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NONAXISYMMETRIC THERMAL STRESS STATE OF LAMINATED ROTATIONAL BODIES OF ORTHOTROPIC MATERIALS UNDER NONISOTHERMIC LOADING

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A procedure for numerical investigation of nonaxisymmetric temperature fields and the elastic stress-strain state of laminated rotational bodies of cylindrically and rectilinearly orthotropic materials under nonisothermal loading is proposed. The deformation of orthotropic materials is described by the equations of anisotropic elasticity theory. The equations of state are written in the form of Hooke's law for homogeneous materials, with additional terms which take into account the thermal deformation, changes in the mechanical properties of materials in the circumferential direction, and their dependence on temperature. A semianalytic finite-element method in combination with the method of successive approximations is used. An algorithm for numerical solution of the corresponding nonlinear boundary problem is elaborated, which is realized as a package of applied FORTRAN programs. Some numerical results are presented.

Introduction

The modern machine- and ship-building, aerospace and rocket engineering, etc., cannot be thought without application of composite materials. Combinations of the light, low-strength, pliable material of the matrix with very strong and rigid reinforcing fibers or grains give light, strong, and rigid materials, which are widely used in different branches of national economy. A specific feature of composite materials is that their properties in different directions can be specified beforehand, i.e., they initially possess a pronounced prescribed anisotropy of mechanical and thermophysical properties. The use of composite materials in multilayer structural elements requires the development of efficient numerical methods for investigating the temperature fields and the stress state of these elements under force loading and heating. In studying the stress state of elements of composite structures, their heterogeneous structure is often neglected, and the material behavior is described phenomenologically, using the relations of anisotropic theory of elasticity [1].

By now, methods for investigating the thermal stress state of anisotropic structural elements have been elaborated rather completely only for laminated plates and shells made of curvilinearly orthotropic elastic materials with symmetry axes aligned with the coordinate axes in which the corresponding solutions are constructed [2-8]. However, in view of the current interest in carbon composite materials with a three-dimensional carbon-fiber reinforcement, which are widely used, e.g., in the elements of nose parts of descent space vehicles, in nozzle blocks of rocket engines, and in parabolic antennas of communica-

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tion satellites, the need to develop new methods for investigating laminated bodies has appeared. The application of three-dimensional finite elements to calculating the stress-strain state of such structures is labor-consuming and unprofitable.

The efficiency of the finite-element method in solving three-dimensional problems for bodies of revolution can be raised considerably by combining it with the method of separation of variables, i.e., by using the so-called semianalytic finite-element method. The essence of the latter consists in representing the resolving functions in the circumferential direction in the form of trigonometric series, whose coefficients are determined based on a finite-element discretization in the plane perpendicular to this direction. The method suggested allows us to reduce the solution of the three-dimensional thermoelasticity problem for compound rotational bodies of orthotropic materials whose symmetry axes are not aligned with the directions of cylindrical coordinates to the solution of a number of two-dimensional problems in the meridional cross section of the bodies.

1. Basic Relations of Thermoelasticity Theory for Compound Rotational Bodies of Orthotropic Materials

Let us examine in the cylindrical coordinates z, r, and φ the stress state of compound rotational bodies of orthotropic elastic materials under volume $\mathbf{K}(K_z, K_r, K_\varphi)$ and surface $\mathbf{t}_n(t_{nz}, t_{nr}, t_{n\varphi})$ forces and nonuniform heating. The loading level is such that the rheological properties of layer materials are not manifested, but their mechanical characteristics depend on temperature. By a compound body we understand a discretely inhomogeneous rotational body each component of which is also a rotational body. In this case, both the whole body and its individual parts have a common axis of revolution coinciding with the coordinate axis z. It is assumed that the components of the body, which are made of different materials, are fastened together at a temperature T_0 without interference fit, and the conditions of ideal force and thermal contacts are satisfied on their common boundary. The investigation of the stress-strain state of such bodies is reduced to successively solving the problem of nonstationary heat conduction to determine the temperature T and the problem of thermoelasticity to determine the components of displacements u_i, deformations ε_{ij}, and stresses σ_{ij} $(i, j = z, r, \varphi)$ at fixed instants of time.

For an elastic orthotropic material whose symmetry axes of mechanical and thermophysical characteristics are aligned with the directions of orthogonal coordinates X_1, X_2, and X_3, the strain–stress relation can be written as follows [1]:

$$\begin{Bmatrix} \varepsilon_{11} - \varepsilon_{11}^{\mathrm{T}} \\ \varepsilon_{22} - \varepsilon_{22}^{\mathrm{T}} \\ \cdots \\ \varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{23}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \cdots \\ \sigma_{23} \end{Bmatrix}, \tag{1}$$

where E_i is the elastic modulus in the symmetry axes, which coincide with the coordinate axes; G_{ij} is the shear modulus in the corresponding coordinate plane; ν_{ij} is the Poisson ratio, describing the deformation of the material in the X_j-direction under loading in the X_i-direction; $\varepsilon_{ii}^{\mathrm{T}} = \alpha_{ii}^{\mathrm{T}}(T - T_0)$, where α_{ii}^{T} is the coefficient of linear thermal expansion of the material along the corresponding symmetry axis. It follows from the existence of a positive definite function of potential energy that $\nu_{ij}/E_i = \nu_{ji}/E_j$ and the compliance matrix in Eq. (1) is symmetric.

Let us consider an elastic orthotropic material whose symmetry axes are aligned with the axes of a cylindrical (z, r, φ) or Cartesian (z, x, y) system of coordinates. Solving the system of equations (1) for stress components, we obtain in the cylindrical coordinates the relation

$$\sigma_{ij} = A_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{\mathrm{T}}), \tag{2}$$

where the expressions for A_{ijkl} $(i, j, k, l = z, r, \varphi)$ depend on the type of the material considered [9, 10]. For a cylindrically orthotropic material, they are

$$A_{zzzz} = \frac{\Delta_{11}}{\Delta}, \quad A_{zzrr} = A_{rrzz} = \frac{\Delta_{12}}{\Delta}, \quad A_{zz\varphi\varphi} = A_{\varphi\varphi zz} = \frac{\Delta_{13}}{\Delta}, \quad A_{rrrr} = \frac{\Delta_{22}}{\Delta},$$

$$A_{rr\varphi\varphi} = A_{\varphi\varphi rr} = \frac{\Delta_{23}}{\Delta}, \; A_{\varphi\varphi\varphi\varphi} = \frac{\Delta_{33}}{\Delta}, \; A_{zrzr} = G_{zr}, \; A_{z\varphi z\varphi} = G_{z\varphi}, \; A_{r\varphi r\varphi} = G_{r\varphi}, \tag{3}$$

$$A_{zzzr} = A_{zzz\varphi} = A_{zzr\varphi} = A_{rrzr} = A_{rrz\varphi} = A_{rrr\varphi} = A_{\varphi\varphi zr} = A_{\varphi\varphi z\varphi}$$

$$= A_{\varphi\varphi r\varphi} = A_{zrzz} = A_{zrrr} = A_{zr\varphi\varphi} = A_{zrz\varphi} = A_{zrr\varphi} = A_{z\varphi zz} = A_{z\varphi rr} = A_{z\varphi\varphi\varphi} = A_{z\varphi zr}$$

$$= A_{z\varphi r\varphi} = A_{r\varphi zz} = A_{r\varphi rr} = A_{r\varphi\varphi\varphi} = A_{r\varphi zr} = A_{r\varphi z\varphi} = 0,$$

where

$$\Delta_{11} = \frac{\dfrac{1}{E_\varphi} - \dfrac{\nu_{r\varphi}^2}{E_r}}{E_r}, \quad \Delta_{12} = \frac{\nu_{z\varphi}\left(\dfrac{\nu_{r\varphi}}{E_r} + \dfrac{\nu_{zr}}{E_\varphi}\right)}{E_z}, \quad \Delta_{13} = \frac{\nu_{zr}\nu_{r\varphi} + \nu_{z\varphi}}{E_z E_r},$$

$$\Delta_{22} = \frac{\dfrac{1}{E_\varphi} - \dfrac{\nu_{z\varphi}^2}{E_z}}{E_z}, \qquad \Delta_{23} = \frac{\nu_{zr}\left(\dfrac{\nu_{z\varphi}}{E_z} + \dfrac{\nu_{r\varphi}}{E_r}\right)}{E_z}, \tag{4}$$

$$\Delta_{33} = \frac{\dfrac{1}{E_r} - \dfrac{\nu_{zr}^2}{E_z}}{E_z}, \qquad \Delta = \frac{\Delta_{11} - \nu_{zr}\Delta_{12} - \nu_{z\varphi}\Delta_{13}}{E_z},$$

$$\varepsilon_{zz}^{\mathrm{T}} = \alpha_{zz}^{\mathrm{T}}(T - T_0), \quad \varepsilon_{rr}^{\mathrm{T}} = \alpha_{rr}^{\mathrm{T}}(T - T_0), \quad \varepsilon_{\varphi\varphi}^{\mathrm{T}} = \alpha_{\varphi\varphi}^{\mathrm{T}}(T - T_0),$$
$$\varepsilon_{zr}^{\mathrm{T}} = \varepsilon_{z\varphi}^{\mathrm{T}} = \varepsilon_{r\varphi}^{\mathrm{T}} = 0. \tag{5}$$

For a rectilinearly orthotropic material, the coefficients A_{ijkl} and the thermal strains $\varepsilon_{kl}^{\mathrm{T}}$ are described by the expressions

$$A_{zzzz} = \Delta_{11}^*, \quad A_{zzr\varphi} = (\Delta_{13}^* - \Delta_{12}^*)\sin 2\varphi/2,$$

$$\left.\begin{matrix} A_{zzrr} \\ A_{zz\varphi\varphi} \end{matrix}\right\} = \frac{\Delta_{12}^* + \Delta_{13}^* \pm (\Delta_{12}^* - \Delta_{13}^*)\cos 2\varphi}{2},$$

$$\left.\begin{matrix} A_{rrrr} \\ A_{\varphi\varphi\varphi\varphi} \end{matrix}\right\} = \frac{(2\Delta_{22}^* + 3\Delta_{33}^* + 2\Delta_{23}^* + 4G_{xy})}{8} \tag{6}$$

$$\pm \frac{4(\Delta_{22}^* - \Delta_{33}^*)\cos 2\varphi + (\Delta_{22}^* + \Delta_{33}^* - 2\Delta_{23}^* - 4G_{xy})\cos 4\varphi}{8},$$

$$A_{rr\varphi\varphi} = \frac{(\Delta_{22}^* + \Delta_{33}^* + 6\Delta_{23}^* - 4G_{xy}) - (\Delta_{22}^* + \Delta_{33}^* - 2\Delta_{23}^* - 4G_{xy})\cos 4\varphi}{8},$$

$$\left.\begin{matrix} A_{rrr\varphi} \\ A_{\varphi\varphi r\varphi} \end{matrix}\right\} = \frac{2(\Delta_{33}^* - \Delta_{22}^*)\sin 2\varphi \pm (\Delta_{22}^* + \Delta_{33}^* - 2\Delta_{23}^* - 4G_{xy})\sin 4\varphi}{8},$$

$$\left.\begin{matrix} A_{zrzr} \\ A_{z\varphi z\varphi} \end{matrix}\right\} = \frac{(G_{zx} + G_{zy}) \pm (G_{zx} - G_{zy})\cos 2\varphi}{2},$$

$$A_{r\varphi r\varphi} = \frac{(\Delta_{22}^* + \Delta_{33}^* - 2\Delta_{23}^* + 4G_{xy}) - (\Delta_{22}^* + \Delta_{33}^* - 2\Delta_{23}^* - 4G_{xy})\cos 4\varphi}{8},$$

$$A_{zzzr} = A_{zzz\varphi} = \cdots = A_{zrzz} = A_{zrrr} = A_{zr\varphi\varphi} = A_{zrr\varphi} = \cdots = A_{r\varphi z\varphi} = 0,$$

$$\Delta_{ij}^* = \frac{\Delta_{ij}}{\Delta}, \quad \Delta_{11} = \frac{\dfrac{1}{E_y} - \dfrac{\nu_{xy}^2}{E_x}}{E_x}, \quad \Delta_{12} = \frac{\nu_{zy}^2 \dfrac{\nu_{xy}}{E_x} + \dfrac{\nu_{zx}}{E_y}}{E_z}, \quad \Delta_{13} = \frac{\nu_{zx}\nu_{xy} + \nu_{zy}}{E_z E_x},$$

$$\Delta_{22} = \frac{\dfrac{1}{E_y} - \dfrac{\nu_{zy}^2}{E_z}}{E_z}, \qquad \Delta_{23} = \frac{\nu_{zx} \dfrac{\nu_{zy}}{E_z} + \dfrac{\nu_{xy}}{E_x}}{E_z},$$

$$\Delta_{33} = \frac{\dfrac{1}{E_x} - \dfrac{\nu_{zx}^2}{E_z}}{E_z}, \qquad \Delta = \frac{\Delta_{11} - \nu_{zx}\Delta_{12} - \nu_{zy}\Delta_{13}}{E_z}, \tag{7}$$

$$\varepsilon_{ij}^{\text{T}} = \alpha_{ij}^{\text{T}}(T - T_0) \quad (i, j = z, r, \varphi),$$

$$\left.\begin{matrix} \alpha_{rr}^{\text{T}} \\ \alpha_{\varphi\varphi}^{\text{T}} \end{matrix}\right\} = \frac{\alpha_{xx}^{\text{T}} + \alpha_{yy}^{\text{T}}}{2} \pm \frac{(\alpha_{xx}^{\text{T}} - \alpha_{yy}^{\text{T}})\cos 2\varphi}{2}, \tag{8}$$

$$\alpha_{r\varphi}^{\text{T}} = \frac{(\alpha_{yy}^{\text{T}} - \alpha_{xx}^{\text{T}})\sin 2\varphi}{2}, \quad \alpha_{zr}^{\text{T}} = \alpha_{z\varphi}^{\text{T}} = 0.$$

Relationship (2) can be written in the form of Hooke's law for homogeneous isotropic materials. For this purpose, we present the coefficients A_{ijkl} as $A_{ijkl} = A_{ijkl}^0(1 - \omega_{ijkl})$, where A_{ijkl}^0 are some averaged temperature-independent values of the corresponding factors (3) and (6), and $A_{ijkl}^0\omega_{ijkl}$ are functions characterizing the change in A_{ijkl} and taking into account their dependence on temperature. Then, the relation between the stresses and strains can be presented as

$$\begin{Bmatrix} \sigma_{zz} \\ \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \sigma_{zr} \\ \sigma_{z\varphi} \\ \sigma_{r\varphi} \end{Bmatrix} = \begin{bmatrix} A_{zzzz}^0 & A_{zzrr}^0 & A_{zz\varphi\varphi}^0 & 0 & 0 & 0 \\ A_{zzrr}^0 & A_{rrrr}^0 & A_{rr\varphi\varphi}^0 & 0 & 0 & 0 \\ A_{zz\varphi\varphi}^0 & A_{rr\varphi\varphi}^0 & A_{\varphi\varphi\varphi\varphi}^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{zrzr}^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{z\varphi z\varphi}^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{r\varphi r\varphi}^0 \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{rr} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{zr} \\ \varepsilon_{z\varphi} \\ \varepsilon_{r\varphi} \end{Bmatrix} - \begin{Bmatrix} \sigma_{zz}^* \\ \sigma_{rr}^* \\ \sigma_{\varphi\varphi}^* \\ \sigma_{zr}^* \\ \sigma_{z\varphi}^* \\ \sigma_{r\varphi}^* \end{Bmatrix} \tag{9}$$

where

$$\sigma^*_{zz} = A_{zzzz}\varepsilon^{\mathrm{T}}_{zz} + A_{zzrr}\varepsilon^{\mathrm{T}}_{rr} + A^{\mathrm{T}}_{zz\varphi\varphi}\varepsilon^{\mathrm{T}}_{\varphi\varphi}$$

$$+ A^0_{zzzz}\omega_{zzzz}\varepsilon_{zz} + A^0_{zzrr}\omega_{zzrr}\varepsilon_{zr} + A^0_{zz\varphi\varphi}\omega_{zz\varphi\varphi}\varepsilon_{\varphi\varphi},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (10)$$

$$\sigma^*_{r\varphi} = 2A^0_{r\varphi r\varphi}\omega_{r\varphi r\varphi}\varepsilon_{r\varphi}$$

for a cylindrically orthotropic material and

$$\sigma^*_{zz} = A^0_{zzzz}\omega_{zzzz}\varepsilon_{zz} + A^0_{zzrr}\omega_{zzrr}\varepsilon_{rr} + A^0_{zz\varphi\varphi}\omega_{zz\varphi\varphi}\varepsilon_{\varphi\varphi}$$

$$-2A_{zzr\varphi}(\varepsilon_{r\varphi} - \varepsilon^{\mathrm{T}}_{r\varphi}) + A_{zzzz}\varepsilon^{\mathrm{T}}_{zz} + A_{zzrr}\varepsilon^{\mathrm{T}}_{rr} + A_{zz\varphi\varphi}\varepsilon^{\mathrm{T}}_{\varphi\varphi},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (11)$$

$$\sigma^*_{r\varphi} = 2A_{r\varphi r\varphi}\omega_{r\varphi r\varphi} + 2A_{r\varphi r\varphi}\varepsilon^{\mathrm{T}}_{r\varphi} - A_{zzr\varphi}(\varepsilon_{zz} - \varepsilon^{\mathrm{T}}_{zz})$$

$$-A_{rrr\varphi}(\varepsilon_{rr} - \varepsilon^{\mathrm{T}}_{rr}) - A_{\varphi\varphi r\varphi}(\varepsilon_{\varphi\varphi} - \varepsilon^{\mathrm{T}}_{\varphi\varphi}).$$

for a rectilinearly orthotropic material.

A comparison between state equations (9) for an orthotropic material and the corresponding relation for an isotropic material shows that, if we assume that

$$A^0_{zzzz} = A^0_{rrrr} = A^0_{\varphi\varphi\varphi\varphi} = \frac{4G_0 + K_0}{3}, \quad A^0_{zzrr} = A^0_{zz\varphi\varphi} = A^0_{rr\varphi\varphi} = \frac{K_0 - 2G_0}{3},$$

$$A^0_{zrzr} = A^0_{z\varphi z\varphi} = A^0_{r\varphi r\varphi} = G_0, \quad (12)$$

where G_0 and K_0 are the shear and bulk moduli of the isotropic material at a temperature T_0, the form of stress—strain relationship is the same for isotropic and orthotropic materials. This allows us to apply the known algorithms of solving three-dimensional problems of thermoviscoelasticity to laminated rotational bodies consisting of inelastic isotropic materials under nonaxisymmetric thermal and force loadings [10, 11].

2. Method for Investigating the Nonaxisymmetric Thermal Stress State of Compound Rotational Bodies

As already mentioned, the problem of investigating the thermal stress state of compound rotational bodies can be reduced to successively solving the problem of nonstationary heat conduction to determine the temperature fields and the problem of thermoelasticity to determine the stress-strain state of the bodies. As basic unknowns, we assume the temperature T and displacements u_z, u_r, and u_φ. They can be found from the variational heat equations

$$\int_V \left[c\rho \frac{\partial T}{\partial \tau}\delta T - q_z \delta\!\left(\frac{\partial T}{\partial z}\right) - q_r\left(\frac{\partial T}{\partial r}\right) - q_\varphi\left(\frac{1}{r}\frac{\partial T}{\partial \varphi}\right)\right] dV + \int_\Sigma \alpha(T-\theta)\delta T d\Sigma \quad (13)$$

and the Lagrange variational equation

$$\int_V (\sigma_{ij}\delta\varepsilon_{ij} - K_i \delta u_i)dV - \int_{\Sigma_t} t_{ni}\delta u_i d\Sigma = 0 \quad (i, j = z, r, \varphi), \tag{14}$$

where V is the volume of a rotational body bounded by a surface Σ; Σ_t is the part of the surface Σ on which the surface load $\mathbf{t}_n$ is specified; c is the coefficient of specific mass heat capacity of the body material; ρ is the material density; α is the coefficient of convective heat exchange at the surface of the body contacting a medium with a temperature θ; t is the current time of heating and loading the body; q_z, q_r, and q_φ are the thermal flows in corresponding directions:

$$q_z = -\left(\lambda_{zz}\frac{\partial T}{\partial z} + \lambda_{zr}\frac{\partial T}{\partial r} + \lambda_{z\varphi}\frac{1}{r}\frac{\partial T}{\partial \varphi}\right), \quad q_r = -\left(\lambda_{zr}\frac{\partial T}{\partial z} + \lambda_{rr}\frac{\partial T}{\partial r} + \lambda_{r\varphi}\frac{1}{r}\frac{\partial T}{\partial \varphi}\right),$$

$$q_\varphi = -\left(\lambda_{z\varphi}\frac{\partial T}{\partial z} + \lambda_{r\varphi}\frac{\partial T}{\partial r} + \lambda_{\varphi\varphi}\frac{1}{r}\frac{\partial T}{\partial \varphi}\right), \tag{15}$$

where λ_{ij} are components of the heat conductivity tensor; for an isotropic material $\lambda_{ij} = \lambda\delta_{ij}$, whereas for a rectilinearly orthotropic material, they are determined from the principal values λ_{zz}, λ_{xx}, and λ_{yy} according to the formulas of transformation from the coordinates z, x, and y to z, r, and φ.

The discretization of variational equations (13) and (14) is carried out based on the semianalytic finite-element method, which reduces the initial three-dimensional problem to a series of two-dimensional ones in the meridional cross section of the body. For this purpose, the solution must be sought in the form of trigonometric series

$$T(z, r, \varphi, t) = \sum_{m=0}^{\infty} \overline{T}_m(z, r, t)\cos m\varphi + \sum_{m=1}^{\infty} \overline{\overline{T}}_m(z, r, t)\sin m\varphi, \tag{16}$$

$$u_z(z, r, \varphi, t) = \sum_{m=0}^{\infty} \overline{u}_z^{(m)}(z, r, t)\cos m\varphi + \sum_{m=1}^{\infty} \overline{\overline{u}}_z^{(m)}(z, r, t)\sin m\varphi,$$

$$u_\varphi(z, r, \varphi, t) = \sum_{m=1}^{\infty} \overline{u}_\varphi^{(r)}(z, r, t)\sin m\varphi + \sum_{m=0}^{\infty} \overline{\overline{u}}_\varphi^{(m)}(z, r, t)\cos m\varphi, \tag{17}$$

whose coefficients are found from variational equations (13) and (14) with the use of finite elements in the meridional cross section of the body.

The coefficients λ_{ij} of heat conduction entering into variational equation (13) depend on temperature and vary both in the circumferential direction and in the meridional cross section. After representing λ_{ij} in the form $\lambda_{ij} = \lambda_{ij}^0(1 - \omega_{ij}^{\mathrm{T}})$ and assuming that, at some fixed instant of time, the heat-transfer factor α, the ambient temperature θ, and the product $c\rho$ are known functions of coordinates and do not change, variational equation (13) can be written as

$$\delta\left\{\int_V \left[\int_{T_0}^{T} c\rho\frac{\partial T}{\partial t}dT + \frac{\lambda_{zz}^0}{2}\left(\frac{\partial T}{\partial z}\right)^2 + \frac{\lambda_{rr}^0}{2}\left(\frac{\partial T}{\partial r}\right)^2 + \frac{\lambda_{\varphi\varphi}^0}{2}\left(\frac{1}{r}\frac{\partial T}{\partial \varphi}\right)^2\right.\right.$$

$$\left.\left. - q_z^*\frac{\partial T}{\partial z} - q_r^*\frac{\partial T}{\partial r} - q_\varphi^*\frac{1}{r}\frac{\partial T}{\partial \varphi}\right] rdzdrd\varphi + \int_\Sigma \alpha\frac{T}{2}(T - 2\theta)rdsd\varphi\right\} = 0. \tag{18}$$

Here,

$$q_z^* = \lambda_{zz}^0 \omega_{zz}^{\text{т}} \frac{\partial T}{\partial z}, \qquad q_r^* = \lambda_{rr}^0 \omega_{rr}^{\text{т}} \frac{\partial T}{\partial r} + \lambda_{r\varphi} \frac{1}{r}\frac{\partial T}{\partial \varphi},$$

$$q_\varphi^* = \lambda_{r\varphi} \frac{\partial T}{\partial r} + \lambda_{\varphi\varphi}^0 \omega_{\varphi\varphi}^{\text{т}} \frac{1}{r}\frac{\partial T}{\partial \varphi}, \tag{19}$$

which are assumed to be known functions of coordinates.

Representing the ambient temperature and additional terms (19) as trigonometric Fourier series, we reduce the initial three-dimensional problem of heat conduction (18) to a series of two-dimensional variational problems with respect to unknown values of the coefficients in series (17), which can be found by the finite-element method. In the meridional cross section of the body, we take triangular finite elements with linearly changing coefficients $\overline{T}_m$ and $\overline{\overline{T}}_m$. To determine the values of coefficients $\overline{T}_m$ at the nodes (i, j, k) of the elements where the side ij coincides with the surface of the body, we repeat the manipulations used in [11] and come to recurrent relations for calculating the coefficients $\overline{T}_m$ at the instant of time $t + \Delta t$ in terms of their values at the instant t:

$$\overline{T}_{mi}(t+\Delta t) = \overline{T}_{mi}(t) + \frac{\Delta t}{\sum_{q=1}^{M} \langle c\rho \rangle_q H_i^{(q)}} \sum_{q=1}^{M} [A_{ij}\overline{\theta}_{mi} + B_{ij}\overline{\theta}_{mj}$$

$$-(D_{ij} + m^2 N_{ij} + A_{ij})\overline{T}_{mi}(t) - (D_{ij} + m^2 N_{ij} + B_{ij})\overline{T}_{mj}(t)$$

$$-(D_{ik} + m^2 N_{ik})\overline{T}_{mk}(t) + L_i\langle q_z^* \rangle + P_i \langle q_r^* \rangle - mR_i \langle q_\varphi^* \rangle]_q \quad (i = 1, 2, ..., N). \tag{20}$$

Here, m is the harmonic number, N is the number of nodal points, M is the number of triangular elements in the meridional cross section of the body,

$$H_i = b_{1i}\int_\Delta r dz dr + b_{2i}\int_\Delta zr dz dr + b_{3i}\int_\Delta r^2 dz dr,$$

$$D_{ij} = \lambda_{zz}^0 b_{2i} b_{2j}\int_\Delta r dz dr + \lambda_{rr}^0 b_{3i} b_{3j}\int_\Delta r dz dr,$$

$$N_{ij} = \lambda_{\varphi\varphi}^0\left[b_{1i}\left(b_{1j}\int_\Delta \frac{1}{r} dz dr + b_{2j}\int_\Delta \frac{z}{r} dz dr + b_{3j}\int_\Delta dz dr\right)\right.$$

$$+ b_{2i}\left(b_{1j}\int_\Delta \frac{z}{r} dz dr + b_{2j}\int_\Delta \frac{z^2}{r} dz dr + b_{3j}\int_\Delta z dz dr\right) +$$

$$\left. + b_{3i}\left(b_{1j}\int_\Delta dz dr + b_{2j}\int_\Delta z dz dr + b_{3j}\int_\Delta r dz dr\right)\right],$$

$$A_{ij} = \operatorname{sign} F_\Delta \frac{l_{ij}}{10}\left[\alpha_i\left(2r_i + \frac{r_j}{2}\right) + \alpha_j\left(\frac{r_i}{2} + \frac{r_j}{3}\right)\right], \tag{21}$$

$$B_{ij} = \operatorname{sign} F_\Delta \frac{l_{ij}}{10}\left[\alpha_i\left(\frac{r_i}{2}+\frac{r_j}{3}\right)+\alpha_j\left(\frac{r_i}{3}+\frac{r_j}{2}\right)\right],$$

$$L_i = b_{2i}\int_\Delta rdzdr,\ P_i = b_{3j}\int_\Delta rdzdr,\ R_i = b_{1i}\int_\Delta dzdr + b_{2i}\int_\Delta zdzdr + b_{3i}\int_\Delta rdzdr,$$

$$b_{1j} = \frac{z_k r_i - z_i r_k}{2F_\Delta},\quad b_{2j} = \frac{r_k - r_i}{2F_\Delta},\quad b_{3j} = \frac{z_i - z_k}{2F_\Delta}, \tag{22}$$

$$F_\Delta = \frac{1}{2}[z_i(r_j - r_k) + z_j(r_k - r_i) + z_k(r_i - r_j)],$$

$$l_{ij} = \sqrt{(z_i - z_j)^2 + (r_i - r_j)^2}.$$

In constructing the variational equations for calculating all amplitude values of temperature and in their subsequent discretization by the finite-element method, the method of successive approximations was used. It was assumed that the additional terms q_z^*, q_r^*, and q_φ^* were known from the previous time step or the previous approximation. However, since the time step Δt in integrating the heat equation was rather small, the amplitude values of temperature belonging to the right-hand part of this equation and the additional terms averaged over each finite element were determined on the previous time step. The coefficients $\overline{\overline{T}}_m$ were found from expressions similar to Eqs. (20), where m must be replaced by $-m$, and all the quantities with overbars must be replaced by the corresponding quantities with double overbars. Taking into account the values of $\overline{T}_m$ and $\overline{\overline{T}}_m$ at all points of finite-element division of the meridional cross section of the body, we can calculate the temperature in the body from trigonometric series (16).

Substituting Eq. (17) into variational equation (14) and assuming that the additional stress σ_{ij}^* is not varied, we come to the equation for the stress-strain state of rotational bodies

$$\delta Q = \delta\left\{\int_V\left[\frac{1}{2}\ (A_{zzzz}^0\varepsilon_{zz}^2 + A_{rrrr}^0\varepsilon_{rr}^2 + A_{\varphi\varphi\varphi\varphi}^0\varepsilon_{\varphi\varphi}^2) + 2(A_{zrzr}^0\varepsilon_{zr}^2 + A_{z\varphi z\varphi}^0\varepsilon_{z\varphi}^2\right.\right.$$

$$+ A_{r\varphi r\varphi}^0\varepsilon_{r\varphi}^2) + A_{zzrr}^0\varepsilon_{zz}\varepsilon_{rr} + A_{zz\varphi\varphi}^0\varepsilon_{zz}\varepsilon_{\varphi\varphi} + A_{rr\varphi\varphi}^0\varepsilon_{rr}\varepsilon_{\varphi\varphi}$$

$$-\sigma_{zz}^*\varepsilon_{zz} - \sigma_{rr}^*\varepsilon_{rr} - \sigma_{\varphi\varphi}^*\varepsilon_{\varphi\varphi} - 2\sigma_{zr}^*\varepsilon_{zr} - 2\sigma_{z\varphi}^*\varepsilon_{z\varphi} - 2\sigma_{r\varphi}^*\varepsilon_{r\varphi} - K_z u_z$$

$$\left.\left. - K_r u_r - K_\varphi u_\varphi\right] rdzdrd\varphi - \int_{\Sigma_t}(t_{nz}u_z + t_{nr}u_r + t_{n\varphi}u_\varphi)rdsd\varphi\right\} = 0. \tag{23}$$

If the coefficients A_{ijkl} are determined from Eqs. (3) and the additional stresses σ_{ij}^* from expressions (10), Eq. (23) is the variational equation for describing the stress-strain state of a cylindrically orthotropic material. For a rectilinearly orthotropic material, the coefficients A_{ijkl} in Eq. (23) are defined by relations (6) and the additional stresses by expressions (11).

To obtain a solution to variational equation (23) in trigonometric series (17), the projections of the surface t_{ni} and volume K_i forces and the function σ_{ij}^* are presented in the form of similar series. Substituting these series, Eqs. (17), and the expressions for strain components obtained from Eqs. (17) by using the Cauchy relation into variational equation (23), we arrive at the following system of equations for determining the amplitude values $\overline{u}_i^{(m)}$ and $\overline{\overline{u}}_i^{(m)}$ of sought-for displacements:

$$(1+\delta_{0m})\pi\delta\overline{\mathrm{Q}}_m = 0, \quad (1+\delta_{0m})\pi\delta\overline{\overline{\mathrm{Q}}}_m = 0, \tag{24}$$

where

$$\overline{\mathrm{Q}}_m = \int_F \Bigg[\frac{1}{2} (A^0_{zzzz}\overline{\varepsilon}^{(m)2}_{zz} + A^0_{rrrr}\overline{\varepsilon}^{(m)2}_{rr} + A^0_{\varphi\varphi\varphi\varphi}\overline{\varepsilon}^{(m)2}_{\varphi\varphi}) + 2(A^0_{zrzr}\overline{\varepsilon}^{(m)2}_{zr} + A^0_{z\varphi z\varphi}\overline{\varepsilon}^{(m)2}_{z\varphi}$$

$$+ A^0_{r\varphi r\varphi}\overline{\varepsilon}^{(m)2}_{r\varphi}) + A^0_{zzrr}\overline{\varepsilon}^{(m)}_{zz}\overline{\varepsilon}^{(m)}_{rr} + A^0_{zz\varphi\varphi}\overline{\varepsilon}^{(m)}_{zz}\overline{\varepsilon}^{(m)}_{\varphi\varphi} + A^0_{rr\varphi\varphi}\overline{\varepsilon}^{(m)}_{rr}\overline{\varepsilon}^{(m)}_{\varphi\varphi} - \overline{\sigma}^*_{zz}\overline{\varepsilon}^{(m)}_{zz}$$

$$-\overline{\sigma}^*_{rr}\overline{\varepsilon}^{(m)}_{rr} - \overline{\sigma}^*_{\varphi\varphi}\overline{\varepsilon}^{(m)}_{\varphi\varphi} - 2\overline{\sigma}^*_{zr}\overline{\varepsilon}^{(m)}_{zr} - 2\overline{\sigma}^*_{z\varphi}\overline{\varepsilon}^{(m)}_{z\varphi} - 2\overline{\sigma}^*_{r\varphi}\overline{\varepsilon}^{(m)}_{r\varphi} - \overline{K}^{(m)}_z\overline{u}^{(m)}_z - \overline{K}^{(m)}_r\overline{u}^{(m)}_r$$

$$- \overline{K}^{(m)}_\varphi\overline{u}^{(m)}_\varphi \Bigg] rdzdr - \int_S (\overline{t}^{(m)}_{nz}\overline{u}^{(m)}_z + \overline{t}^{(m)}_{nr}\overline{u}^{(m)}_r + \overline{t}^{(m)}_{n\varphi}\overline{u}^{(m)}_\varphi) rds \quad (m = 0, 1, \ldots), \tag{25}$$

$$\overline{\varepsilon}^{(m)}_{zz} = \frac{\partial\overline{u}^{(m)}_z}{\partial z}, \quad \overline{\varepsilon}^{(m)}_{rr} = \frac{\partial\overline{u}^{(m)}_r}{\partial r}, \quad \overline{\varepsilon}^{(m)}_{\varphi\varphi} = \frac{1}{r}(\overline{u}^{(m)}_r - m\overline{u}^{(m)}_\varphi),$$

$$\overline{\varepsilon}^{(m)}_{zr} = \frac{1}{2}\left(\frac{\partial\overline{u}^{(m)}_z}{\partial r} + \frac{\partial\overline{u}^{(m)}_r}{\partial z}\right), \quad \overline{\varepsilon}^{(m)}_{z\varphi} = \frac{1}{2}\left(\frac{\partial\overline{u}^{(m)}_\varphi}{\partial z} - \frac{m}{r}\overline{u}^{(m)}_z\right), \tag{26}$$

$$\overline{\varepsilon}^{(m)}_{r\varphi} = \frac{1}{2}\left(\frac{\partial\overline{u}^{(m)}_\varphi}{\partial r} - \frac{m}{r}\overline{u}^{(m)}_r - \frac{1}{r}\overline{u}^{(m)}_\varphi\right).$$

Here, $\overline{K}^{(m)}_i$, $\overline{t}^{(m)}_{ni}$, and $\overline{\sigma}^{*(m)}_i$ are coefficients in the expansion of corresponding values into trigonometric series with respect to the circumferential coordinate; F is the area of half the meridional cross section of the body, and S is its contour. The expressions for $\overline{\mathrm{Q}}_m$ are written in a similar form: the variational equations relative to $\overline{\mathrm{Q}}_0$ and $\overline{\overline{\mathrm{Q}}}_0$ describe the axisymmetric stress state without and with torsion, respectively.

For calculating the stationary values of functionals (25), we use the finite-element method. As in solving the problem of heat conduction, for a finite element in the meridional cross section of the body, we take a triangular element with linear approximation of the coefficients of series (17). To determine the coefficients $\overline{u}^{(m)}_\alpha$ and $\overline{\overline{u}}^{(m)}_\alpha$ $(\alpha = z, r, \varphi)$ at the nodes (i, j, k) of triangular elements q of the meridional cross section of the body in trigonometrical series (17) in each approximation, we use the procedure described in [11] and come to a system of $3N$ linear algebraic equations for each harmonic separately:

$$\sum_{q=1}^{M} (B^{zi(q)}_{zp}u_{zp} + B^{zi(q)}_{rp}u_{rp} + B^{zi(q)}_{\varphi p}u_{\varphi p}) = D_{zi},$$

$$\sum_{q=1}^{M} (B^{ri(q)}_{zp}u_{zp} + B^{ri(q)}_{rp}u_{rp} + B^{ri(q)}_{\varphi p}u_{\varphi p}) = D_{ri}, \tag{27}$$

$$\sum_{q=1}^{M}(B_{zp}^{\varphi i(q)}u_{zp}+B_{rp}^{\varphi i(q)}u_{rp}+B_{\varphi p}^{\varphi i(q)}u_{\varphi p})=D_{\varphi i}\quad (p=i,j,k),\ <i=1,2,\ldots,N>.$$

The number of such systems is equal to the number of terms retained in solution (17).

The matrix elements of system (27) are found from the coefficients of physical equations (9) and nodal coordinates of finite elements in the meridional plane. The right-hand part of the system is determined from the amplitude values of additional stresses σ_{ij}^{*} in Eq. (9) and the volume and surface loads at corresponding points of the meridional cross section. For an individual triangular element with nodes i, j, and k, they have the form

$$B_{zj}^{zi}=\int_{F_\Delta}(A_{11}^{0}b_{2i}b_{2j}+A_{44}^{0}b_{3i}b_{3j}+m^2A_{55}^{0}\Delta_{1i}\Delta_{1j})rdzdr,$$

$$B_{rj}^{zi}=\int_{F_\Delta}(A_{44}^{0}b_{3i}b_{2j}+A_{12}^{0}b_{2i}b_{3j}+A_{13}^{0}b_{2i}\Delta_{1j})rdzdr,$$

$$B_{\varphi j}^{zi}=m\int_{F_\Delta}(-A_{55}^{0}b_{2j}\Delta_{1i}+A_{13}^{0}b_{2i}\Delta_{1j})rdzdr,$$

$$B_{zj}^{ri}=\int_{F_\Delta}(A_{44}^{0}b_{2i}b_{3j}+A_{12}^{0}b_{3i}b_{2j}+A_{13}^{0}b_{2j}\Delta_{1i})rdzdr,$$

$$B_{rj}^{ri}=\int_{F_\Delta}\{(A_{22}^{0}b_{3i}+A_{23}^{0}\Delta_{1i})b_{3j}+A_{44}^{0}b_{2i}b_{2j}$$

$$+[A_{23}^{0}b_{3i}+(A_{33}^{0}+m^2A_{66})\Delta_{1j}]\Delta_{1j}\}rdzdr, \tag{28}$$

$$B_{\varphi j}^{ri}=m\int_{F_\Delta}[(A_{33}^{0}\Delta_{1i}+A_{23}^{0}b_{3i})\Delta_{1j}+A_{66}^{0}\Delta_{1i}\Delta_{2j}]rdzdr,$$

$$B_{zj}^{\varphi i}=m\int_{F_\Delta}(-A_{55}^{0}b_{2i}\Delta_{1j}+A_{13}^{0}b_{2j}\Delta_{1i})rdzdr,$$

$$B_{rj}^{\varphi i}=m\int_{F_\Delta}[(A_{33}^{0}\Delta_{1i}+A_{66}^{0}\Delta_{2i})\Delta_{1j}+A_{23}^{0}b_{3j}\Delta_{1i}]rdzdr,$$

$$B_{rj}^{\varphi i}=\int_{F_\Delta}(A_{55}^{0}b_{2i}b_{2j}+A_{66}^{0}\Delta_{2i}\Delta_{2j}+m^2A_{33}^{0}\Delta_{1i}\Delta_{1j})rdzdr;$$

$$D_{zi}=\sum_{q=1}^{M}\int_{F_\Delta}[\sigma_{zz}^{*}b_{2i}+\sigma_{zr}^{*}b_{3i}-(m\sigma_{z\varphi}^{*}-K_z z)\Delta_{1i}]rdzdr$$

$$+\sum_{l=1}^{L}\frac{l_{ij}}{12}\operatorname{sign}F_\Delta[t_{nz_i}(3r_i+r_j)+t_{nz_j}(r_i+r_j)],$$

$$D_{ri}=\sum_{q=1}^{M}\int_{F_\Delta}[\sigma_{rr}^{*}b_{3i}+\sigma_{zr}^{*}b_{2i}+(\sigma_{\varphi\varphi}^{*}-m\sigma_{r\varphi}^{*}+K_r r)\Delta_{1i}]rdzdr$$

$$+\sum_{l=1}^{L}\frac{l_{ij}}{12}\operatorname{sign}F_\Delta\,[t_{nr_i}(3r_i+r_j)+t_{nr_j}(r_i+r_j)],\tag{29}$$

$$D_{\varphi i}=\sum_{q=1}^{M}\int_{F_\Delta}[\sigma^*_{z\varphi}b_{ri}-\sigma^*_{r\varphi}\Delta_{ri}+(m\sigma^*_\varphi+K_\varphi r)\Delta_{1i}]rdzdr$$

$$+\sum_{l=1}^{L}\frac{l_{ij}}{12}\operatorname{sign}F_\Delta\,[t_{n\varphi_i}(3r_i+r_j)+t_{n\varphi_j}(r_i+r_j)];$$

$$\Delta_{1j}=b_{3j}+b_{2j}\frac{z}{r}+b_{1j}\frac{1}{r},\quad \Delta_{2j}=b_{2j}\frac{z}{r}+b_{1j}\frac{1}{r}.\tag{30}$$

The first summation in Eq. (29) is performed over all finite elements and the second one over all sides of the corresponding elements, coinciding with the contour of meridional cross section of the body, one node of which has the number i. The remaining coefficients in Eq. (27) can be obtained from Eq. (28) by replacing the subscript j with i or k.

After determining the amplitude values of displacements from systems (27), we calculate from Eqs. (17), (26), and (9) the components of displacements, deformations, and stresses in each approximation for the instant of time considered. The number of required approximations is found from the condition that the relative change in the stress-strain state for two subsequent solutions is smaller than a given value.

3. Examples of Calculation

To check the convergence of the method suggested for investigating the stress-strain state of rotational bodies made of rectilinearly orthotropic elastic materials, we considered some test problems with known analytical solutions obtained by other authors: the stress state of a rotating solid, thin, rectilinearly orthotropic disk [12] and of a solid, thin, rectilinearly orthotropic disk whose temperature varies along the radius according to a square law [13].

It is known [12] that the stress state of a rotating solid, thin, rectilinearly orthotropic disk, which is in a plane stress state, is axisymmetric and, in the cylindrical coordinates, is described by the relation

$$\sigma_{rr}=\frac{\omega^2\rho R^2}{2}(1-\beta)\left(1-\frac{r^2}{R^2}\right),\quad \sigma_{\varphi\varphi}=\sigma_{rr}+\omega^2\rho\beta r^2,\tag{31}$$

where ω is the angular speed of the disk and R is its external radius.

The anisotropy of the mechanical characteristics of the disk material is characterized by a nondimensional parameter β, which is defined as

$$\beta=\left(\frac{1}{E_x}+\frac{1}{E_y}-\frac{2\nu_{xy}}{E_x}\right)\Bigg/\left(\frac{3}{E_x}+\frac{3}{E_y}-\frac{2\nu_{xy}}{E_x}+\frac{1}{G_{xy}}\right).\tag{32}$$

In the case of an isotropic material, $\beta=(1-\nu)/4$.

The calculations were performed for a disk having an outer radius $R = 0.3$ m and thickness 0.005 m, with the following mechanical characteristics of the material: $E_z = 4.5\cdot10^4$ MPa, $\nu_{zx}=\nu_{zy}=0.1$, $\nu_{xy} = 0.05$, $G_{zx}=G_{zy}=0.5\cdot10^4$ MPa, and ρ = 0.0016 (MPa · s^2)/m^2, which were assumed the same in all the cases considered, whereas the values of E_x, E_y, and G_{xy} were varied. In particular, calculations were carried out with $E_x=E_y=3\cdot10^4$ MPa and $G_{xy} = 0.5\cdot10^3$ or $0.06\cdot10^3$ MPa and

TABLE 1. Stress State of a Rotating Disk, MPa

r, m	$E_x = E_y = 3 \cdot 10^4$ MPa				$E_x = 3 \cdot 10^4$ MPa $G_{xy} = 0.6 \cdot 10^4$ MPa				Isotropic material	
	$E_x/G_{xy} = 60$		$E_x/G_{xy} = 500$		$E_x/E_y = 100$		$E_x/E_y = 1000$		$E/G = 2.1$	
	σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$
0.005	69.9	69.9	71.7	71.7	48.4	48.4	48.0	48.0	54.9	54.9
0.025	69.4	69.4	71.2	71.3	48.1	48.4	47.7	48.0	54.5	54.7
0.045	68.3	68.47	70.1	70.1	47.3	48.4	46.9	48.0	53.7	54.5
0.255	19.4	22.4	19.9	20.3	13.4	47.5	13.3	48.0	15.2	39.9
0.275	11.2	14.6	11.5	11.9	7.7	47.4	7.7	48.0	8.8	37.5
0.295	2.3	6.3	2.4	2.9	1.6	47.2	1.6	48.0	1.8	34.9

with $E_x = 3 \cdot 10^4$ MPa and $G_{xy} = 0.6 \cdot 10^4$ MPa, and $E_y = 0.3 \cdot 10^3$ or $0.03 \cdot 10^3$ MPa. The disk was in the field of centrifugal forces caused by an angular speed $\omega = 1000$ 1/s at $T = T_0 = 20°C$.

During the calculations, seven terms of trigonometric series (17) were retained for displacement components, i.e., in each approximation, the problem was reduced to a solution of seven linear systems of algebraic equations. In determining the right-hand parts of systems (27), for approximate calculation of integrals in the circumferential direction, it was assumed that $\Delta\varphi = 1°$.

The results of calculations after a twentieth approximation and the data on the isotropic material are given in Table 1. The exact solution is not presented, because it completely coincides with that obtained according to the method suggested here.

We also considered the stress state of a thin rectilinearly orthotropic disk of radius R caused by nonuniform axisymmetric heating. In the case of plane stress state, if the temperature of the disk varies according to the law $T(r) = T_0 + T_1 (r^2/R^2)$ and $E_x = E_y$, $\alpha^{\mathrm{T}}_{xx} = \alpha^{\mathrm{T}}_{yy}$, the axisymmetric stress state realized in it, in cylindrical coordinates, is found from the relations [13]

$$\sigma_{rr} = \beta \frac{E_x \alpha^{\mathrm{T}}_{xx} T_1}{(1-\nu_{xy})} \left(1 - \frac{r^2}{R^2} \right), \quad \sigma_{\varphi\varphi} = \beta \frac{E_x \alpha^{\mathrm{T}}_{xx} T_1}{1-\nu_{xy}} \left(1 - \frac{3r^2}{R^2} \right), \tag{33}$$

where

$$\beta = \frac{1-\nu_{xy}}{3 - \nu_{xy} + \dfrac{E_x}{2G_{xy}}}. \tag{34}$$

For an isotropic material, $\beta = (1-\nu)/4$.

The disk material had the following mechanical characteristics: $E_z = 4.5 \cdot 10^4$ MPa, $E_x = E_y = 3 \cdot 10^4$ MPa, $G_{zx} = G_{zy} = 1 \cdot 10^4$ MPa, $\nu_{zx} = \nu_{zy} = 0.1$, $\nu_{xy} = 0.05$, $\alpha^{\mathrm{T}}_{zz} = 2 \cdot 10^{-4}$ 1/K, and $\alpha^{\mathrm{T}}_{xx} = \alpha^{\mathrm{T}}_{yy} = 1 \cdot 10^{-4}$ 1/K. The stress state of the disk ($R = 0.3$ m and thickness 0.005 m) was calculated for $T_1 = 450°C$ at different values of G_{xy} varying from $1 \cdot 10^3$ to $0.06 \cdot 10^3$ MPa. In the calculations, we restricted ourselves to seven terms of series (18), with a step $\Delta\varphi = 1°$ along the circumferential coordinate. In Table 2, the greatest values of components of σ_{rr} and $\sigma_{\varphi\varphi}$ stresses for the corresponding points of the radius are given. The table presents also the values of stress components calculated by Eqs. (33) and (34): in the cases where they do not

TABLE 2. Stress State of a Disk Heated Nonuniformly Along the Radius, MPa

r, m	T, °C	Isotropic material		E_x/G_{xy}					
				30		100		500	
		σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$
0.005	20	337.4	337.2	75.3	75.4	25.7	25.7	5.6	5.6
				75.2	75.1	25.5	25.5	5.3	5.3
0.025	23	335.2	330.5	74.8	73.8	25.5	25.2	5.5	5.5
				74.7	73.7	25.3	25.0	5.3	5.2
0.045	30	329.9	314.7	73.7	70.3	25.1	24.0	5.5	5.2
				73.5	70.1	24.9	23.8	5.2	5.0
0.255	345	93.6	–394.0	20.9	–88.0	7.1	–30.0	1.6	–6.5
					–87.8		–29.8	1.5	–6.2
0.275	398	53.9	–513.3	12.0	–114.6	4.1	–39.1	0.9	–8.5
					–114.4		–38.8	0.8	–8.1
0.295	455	11.2	–641.5	2.5	–143.2	0.8	–48.8	0.2	–10.5
					–142.9		–48.5		–10.2

TABLE 3. Variation in Calculation Results (MPa) with Integration Step with Respect to φ

r, m	T, °C	$\Delta\varphi$, deg						Exact solution	
		1		2		3			
		σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$	σ_{rr}	$\sigma_{\varphi\varphi}$
0.005	20	9.0	9.0	9.8	9.8	11.0	11.0	8.8	8.8
0.025	23	9.0	8.9	9.7	9.6	11.0	10.8	8.7	8.6
0.045	30	8.9	8.5	9.6	9.2	10.8	10.3	8.6	8.2
0.255	345	2.5	–10.6	2.7	–11.5	3.1	–12.9	2.4	–10.2
0.275	398	1.5	–13.8	1.6	–14.9	1.8	–16.8	1.4	–13.4
0.295	455	0.3	–17.1	0.3	–18.6	0.4	–20.9	0.3	–16.7

differ from those calculated by the method suggested, only one value is given. A comparison of the results shows their satisfactory coincidence over the whole range of the shear modulus G_{xy}, which allows us to use the method developed for calculating the thermal stress state of particular structural elements in the form of orthotropic bodies of revolution.

To estimate the influence of the step of integration with respect to the circumferential coordinate on the accuracy of determining the stresses, calculations with $G_{xy} = 0.1 \cdot 10^3$ MPa and different values of $\Delta\varphi$ were performed according to the procedure suggested (Table 3). As seen from data in the table, the calculation accuracy decreases with increasing $\Delta\varphi$.

A comparison between the results obtained and the analytical solution shows the high efficiency and accuracy of the procedure offered for solving the problems of thermoelasticity for rotational bodies made of rectilinearly orthotropic materials.

Using the procedure suggested for determining nonaxisymmetric temperature fields and the thermal stress state of compound rotational bodies, we investigated the nonstationary temperature field and the stress-strain state of a two-layer cylin-

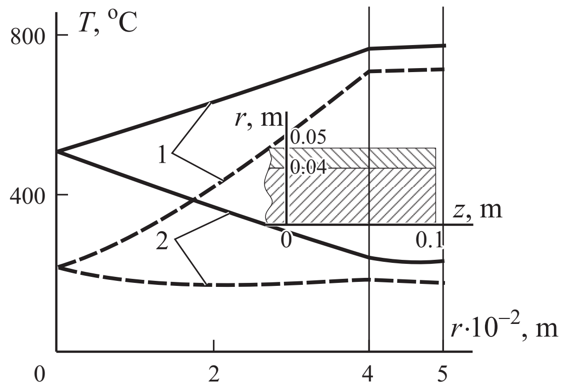

Fig. 1. Radial variation of temperature in the midsection of a cylinder.

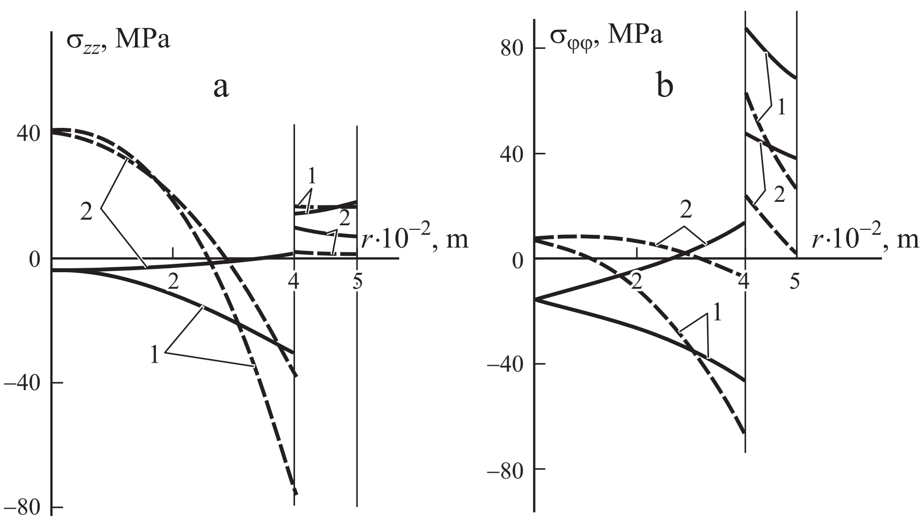

Fig. 2. Radial variation of the stresses σ_{zz} (a) and $\sigma_{\varphi\varphi}$ (b) in the midsection of a cylinder.

der under convective heat exchange with the surrounding medium (Fig. 1). The external layer was made of a cylindrically orthotropic material, and the internal part of the cylinder was manufactured from a rectilinearly orthotropic material whose one symmetry axis, *Ox* or *Oy*, corresponded to $\varphi = 0$ of the cylindrical coordinates.

At $t = 0$, the temperature of the cylinder was $T_0 = 20°C$. Then its cylindrical surface was heated by a surrounding medium whose temperature varied according to the law $\theta = (520 + 500\cos\varphi)°C$, and the end faces $z = \pm 0.1$m were heated by a medium of constant temperature $\theta = 300°C$. The coefficient of heat transfer α between the surrounding media and the cylinder material was assumed constant and equal to 1 kW/(m^2 · K) in our calculations.

As an axisymmetrically orthotropic material, we selected one with the following temperature-independent mechanical and thermophysical characteristics: $E_r = E_\varphi = 2.5 \cdot 10^4$ MPa, $E_z = 0.46 \cdot 10^4$ MPa, $\nu_{zr} = \nu_{z\varphi} = 0.11$, $\nu_{r\varphi} = 0.1$, $G_{zr} = G_{z\varphi} = 0.33 \cdot 10^4$ MPa, and $G_{r\varphi} = 1.14 \cdot 10^4$ MPa. In calculating temperature fields, the thermophysical characteristics were $c\rho = 2.93$

MJ/(m^3 · K) and $\lambda_{zz} = \lambda_{rr} = \lambda_{\varphi\varphi} = 0.149$ kW/(m · K); in calculating the thermal strains, it was assumed that $\alpha^{\text{T}}_{zz} = \alpha^{\text{T}}_{rr} = \alpha^{\text{T}}_{\varphi\varphi} = 0.65 \cdot 10^{-5}$ 1/K.

The rectilinearly orthotropic material was calculated with E_z = 4.5 ·10^4 MPa, $E_x = E_y$ = 3.1 ·10^4 MPa, $\nu_{zx} = \nu_{zy} = \nu_{xy} = 0.23$, $G_{zx} = G_{zy} = 1.5 \cdot 10^3$ MPa, G_{xy} = 2 ·10^3 MPa, $\alpha_{zz} = \alpha^{\text{T}}_{xx} = \alpha^{\text{T}}_{yy}$ = 1.2 ·10^{-5} 1/K, $\lambda_{zz} = \lambda_{xx} = \lambda_{yy}$ = 0.0067 kW/(m · K), and $c\rho$ = 4.129 MJ/(m^3 · K).

In solving the problem, half the meridional cross section of the body (see Fig. 1) was divided into 1600 triangular finite elements with 841 nodal points. In the trigonometric series, the first eleven members were used and, for determining displacement components, eleven systems of 2523-rd order algebraic equations were solved in each approximation.

Some results on calculating the temperature fields and the stress-strain state in the middle cross section of the cylinder are presented in Figs. 1 and 2. The figures show the radial variation of the temperature field (see Fig. 1) and the stresses σ_{zz} (Fig. 2a) and $\sigma_{\varphi\varphi}$ (Fig. 2b) on the 150-th second of heating (dashed lines) and upon achieving a steady temperature distribution (after 15 min of heating) (continuous lines). Here, results for the angular coordinates $\varphi = 0$ (curves 1) and $\varphi = \pi$ (curves 2) are given. These stress components were chosen because they are maximum. At the same time, an analysis of the calculation results shows that the account of nonaxisymmetry of loading and mechanical characteristics has led to the appearance of tangential stresses $\sigma_{r\varphi}$ equal to 7.5 MPa in the cross sections $\varphi = \pi/4 + k\pi/2$ (k = 0, 1, 2) close to the contact surface of materials, which makes about 10% of the magnitude of normal stresses.

Thus, the procedure suggested allows us to investigate the nonaxisymmetric stress-strain state of compound bodies of revolution made of elastic orthotropic materials and to determine the optimum — from the viewpoint of strength — directions for the symmetry axes of rectilinearly orthotropic materials under nonaxisymmetric loading [14].

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