k)} \neq \lambda_3^{(k)}, \quad \lambda_m^{(k)} = \text{const}, \tag{1}$$

$$\lambda_1^{(k)}\lambda_2^{(k)}\lambda_3^{(k)} = 1; \quad m = 1, 2, 3; \quad k = 1, 2,$$

where $u_m^{(k),0}$ is the displacement and $\lambda_m^{(k)}$ is the elongation along the Oy_m axis. Hereafter, the designations

$$\lambda_1^{(k)} = \lambda_2^{(k)} = \lambda^{(k)}, \quad \lambda_3^{(k)} = (\lambda^{(k)})^{-2} \tag{2}$$

will be used.

Let us investigate the forced vibration of the slab in the case where a normal point force varying harmonically with time is applied to the upper layer.

It follows from Eq. (1) that

$$y_i' = \lambda_i^{(k)} y_i, \quad r' = \lambda^{(k)} r, \quad h_1' = (\lambda^{(1)})^{-2} h_1, \quad h_2' = (\lambda^{(2)})^{-2} h_2,$$

where the primed quantities are associated with the initial state, i.e., with the coordinate system $O' y_1' y_2' y_3'$.

According to [1], we write the basic relations of the TLTEWISB for an incompressible body in an axisymmetric stress-strain state. These relations are satisfied for each layer, because a piecewise homogeneous body model is used.

We have the equations of motion

$$\frac{\partial}{\partial r'} Q_{rr}'^{(k)} + \frac{\partial}{\partial y_3'} Q_{r'3}'^{(k)} + \frac{1}{r'} (Q_{r'r'}'^{(k)} - Q_{\theta'\theta'}'^{(k)}) = \rho'^{(k)} \frac{\partial^2}{\partial t^2} u_{r'}'^{(k)},$$

$$\frac{\partial}{\partial r'} Q_{3r'}'^{(k)} + \frac{\partial}{\partial y_3'} Q_{33}'^{(k)} + \frac{1}{r'} Q_{3r'}'^{(k)} = \rho'^{(k)} \frac{\partial^2}{\partial t^2} u_3'^{(k)}, \tag{3}$$

and the mechanical relations

$$Q_{r'r'}'^{(k)} = \chi_{1111}'^{(k)} \frac{\partial u_{r'}'^{(k)}}{\partial r'} + \chi_{1122}'^{(k)} \frac{u_{r'}'^{(k)}}{r'} + \chi_{1133}'^{(k)} \frac{\partial u_3'^{(k)}}{\partial y_3'} + p'^{(k)},$$

$$Q_{\theta'\theta'}'^{(k)} = \chi_{2211}'^{(k)} \frac{\partial u_{r'}'^{(k)}}{\partial r'} + \chi_{2222}'^{(k)} \frac{u_{r'}'^{(k)}}{r'} + \chi_{2233}'^{(k)} \frac{\partial u_3'^{(k)}}{\partial y_3'} + p'^{(k)}, \tag{4}$$

$$Q_{33}'^{(k)} = \chi_{3311}'^{(k)} \frac{\partial u_{r'}'^{(k)}}{\partial r'} + \chi_{3322}'^{(k)} \frac{u_{r'}'^{(k)}}{r'} + \chi_{3333}'^{(k)} \frac{\partial u_3'^{(k)}}{\partial y_3'} + p'^{(k)},$$

$$Q_{r'3}'^{(k)} = \chi_{1313}'^{(k)} \frac{\partial u_{r'}'^{(k)}}{\partial y_3'} + \chi_{1331}'^{(k)} \frac{\partial u_3'^{(k)}}{\partial r'}, \quad Q_{3r'}'^{(k)} = \chi_{3113}'^{(k)} \frac{\partial u_{r'}'^{(k)}}{\partial y_3'} + \chi_{3131}'^{(k)} \frac{\partial u_3'^{(k)}}{\partial r'}.$$

By using the quantities $Q_{r'r'}'^{(k)}, \ldots, Q_{3r'}'^{(k)}$, perturbations of the components of the Kirchhoff stress tensor are determined; $u_{r'}'^{(k)}$ and $u_3'^{(k)}$ are perturbations of the components of the displacement vector, and $p'^{(k)} = p'^{(k)}(r', y_3', t)$ is an unknown function (Lagrange multiplier). The constants $\chi_{1111}'^{(k)}, \ldots, \chi_{3333}'^{(k)}$ of Eqs. (3) and (4) are determined in terms of mechanical constants of layer materials and components of the initial stress state, and $\rho'^{(k)}$ is the material density of a kth layer. For the initial strain state, the constants $\chi_{1111}'^{(k)}, \ldots, \chi_{3333}'^{(k)}$ are expressed in terms of the corresponding quantities in the coordinate system $Or\theta y_3$ (we denote them by $\chi_{1111}^{(k)}, \ldots, \chi_{3333}^{(k)}, \rho^{(k)}$):

$$\chi_{1111}'^{(k)} = (\lambda^{(k)})^2 \chi_{1111}^{(k)}, \quad \chi_{1122}'^{(k)} = (\lambda^{(k)})^2 \chi_{1122}^{(k)}, \quad \chi_{1133}'^{(k)} = (\lambda^{(k)})^{-1} \chi_{1133}^{(k)},$$

$$\chi_{2222}'^{(k)} = (\lambda^{(k)})^2 \chi_{2222}^{(k)}, \quad \chi_{1221}'^{(k)} = (\lambda^{(k)})^2 \chi_{1221}^{(k)}, \quad \chi_{1313}'^{(k)} = (\lambda^{(k)})^{-1} \chi_{1313}^{(k)}, \tag{5}$$

$$\chi'^{(k)}_{1331} = (\lambda^{(k)})^2 \chi^{(k)}_{1331}, \quad \chi'^{(k)}_{3131} = \chi'^{(k)}_{1313}, \quad \chi'^{(k)}_{2211} = \chi'^{(k)}_{1122}, \quad \chi'^{(k)}_{2233} = \chi'^{(k)}_{1133},$$

$$\chi'^{(k)}_{3311} = \chi'^{(k)}_{1133} = \chi'^{(k)}_{3322} = \chi'^{(k)}_{2233}, \quad \chi'^{(k)}_{3113} = (\lambda^{(k)})^2 \chi^{(k)}_{3113},$$

$$\chi'^{(k)}_{3333} = (\lambda^{(k)})^{-4} \chi^{(k)}_{3333}, \quad \rho'^{(k)} = \rho^{(k)}.$$

In the present investigation, we assume that the elasticity relations of layer materials are given by Treloar's potential

$$\Phi = C_{10}(I_1 - 3), \quad I_1 = 3 + 2A_1, \quad A_1 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{33}, \tag{6}$$

where C_{10} is the elastic constant; A_1 is the first invariant of Green's strain tensor, but ε_{rr}, $\varepsilon_{\theta\theta}$, and ε_{33} are the components of this tensor. For the axisymmetric case considered, the components of Green's strain tensor are determined by components of the displacement vector in the following way:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{2}\left(\frac{u_r}{r}\right)^2,$$

$$\varepsilon_{r3} = \frac{1}{2}\left(\frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial y_3} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial y_3} + \frac{\partial u_3}{\partial r}\frac{\partial u_3}{\partial y_3}\right), \tag{7}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial y_3} + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial y_3}\right)^2.$$

In this case, the components S_{ij} of the Lagrange stress tensor are

$$S_{rr} = \frac{\partial \Phi}{\partial \varepsilon_{rr}} + p g^*_{rr}, \; S_{\theta\theta} = \frac{\partial \Phi}{\partial \varepsilon_{\theta\theta}} + p g^*_{\theta\theta}, \; S_{33} = \frac{\partial \Phi}{\partial \varepsilon_{33}} + p g^*_{33},$$

$$S_{r3} = \frac{\partial \Phi}{\partial \varepsilon_{r3}}, \quad S_{r3} = S_{3r}, \quad g^*_{rr} = 1 + 2\frac{\partial u_r}{\partial r} + \left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_3}{\partial r}\right)^2, \tag{8}$$

$$g^*_{33} = 1 + 2\frac{\partial u_3}{\partial y_3} + \left(\frac{\partial u_3}{\partial y_3}\right)^2 + \left(\frac{\partial u_r}{\partial y_3}\right)^2.$$

Note that Eqs. (6)-(8) are written in an arbitrary system of cylindrical coordinates associated neither with the natural nor the initial state of the two-layer slab considered.

For the case considered, the relations between the perturbations of the Kirchhoff and Lagrange stress tensors can be written in the form

$$Q'^{(k)}_{r'r'} = \lambda^{(k)} S^{(k)}_{r'r'} + S^{0(k)}_{rr} \frac{\partial u'^{(k)}_{r'}}{\partial r'}, \quad Q'^{(k)}_{\theta'\theta'} = \lambda^{(k)} S^{(k)}_{\theta'\theta'} + S^{0(k)}_{rr} \frac{u'^{(k)}_{r'}}{r'},$$

$$Q'^{(k)}_{33} = (\lambda^{(k)})^{-2} S^{(k)}_{33}, \quad Q'^{(k)}_{r'3} = \lambda^{(k)} S^{(k)}_{r'3} + S^{0(k)}_{rr} \frac{\partial u'^{(k)}_{3}}{\partial r'}, \tag{9}$$

$$Q'^{(k)}_{3r'} = (\lambda^{(k)})^{-2} S^{(k)}_{3r'}.$$

According to [1], by linearizing Eq. (8) and taking into account Eqs. (9), (1), and (2), we obtain the following expressions for the constants $\chi_{1111}^{(k)},\ldots,\chi_{3333}^{(k)}$ of Eq. (5):

$$\chi_{1111}^{(k)}=2C_{10}^{(k)}(\lambda^{(k)})^{-2}[(\lambda^{(k)})^{2}+(\lambda^{(k)})^{-4}],$$

$$\chi_{1122}^{(k)}=\chi_{1133}^{(k)}=\chi_{2233}^{(k)}=\chi_{3311}^{(k)}=\chi_{2211}^{(k)}=\chi_{3322}^{(k)}=0, \tag{10}$$

$$\chi_{1331}^{(k)}=2C_{10}^{(k)},\quad \chi_{1221}^{(k)}=2C_{10}^{(k)},\quad \chi_{3333}^{(k)}=4C_{10}^{(k)},$$

$$\chi_{1313}^{(k)}=2C_{10}^{(k)}(\lambda^{(k)})^{-3},\quad \chi_{3113}^{(k)}=2C_{10}^{(k)}.$$

The above-written equations must be supplemented with the incompressibility conditions of layers materials, which can be written in the form

$$\frac{1}{\lambda^{(k)}}\left(\frac{\partial u_{r'}^{\prime(k)}}{\partial r'}+\frac{u_{r'}^{\prime(k)}}{r'}\right)+(\lambda^{(k)})^{2}\frac{\partial u_{3}^{\prime(k)}}{\partial y_3'}=0. \tag{11}$$

Thus, the stress state in the two-layer slab will be investigated with the use of Eqs. (3)-(11). In addition, we assume that the following boundary and contact conditions are satisfied:

$$Q_{33}^{\prime(1)}\Big|_{y_3'=0}=-P_0\delta(r')e^{i\omega t}\frac{1}{(\lambda^{(1)})^{2}},\quad Q_{3r'}^{\prime(1)}\Big|_{y_3'=0}=0,$$

$$Q_{33}^{\prime(1)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2}=Q_{33}^{\prime(2)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2},$$

$$Q_{3r'}^{\prime(1)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2}=Q_{3r'}^{\prime(2)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2},$$

$$u_{r'}^{\prime(1)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2}=u_{r'}^{\prime(2)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2},$$

$$u_{3}^{\prime(1)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2}=u_{3}^{\prime(2)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2}, \tag{12}$$

$$u_{r'}^{\prime(2)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2-h_2/(\lambda^{(2)})^2}=0,\quad u_{3}^{\prime(2)}\Big|_{y_3'=-h_1/(\lambda^{(1)})^2-h_2/(\lambda^{(2)})^2}=0,$$

$$\left\{\left|Q_{33}^{\prime(2)}\right|;\left|Q_{3r'}^{\prime(2)}\right|;\left|Q_{r'r'}^{\prime(2)}\right|;\left|Q_{\theta'\theta'}^{\prime(2)}\right|;\left|Q_{r'3}^{\prime(2)}\right|;\left|u_{3}^{\prime(2)}\right|;\left|u_{r'}^{\prime(2)}\right|\right\}<M=\text{const},\quad y_3\to-\infty.$$

It should be noted that, in the case $\lambda^{(k)}=1$, $k=1, 2$, Eqs. (3)-(5) and (9)-(11) and conditions (12) for a kth layer transform into the corresponding relations of the classical linear theory of elasticity for incompressible bodies.

3. Solution Procedure

Substituting Eq. (4) in Eq. (3) we obtain the equations of motion in terms of displacements

$$\chi_{1111}^{\prime(k)}\frac{\partial^2 u_{r'}^{\prime(k)}}{\partial r'^2}+\chi_{1122}^{\prime(k)}\frac{\partial}{\partial r'}\left(\frac{u_{r'}^{\prime(k)}}{r'}\right)+(\chi_{1133}^{\prime(k)}+\chi_{1331}^{\prime(k)})\frac{\partial^2 u_{3}^{\prime(k)}}{\partial r'\partial y_3'}$$

$$+\chi'^{(k)}_{1313}\frac{\partial^2 u'^{(k)}_{r'}}{\partial y'^2_3}+\frac{1}{r'}(\chi'^{(k)}_{1111}-\chi'^{(k)}_{2211})\frac{\partial u'^{(k)}_{r'}}{\partial r'}+(\chi'^{(k)}_{1122}-\chi'^{(k)}_{2222})\frac{u'^{(k)}_{r'}}{r'^2}$$

$$+(\chi'^{(k)}_{1133}-\chi'^{(k)}_{2233})\frac{1}{r'}\frac{\partial u'^{(k)}_3}{\partial y'_3}=\rho'^{(k)}\frac{\partial^2 u'^{(k)}_{r'}}{\partial t^2}-\frac{\partial p'^{(k)}}{\partial r'}, \tag{13}$$

$$\chi'^{(k)}_{3133}\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'\partial y'_3}+\chi'^{(k)}_{3131}\frac{\partial^2 u'^{(k)}_3}{\partial r'^2}+\frac{1}{r'}\chi'^{(k)}_{3113}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3}+\frac{1}{r'}\chi'^{(k)}_{3131}\frac{\partial u'^{(k)}_3}{\partial r'}$$

$$+\chi'^{(k)}_{3311}\frac{\partial^2 u'^{(k)}_{r'}}{\partial y'_3\partial r'}+\chi'^{(k)}_{3322}\frac{1}{r'}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3}+\chi'^{(k)}_{3333}\frac{\partial^2 u'^{(k)}_3}{\partial y'^2_3}=\rho'^{(k)}\frac{\partial^2 u'^{(k)}_3}{\partial t^2}-\frac{\partial p'^{(k)}}{\partial y'_3}.$$

Equations (11) and (13) present a complete system with respect to the unknown functions $u'^{(k)}_{r'}$, $u'^{(k)}_3$, and $p'^{(k)}$. According to [1], we use the following representation for the displacements and the unknown function $p'^{(k)}$:

$$u'^{(k)}_{r'}=-\frac{\partial^2}{\partial r'\partial y'_3}X'^{(k)},\quad u'^{(k)}_3=\Delta'_1 X'^{(k)}, \tag{14}$$

$$p'^{(k)}=\left[(\chi'^{(k)}_{1111}-\chi'^{(k)}_{1133}-\chi'^{(k)}_{1313})\Delta'_1+\chi'^{(k)}_{3113}\frac{\partial^2}{\partial y'^2_3}-\rho'^{(k)}\frac{\partial^2}{\partial t^2}\right]\frac{\partial}{\partial y'_3}X'^{(k)},$$

where

$$\Delta'_1=\frac{d^2}{dr'^2}+\frac{1}{r'}\frac{d}{dr'}. \tag{15}$$

The function $X'^{(k)}$ satisfies the equation

$$\left[\left(\Delta'_1+(\xi'^{(k)}_2)^2\frac{\partial^2}{\partial y'^2_3}\right)\left(\Delta'_1+(\xi'^{(k)}_3)^2\frac{\partial^2}{\partial y'^2_3}\right)-\frac{\rho'^{(k)}}{\chi'^{(k)}_{1331}}\left(\Delta'_1+\frac{\partial^2}{\partial y'^2_3}\right)\frac{\partial^2}{\partial t^2}\right]X'^{(k)}=0, \tag{16}$$

where, for the case considered,

$$(\xi'^{(k)}_2)^2=1,\quad (\xi'^{(k)}_3)^2=(\lambda^{(k)})^{-6}.$$

Since the point force is harmonic in time, only a stationary case is to be considered. Then all the dependent variables become harmonic in time and can be represented as

$$\{Q'^{(k)}_{r'r'},\ldots,Q'^{(k)}_{33},u'^{(k)}_{r'},u'^{(k)}_3,p'^{(k)},X'^{(k)}\}=\{\bar{Q}'^{(k)}_{r'r'},\ldots,\bar{Q}'^{(k)}_{33},\bar{u}'^{(k)}_{r'},\bar{u}'^{(k)}_3,\bar{p}'^{(k)},\bar{X}'^{(k)}\}e^{i\omega t}, \tag{17}$$

where the overbar denotes the amplitude of the corresponding quantity. Hereafter, this overbar will be omitted.

Substituting Eqs. (13)-(16) in Eq. (17) and replacing the operator $\partial^2/\partial t^2$ with $-\omega^2$, we obtain equations and conditions for the amplitudes of the sought-for quantities. Consequently, introducing the dimensionless coordinates $r'\to r'/h_1$ and $y'_3\to y'_3/h_1$ and the dimensionless frequency

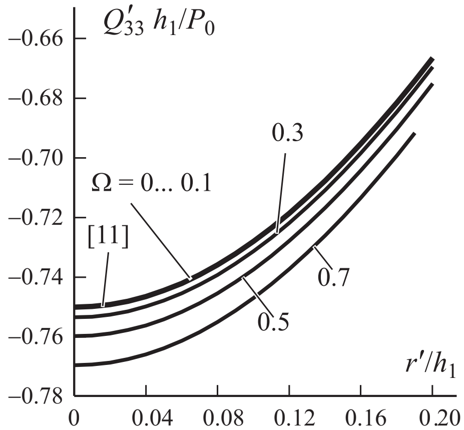

Fig. 2

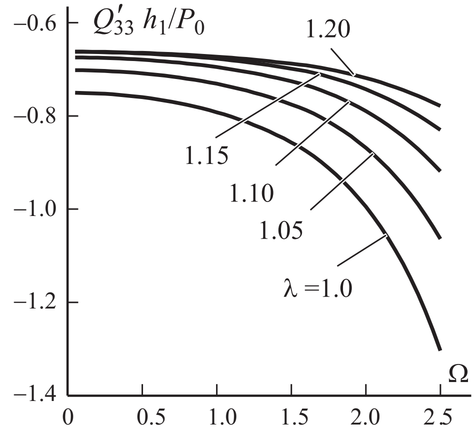

Fig. 3

Fig. 2. Comparison of the results obtained for a single-layer slab with those given in [11] as $\Omega \to 0$.

Fig. 3. Influence of prestretching the single-layer slab on the relation between the normal stress Q'_{33} and the dimensionless frequency Ω. $r'/h_1 = 0$.

$$\Omega^2 = \frac{(\omega h_1)^2 \rho'^{(2)}}{2C_{10}^{(2)}}, \tag{18}$$

we obtain the equation for the potential $X'^{(k)}$

$$\left[\left(\Delta'_1 + (\xi_2'^{(k)})^2 \frac{\partial^2}{\partial y_3'^2}\right)\left(\Delta'_1 + (\xi_3'^{(k)})^2 \frac{\partial^2}{\partial y_3'^2}\right) - \frac{\Omega^2}{(\lambda^{(k)})^{(2)}}\left(\Delta'_1 + \frac{\partial^2}{\partial y_3'^2}\right)\frac{C_{10}^{(2)}\rho'^{(k)}}{C_{10}^{(k)}\rho'^{(2)}}\right] X'^{(k)} = 0. \tag{19}$$

To solve Eq. (19), we use the Hankel integral representation for the function $X'^{(k)}$

$$X'^{(k)} = \int_0^\infty F_1^{(k)} e^{\gamma^{(k)} y'_3} J_0(sr) s ds, \tag{20}$$

where $J_0(sr)$ is the zero-order Bessel function.

Substituting Eq. (20) into Eq. (19), we obtain the algebraic equation for $\gamma^{(k)}$

$$A^{(k)}(\gamma^{(k)})^4 + B^{(k)}(\gamma^{(k)})^2 + C^{(k)} = 0, \tag{21}$$

where

$$A^{(k)} = (\xi_2'^{(k)})^2 (\xi_3'^{(k)})^2,$$

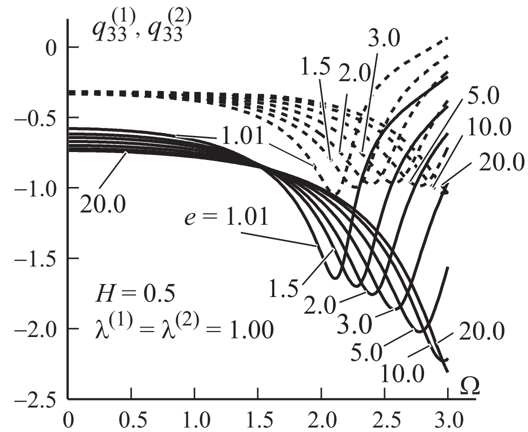

Fig. 4. Influence of the parameter $e = C_{10}^{(2)} / C_{10}^{(1)}$ on relations between the normal stresses $q_{33}^{(1)}$ and $q_{33}^{(2)}$ and the frequency Ω. (—) — $q_{33}^{(1)}$; (- - -) — $q_{33}^{(2)}$.

$$B^{(k)} = \frac{1}{(\lambda^{(k)})^2} \frac{C_{10}^{(2)}}{C_{10}^{(k)}} \frac{\rho'^{(k)}}{\rho'^{(2)}} \Omega^2 - [(\xi_2'^{(k)})^2 + (\xi_3'^{(k)})^2] s^2,$$

$$C^{(k)} = s^4 - s^2 \frac{C_{10}^{(2)}}{C_{10}^{(k)}} \frac{\rho'^{(k)}}{\rho'^{(2)}} \frac{1}{(\lambda^{(k)})^2}.$$

It follows from Eq. (21) that

$$(\gamma^{(k)})^2 = \frac{-B^{(k)} \pm \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}}.$$

By direct verification, it was found that

$$(\gamma^{(k)})^2 = \frac{-B^{(k)} + \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}} > 0.$$

However, for $(\gamma^{(k)})^2 = \frac{-B^{(k)} - \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}}$, the following two cases are possible:

case 1,

$$(\gamma^{(k)})^2 = \frac{-B^{(k)} - \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}} > 0;$$

case 2,

$$(\gamma^{(k)})^2 = \frac{-B^{(k)} - \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}} < 0.$$

In case 1, the solution to Eq. (19) is

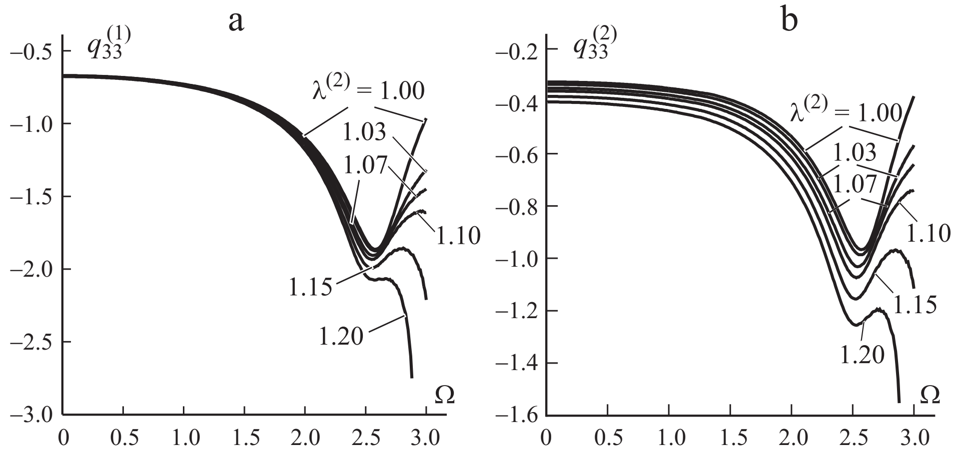

Fig. 5. Influence of prestretching the lower layer on the stress–frequency relations $q_{33}^{(1)}(\Omega)$ (a) and $q_{33}^{(2)}(\Omega)$ (b). $H = 0.54$, $e = 3.04$, and $\lambda^{(1)} = 1$.

$$X'^{(k)} = \int_0^\infty \left[F_1^{(k)} e^{\gamma_1^{(k)} y_3'} + F_2^{(k)} e^{-\gamma_1^{(k)} y_3'} + F_3^{(k)} e^{\gamma_2^{(k)} y_3'} + F_4^{(k)} e^{-\gamma_2^{(k)} y_3'} \right] J_0(sr') s ds, \tag{22}$$

where

$$\gamma_1^{(k)} = \sqrt{\frac{-B^{(k)} + \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}}},$$

$$\gamma_2^{(k)} = \sqrt{\frac{-B^{(k)} - \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}}}.$$

In case 2, the solution is

$$X'^{(k)} = \int_0^\infty \left[F_1^{(k)} e^{\gamma_1^{(k)} y_3'} + F_2^{(k)} e^{-\gamma_1^{(k)} y_3'} + F_3^{(k)} e^{i\gamma_2^{(k)} y_3'} + F_4^{(k)} e^{-i\gamma_2^{(k)} y_3'} \right] J_0(sr') s ds, \tag{23}$$

where

$$\gamma_2^{(k)} = \sqrt{\frac{B^{(k)} + \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}}} > 0. \tag{24}$$

By the use of Eqs. (14), (23), (24), and (4), integral expressions for stresses and displacements, which are similar to Eqs. (22) and (23), were also obtained, but they are too cumbersome and therefore are omitted here.

To find the unknowns $F_1^{(k)}(s), \ldots, F_4^{(k)}(s)$, boundary and contact conditions (12) are utilized. For this purpose, we must find the Hankel transform of the right-hand side of the first condition in (12). Using the equality $P_0 \delta(r') = \lim_{r' \to 0} (P_0 / \pi r'^2)$,

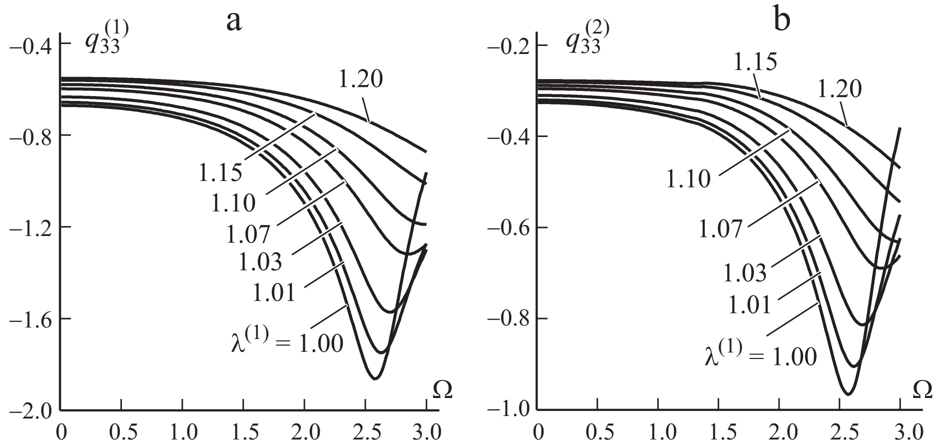

Fig. 6. Influence of prestretching the upper layer on the stress–frequency relations $q_{33}^{(1)}(\Omega)$ (a) and $q_{33}^{(2)}(\Omega)$ (b).

we obtain from $\lim_{\varepsilon \to 0} \int_0^{\varepsilon} \frac{P_0}{\pi \varepsilon^2} r' J_0(sr') dr'$ that the Hankel transform of $P_0 \delta(r')$ is $P_0 / 2\pi$. Thus, the following equations for the above-listed unknowns were derived:

$$F_j^{(1)}(s)\alpha_{ij}^{(1)}(s) = \frac{P_0}{2\pi(\lambda^{(1)})^2}\delta_i^1, \quad i = 1, 2; \quad j = 1, 2, 3, 4,$$

$$F_j^{(1)}(s)\alpha_{ij}^{(1)}(s) + F_j^{(2)}(s)\alpha_{ij}^{(2)}(s) = 0, \ i = 3, 4, 5, 6; \ F_j^{(2)}(s)\alpha_{ij}^{(2)}(s) = 0, \ i = 7, 8, \tag{25}$$

where summation is carried out over the repeated subscript j.

The coefficients of unknowns in Eq. (25) are determined through the use of expressions for stresses and displacements. Thus, the unknowns $F_1^{(k)}(s), \ldots, F_4^{(k)}(s)$ are found from Eq. (25), after which the stresses and displacements can be calculated by using the corresponding integral expressions. In calculating the integrals, we employed the algorithm proposed in [8-10].

Now we will analyze some numerical results obtained within the framework of the solution procedure utilized and clarify the influence of prestretching the layers on the distribution of the stresses Q'_{33} acting on the interface between the layers and between the lower layer and the rigid foundation.

4. Numerical Results and Discussions

In order to test the calculation algorithm, we will consider a single-layer slab and analyze the distribution of the stress Q'_{33} on the plane between the rigid foundation and the slab. In this case, we will examine the influence of Ω (18) on this distribution. When $\Omega \to 0$ in the absence of initial stretching of the slab, Q'_{33} must approach the value obtained for the corresponding static problem studied in [11]. In that work, the expression for the stress $Q'_{33} = \sigma_{33}$ was obtained in an integral form, the slab material was compressible, and the problem was solved within the framework of the classical linear theory of elasticity.

Let us compare our results with those obtained by using the integral expression given for σ_{33} in [11], assuming that the Poisson ratio ν of the slab material is 0.499. Figure 2 shows relationships between $Q'_{33} h_1 / P_0$ and r'/h_1 (h_1 is the slab thickness) for various values of Ω. It is seen that the value of $Q'_{33} h_1 / P_0$ obtained for the dynamical problem tends to the that for the static problem as $\Omega \to 0$, which indicates that the algorithm and programs used are correct.

Let us look at the influence of initial prestretching of the single-layer slab on the relations between $Q'_{33} h_1 / P_0$ (at $r'/h_1 = 0$) and Ω. The graphs of these relations, which are given in Fig. 3, show that, for $0 < \Omega \le 2.5$, the absolute value of $Q'_{33} h_1 / P_0$ increases monotonically with Ω. As a result of prestretching the slab, $Q'_{33} h_1 / P_0$ decreases monotonically with λ. An explanation for these results will be considered below.

Now, we return to examining the stress distribution in the two-layer slab and introduce the designations

$$e = \frac{C_{10}^{(2)}}{C_{10}^{(1)}}, \quad H = \frac{h_2}{h_1}, \quad q_{33}^{(1)} = \left(\frac{Q'^{(1)}_{33} h_1}{P_0} \right) \Bigg|_{y'_3 = -h_1 / (\lambda^{(1)})^2},$$

$$q_{33}^{(2)} = \left(\frac{Q'^{(2)}_{33} h_1}{P_0} \right) \Bigg|_{y'_3 = -h_1 / (\lambda^{(1)})^2 - h_2 / (\lambda^{(2)})^2}.$$

Let us assume that $e \ge 1$, $0 \le \Omega \le 3.0$, $\rho^{(2)} / \rho^{(1)} = 1.0$, and $H = 0.5$. Figure 4 shows the influence of the parameter e on the character of relations between $q_{33}^{(1)}$, $q_{33}^{(2)}$, and Ω for the case where the initial stretching in the layers is absent, i.e., $\lambda^{(1)} = \lambda^{(2)} = 1.0$. It is seen that the absolute values of $q_{33}^{(1)}$ and $q_{33}^{(2)}$ have extrema at certain values of Ω, which we will call the "resonance" values of Ω. Moreover, the "resonance" values of Ω and $\max|q_{33}^{(1)}|$ increase with , but $\max|q_{33}^{(2)}|$ decreases while $e < 3.0$ and increases when $e > 3.0$.

According to [12-14], the behavior of the half-space or half-plane under forced vibrations is similar to the behavior of the system consisting of a mass, a parallel spring, and a dashpot. A similar behavior is also observed in dynamic contact problems [15]. The numerical results given in Fig. 4 show that the behavior of the two-layer slab under forced vibration is also similar to the behavior of the above-mentioned system of a mass, a spring, and a dashpot. Consequently, the arise of the "resonance" values of Ω follows from the nature of the mechanical object considered. In the cases considered, the stiffness of the system investigated increases with e, therefore, the "resonance" values of $\max|q_{33}^{(1)}|$ and Ω increase with e, as seen from the graphs in Fig. 4. Note that, in Fig. 4, the "resonance" values of Ω are absent for $e = 10$ and 20 and for the system consisting of the single-layer slab and rigid foundation (Fig. 3), because the corresponding "resonance" values of Ω for these cases are out of the interval of Ω considered.

Let us consider now the influence of prestretching the layers of the slab on the above-discussed relationships. Figure 5 depicts the influence of $\lambda^{(2)}$, i.e., the prestretching of the lower layer of the slab, on the relations $q_{33}^{(1)}(\Omega)$ and $q_{33}^{(2)}(\Omega)$ for $e = 3.0$ and $\lambda^{(1)} = 1.0$. It is evident that the prestretching of the lower layer causes a qualitative change in these dependencies, which shows up for $\Omega > \Omega_*$ (Ω_* is the "resonance" value) and is seen more clearly with growing $\lambda^{(2)}$. Moreover, the absolute values of $q_{33}^{(1)}$ and $q_{33}^{(2)}$ increase with $\lambda^{(2)}$.

Figure 6 shows the influence of prestretching the upper layer of the slab, i.e., the influence of $\lambda^{(1)}$ on the relations $q_{33}^{(1)}(\Omega)$ and $q_{33}^{(2)}(\Omega)$ for $e = 3.0$ and $\lambda^{(2)} = 1.0$. It follows from these results that the absolute values of $q_{33}^{(1)}$ and $q_{33}^{(2)}$ decrease, but the "resonance" values of Ω increase with $\lambda^{(1)}$, which agrees with the well-known mechanical considerations.

It should be noted that similar results were also obtained for other values of the parameters e and H.

5. Conclusions

In the present paper, within the framework of a piecewise homogeneous body model, with the use of the TLTEWISB, the axisymmetric forced vibration of an initially prestretched two-layer slab resting on a rigid foundation has been studied. The vibration was initiated by a harmonic point force applied to the upper plane of the slab. The elasticity relations for layer materials was described by Treloar's potential. The numerical results were presented for the case $C_{10}^{(2)} > C_{10}^{(1)}$, where $C_{10}^{(1)}$ and $C_{10}^{(2)}$ are material constants of the upper and lower layer of the slab, respectively. From the results obtained, the following conclusions can be drawn:

— under forced vibration, the mechanical behavior of a two-layer slab resting on a rigid foundation is similar to that of the system consisting of a mass, a spring, and a dashpot;

— the "resonance" frequency of the external force increases with $C_{10}^{(2)} / C_{10}^{(1)}$;

— as a result of prestretching the lower layer, the absolute values of normal stresses decrease;

— as a result of prestretching the upper layer, the absolute values of normal stresses decrease, but the "resonance" frequency of the external force increases;

— the influence of prestretching the layers on the stress distribution and on the "resonance" frequency is significant, both quantitatively and qualitatively, and must be taken into account.

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