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RADIO ENGINEERING MEASUREMENTS

SIGNALS RECOVERY BY THE AMPLITUDE OF THE SPECTRUM

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A two-stage approximate method for recovering the phase of the signal spectrum from the amplitude of the spectrum is proposed. In the first stage, the signal is recovered by a numerical method (in the one-dimensional and two-dimensional cases) from the known modulus of the spectrum; in the second stage, the spectrum of the recovered signal is determined and the phase of the spectrum is calculated. signal recovery from a known modulus of the spectrum is modeled by a nonlinear Fredholm equation of the first kind, which is solved using the spline-collocation method with splines of zero and first orders and a generalization of the continuous method for solving nonlinear operator equations. Model examples of the recovery of one-dimensional and two-dimensional signals are given. The accuracy of signal recovery for various perturbations in the input signals and in computational frameworks has been studied. The absolute and relative values of spikes at the leading and trailing edges of the signals are estimated. methods for suppressing the Gibbs effect are considered. The proposed method can be used in optics, astrophysics, biology, and medicine.

Keywords: *signal recovery, amplitude-phase problem, nonlinear Fredholm integral equation of the first kind, regularization method, continuous method for solving operator equations.*

Introduction. In the solution of numerous problems in measurement technique, automation, physics, biology, and medicine, there arise situations when only amplitudes of signal spectra are accessible to measurement. Determining the amplitude of a spectrum from the phase of a signal spectrum and the phase of a spectrum from the amplitude of a signal spectrum is the amplitude–phase problem. A great number of works, in which various analytical and numerical methods such as [1–6] were proposed, have been devoted to a solution of this problem. In [1–3], the apparatus of nonlinear singular integral equations (one-dimensional case) and nonlinear bisingular integral equations (two-dimensional case) is applied to an approximate solution of the amplitude–phase problem.

The problem of recovering one-dimensional and two-dimensional functions from the known amplitudes of spectra is simulated by one-dimensional and two-dimensional Fredholm integral equations of the first kind. It is known [7] that the solution of Fredholm integral equations of the first kind is an ill-posed problem according to Hadamard, i.e., small variations in the kernels and the right sides of the equations may result in large errors in the solutions. Therefore, it is necessary to develop the appropriate methods of regularization. In this article, a continuous method of solving nonlinear operator equations [8] is used as the algorithm for regularization. This method is based on Lyapunov's theories of the stability of solutions of differential equations. According to this method, a stable system of ordinary differential equations is put in correspondence asymptotically to the prototype system of algebraic equations (linear and nonlinear), which ensures the convergence and stability of the method with perturbations of the coefficients of the equations and right sides. It is possible to find the spectrum phase after recovering the desired function and calculating the Fourier transform of this function.

The objective of this article is to construct approximation methods for the recovery of one-dimensional and two-dimensional signals from the known amplitudes of the spectra.

Continuous method of solving nonlinear operator equations. We introduce designations and definitions that will be used later in the article. Let B be a Banach space; K is an operator acting from B into B, $B(a, r) = \{x, a \in X: \|x - a\| \leq r\}$ is a sphere in the Banach space B; a and r are the center and radius of a sphere; $\Lambda(K) = \lim_{v \downarrow 0} (\|I + vK\| - 1)/v$

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is the logarithmic norm [9] of the linear operator K; and I is the unity operator in the Banach space. The symbol $v \downarrow 0$ signifies that the real variable v will converge to zero.

We will describe formulas to calculate the logarithmic norms in the frequently used Banach spaces.

Let $C = \{c_{ij}\},\ i,\ j = \overline{1,\ n}$ be a real matrix in the n-dimensional space R_n of vectors $\mathbf{x} = (x_1, \ldots, x_n)$ with norms $\|x\|_1 = \sum_{k=1}^{n} |x_k|$; $\|x\|_2 = \left[\sum_{k=1}^{n} |x_k|^2\right]^{1/2}$; $\|x\|_3 = \max_{1 \le k \le n} |x_k|$. The logarithmic norms of the matrix C is defined as [10]:

$$\Lambda_1(C) = \max_j \left(c_{jj} + \sum_{i=1, i \ne j}^{n} \left| c_{ij} \right| \right); \quad \Lambda_2(C) = \lambda\left(\left[C + C^* \right] / 2 \right)_{\max}; \quad \Lambda_3(C) = \max_i \left(c_{ii} + \sum_{j=1, j \ne i}^{n} \left| c_{ij} \right| \right),$$

where $\lambda([C + C^*]/2)_{\max}$ is the largest eigenvalue of the matrix $[C + C^*]/2$; and C^* is the matrix conjugate to the matrix C.

We consider in the Banach space B the nonlinear equation

$$A(x) - f = 0, \tag{1}$$

where $A: B \to B$ is a nonlinear operator mapping space B onto itself.

We will assume that the nonlinear operator $A(x)$ has a Fréchet or Gateaux derivative .

The operator $A(x)$ is referred to as Fréchet differentiable at the point $x \in B$ if in a neighborhood of this point for any $u \in B$ the equality $\|A(x + u) - A(x) - A'(x)u\| = \|\beta(x, u)\|$ is satisfied, where $\lim_{\|u\| \to 0} \frac{\|\beta(x, u)\|}{\|u\|} = 0$ [11]. The linear operator $A'(x)$ is referred to as the Fréchet derivative (strong derivative) of the operator $A(x)$ at the point $x \in B$. The operator $A(x)$ is referred to as Gateaux differentiable at the point $x \in B$ if for any $u \in B$ the equality $\lim_{\lambda \to 0} \frac{A(x + \lambda u) - A(x)}{\lambda} = A'(x)$ [11] is satisfied. The linear operator $A'(x)$ defined in this manner is referred to as a Gateaux derivative (weak derivative) of the operator $A(x)$ at the point $x \in B$. The Fréchet and Gateaux derivatives are designated by the same numeral. If there is a Fréchet derivative, there is also a Gateaux derivative, and they are equal.

If the operator equation (1) is a system of nonlinear algebraic equations, then the definitions of the Gateaux and Fréchet derivatives coincide with the definition of a Jacobian.

We assign to Eq. (1) the Cauchy problem

$$\left. \begin{aligned} \frac{dx(t)}{dt} &= A(x(t)) - f; \\ x(t_0) &= x_0 \end{aligned} \right\}. \tag{2}$$

We present the following assertions.

Theorem 1 [8]. Let Eq. (1) have a solution x^* and on any differentiable curve $g(t)$ located in the Banach space B the inequality be satisfied

$$\lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \Lambda\left[A'(g(\tau)) \right] d\tau \le -\alpha_g, \tag{3}$$

where some magnitude $\alpha_g > 0$.

Then the solution of the Cauchy problem (2) as $t \to \infty$ converges to the solution x^* of Eq. (1) for any initial value $x_0 \in B$.

Theorem 2 [8]. Let Eq. (1) have a solution x^* and on any differentiable curve $g(t)$ located in the sphere $B(x^*, r)$, $r > 0$ the following conditions be satisfied:

1) for any $t \ge t_0$, the inequality $\int_{t_0}^{t} \Lambda(A'[g(\tau)]) d\tau \le 0$ is valid;

2) inequality (3) is satisfied.

Then the solution of the Cauchy problem (2) as $t \to \infty$ converges to the solution x^* of Eq. (1) for any initial value $x(t_0) \in B(x^*, r)$.

Note. The constant α_g in Theorems 1 and 2 depends on the curve g. The requirement that $\alpha_g > 0$ for each curve $g(t)$ is general.

If the conditions of Theorems 1 and 2 are not satisfied, then we assign to Eq. (1) the Cauchy problem

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= -\left[A'(x(t))\right]^* \left[A(x(t)) - f\right]; \\ x(t_0) &= x_0 \end{aligned}\right\}, \tag{4}$$

where $[A'(x(t))]$ is the operator conjugate to the operator $A'(x(t))$.

By analogy with proofs of Theorems 1 and 2, the following Theorems 3 and 4 are proved.

Theorem 3. Let Eq. (1) have a solution x^* and on any differentiable curve $q(t)$ in the space B the condition be satisfied:

$$\lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} \Lambda\left\{\left[A'(q(\tau))\right]^* A'(q(\tau))\right\} d\tau \geq \alpha_q, \tag{5}$$

where $\alpha_q > 0$.

Then the solution of the Cauchy problem (4) converges as $t \to \infty$ to the solution x^* of Eq. (1) for any initial approximation $x(t_0) \in B$**.**

Theorem 4. Let Eq. (1) have a solution x^* and on any differentiable curve $q(t)$ located in the sphere $B(x^*, r)$, $r > 0$ the following conditions be safisfied:

1) the inequality $\int_{t_0}^{t} \Lambda\{[A'(q(\tau))]^* A'(q(\tau))\}\, d\tau \geq 0$ is valid;

2) inequality (5) is satisfied.

Then the solution of the Cauchy problem (4) converges as $t \to \infty$ to the solution x^* of Eq. (1) for any initial approximation $x(t_0) \in B(x^*, r)$.

If the conditions of Theorems 3 and 4 are not satisfied, then it is necessary to introduce a regularizing parameter $\alpha > 0$ and pass to the Cauchy problem

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= -\alpha x(t) - \left[A'(x(t))\right]^* \left[A(x(t)) - f\right] \\ x(t_0) &= x_0 \end{aligned}\right\}. \tag{6}$$

It is possible to show that the Cauchy problem (6) for any initial approximation $x(t_0) \in B$ converges to a solution of the equation

$$\alpha x(t) + \left(A'(x)\right)^* \left(A(x) - f\right) = 0.$$

In the recovery of signals, the Cauchy problems (2), (4), and (6) are solved by numerical methods. Here it is possible to use any numerical method of a solution of differential equations [12].

Later, a modified Euler method is used to solve the model examples. Let the operator equation (1) be an n-dimensional system of nonlinear algebraic equations. We consider the Cauchy problem (2). We perform the calculations according to the iterative scheme

$$x(m+1) = x(m) + hG(m)\left[A(x(m)) - f\right], \quad m = m_0, m_0 + 1, \ldots; \quad x(m_0) = x_0,$$

where h is the step size of the Euler method; $G(m)$ is the diagonal matrix, the elements $\gamma_{ii} = \pm 1$ $(i = \overline{1, n})$ of which are selected in such a manner that the logarithmic norm of the matrix $G(m)\, A'(x(m))$ is negative.

Methods of signal recovery. ***One-dimensional signal.*** Let $f(t)$, $t \in [0, a]$ be a signal subject to recovery. It is required to recover the function $f(t)$ if it is known that the amplitude of its spectrum is $A(\omega) = |F(\omega)|$, $-\infty < \omega < \infty$, where $F(\omega)$ is the Fourier transform of the function $f(t)$.

In order to recover the function $f(t)$, we solve the nonlinear integral equation

$$\left| \frac{1}{\sqrt{2\pi}} \int_0^a f(t) e^{i\omega t} dt \right| = A(\omega), \quad -\infty < \omega < \infty . \tag{7}$$

For an approximate solution of Eq. (7), we construct spline-collocation computational schemes with splines of zero and first order.

First computational scheme. We introduce the nodes

$$t_k = ak/N, \quad k = \overline{0, N}; \quad \overline{t_k} = (t_k + t_{k+1})/2, \quad k = \overline{0, N-1};$$

$$\omega_k = -D + 2Dk/N, \quad k = \overline{0, N}; \quad \overline{\omega_k} = (\omega_k + \omega_{k+1})/2, \quad k = \overline{0, N-1}$$

and intervals

$$\Delta_k = [t_k, t_{k+1}), \quad k = \overline{0, N-2}, \quad \Delta_{N-1} = [t_{N-1}, t_N] .$$

Here D is a sufficiently large number, determined by the inequality

$$\max_{\omega \in (-\infty, -D) \cup (D, \infty)} A(\omega) \le N^{-1}.$$

We seek an approximate solution of Eq. (7) in the form of a piecewise-continuous function

$$x_N(t) = \sum_{k=0}^{N-1} \alpha_k \psi_k(t),$$

where

$$\psi_k(t) = \begin{cases} 1, & t \in \Delta_k \\ 0, & t \in [0, a] \setminus \Delta_k \end{cases}; \quad l = \overline{0, N-1} .$$

The unknown coefficients $\{\alpha_k\}$ are defined from the system of equations

$$\left| \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{N-1} \alpha_k \int_{\Delta_k} f(t) e^{i\omega_l t} dt \right| = A(\omega_l); \quad l = \overline{0, N-1} . \tag{8}$$

We use the continuous method of solution of nonlinear operator equations for the numerical implementation of the system of equations (8). We find the values of $\{\alpha_k\}$ from the system of ordinary differential equations

$$\frac{d\alpha_l(u)}{du} = G(u) \left(\left| \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{N-1} \alpha_j(u) \int_{\Delta_j} e^{i\omega_l t} dt \right| - A(\omega_l) \right), \tag{9}$$

$$\alpha_l(u_0) = \overline{\alpha_l}, \quad l = \overline{0, N-1} . \tag{10}$$

We solve the system (9)–(10) by a modified Euler method:

$$\alpha_l\left(n+1\right)=\alpha_l\left(n\right)+G\left(n\right)h\left(\left|\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{N-1}\alpha_j\left(n\right)\int\limits_{\Delta_j}\mathrm{e}^{\mathrm{i}\omega_l t}dt\right|-A\left(\omega_l\right)\right);\quad \alpha_l\left(0\right)=\overline{\alpha_l},\quad l=\overline{0,\,N-1}\;.$$

where the diagonal matrix $G(n)$ must be such that the logarithmic norm $\Lambda(G(n)J(n))$ is negative in the corresponding space of N-dimensional vectors; and $J(n)$ is the Jacobian of the vector

$$\left(\left|\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{N-1}\alpha_j(n)\int\limits_{\Delta_j}\mathrm{e}^{\mathrm{i}\,\omega_l t}\,dt\right|-A\left(\omega_l\right)\right)_0^{N-1}.$$

The diagonal matrix $G(n)$ is selected such that system (9) is asymptotically stable.

Computational schemes for solving the Cauchy problems (4) and (6) are built in an analogous manner.

Second computational scheme. We find an approximate solution of Eq. (7) by the spline-collocation method with splines of first order.

The approximate solution is written as a polygon

$$x_N\left(t\right)=\sum_{k=0}^{N-1}\alpha_k\psi_k\left(t\right)$$

with the basis functions

$$\psi_0\left(t\right)=1-tn/a\,,\;0\le t\le a/n\;;\quad \psi_1\left(t\right)=\begin{cases}tn/a\,, & 0\le t\le a/n;\\ 2-tn/a\,, & a/n\le t\le 2a/n\,;\end{cases}$$

$$\psi_k\left(t\right)=\begin{cases}-k+1+tn/a\,, & (k-1)a/n\le t\le ka/n\,;\\ k+1-tn/a\,, & ka/n\le t\le (k+1)\,a/n\,,\end{cases}\quad k=\overline{2,\,n-1};$$

$$\psi_n\left(t\right)=-n+1+tn/a\,,\quad (n-1)\le t\le a\;.$$

We find the coefficients $\{\alpha_k\}$ from the system of equations

$$\left|\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{N-1}\alpha_k\int\limits_0^a\psi_k\left(t\right)\mathrm{e}^{\mathrm{i}\omega_l t}dt\right|=A\left(\omega_l\right),\quad l=\overline{0,\,N-1}\,.\tag{11}$$

We introduce the function $g(k,l)=\int\limits_0^a\psi_k\left(t\right)\mathrm{e}^{\mathrm{i}\omega_l t}dt$ and after elementary computations derive

$$g\left(0,\,l\right)=\int\limits_0^{a/n}\psi_0(t)\,\mathrm{e}^{\mathrm{i}\,\omega_l t}dt=\frac{n}{a\omega_l^2}\left(1-\cos\frac{a\omega_l}{n}\right)+\frac{\mathrm{i}}{\omega_l}\left(1-\frac{n}{a\omega_l}\sin\frac{a\omega_l}{n}\right);$$

$$g\left(1,\,l\right)=\int\limits_0^{a/n}\psi_1(t)\,\mathrm{e}^{\mathrm{i}\,\omega_l t}dt=\frac{n}{a\omega_l^2}\left[2\cos\frac{a\omega_l}{n}-\cos\frac{a\omega_l}{n}-1+\mathrm{i}\left(2\sin\frac{a\omega_l}{n}-\sin\frac{2a\omega_l}{n}\right)\right];$$

$$g\left(k,\,l\right)=\int\limits_{(k-1)a/n}^{(k+1)a/n}\psi_k(t)\,\mathrm{e}^{\mathrm{i}\,\omega_l t}dt=\frac{-2n}{a\omega_l^2}\left[\cos\frac{ak\omega_l}{n}\cos\frac{a\omega_l}{n}-\cos\frac{ak\omega_l}{n}+\mathrm{i}\left(\sin\frac{ak\omega_l}{n}\cos\frac{ak\omega_l}{n}-\sin\frac{ak\omega_l}{n}\right)\right];$$

$$g(n, l) = \int_{(n-1)a/n}^{a} \psi_n(t)\, e^{i\,\omega_l t} dt = \frac{-1}{a\omega_l^2} \Big[n \cos a\omega_l \cos \frac{a\omega_l}{n} + n \sin a\omega_l \sin \frac{a\omega_l}{n}$$

$$- a\omega_l \sin a\omega_l - a\omega_l \cos a\omega_l + i \left(n \sin a\omega_l \cos \frac{a\omega_l}{n} - n \cos a\omega_l \sin \frac{a\omega_l}{n} + a\omega_l \cos a\omega_l - n \sin a\omega_l \right) \Big] .$$

After substituting these values into Equation (11), we derive a system of nonlinear algebraic equations approximating the posed problem (the system is not given here because of its cumbersomeness). The system is implemented by the continuous method of the solution of nonlinear operator equations by analogy with the first computational scheme.

After recovery of the signal $f(t)$, $t \in [0, a]$, we will calculate its phase. For this purpose, we find the Fourier transform $F_n(\omega)$ of the function $f_n(t)$, approximating the function $f(t)$. We present $F_n(\omega)$ in the form $F_n(\omega) = U_n(\omega) + iV_n(\omega)$ and define the phase of the spectrum of the signal $f(t)$ by the formula $\varphi_n(\omega) = \arctan [V_n(\omega)/U_n(\omega)]$.

Two-dimensional signal. Statement of the problem. Let $f(t_1, t_2)$, $(t_1, t_2) \in [0, a]^2$ be a signal that is subject to recovery. It is required to recover the function $f(t_1, t_2)$, having information on the modulus of its spectrum $A(\omega_1, \omega_2) = |F(\omega_1, \omega_2)|$, where $F(\omega_1, \omega_2)$ is the Fourier transform of the function $f(t_1, t_2)$. Hence, the problem reduces to the solution of the nonlinear Fredholm integral equation

$$\left| \frac{1}{2\pi} \int_0^a \int_0^a f(t_1, t_2)\, e^{i(\omega_1 t_1 + \omega_2 t_2)} dt_1 dt_2 \right| = A(\omega_1, \omega_2) . \quad (12)$$

We solve Eq. (12) by the spline-collocation method with splines of order zero. We find an approximate solution of Eq. (12) as a piecewise constant function

$$f_n(t_1, t_2) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \alpha_{kl} \psi_{kl}(t_1, t_2) ,$$

where

$$\psi_{kl}(t_1, t_2) = \begin{cases} 1, & (t_1, t_2) \in \Delta_{kl}; \\ 0, & (t_1, t_2) \in [0, a]^2 \setminus \Delta_{kl}, \end{cases} \qquad k, l = \overline{0, n-1};$$

$$\Delta_{kl} = \{(t_1, t_2) : t_1 \in [x_k, x_{k+1}), t_2 \in [x_l, x_{l+1})\} , k, l = \overline{0, n-2} ;$$

$$\Delta_{k, n-1} = \{(t_1, t_2) : t_1 \in [x_k, x_{k+1}), t_2 \in [x_{n-1}, x_n]\} , k = \overline{0, n-2} ;$$

$$\Delta_{n-1, l} = \{(t_1, t_2) : t_1 \in [x_{n-1}, x_n], t_2 \in [x_l, x_{l+1})\} , l = \overline{0, n-2} ;$$

$$\Delta_{n-1, n-1} = [x_{n-1}, x_n; x_{n-1}, x_n] ; \quad x_k = ak/n , \quad k = \overline{0, n} .$$

We define the coefficients $\{\alpha_{kl}\}$, $k, l = \overline{0, n-1}$ from the system of equations written in operator form:

$$\left| \frac{1}{2\pi} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \alpha_{kl} \iint_{\Delta_{kl}} \exp\left[i \left(v_{k_1} t_1 + v_{k_2} t_2 \right) \right] dt_1 dt_2 \right| = A\left(v_{k_1}, v_{k_2} \right) , \ k_1, k_2 = \overline{0, n-1} , \quad (13)$$

where $v_k = -D_1 + k2D_1/n$, $k = \overline{0, n}$; D_1 is a sufficiently large positive number defined from the same considerations as the constant D in the one-dimensional case.

The integrals $\iint_{\Delta_{kl}} \exp [i (v_{k_1} t_1 + v_{k_2} t_2)]\, dt_1 dt_2$ are calculated analytically or less accurately by the quadrature formulas of rectangles. In this case, (13) has a more compact form, but the formula will be less pecise. By analogy with the one-dimensional signal, a continuous method for the solution of nonlinear operator equations is applied to system (13).

Model examples. *Example 1*. Let us examine the recovery of the one-dimensional signal specified by the function

$$f(t)=\begin{Bmatrix}\sin 2t,\ t\in[0,\ \pi/2]\\ 0,\ t\in[-\infty,\infty]\setminus[0,\ \pi/2]\end{Bmatrix}. \tag{14}$$

It is required to recover the signal and phase of its spectrum from the amplitude of the spectrum of function $f(t)$.

A perturbation that is set by a random number generator is superimposed, for the purpose of demonstrating the stability of the method proposed in this article for recovery of signals from the amplitude of the spectrum of the signal:

$$A_k(\omega_l)=A(\omega_l)+\xi_l(k),\ l=\overline{0,\ N},\ k=1,2,3;\quad \max_{0\le l\le N}|\xi_l(1)|=0.001;\quad \max_{0\le l\le N}|\xi_l(2)|=0.01;\quad \max_{0\le l\le N}|\xi_l(3)|=0.1\ .$$

The computational scheme in (8) and (9), in which the nodes were determined by the formulas $t_k=ak/N,\ k=\overline{0,\ N}$; $\overline{t_k}=(t_k+t_{k+1}/2),\ k=\overline{0,\ N-1}$; $\omega_k=-D+2Dk/N,\ k=\overline{0,\ N}$; $\overline{\omega_k}=(\omega_k+\omega_{k+1})/2,\ k=\overline{0,\ N-1}$ was used in the solution of example 1.

In Fig. 1a, the results of the recovery of function (14) are shown, where $f(t)$ and $f_n(t)$ are the precise and recovered signals, respectively; for $D=5$, the number of nodes in the computational scheme is $N=99$, the step size is $h=0.001$, the number of iterations $m=1000$, and the perturbation $A_2(\omega_l)=A_2(\omega_l)+\xi_l(2),\ l=\overline{0,\ N}$; $\max_{0\le l\le N}|\xi_l(2)|=0.01$.

Since the function $A(\omega)$ is nonnegative, then in the calculation of the function $A_k(\omega)$, filtering takes place at the nodes ω_l; $l=\overline{0,\ N}$.

$$A_k(\omega_l)=\begin{cases}A(\omega_l)+\xi_l(k), & A(\omega_l)+\xi_l(k)>0\\ 0, & A(\omega_l)+\xi_l(k)\le 0\end{cases}.$$

Figure 1b shows the results of recovery of a signal after filtering and averaging on five values of the originally recovered signal Tables 1 and 2 present the results of the recovery (for D, N, h, m specified above) of function (14) with perturbation of the modulus of its spectrum by a random function with amplitude ε: maximum ε_{max}, mean ε_{mean}, quadratic ε_{quad}, and root mean square ε_{rms} errors, as well as the steepness of the fronts of the precise $\varepsilon_{st.p}$ and approximate $\varepsilon_{st.a}$ functions.

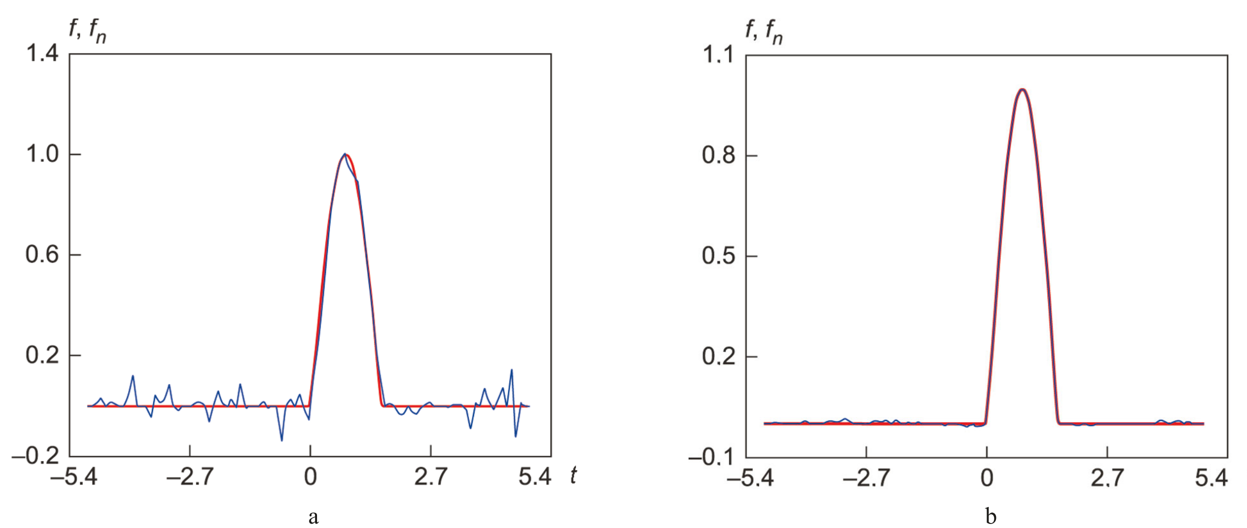

Fig. 1. Recovery of the signal in (14) when the amplitude of its spectrum is perturbed by a random function with amplitude $\varepsilon = 0.01$: a, initial signal f (red line); b, signal after filtering and averaging over five values of the originally recovered signal f_n (blue line).

TABLE 1. Errors of the Recovery of Function (14) with Perturbation of the Amplitude of the Spectrum

ε	ε_{max}	ε_{mean}	ε_{quad}	ε_{rms}
0.001	0.052	0.021	0.253	$2.553 \cdot 10^{-3}$
0.010	0.149	0.023	0.398	$4.016 \cdot 10^{-3}$
0.100	0.194	0.056	0.723	$7.301 \cdot 10^{-3}$

TABLE 2. Steepness of the Front of the Precise and Recovered Function (14) with Perturbation of the Amplitude of the Spectrum

l	$\varepsilon_{st\ p}$	$\varepsilon_{st\ a}$		
		0.001	0.01	0.1
0–10	0	0.023	0.208	0.764
11–20	0	0.017	0.428	0.823
21–30	0	0.012	0.279	1.103
31–40	0	0.044	0.298	0.521
41–49	0	0.043	0.528	0.480
50	1.010	1.240	1.548	1.660
51	1.970	1.960	1.156	0.010
52	1.860	1.850	1.676	4.780
53	1.660	1.640	2.254	0.450
54	1.390	1.390	1.798	2.340
55	1.070	1.070	1.073	1.820
56	0.710	0.710	0.708	0.720
57	0.310	0.320	0.315	0.430
58	0.090	0.090	0.515	0.340
59	0.490	0.490	0.371	0.240
60	0.880	0.860	0.176	0.130
61	1.220	1.210	1.623	1.470
62	1.520	1.510	1.799	2.700
63	1.750	1.740	1.477	1.040
64	1.920	1.900	1.919	0.600
65	2.010	1.990	1.109	1.440
66–70	0	0.032	0.314	0.558
71–80	0	0.016	0.119	0.308
81–90	0	0.004	0.363	0.721
91–99	0	0.016	0.807	0.982

The steepness of the front of a function is understood to be the first finite difference of the corresponding function (the finite differences of the oscillatory processes caused by the Gibbs effect are imported into Table 2).

Figure 2 shows the precise $\varphi(\omega)$ and recovered $\varphi_n(\omega)$ values of the phase.

Example 2. We examine the recovery of the one-dimensional signal specified by the function

$$f(t) = \left. \begin{cases} 1, & t \in [0, 3] \\ 0, & t \in [-\infty, \infty] \setminus [0, 3] \end{cases} \right\} . \tag{15}$$

It is required to recover the signal and the phase of its spectrum from the amplitude of the spectrum of function $f(t)$.

Figure 3 reflects the input function and the function recovered after filtering and averaging on five points (15). A perturbation with amplitude $\varepsilon = 0.01$ is superimposed on the amplitude of the spectrum of the input function. Tables 3 and 4 present the numerical characteristics of the recovery results for the same D, N, h, m, and ε as in Tables 1 and 2.

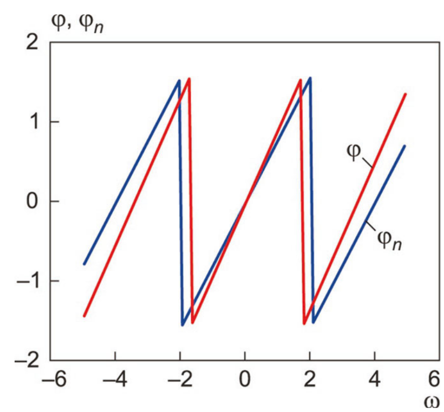

Fig. 2. Recovery of the phase of the spectrum of function (14): φ (—) initial phase; φ_n (—) recovered phase.

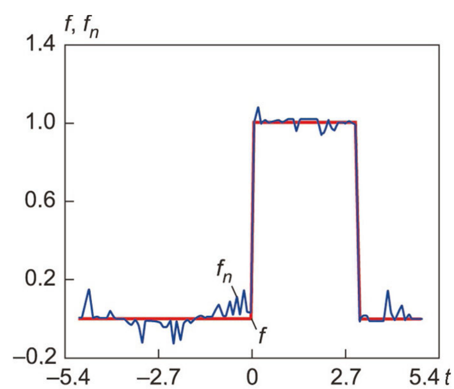

Fig. 3. Recovery of the signal (15) (after filtering and averaging of the originally recovered signal over five points) when the amplitude of its spectrum is perturbed by a random function with amplitude $\varepsilon = 0.01$: f(—) initial signal; f_n(—) recovered signal.

TABLE 3. Errors of the Recovery of Function (15) with Perturbation of the Amplitude of the Spectrum

ε	ε_{max}	ε_{mean}	ε_{quad}	ε_{rms}
0.001	0.043	0.001	0.148	$1.497 \cdot 10^{-3}$
0.010	0.152	0.026	0.43	$4.342 \cdot 10^{-3}$
0.100	0.194	0.056	0.723	$7.301 \cdot 10^{-3}$

TABLE 4. Steepness of the Front of the Precise and Recovered Function (14) with Perturbation of the Amplitude of the Spectrum

l	$\varepsilon_{st\ p}$	$\varepsilon_{st\ a}$		
		0.001	0.01	0.1
0–10	0	0.006	0.371	1.114
11–20	0	0.021	0.29	0.476
21–30	0	0.013	0.487	1.098
31–40	0	0.039	0.138	0.505
41–49	0	0.026	0.697	0.854
50	10	9.740	9.585	9.474
51	0	0.015	0.872	0.991
52	0	0.017	0.824	1.175
53	0	0.016	0.168	0.514
54	0	0.015	0.115	0.120
55	0	0.013	0.027	0.136
56	0	0.009	0.025	0.133
57	0	0.005	0.024	0.189
58	0	0.000	0.102	0.106
59	0	0.005	0.143	0.172
60	0	0.010	0.016	0.607
61	0	0.013	0.013	0.117
62	0	0.016	0.579	1.067
63	0	0.018	0.593	0.706
64	0	0.018	0.001	1.466
65	0	0.017	0.002	0.730
66	0	0.014	0.006	1.216
67	0	0.011	0.010	0.270
68	0	0.006	0.013	0.471
69	0	0.000	0.759	1.143
70	0	0.006	0.097	0.096
71	0	0.023	0.609	1.120
72	0	0.025	0.409	0.921
73	0	0.021	0.064	0.068
74	0	0.018	0.410	0.439
75	0	0.015	0.021	0.062
76	0	0.012	0.020	0.178
77	0	0.010	0.085	0.260
78	0	0.008	0.120	1.127
79	0	0.006	0.015	0.594
80–90	0.909	0.913	1.238	1.771
91–99	0	0.080	0.215	0.392

Example 3. We examine the recovery of a two-dimensional signal specified by the function

$$f\left(t_1, t_2\right) = \begin{Bmatrix} 1, & 0 \le t_1, t_2 \le 1 \\ 0, & \left(-\infty \le t_1, t_2 \le \infty\right) \setminus \left(0 \le t_1, t_2 \le 1\right) \end{Bmatrix}. \tag{16}$$

We use the computational scheme (13) for the following values of parameters: $D_1 = 3$, number of collocation sites $n = 50$, step size $h = 0.001$, and number of iterations of the Euler method $m = 500$.

Figures 4 and 5, respectively, show the initial $f(t_1, t_2)$ and recovered $f_n(t_1, t_2)$ signals (16) and the phases of these signals $\varphi(u_1, u_2)$ and $\varphi_n(\omega_1, \omega_2)$, determined from the known modulus of the spectrum, with perturbation of the amplitude of the spectrum of the input signal by a random function with amplitude $\varepsilon = 0.01$. Table 5 shows the errors of the signal recovery before and after averaging, and Table 6 shows the steepness of the fronts of the precise and recovered signals, where FF, CF, and RF are the forward, central, and reverse fronts, respectively.

The model examples illustrate the effectiveness of the proposed two-stage method of recovery of the signal and phase of a spectrum, and the numerical algorithms implementing the method.

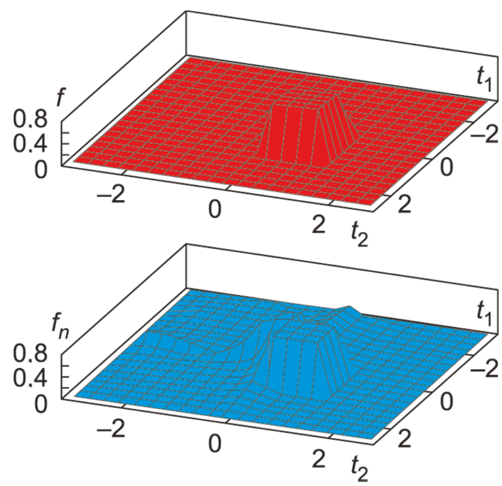

Fig. 4. Recovery of the function (16) when the amplitude of the spectrum of the original signal is perturbed by a random function with amplitude ε = 0.01.

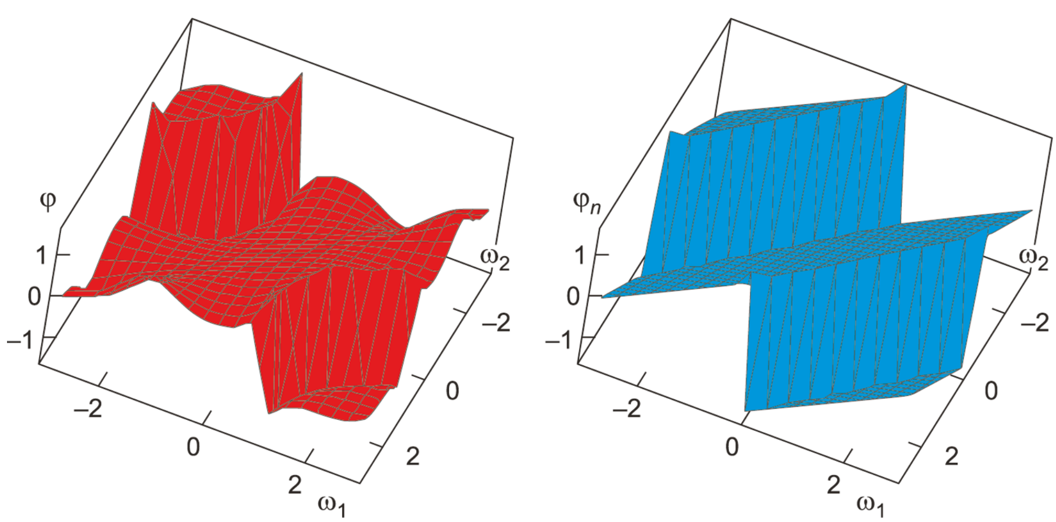

Fig. 5. Recovery of the phase of the spectrum of a signal of function (16).

TABLE 5. Errors of the Recovery of Signal (16) before Averaging

Signal front	ε_{max}	ε_{mean}	ε_{quad}	ε_{rms}
Errors before averaging				
Forward	0.400	0.032	0.570	0.012
Central	0.068	$7.241 \cdot 10^{-3}$	0.117	$2.381 \cdot 10^{-3}$
Reverse	0.048	$6.841 \cdot 10^{-3}$	0.093	$1.893 \cdot 10^{-3}$
Errors after averaging				
Forward	0.156	0.019	0.236	$5.018 \cdot 10^{-3}$
Central	0.028	$6.011 \cdot 10^{-3}$	0.061	$1.307 \cdot 10^{-3}$
Reverse	0.033	$6.540 \cdot 10^{-3}$	0.070	$1.499 \cdot 10^{-3}$

TABLE 6. Steepness of the Fronts of the Precise and Recovered Signals (16)

l	$\varepsilon_{st\ p}$	$\varepsilon_{st\ a}$		
		FF	CF	RF
0–7	0	0.110	0.182	0.075
7–15	0	0.343	0.087	0.142
16–22	0	0.275	0.111	0.083
23	0	1.059	0.195	0.216
24	0	1.622	0.475	0.357
25	10	11.310	9.320	9.990
26	0	1.700	0.001	0.001
27	0	2.300	0	0
28	0	0	0	0
29	0	0	0	0
30	0	0	0	0
31	0	0	0	0.001
32	0	0	0.001	0.001
33–40	1.25	1.301	1.301	1.264
41–49	0	0.011	0.021	0.018

Conclusion. The method proposed in the article makes it possible to recover the signal phase from the amplitude of the spectrum without introducing additional information about the signal. The method is based on a preliminary recovery of the signal by the amplitude of the spectrum, which is of independent interest. The examples that were presented show the effectiveness of the proposed method. The method may be used for the solution of many inverse problems of informational measurement technique, when measurements of just the operators (functionals) from signals subject to definition are possible.

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