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GENERALIZED INTERVAL METHOD OF BISECTION FOR METROLOGICALLY BASED SEARCH FOR SOLUTIONS OF SYSTEMS OF EQUATIONS WITH INACCURATELY SPECIFIED INITIAL DATA

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A method of solving systems of nonlinear equations for use in metrological applications based on the generalized method of bisection is presented. It is shown that through the use of the proposed approach it is possible to satisfy requirements imposed on the solution as regards the set of results of indirect measurements and to take into account data on the inaccuracy of the coefficients of the equations to be solved, which are the results of direct measurements. The resulting solution is automatically tracked by a reliable estimate of the limiting error of its components.
Keywords: solution of systems of nonlinear equations, bisection, indirect measurements, metrological self-tracking of calculations with inaccurate data.

Many physical quantities may be measured only indirectly. Such a situation arises in estimation of the parameters of the state of complex objects which it is impossible to measure directly. The values of these parameters are, therefore, determined from the results of direct measurements of the characteristics of the object by solving systems of equations that form a known mathematical model of its behavior. Since any measurement is always performed with error, the resulting characteristics prove to be distorted. Since the latter are present in the system of equations as the coefficients of these equations, when estimating the precision of solutions the error inherited from these characteristics must be taken into account. Such an error is generally the most substantial component of the uncertainty of the results of indirect measurements and significantly exceeds the components of the uncertainty due to the use of particular computational methods of solving systems of equations [1]. Thus, ignoring the error of the results of direct measurements leads to an inadmissible understatement of the estimated error of the desired results of indirect measurements.

There are several different iteration methods that are usually used in practical applications to solve systems of equations. These methods form a mathematical model of a test object and are often nonlinear. The essence of the approach realized by these methods is as follows. An initial approximation of the solution of the system is first specified, and then successive, step-by-step refinement of this approximation is realized by means of a computational algorithm until the necessary precision is achieved. The error of the coefficients of the equations, which are the results of direct measurements, are often not taken into account in the course of solving the system. An estimate of the error of the solution is carried out even after the solution is found. The estimation is performed either from a previously found analytic expression or by statistical stimulation based on the Monte-Carlo method. The techniques that underlie the first approach usually assume that the error of the results of direct measurements within the framework of the particular problem are quite small, which it is difficult to verify. The second approach is time-consuming and may involve situations in which the iteration procedures used for the solution of the system of equations diverge.

In the present article, we wish to propose a method of solution of systems of equations of indirect measurement which, on the one hand, do not require substantial computational resources and, on the other hand, do not impose any requirements

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whatsoever on the error of the results of direct measurements. Such a modification of the method of bisection by means of which an estimate of the root of an individual equation may be obtained with certainty with specified precision is described in [2]. The modification takes into account the error of the coefficients of equation. In the present article the algorithm presented in [2] is generalized to the case in which a solution of a system of equations must be found.

Method of bisection for finding solutions of systems of nonlinear equations. The method of bisection for the solution of systems of equations of indirect measurements basically comprises the following steps. Suppose we are given a mathematical model of a test object that constitutes a system of nonlinear equations [3]:

$$f_1(\mathbf{x}, \mathbf{p}) = 0, ..., f_k(\mathbf{x}, \mathbf{p}) = 0,$$

where $f_1, ..., f_k$ are monotone functions that describe the relationship between the desired results of indirect measurements $\mathbf{x} = (x_1, ..., x_n)$, i.e., the parameters of the state of the object, and the results of direct measurements $\mathbf{p} = (p_1, ..., p_m)$, i.e., the measured characteristics of the object.

Suppose a sub-domain Ω_i is specified for each function f_i within its domain of definition such that the sign of the values of this function is different at different points of the boundary $\delta\Omega_i$. Then, since f_i is monotone, there exists at least one root within the sub-domain Ω_i that most likely satisfies all the equations of the system. Thus, the possible solutions of the system are determined within the specified domain of localization. On the first step, this domain comprises $\Omega^{(1)} = \Omega_1 \times \Omega_2 \times \times ... \times \Omega_n$, i.e., an n-dimensional rectangular parallelepiped. On the second stage, the boundaries of the domain are refined to values $\Omega^{(2)} \subseteq \Omega^{(1)}$, on the third stage, to $\Omega^{(3)} \subseteq \Omega^{(2)}$, and so on. A refinement on an arbitrary step of the method is produced by partitioning the current domain which localizes a solution into parts from which those parts are discarded which obviously do not contain any solutions. The given partitioning process may be realized by any one of a number of different methods. In the method we wish to propose, a division into rectangular sub-domains (n-dimensional parallelepipeds) $\Omega^{(jk)}$, $k = 1, 2, ..., K$, each of the same volume is produced on each jth step. The signs of the values of the functions f_i at all vertices of the n-dimensional parallelepipeds) $\Omega^{(jk)}$ are calculated on each iteration of the method. The errors of the coefficients $\mathbf{p}$ are also taken into account, since it is because of these errors that the sign of the function may be undefined; that is, the sign may be positive for certain possible values of the parameters $\mathbf{p}$ (within the limits of the error) and may be negative for other values. Next, those parallelepipeds which are certain not to contain any solutions of the system are determined from the combination of the signs of the obtained values (i.e., parallelepipeds at the boundaries of which there does not occur any change whatsoever in the sign of any of the functions f_i) and are discarded. A transition to the next iteration of the algorithm is then performed. The boundaries of a domain that localizes a solution of the system are refined until the dimensions of the domain satisfy a metrologically based criterion for a halt in the iteration process presented in [2]. The process halts when the limits (boundaries) of the possible values of the components of the solution found in the current and previous iterations of the algorithm coincide (with rounding according to the rules adopted in metrology).

Let us now explain a generalization of the method of bisection for the case of a system of equations for two unknown results of indirect measurements (x, y) that contain two inaccurate parameters p_1 and p_2 with measured values. Such a system has the form

$$\begin{cases} f_1\left(x, y, p_1, p_2\right) = 0; \\ f_2\left(x, y, p_1, p_2\right) = 0. \end{cases}$$

Suppose intervals $Sx = [Sx_1, Sx_2]$ and $Sy = [Sy_1, Sy_2]$ are specified from a priori considerations within which it is necessary to find values $x = x_0$ and $y = y_0$ that satisfy the given system of equations. The functions f_1 and f_2 must be monotone in the domain $Sx \times Sy$. On the first step of the algorithm we verify that the functions on the boundaries of this domain in fact do take values with different signs. Then the algorithm certainly converges to a solution of the system of equations. We will use the rectangle $\Omega^{(1)} = Sx \times Sy$ (Fig. 1) as the domain of localization of the solution on the first step of the algorithm and divide it into four equal rectangles by means of the lines $x = m_x$ and $y = m_y$, where $m_x = 0.5(Sx_1 + Sx_2)$ and $m_y = 0.5(Sy_1 + Sy_2)$ are the midpoints of the intervals Sx and Sy, respectively:

$$\Omega^{(21)} = [Sx_1, m_x] \times [Sy_1, m_y];$$

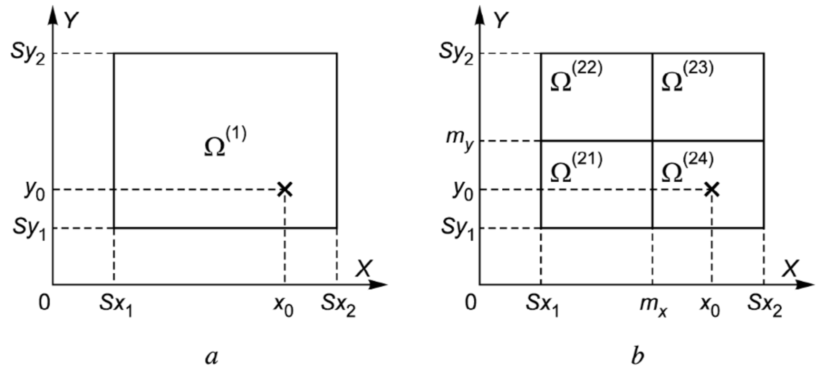

Fig. 1. Domains $\Omega^{(1)}$ and $\Omega^{(21)}$, $\Omega^{(22)}$, $\Omega^{(23)}$, and $\Omega^{(24)}$ of localization of a solution of a system of two equations on the first (*a*) and second (*b*) steps of implementation of the algorithm.

$$\Omega^{(22)} = [Sx_1, m_x] \times [m_y, Sy_2];$$
$$\Omega^{(23)} = [m_x, Sx_2] \times [m_y, Sy_2];$$
$$\Omega^{(24)} = [m_x, Sx_2] \times [Sy_1, m_y].$$

For each rectangle $\Omega^{(21)}$, $\Omega^{(22)}$, $\Omega^{(23)}$, and $\Omega^{(24)}$, we calculate at all the vertices an estimate of the values of the functions f_1 and f_2 with allowance made for the errors of the parameters p_1 and p_2. For example, the following four intervals are obtained for the rectangle $\Omega^{(21)}$ and function f_1:

$$z_{211} = f_1\left(Sx_1, Sy_1, p_1, p_2\right) \pm \left[\left|\frac{\partial f_1\left(Sx_1, Sy_1, p_1, p_2\right)}{\partial p_1}\right| \Delta p_1 + \left|\frac{\partial f_1\left(Sx_1, Sy_1, p_1, p_2\right)}{\partial p_2}\right| \Delta p_2\right]; \quad (1)$$

$$z_{212} = f_1\left(m_x, Sy_1, p_1, p_2\right) \pm \left[\left|\frac{\partial f_1\left(m_x, Sy_1, p_1, p_2\right)}{\partial p_1}\right| \Delta p_1 + \left|\frac{\partial f_1\left(m_x, Sy_1, p_1, p_2\right)}{\partial p_2}\right| \Delta p_2\right]; \quad (2)$$

$$z_{213} = f_1\left(Sx_1, m_y, p_1, p_2\right) \pm \left[\left|\frac{\partial f_1\left(Sx_1, m_y, p_1, p_2\right)}{\partial p_1}\right| \Delta p_1 + \left|\frac{\partial f_1\left(Sx_1, m_y, p_1, p_2\right)}{\partial p_2}\right| \Delta p_2\right]; \quad (3)$$

$$z_{214} = f_1\left(m_x, m_y, p_1, p_2\right) \pm \left[\left|\frac{\partial f_1\left(m_x, m_y, p_1, p_2\right)}{\partial p_1}\right| \Delta p_1 + \left|\frac{\partial f_1\left(m_x, m_y, p_1, p_2\right)}{\partial p_2}\right| \Delta p_2\right], \quad (4)$$

where Δp_1 and Δp_2 are the greatest values of the moduli of the admissible absolute error of the coefficients p_1 and p_2, respectively.

In (1)–(4), estimation of the possible error of the value of the function f_1 caused by the errors of its coefficients is performed by the technique of linearization generally accepted in metrology. The values of the derivatives $\partial f_1/\partial p_1$ and $\partial f_1/\partial p_2$ may be determined by any one of a number of methods. In the general case, they may be calculated by the method of metrological self-tracking [4]. By means of this method, a required estimate of the possible error of the value of any one of the functions f_i with any (systematic, random, or mixed) type of error in the coefficients p_1 and p_2 may be obtained automatically and an individual calculation of each of the derivatives is not required. If both limits of any of the obtained intervals z_{211}, z_{212}, z_{213}, and z_{214} of values of the function f_1 prove to be of the same sign, then it is obvious which sign this is. But if it turns out that the signs are different due to the influence of the error of the coefficients p_1 and p_2, the sign of the function f_1 cannot be precisely specified at the corresponding point.

After processing the rectangles $\Omega^{(21)}$, $\Omega^{(22)}$, $\Omega^{(23)}$, and $\Omega^{(24)}$ in this way, we may conclude whether or not these rectangles contain a solution of the system. For this purpose, we calculate the sum of the values of the functions f_1 and f_2 for values of *x* and *y* corresponding to the vertices of each rectangle. If it is impossible to precisely determine the sign of the

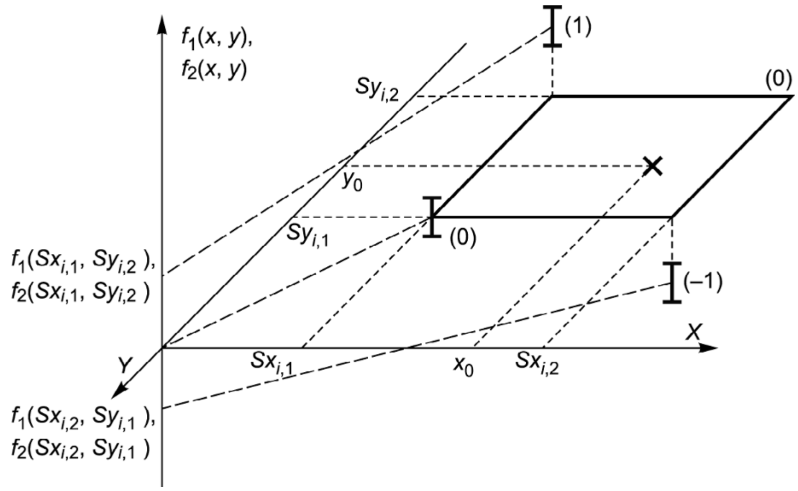

Fig. 2. Determination of the signs of the functions f_1 and f_2 at the vertices of a rectangle based on the errors of the parameters of the solved equations.

function because of the errors of the coefficients, it is assumed that the function is equal to zero. If the modulus of the sum which has been found for f_1 and for f_2 proves to be less than 4, i.e., less than the number of vertices of a rectangle, the solution of the system is contained precisely in the analyzed rectangle. Otherwise, there is no solution present in this domain of values of the arguments x and y and this rectangle is eliminated from further consideration. On the next step of the algorithm, rectangles from the list $\Omega^{(21)}$, $\Omega^{(22)}$, $\Omega^{(23)}$, and $\Omega^{(24)}$ that have not been previously discarded must be considered in the same sequence of operations. Systems that contain more than two equations may be solved in the same way as in this description. The iteration process is continued until a halt condition that satisfies a series of considerations typical in metrological applications is satisfied. A rule is presented in [2] for halting the algorithm in the case in which the root of only one equation is found. For the system of equations, an estimate of the solution must be refined until the limits of the possible error of the components (the coordinates x and y in the two-dimensional case) no longer decrease as new iterations are performed. On each iteration we suggest rounding the limits of the error according to rules adopted in metrology, that is, rounding only in increasing direction and to one and no more than two significant digits. The limits of the error of the components of the solution on the jth iteration must be estimated through a search for their minimal and maximal values relative to the boundaries of all n-dimensional parallelepipeds $\Omega^{(jk)}$, $k = 1, 2, ..., K$, that realize a localization of the solution.

The approach which we have proposed is based on the fact that if some domain of values of arguments of a monotone function contains a root of this function, the function assumes values of different signs on the boundaries of this domain. The inaccuracy of the parameters that occur in the equations of the system, on the other hand, may lead to uncertainty of the sign of the function. In such a situation, it makes sense to assume the function is equal to zero. In fact, if at some point the value of the function proves to be less in modulus to an estimate of its limiting error, this will mean that there is no reason to consider the function to be nonzero. But if such a situation occurs when determining the sign of the function at the vertices of n-dimensional parallelepipeds that localize the solution of the analyzed system, such n-dimensional parallelepipeds must not be eliminated from the search domain. An example of a situation with a determination of the signs of the values of the functions f_1 and f_2 at the vertices of one of the rectangles $\Omega^{(21)}$, $\Omega^{(22)}$, $\Omega^{(23)}$, and $\Omega^{(24)}$, taking into account the error in the parameters p_1 and p_2, is shown in Fig. 2.

Examples illustrating the use of the generalized interval method of bisection. The present approach was applied to several problems borrowed from metrological applications.

Example 1. Let us consider a problem from applied hydrodynamics. We wish to determine the height of a wave on a water surface from the values of the pressure in the liquid beneath the surface. As the initial values we use the results of measurements of the range of the hydrodynamic pressure Δp in the liquid at a depth z induced by propagation of waves with

TABLE 1. Results of a Solution of System (5) Represented by the Present Method

Experiment No.	Initial data [6]		Solution of system (5)	
	Δp, cm	T, s	k, m^{-1}	a, m
1	10.1 ± 0.5	1.930 ± 0.005	1.435 ± 0.009	0.063 ± 0.004
2	6.0 ± 0.5	1.928 ± 0.003	1.447 ± 0.009	0.038 ± 0.004
3	16.8 ± 1.0	1.938 ± 0.007	1.390 ± 0.05	0.103 ± 0.010
4	5.97 ± 0.35	1.697 ± 0.005	1.710 ± 0.02	0.042 ± 0.004
5	10.60 ± 0.35	1.702 ± 0.003	1.683 ± 0.015	0.071 ± 0.005

period T on the surface of a layer of water at a depth $d \geq z$. The height of the waves h must be determined indirectly based on the results of direct measurements of the quantities Δp, T, d, and z.

To solve the problem, we use a mathematical model of the propagation of waves on the surface of a fluid that describes the relation between the quantities h and Δp by means of the equations [5]

$$\left.\begin{aligned} &\frac{\Delta p}{\rho g} = 2a\frac{\cosh\left[k(d-z)\right]}{\cosh(kd)} + \frac{3}{4}\frac{a^3k^2}{\sinh^2(kd)\sinh(2kd)}\times \\ &\times\left\{-2\cosh\left[3k(d-z)\right] + \frac{13-4\cosh^2(kd)}{4\sinh^2(kd)}\cosh^2\left[3k(d-z)\right] + \right. \\ &\left. +\left(\frac{8}{3}\cosh^4(kd) - \frac{8}{3}\cosh^2(kd)+1\right)\cosh\left[k(d-z)\right]\right\}; \\ &T^2gk\tanh(kd)\left(1+\frac{8\cosh^4(kd)-8\cosh^2(kd)+9}{16\sinh^4(kd)}a^2k^2\right)^2 = 4\pi^2; \end{aligned}\right\} \tag{5}$$

$$h = 2a\left[1+\frac{a^2k^2}{64\sinh^6(kd)}\left[32\cosh^6(kd)+32\cosh^4(kd)-76\cosh^2(kd)+39\right]\right], \tag{6}$$

where ρ is the density of water; g, free-fall acceleration; a, amplitude factor; and k, wave number.

In order to determine the unknown quantity h, we solve the system of equations (5) relative to the unknown parameters k and a, which we then substitute into formula (6). For this purpose, we use the method proposed in the current article. To estimate the reliability of the obtained estimates of the metrological characteristics of the solution of the system, we compare the present results to the results of joint measurements of the quantities Δp, T, d, z, and h. These measurements were performed by one of the present authors in the course of special experiments described in [6]. The experiments were conducted in a hydraulic wave laboratory with the use of sensors for finding the current values of the pressure in a fluid along with fluid-level gauges that measure the current values of the water level.

We indirectly calculate the values of Δp, T, and h by mathematical processing of the results of the above measurements, relying on the readings of the devices. We perform an estimation of the error of the results inherited from the error of the measuring instruments employed. It is known that the depth of the pressure sensors $z = 26.5 \pm 0.2$ cm, while the depth at the installation site $d = 65.7 \pm 0.2$ cm. The initial data for the solution of the system of equations (5) of the mathematical model are as follows: intervals of initial approximations for the arguments k and a relative to which the system is solved, correspondingly Sk = [0.5; 2.0] m^{-1} and Sa = [0; 0.15] m. The results of measurements of the range of the hydrodynamic pressure Δp and the period of regular seas T as well as the limits of their admissible absolute error are presented in Table 1.

In solving the system of equations, we round off the estimates of the error of the quantities a and k inherited from the quantities Δp, T, d, and z to the next significant digit. Then we substitute the values of a and k thus obtained into formula (6)

TABLE 2. Values of Height of Seas h, cm, Obtained as a Result of Measurements and from Calculations by the Monte-Carlo Method and by the Proposed Method

Experiment No.	Direct measurements [6]	Monte-Carlo method [6]	Proposed method
1	12.93 ± 0.25	13.2 ± 0.7	13.1 ± 0.8
2	7.7 ± 0.2	7.8 ± 0.7	7.7 ± 0.7
3	22.23 ± 0.25	22.7 ± 1.6	22.2 ± 2.0
4	8.0 ± 0.2	8.3 ± 0.7	8.3 ± 0.5
5	14.87 ± 0.25	14.9 ± 1.1	14.7 ± 1.3

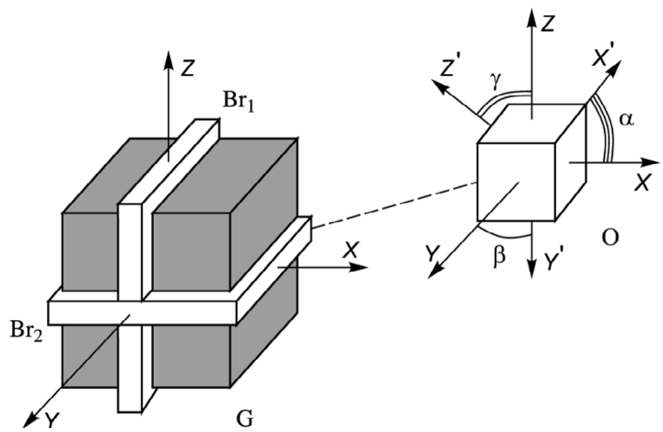

Fig. 3. Coordinate systems in positioning in an external magnetic field: G – generator of field; O – positioned object; Br_1 and Br_2 – brackets of generator.

in order to calculate the height of the seas h. We compare the results with the measurement data and the resulting estimate of the limit of its absolute error with the results of calculations from [6] performed by the Monte-Carlo method (Table 2).

The data presented in Table 2 demonstrate the reliability of the results obtained by the proposed method. That is, all the computed limits of the possible values of the height of the wave h are in good agreement with the results of the Monte-Carlo method and contain the results of direct measurements of this quantity.

Example 2. We wish to determine the coordinates of an object in an external magnetic field created by a special generator. The essence of such a problem of positioning and the technical instruments for its solution are described in [7–9].

A special generator creates an alternating magnetic field by means of two perpendicularly oriented brackets Br_1 and Br_2 through which electric current is alternately transmitted. Positioning of the object in this field is achieved in the *OXYZ* coordinate system with origin at the center of the generator G (Fig. 3). In order to find the coordinates of the object, we measure three mutually orthogonal components of the magnetic induction at the point where the object is situated in different phases of operation of the generator. We determine the linear coordinates (x_0; y_0; z_0) of the object and the angles (α_0; β_0; γ_0) of its orientation in the *OXYZ* coordinate system by means of a mathematical model that relates the induction of the magnetic field at different points in space with the known parameters of the generator. We measure the components (B_x; B_y; B_z) of the magnetic induction vector **B** by means of three Hall sensors situated on the object the axes of sensitivity of which specify a coordinate system $OX'Y'Z'$. The problem of finding the values of $x_0, y_0, z_0, \alpha_0, \beta_0, \gamma_0$ reduces to solving the system of nonlinear equations [9]

$$\begin{cases} \mathbf{B}_1^{\mathrm{T}}\mathbf{E} = \tilde{\mathbf{B}}_1; \\ \mathbf{B}_2^{\mathrm{T}}\mathbf{E} = \tilde{\mathbf{B}}_2, \end{cases} \tag{7}$$

where $\mathbf{B}_1^{\mathrm{T}} = (B_{x1}; B_{y1}; B_{z1})$ and $\mathbf{B}_2^{\mathrm{T}} = (B_{x2}; B_{y2}; B_{z2})$ are the magnetic induction vectors of the field created by the first, respectively, second bracket of the generator at a point with the coordinates $(x_0; y_0; z_0)$ according to the mathematic model of the generator; B_{x1}, B_{y1}, B_{z1} and B_{x2}, B_{y2}, B_{z2}, functions of the parameters of the generator; $\mathbf{E}$, Euler matrix, which determines the orientation of the Hall sensors situated on the object in the $OXYZ$ coordinate system; and $\tilde{\mathbf{B}}_1^{\mathrm{T}} = (\tilde{B}_{x'1}; \tilde{B}_{y'1}; \tilde{B}_{z'1})$ and $\tilde{\mathbf{B}}_2^{\mathrm{T}} = (\tilde{B}_{x'2}; \tilde{B}_{y'2}; \tilde{B}_{z'2})$, vectors of the results of measurements of the components of the magnetic induction at the point $(x_0; y_0; z_0)$ in the coordinate system $OX'Y'Z'$ adopted on the object.

The values of B_{x1}, B_{y1}, B_{z1} and B_{x2}, B_{y2}, B_{z2} depend on the geometric dimensions of the generator G and the force of the transmitted electric current. We next write out equations that relate these parameters to the components of the vector $\mathbf{B_1}$:

$$\frac{B_{x1}}{\mu_0 I_1} = r_{14z}\left(\frac{r_{14y}}{\|\mathbf{r}_{14}\|}+\frac{r_{12y}}{\|\mathbf{r}_{12}\|}\right)\Big/\left(x_0^2+r_{14z}^2\right)+r_{14y}\left(\frac{r_{13z}}{\|\mathbf{r}_{13}\|}+\frac{r_{14z}}{\|\mathbf{r}_{14}\|}\right)\Big/\left(x_0^2+r_{14y}^2\right)+$$

$$+r_{11z}\left(\frac{r_{11y}}{\|\mathbf{r}_{11}\|}+\frac{r_{13y}}{\|\mathbf{r}_{13}\|}\right)\Big/\left(x_0^2+r_{11z}^2\right)+r_{11y}\left(\frac{r_{11z}}{\|\mathbf{r}_{11}\|}+\frac{r_{12z}}{\|\mathbf{r}_{12}\|}\right)\Big/\left(x_0^2+r_{11y}^2\right);$$

$$\frac{B_{y1}}{\mu_0 I_1} = x_0\left(\frac{r_{13z}}{\|\mathbf{r}_{13}\|}+\frac{r_{14z}}{\|\mathbf{r}_{14}\|}\right)\Big/\left(x_0^2+r_{14y}^2\right)+x_0\left(\frac{r_{11z}}{\|\mathbf{r}_{11}\|}+\frac{r_{12z}}{\|\mathbf{r}_{12}\|}\right)\Big/\left(x_0^2+r_{11y}^2\right);$$

$$\frac{B_{z1}}{\mu_0 I_1} = x_0\left(\frac{r_{14y}}{\|\mathbf{r}_{14}\|}+\frac{r_{12y}}{\|\mathbf{r}_{12}\|}\right)\Big/\left(x_0^2+r_{14z}^2\right)+x_0\left(\frac{r_{11y}}{\|\mathbf{r}_{11}\|}+\frac{r_{13y}}{\|\mathbf{r}_{13}\|}\right)\Big/\left(x_0^2+r_{11z}^2\right),$$

where I_1 is the force of the electric current travelling through the first bracket; μ_0, magnetic constant; r_{11}, r_{12}, r_{13}, and r_{14}, vectors formed by the point $(x_0; y_0; z_0)$ and the vertices of the first bracket, which is assumed to be thin; $\|r_{11}\|$, $\|r_{12}\|$, $\|r_{13}\|$, and $\|r_{14}\|$, lengths of the given vectors; and r_{11x}, r_{11y}, r_{11z}, r_{12y}, r_{12y}, r_{12z}, etc., projections of the vectors onto the OX, OY, and OZ axes.

We determine the components of the vector $\mathbf{B}_2$, which describes the magnetic field produced by the second bracket of the generator, from analogous relationships in which the coordinates of the vertices of this bracket must now be used.

Since the positioned object may be oriented in space in such a way that the axes of sensitivity of the Hall sensors installed on it may be rotated relative to the axes of the generator, we apply the matrix $\mathbf{E}$ of the rotation of the $OXYZ$ coordinate system into the $OX'Y'Z'$ coordinate system:

$$\mathbf{E} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$a_{11} = \cos\beta_0\cos\alpha_0; \quad a_{12} = \cos\beta_0\sin\alpha_0; \quad a_{13} = -\sin\beta_0; \quad a_{21} = \sin\gamma_0\sin\beta_0\cos\alpha_0 - \cos\gamma_0\sin\alpha_0;$$

$$a_{22} = \sin\gamma_0\sin\beta_0\sin\alpha_0 + \cos\gamma_0\cos\alpha_0; \quad a_{23} = \sin\gamma_0\cos\beta_0; \quad a_{31} = \cos\gamma_0\sin\beta\cos\alpha_0 + \sin\gamma_0\sin\alpha_0;$$

$$a_{32} = \cos\gamma_0\sin\beta_0\sin\alpha_0 - \sin\gamma_0\cos\alpha_0; \quad a_{33} = \cos\gamma_0\sin\beta_0.$$

After multiplication of $\mathbf{B}_1^{\mathrm{T}}\mathbf{E}$ and $\mathbf{B}_2^{\mathrm{T}}\mathbf{E}$, we obtain expressions in which the desired unknowns – the coordinates x_0, y_0, and z_0 and the orientation angles α_0, β_0, γ_0 – occur nonlinearly.

The positioning precision in the magnetic field created in the approach that has been described here depends substantially on the precision of the generator G, in particular, on the precision with which its brackets are produced, the stability of the force of the current transmitted through the brackets, the precision of the positioning of the Hall sensors on the object O, noise, and other factors the influence of which may be substantially reduced with where the required level of technological efficiency of the article is achieved. The inaccuracy of the Hall sensors employed is an unavoidable source of the error in the determination of the coordinates of the positioned object. The potentially attainable precision of positioning in the present method of positioning depends above all on their metrological characteristics.

TABLE 3. Results of a Solution of the System of Equations (7) for the Problem of Positioning of an Object in a Magnetic Field

γ, %	Experiment No.	Specified model values of coordinates			Results of calculations by proposed method		
		x, m	*y*, m	α, rad	*x*, m	*y*, m	α, rad
0.1	1	1.90	1.20	–0.38	1.90 ± 0.01	1.20 ± 0.01	–0.381 ± 0.005
	2	1.50	1.75	–0.10	1.500 ± 0.008	1.75 ± 0.02	–0.100 ± 0.005
	3	1.85	1.20	–0.30	1.85 ± 0.01	1.20 ± 0.01	–0.300 ± 0.004
0.01	1	1.90	1.20	–0.38	1.900 ± 0.001	1.200 ± 0.001	0.3800 ± 0.0005
	2	1.50	1.75	–0.10	1.5000 ± 0.0005	1.75 ± 0.001	0.1000 ± 0.0004
	3	1.85	1.20	–0.30	1.850 ± 0.001	1.20 ± 0.001	0.3000 ± 0.0004

The present authors carried out calculations that demonstrate the limits of the precision of a determination of the coordinates in similar positioning which is attainable with the use of modern serially produced Hall sensors. For this purpose, it was assumed that all the components of the error of the sensors (caused by nonlinearity, hysteresis, intrinsic noise, etc.) collectively lead to an admissible relative error that does not exceed $\gamma = 0.1\%$ in modulus. With such an assumption, it becomes possible to obtain a quantitative representation of the limiting positioning error attainable within the framework of the approach used in [7–9] in the modern state of technology. The following parameters were used in the calculations: length and width of first and second brackets, 0.25 and 0.50 m; 0.50 and 0.25 m, respectively; the values of the forces of the currents travelling through the brackets were borrowed from [10]: $I_1 = 0.5$ A and $I_2 = 1.0$ A.

The system of equations (7) for the three unknowns – the coordinates x_0 and y_0 and the angle α_0 – are solved by the proposed method in the case in which the positioned object does not travel along the *Z* axis (travel of the object O in the *OXY* plane is considered). The values of z_0, β_0, and γ_0 are assumed to be fixed and equal to 2.5 m, 0.12 rad, and 0.24 rad, respectively. The calculations were carried out for different initial approximations of the coordinates: $Sx = [0; 2]$ m – the interval for x_0; $Sy = [0; 2]$ m – the interval for y_0; and $S\alpha = [-0.75; 0]$ rad – the interval for α_0. The final results, moreover, coincide.

Estimates of the desired coordinates and their errors calculated by the proposed method as well as model values of the coordinates of the positioned object incorporated into the calculations are given in Table 3.

The intervals found by means of the method proposed in the present study contain the specified values of the coordinates. In a limit of the admissible relative error of Hall sensors equal to 0.1%, the limit of the absolute positioning error in the plane with travel of several meters from the field generator reaches around 1 cm, which for certain applications may prove to be quite substantial. But if it is assumed that the limit of the admissible relative error of Hall sensors is one only one-tenth as great and amounts to 0.01%, the error in the determination of the coordinates falls to roughly a millimeter (cf. Table 3).This result points in general to a limiting precision of positioning of the object in a magnetic field that is not very great if the above method of management is used.

Conclusion. A modification of the interval method of bisection is proposed. By means of this modification, it is possible to solve systems of equations in measurement problems and to take into account the characteristics of the error of all the results of the measurements that occur in the system as coefficients of the equations. A rule for halting the iteration process that correlates the halting moment with the precision of the initial data of the problem is presented. The method produces guaranteed generation of results if the equations are specified by functions that are monotone in the solution search domain. The proposed method, together with solutions of the systems of equations, provide estimates of the characteristics of their errors which makes it desirable to use it as a component of a metrologically important software package. An estimate of the limiting precision of a particular method of indirect measurements with known composition of the set of measuring instruments employed may be obtained by means of the method.

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