Measurement Techniques, Vol. 50, No. 1, 2007

LINEAR AND ANGULAR MEASUREMENTS

NUMERICAL STUDY OF A METHOD FOR MEASURING SMALL LINEAR AND ANGULAR DISPLACEMENTS BY LASER INTERFEROMETERS

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UDC 531.715

A measurement method for small linear and angular displacements with laser interferometers is developed; a mathematical model is presented for measurements taking account the vector nature of laser radiation. Results of a calculation-theoretical study on the basis of this method are presented.

Key words: *laser interferometer, linear and angular displacements, holographic diffraction lattice.*

Linear measurements at the nano- and micrometric levels play an important role in developing fundamentally new technical solutions in creating microelectronic instruments and microsystems for technically different purposes, and this places before metrologists the task of advanced development of measuring instruments for small lengths and displacements, and also their metrological provisions.

The high accuracy of measurement, inherent for a laser interferometer, is entirely realized if the axis of sight of the interferometer coincides with measurement lines, i.e., it coincides with the Abbe comparison principle. In the majority of cases, it is very difficult to observe the comparison principle due to construction features of the measuring device. The position of the axis of sight or the comparator straight line in space is determined by the mutual position of movable and immovable assemblies of the converter and it may change during measurements [1]. This leads to the occurrence of additional errors of a nonstationary character that points to the requirement of constantly taking them into account [2].

Under these conditions, an important task is improvement and scientific and procedural equipment for theoretical studies, and substantiation of methods for measuring small displacements; also the development of modern unified measuring instruments making it possible to combine functionally in one device a measuring instrument for linear and angular displacements or to provide monitoring of object angular rotations.

Currently, the most widespread method for analyzing interferometers, which may be generally presented in the form of optically connected and successively arranged source of coherent radiation *1*, an optical system *2*, a photodetector *3*, a light splitter *4*, and a reflector *5* (Fig. 1), is a method based on using a scalar diffraction theorem. However, this approximation is not always a correct description of diffraction phenomena since the distribution of light energy in space is governed by polarization phenomena and the whole of electromagnetic theory that is based on solving Maxwell equations.

New calculation relationships are presented in this article for determining the intensity of light created by a laser interference measuring instrument for small displacements in the zone of photodetector placement taking account of the electromagnetic nature of light diffraction and features of light splitter structure. On the basis of analyzing the results of a numerical experiment, assumptions are formulated for improvement of laser interferometers.

Rostov Military Instiute of Rocket Forces. Translated from Izmeritel'naya Tekhnika, No. 1, pp. 9–13, January, 2007. Original article submitted June 27, 2006.

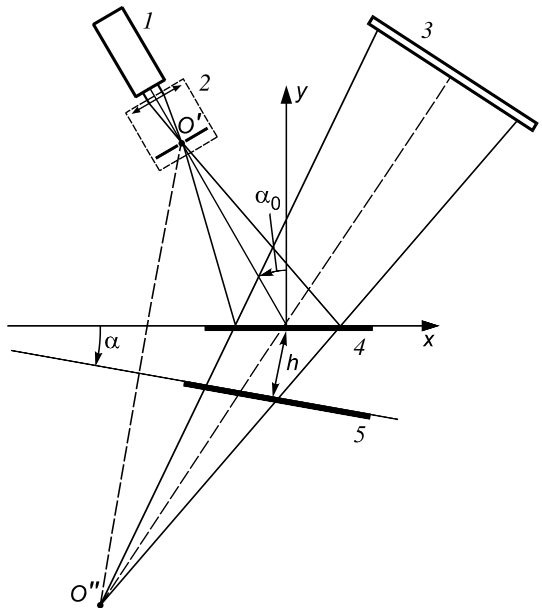

Fig. 1. Functional layout of an interference measuring instrument for small displacements: *1*) coherent radiation source; *2*) optical system; *3*) photodetector; *4*) light splitter; *5*) reflector.

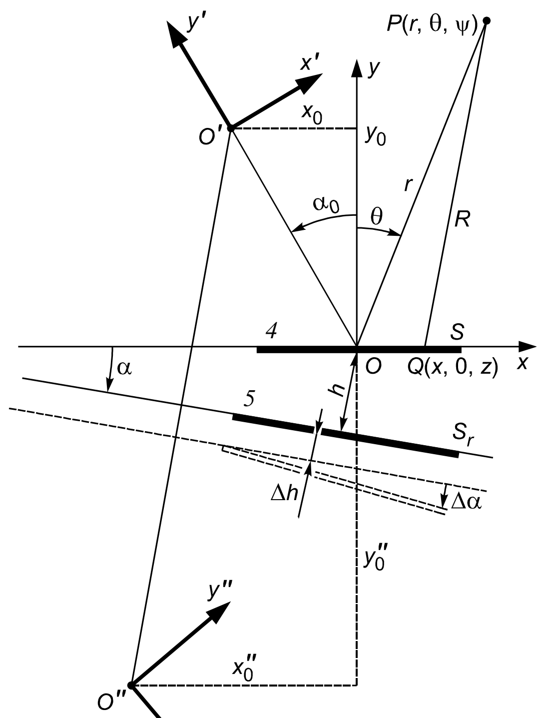

Fig. 2. Measuring instrument layout clarifying the model developed: *4*, *5*) the same as in Fig. 1.

The test measuring instrument is generally represented by a layout (Fig. 2) where light splitter *4* with known transmission amplitude is placed in plane *S*. Cartesian coordinate system *x*, *y*, *z* is connected with this plane. Reflector *5* is in plane S_r, that comprises angle α with plane *S* and is located at distance *h*, determined along the normal, reduced from the center of the light splitter *4* from a point source presented in the form of an electric dipole located at point x_0, y_0, $z_0 = 0$. We consider two polarization cases when the direction of the point in the dipole is perpendicular to plane $z = 0$ (along axis *z*) and parallel to it (along axis x'). Photoreceiving devices or photodetectors *3* (see Fig. 1), accomplishing analysis of the interference pattern, are located at some distance from light splitter *4*, corresponding to the Fresnel zone. This device makes it possible by measuring the interference pattern to determine the displacement of reflecting surface S_r in space, for which it is necessary to find the scattered electromagnetic field of the system in question in the observation zone, i.e., in space $y > 0$. In order to solve this problem, we use the Green theorem [3] that connects the vector field at the points of sources ($y = 0$) and the field at points of observation ($y > 0$).

When using of the Green tensor function with combined Neumann–Dirichlet conditions [4], taking account of the fact that the electromagnetic field has a transverse character and relates to Maxwell equations, in order to determine it at points of observation it is sufficient to know only the tangential components $E_{\tilde{S}}$ of the electric field voltage E_τ at the surface of the light splitter:

$$E_{\tilde{S}} = \frac{1}{4\pi}\int_S \left[E^2 \left(\nabla_1 \overset{D}{G}_{2\tilde{S}} - \nabla_2 \overset{H}{G}_{1\tilde{S}} \right) + E^3 \left(\nabla_1 \overset{D}{G}_{3\tilde{S}} - \nabla_3 \overset{H}{G}_{1\tilde{S}} \right) \right] dS. \tag{1}$$

Here indices with a tilde relate to points of observation that will be determined in a spherical coordinate system *r*, θ, ψ, and indices 1, 2, 3 relate to the points of sources lying at the surface of the light splitter; coordinate 1 coincides (with an opposite sign) with a single normal $\bar{n}$ to *S*, external with respect to the region in question. If E_τ and *S* are determined in terms of the Cartesian coordinate system introduced, then index 1 relates to coordinate *y*, index 2 to *z*, and 3 to *x*. Vector E_S is covariant, E^i, $i = 1, 2, 3$ are contravariant; $\overset{DH}{G}_{i\tilde{S}}$, $i = 1, 2, 3$ is Green tensor function satisfying the inhomogeneous equation

$$\nabla^S \nabla_S G_{i\tilde{S}} + k^2 G_{i\tilde{S}} = -4\pi g_{i\tilde{S}} \delta(r - \tilde{r}), \tag{2}$$

where $g_{i\tilde{S}}$ is parallel transfer operator in Euclidian space [5]:

$$g_{i\tilde{S}} = \frac{\partial \xi_v}{\partial x^i} \frac{\partial \xi_v}{\partial x^{\tilde{S}}}. \tag{3}$$

In (3), summing is carried out with respect to *v*, and coordinates ξ_v are Cartesian.

The following boundary conditions are imposed on the Green function in (1):

$$\nabla^N \overset{H}{G}_{N\tilde{S}} = 0 \in S; \quad \overset{D}{G}_{\tau\tilde{S}} = 0 \in S, \tag{4}$$

where *N* is a fixed index, and it corresponds to the normal to the coordinate surface, τ is tangential coordinate.

Thus, in order to resolve this problem it is desirable to determine the optical field at the surface of the light splitter, and then to calculate the intensity of light in the zone of observation for the interference pattern.

Determination of the Field at the Light Splitter Surface. In order to determine tangential components of the electric field voltage *E* vector, it is possible to use the method of images [6] in accordance with which the whole field is a superposition of fields from the source at point O' and the imaginary mirror source at point O'', and here it is necessary to consider the amplitude transmission of the light splitter.

In solving this problem, it is convenient to introduce additionally local Cartesian coordinate systems and spherical coordinate systems connected with them with an origin at points O' and O''. If the coordinates are shown without primes, then they correspond to the basic system with an origin on the surface of the light splitter at point *O*; coordinates with one and two primes relate correspondingly to coordinate systems with origins at points O' and O'' (see Fig. 2). The coordinate system x', y', z' is obtained by rotation around axis z by angle α_0 in a positive direction and subsequent parallel transfer by x_0, y_0, respectively. The coordinate system x'', y'', z'' is obtained from the system x', y', z' by its mirror reflection and a subsequent change in the direction of axis y into the opposite direction.

By using known relationships for spherical components of the electric dipole field and moving the Cartesian coordinates x, y, z, we express tangential components of the field at surface $y = 0$ in terms of Cartesian components:

for z-polarization (symbol ⊥)

$$E'_{x\perp} = \frac{\mathrm{i}\,k^2}{\omega\,\varepsilon}\frac{z(x+x_0)}{(r')^3}\mathrm{e}^{\mathrm{i}\,kr'}; \qquad E'_{z\perp} = \frac{\mathrm{i}\,k^2}{\omega\,\varepsilon}\frac{z^2-(r')^2}{(r')^3}\mathrm{e}^{\mathrm{i}\,kr'}, \tag{5}$$

for x-polarization (symbol ||)

$$E'_{x\|} = \frac{-\mathrm{i}\,k^2}{\omega\,\varepsilon}\frac{[(x+x_0)\sin\alpha_0 + y_0\cos\alpha_0]y_0 + z^2\cos\alpha_0}{(r')^3}\mathrm{e}^{\mathrm{i}\,kr'};$$

$$E'_{z\|} = \frac{-\mathrm{i}\,k^2}{\omega\,\varepsilon}\frac{[(x+x_0)\cos\alpha_0 - y_0\sin\alpha_0]\,z}{(r')^3}\mathrm{e}^{\mathrm{i}\,kr'}, \tag{6}$$

where ω is radiation cyclic frequency; ε is dielectric permittivity of the medium, and for the Fresnel zone it is possible to adopt

$$r' = r_0 + \frac{(x+x_0)^2 + z^2 - y_0^2}{2r_0}; \qquad r_0 = \sqrt{x_0^2 + y_0^2}\,. \tag{7}$$

In order to determine the field of the imaginary dipole, it is necessary in Eqs. (5)–(7) to substitute x_0 by x_0'', y_0 by y_0'', r' by r'', r_0 by r_0'', α_0 by $(-\alpha_0 - 2\alpha)$, taking account of the following:

$$x_0'' = x_0\cos 2\alpha + 2h\sin\alpha + y_0\sin 2\alpha;$$

$$y_0'' = -x_0\sin 2\alpha + 2h\cos\alpha + y_0\cos 2\alpha. \tag{8}$$

In expressions (5)–(8), values of parameters x_0, y_0, α_0, α, and h determine the position of the dipole and the reflecting surface in space.

We write the complete surface for the light splitter in the form of the sum of the incident field from the dipole and diffracted at the light splitter:

$$E_x = E'_x + T_x(x, z)E''_x; \qquad E_z = E'_z + T_z(x, z)E''_z, \tag{9}$$

where $T(x, z)$ is light splitter transmission coefficient, and it may be different for different polarization of irradiated light.

Determination of the Field in the Observation Zone of the Interference Pattern. By using (1)–(4), changing to the physical components of vector E and taking $y = 0$, we find a relationship for the components E_θ and E_ψ sought:

$$E_\theta = \frac{1}{2\pi}\int_S \frac{\mathrm{e}^{\mathrm{i}\,kR}}{R^2}(1-\mathrm{i}\,kR)\left[E_z\left(\frac{r}{R}\cos\psi - \frac{z}{R}\sin\theta\right) + E_x\left(\frac{r}{R}\sin\psi - \frac{x}{R}\sin\theta\right)\right]dxdz;$$

$$E_\psi = \frac{1}{2\pi}\int_S \frac{\mathrm{e}^{\mathrm{i}\,kR}}{R^2}(1-\mathrm{i}\,kR)\frac{r}{R}\cos\theta\,(E_x\cos\psi - E_z\sin\psi)\,dxdz. \tag{10}$$

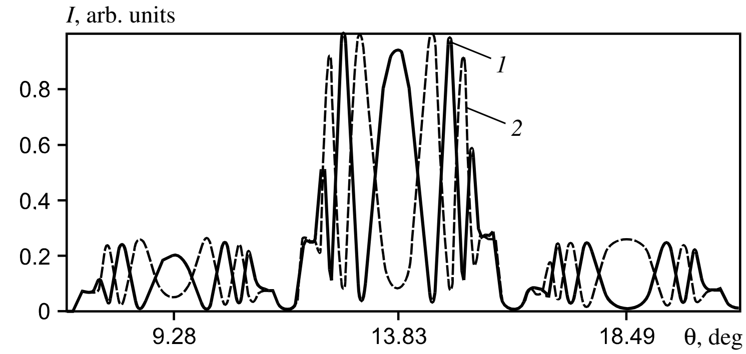

Fig. 3. Distribution of optical field intensity in the Fresnel zone: *1*) for $\alpha_1 = 0.07°$, $h_1 = 0.005$ m; *1*) for $\alpha_2 = \alpha_1 + 0.002°$, $h_2 = h_1 + 0.2\lambda$.

Here R is the distance between the point of observation $P(r, \theta, \psi)$ and the point of sources $Q(x, 0, z)$ (see Fig. 2). When points of observation are in the Fresnel zone, in the amplitude multiplier of expression (10) it is possible to take $r = R = R_0 = \text{const}$, and in the phase multiple to take

$$R = R_0 - (x\sin\psi + z\cos\psi)\sin\theta + (x^2 + z^2)/2R_0.$$

By substituting (5), (6) in (10) and specifying $T(x, z)$, we obtain a relationship for calculating the field at any point in space $y > 0$.

The distribution of radiation intensity in the observation zone may be found by substituting calculated values of physical components of the electric field voltage tensor in the well-known expression

$$I = E_\theta E_\theta^* + E_\psi E_\psi^*,$$

where E_θ^* and E_ψ^* are complex conjugates of E_θ and E_ψ, respectively.

Results of Numerical Studies. The mathematical model developed was implemented in a computer. It makes it possible to perform numerical studies for both individual elements and for interference measuring instruments for small displacements as a whole. In order to reduce calculations and for greater clarity of the results, fields were calculated in plane E ($\psi = \pi/2$) assuming that the irradiation field does not depend on coordinate z. The light splitters studied were diffraction gratings with a transmission coefficient varying by a harmonic rule. As is well known, diffraction at this grating only leads to formation of beams of the first order (±1). In the course of calculations, it was revealed that for spatial distribution of these beams a grating with the period $l \le 13\lambda$ ($\lambda = 0.63 \cdot 10^{-6}$ m is wavelength) is required with the following starting data $y_0 = 0.2$ m, $h = 0.005$ m, $R_0 = 0.3$ m; light splitter aperture size $2d = 0.01$ m.

In considering the particular case $x_0 = 0$, $y_0 = 0.2$ m, $\alpha = \alpha_0 = 0$, the radiation source, light splitter, reflector and photodetector are arranged on the comparator straight line. This mathematical model corresponds to the process of measuring linear displacement of the reflector when the normal to its surface, reduced from point O (see Fig. 2), does not change its direction and it constantly agrees with the comparator straight line. Studies of this model make it possible to connect clearly linear displacements of the reflector within the limits $\Delta h \le \lambda/4$ with the change intensity at any point of the interference pattern. The intensity in side beams of (±1) orders changes identically, and the phase in these beams differs from the zero order by π. In order to place photodetectors it is desirable to use the center of a beam of zero order since the amplitude for the change in intensity in this beam is much greater.

As already noted, of practical importance is accurate determination of small displacements with infringement of the Abbe principle in its original formulation [1] or if displacements of an object are specified by both linear and angular com-

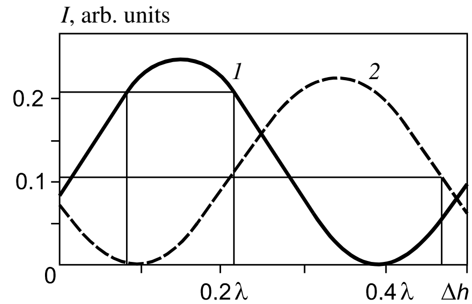

Fig. 4. Change in intensity with reflector displacement by $\Delta h = 0.5\lambda$ measured by a photodetector placed at point θ_1 (*1*) or θ_2 (*2*).

ponents. It follows from expressions (5)–(10) that light field and intensity depends on parameters determining the position in space of both the light source and the reflecting surface. The mathematical model developed makes it possible to study this effect and to substantiate theoretically a new method for measuring linear and angular displacements. We assume that the radiation source and the light splitter are rigidly fixed. Then a change of the position of the comparator straight line in space is caused by angular rotations of the reflector and will contribute an error to the results of measurement. Therefore, it is necessary to study the possibility of simultaneous measurement of linear and angular displacements of the reflector.

In this case, the intensity at a prescribed point of the interference pattern is a function of two variables $I_\theta = f(h, \alpha)$, and therefore ambiguity arises in solving this measurement problem. A study of the particular case above in question showed that with a change in parameters α and h the change in intensity for side beams is uniform in nature and it is impossible to overcome this ambiguity.

From the results of calculations performed with $x_0 = 0.05$ m; $y_0 = 0.2$ m; $h = 0.005$ m; $\alpha = 0.07°$; $R_0 = 0.3$ m; $2d = 0.01$ m (Fig. 3), it follows that the nature of change in intensity in beams of (±1) orders is different with displacement of the reflector. In view of this, it is desirable to place the photodetectors in directions corresponding to the centers of these beams.

In this case instead of one equation with two unknowns we obtain a set of two equations

$$\left.\begin{aligned} I_{\theta 1} &= f_1(h, \alpha); \\ I_{\theta 2} &= f_2(h, \alpha), \end{aligned}\right\} \tag{11}$$

where $I_{\theta 1}$ and $I_{\theta 2}$ are values of optical field intensity obtained as a result of measurement at points θ_1 and θ_2 of beams of (±1) orders, respectively.

Methods for finding roots that satisfy system (11) are quite developed and well known, and therefore it is undesirable in this article to dwell on the procedure for determining small displacements of the reflector $\Delta\alpha$ and Δh from the results of measuring the intensity in two directions ($\theta_1 = 9.28°$, $\theta_2 = 18.47°$). In order to determine displacements of an object rigidly connected with the reflector of an interference measuring instrument, it is necessary to measure intensities I_1 and I_2 at points corresponding to directions θ_1 and θ_2, and then to solve this equation. It is apparent that the algorithm and method for the solution depend upon technical specifications laid down for the measuring instrument developed for small displacements.

It should be noted that if there is only linear displacement of the reflector with constant angle α, then in order to determine small displacements Δh it is sufficient to measure the intensity at one point, for example θ_1. However, with $\Delta h \le \lambda/4$ ambiguity arises in determining linear displacements (Fig. 4). A photodetector placed at point θ_1 records the value $I_1 = 0.21$ if $\Delta h = 0.08\lambda$ or 0.22λ. In order to avoid this ambiguity, it is possible to use the result of measurement at a second point θ_2, where measured value $I_2 = 0.11$ corresponds to $\Delta h = 0.22\lambda$ or 0.47λ. As may be seen, $\Delta h = 0.22\lambda$ is simultaneously satisfied by curves *1* and *2*.

Thus, the mathematical model proposed for an interference measuring instrument for small displacements makes it possible to study the effect on a measured result of the original parameters of the device, governed by fulfilment of the Abbe principle, and the results obtained are a theoretical basis for a potential method of measuring linear and angular displacements of an interferometer reflector.

Use as light splitters for interferometers of diffraction gratings with a transmission coefficient changing by a harmonic rule and measurement of the intensity at corresponding points of the diffracted beams of (±1) orders expand the range of measurements with respect to h and the functional possibilities of laser measuring instruments for small displacements.

In order to resolve measurement problems, it is desirable to use phase sinusoidal gratings since here the intensity of diffracted beams equals the intensity of a beam of zero order.

REFERENCES

1. L. Z. Dich, *Izmer. Tekh.*, No. 9, 23 (1994).
2. V. A. Khanov, "Laser interferometers," *Proc. IAiÉ*, Novosibirsk (1978).
3. M. Morse and G. Feshbach, *Methods of Theoretical Physics, Vol. 2* [Russian translation], Izd. Inostr. Lit., Moscow (1960).
4. N. V. Kovalenko, "Antennae gratings," *Proc. RGU*, Rostov-on-Don (1971).
5. J. Singh, *General Theory of Relativity* [Russian translation], Izd. Inostr. Lit., Moscow (1963).
6. F. M. Morse and G. Feshbach, *Methods of Theoretical Physics, Vol. 1* [Russian translation], Izd. Inostr. Lit., Moscow (1958).