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A Note on the Equivalence of Coherence and Constrained Coherence

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Abstract Constrained coherence is compared to coherence and its role in the behavioural interpretation of coherence is discussed. The equivalence of these two notions is proven for coherent conditional previsions, showing that the same course of reasoning applies to several similar concepts developed in the realm of imprecise probability theory.

Keywords Coherence · Constrained coherence · Subjective probability · Imprecise probabilities

As well known, *coherence* is a fundamental concept in the work of Bruno de Finetti to establish the consistency of a subjective probability assessment (de Finetti 1974). He also defined coherence of previsions for bounded random variables (or *gambles*, following the current terminology), definition extended formally to conditional gambles by his School, cf. Holzer (1985). Several variants appeared later in the realm of imprecise probability theory, ranging from coherent lower/upper probabilities and (conditional) previsions to convex lower/upper conditional previsions (Walley 1991; Williams 2007; Pelessoni and Vicig 2005). These definitions share a common feature: they require that the supremum of a certain gamble, usually termed *gain*, is non-negative, which has a behavioural interpretation referring to some betting scheme. To exemplify in a sufficiently general case, recall the definition of coherent prevision on an arbitrary set D of conditional gambles (Holzer 1985).

Definition 1 (*Coherence*) A map $P : D \rightarrow \mathbb{R}$ is a *coherent conditional prevision* on D if and only if, $\forall n \in \mathbb{N}$, $\forall X_1|B_1, \ldots, X_n|B_n \in D$, $\forall s_1, \ldots, s_n \in \mathbb{R}$, defining

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$$G = \sum_{i=1}^{n} s_i I_{B_i}(X_i - P(X_i|B_i)), \quad B = B_1 \vee \ldots \vee B_n \tag{1}$$

it holds that $\sup\{G|B\} \geq 0$.

Here I_{B_i} is the indicator of event B_i and $I_{B_i}(X_i - P(X_i|B_i))$ is the *elementary gain* of an agent paying the price $P(X_i|B_i)$ to buy $X_i|B_i$ (to bet on $X_i|B_i$), with the proviso that the bet is called off and the money returned if and only if B_i does not occur. Then Definition 1 requires that no finite linear combination G of elementary gains is such that $\sup\{G|B\} < 0$. Conditioning on B is equivalent to at least one elementary bet being not called off.

De Finetti's coherence for unconditional previsions is the special case of Definition 1 where all gambles in D are unconditional (hence $X_i|B_i = X_i|\Omega = X_i$, $\forall i$, and $G|B = G|\Omega = G$). If further they are indicators of events, we obtain coherence for probabilities. Moreover, many coherence concepts in imprecise probability theory can be obtained from Definition 1, typically just imposing additional constraints to the coefficients $s_1, \ldots, s_n$. For instance, requiring that at most one of the $s_1, \ldots, s_n$ may be negative corresponds to Williams' coherence for lower conditional previsions [in the version of Pelessoni and Vicig (2009)], or to the coherence in Walley (1991), Sec. 2.5.4(a), when all gambles in D are unconditional.

In all such cases, an objection commonly raised by researchers approaching these issues is that the gain ($G|B$ in Definition 1) is lower unbounded (at the varying of $s_1, \ldots, s_n$), which might cause a distortion in the agent's price assessments (for $X_i|B_i$, in Definition 1), or an unwillingness to accept whatever $s_1, \ldots, s_n$. Mathematically, the question might be ignored by considering Definition 1 as axiomatical, hence not necessarily supporting any interpretation. Interestingly, there is however a mathematically simple way-out to the question that preserves the betting interpretation, and this is *constrained coherence*. In fact, it is possible to modify the definition of coherence so that each gain is bounded in absolute value by some arbitrarily chosen real $k > 0$ (constrained coherence), and to prove the equivalence coherence–constrained coherence. This clearly allows us to focus on the simpler concept of coherence.

Constrained coherence is not much emphasised in the literature: it is discussed for coherent (unconditional) probabilities in Crisma (2006), and hinted in Pelessoni and Vicig (2005), footnote 6 for (unconditional) convex lower previsions. Here I prove the equivalence in the case of coherent conditional previsions.

Definition 2 (*Constrained coherence*) A map $P : D \to \mathbb{R}$ is a *constrained coherent conditional prevision* on D if and only if, given an arbitrary real $k > 0$, $\forall n \in \mathbb{N}$, $\forall X_1|B_1, \ldots, X_n|B_n \in D$, $\forall s_1, \ldots, s_n \in \mathbb{R}$, defining G and B as in (1), if $\sup\{|G|B|\} \leq k$ then $\sup\{G|B\} \geq 0$.

Proposition 1 *P is coherent if and only if it is constrained coherent on D.*

Proof Coherence trivially implies constrained coherence.

For the converse implication, assume P is constrained coherent. It is then sufficient to prove that if $\sup\{|G|B|\} > k$, then $\sup\{G|B\} \geq 0$.

In fact, define for any such G in (1) the coefficients

$$s_i' = \frac{k}{\sup\{|G|B|\}} \cdot s_i, \quad i = 1, \ldots, n$$

and the corresponding gain

$$G' = \sum_{i=1}^{n} s_i' I_{B_i}(X_i - P(X_i|B_i)) = \frac{k}{\sup\{|G|B|\}} \cdot G.$$

Then $\sup\{|G'|B|\} = \sup\{\frac{k}{\sup\{|G|B|\}} \cdot |G|B|\} = k$, hence $\sup\{G'|B\} \geq 0$ by constrained coherence of P.

Therefore also $\sup\{G|B\} = \frac{\sup\{|G|B|\}}{k} \cdot \sup\{G'|B\} \geq 0$. □

The above result clearly applies to its special cases of coherence for unconditional previsions and (conditional or not) probabilities. By means of essentially the same proof, it is also easy to prove a number of analogous equivalences between further coherence concepts and their constrained definitions, including Williams' coherence and other notions developed for imprecise probabilities and previsions. We may thus conclude that coherence–based theories generally offer a sound solution to interpretation problems arising from the possible unboundedness of (the set of the infima of) the gains.

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