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A variational formulation for three-dimensional linear thermoelasticity with 'thermal inertia'

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Abstract A variational model has been developed to investigate the coupled thermo-mechanical response of a three-dimensional continuum. The linear Partial Differential Equations (PDEs) of this problem are already well-known in the literature. However, in this paper, we avoid the use of the second principle of thermodynamics, basing the formulation only on a proper definition (*i*) of kinematic descriptors (the displacement and the entropic displacement), (*ii*) of the action functional (with kinetic, internal and external energy functions) and (*iii*) of the Rayleigh dissipation function. Thus, a Hamilton–Rayleigh variational principle is formulated, and the cited PDEs have been derived with a set of proper Boundary Conditions (BCs). Besides, the Lagrangian variational perspective has been expanded to analyze linear irreversible processes by generalizing Biot's formulation, namely, including thermal inertia in the kinetic energy definition. Specifically, this implies Cattaneo's law for heat conduction, and the well-known Lord–Shulman model for thermo-elastic anisotropic bodies is then deduced. The developed variational framework is ideal for the perspective of analyzing the thermo-mechanical problems with micromorphic and/or higher-order gradient continuum models, where the deduction of a coherent system of PDEs and BCs is, on the one hand, not straightforward and, on the other hand, natural within the presented variational deduction.

Keywords Thermoelasticity · Hamilton–Rayleigh variational formulation · Heat conduction · Cattaneo's law · Lord–Shulman model

1 Introduction

Thermoelasticity is generally treated with a thermodynamic approach based on a sequence of independent postulates (see, e.g., some reviews on the subject [1–3]). There exist in the literature few examples [4–8], in which the mechanics of deformable bodies coupled with temperature effects is formulated using the setting provided by the variational principles [9].

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Weak formulation of thermoelasticity can be deduced from the strong form of the PDEs [10, 11] but inertial effects of the Lord–Shulman model can be confused with a dissipative term.

Starting from the Biot's fundamental contributions [6, 7], Giorgio [12] was able to show in the 1D case how the thermal phenomenologies could be framed using the same synthetic approach that is also successfully adopted for mechanical phenomena. In this paper, we extend to the more complex 3D case the same approach exploited in [12] for the 1D case considering both the general anisotropic and the simpler isotropic cases.

The mentioned synthetic viewpoint is exploited by introducing ab initio, as fundamental kinematic descriptors, the displacement and the entropic displacement fields, being the first the standard one and the second related to temperature in a way that will be specified in Eq. (12). Besides, some generalized Action functional has been defined as well as a dissipation Rayleigh function [13–15]. Finally, a generalized Rayleigh–Hamilton principle has been assumed in order to derive a system of PDEs and BCs. On the one hand, it is the generalized Rayleigh–Hamilton principle that gives the irreversibility to the system of the derived equations. On the other hand, in the standard thermodynamic approach, the irreversible nature of the thermal phenomena is dealt with by using the Clausius–Duhem inequality [16, 17]. Finally, it is demonstrated also in the 3D case that the classical thermoelastic problem and the well-known generalization due to Lord and Shulman [18], a finite velocity of propagation for thermal waves is deduced [19, 20].

The use of a variational principle is a highly effective and safe conceptual tool for developing complex theories regarding generalized continua [21–28], or approximate theories of structural components, alongside natural boundary conditions [29]. Additionally, it may serve as a preparatory step towards the establishment of a comprehensive thermo-mechanical formulation, a synthetic viewpoint to facilitate the proof of many mathematical theorems, and the theoretical basis for the application of modern numerical techniques like, for instance, the finite element method [30–38]. Variational principles are a widely accepted method of organizing information about dynamic systems and developing their evolution equations [7, 39–46]. By utilizing a singular quantity, such as the Lagrangian or Hamiltonian functional, one can deduce all aspects of a system, including its equations of motion, symmetries, and conservation laws. As a result, contemporary methods of modeling dynamical systems take the variational principle as a fundamental concept and initiate the process by constructing the Lagrangian or the Hamiltonian. Opting for the Lagrangian approach, one can prove that it is advantageous for various practical purposes. The Lagrangian formalism is, for example, a synthetic and comprehensive way to describe a system. For this reason, it becomes a mathematically elegant unifying picture that seems to tell us something profound about our physical universe. It enables straightforwardly stating underlying physical laws in arbitrary curvilinear coordinates. Therefore, it makes it possible to use generalized coordinates that permit the representation of the unique characteristics of any given system aptly and accurately. Besides, constraining forces do not exert work when constraints are bilateral and perfect; hence, there is no need to calculate them in an analysis involving energy and work. In the Lagrangian formulation, internal constraining stresses or reaction forces can be ignored completely. The Lagrangian approach naturally accounts for energy transfer between components of a system, making it a universal choice for physical systems.

With its remarkable features and capabilities, the proposed novel variational formulation proves to be an invaluable asset in tackling even the most complex thermo-elastic problems. Its unmatched versatility make it an ideal solution for a wide range of applications in engineering and beyond. By leveraging the unique advantages of this formulation, engineers and researchers can develop innovative and practical solutions to some of the most challenging problems in their respective fields. Overall, the novel formulation is a highly recommended tool that can help to drive progress and innovation in many areas of research and development, especially in the field of generalized continua. For example, to conceive models capable of describing fiber-reinforced thermo-elastic media [47], porous continua with thermal effects [48], and generalized thermoelasticity with couple-stress solids based on Lord–Shulman thermoelastic theory [49], to name a few.

The current discussion involves an exploration of the interplay between the thermal and elastic behaviors of a continuous body. The resultant equations are hyperbolic in nature and address the paradox of the infinite velocity of propagation intrinsic in the classical theory of thermo-elasticity.

2 Modeling linear elastic problems with thermal conductivity

2.1 Kinematic descriptors

Herein, we apply the generalized continuum theory to model 3D linear thermo-elasticity. Within the classical elasticity framework, a continuum medium is a continuous distribution of material particles represented geometrically in the reference configuration by a point $\boldsymbol{x}$ of a three-dimensional Euclidian space which is characterized kinematically by a vector field representing its position or displacement. In a generalized continuum theory, each particle is still represented by a point $\boldsymbol{x}$ from the macroscopic point of view, which is the point of view of any continuum theory, but its kinematical properties are defined more comprehensively. Therefore, in this generalized context, to develop a theory able to describe the mechanical and thermal response of the body, we can assume that the macroscopic kinematical description of the considered continuum is defined entirely through two fields on a domain $\mathcal{B}$ and over the time t: the usual displacement field, denoted here by $\boldsymbol{u}(\boldsymbol{x}, t)$, and a further field, namely, the *entropy* (or *heat*) *displacement*, $\boldsymbol{s}(\boldsymbol{x}, t)$, capable of describing the thermal behavior of the material particle, in particular, related to heat conduction, in the absence of any process of mass transfer [6, 7]. The proposed choice of entropy displacement, which is a vector field, seems to be more comprehensive than the alternative formulation adopting the temperature as a kinematical descriptor (see, e.g., [50]) on the grounds that the temperature is a scalar quantity and, therefore, has limited capability to describe the phenomenon. In the following, we will connect time and space derivative of the entropy displacement with the well-known concepts of the heat flux $\boldsymbol{q}$ and of the entropy increment η. Thus, the entropy displacement can be defined as the time integral of the heat flux or thermal flux, $\boldsymbol{q}$, divided by a reference temperature T_r; therefore, the expression

$$\frac{\partial \boldsymbol{s}}{\partial t}(\boldsymbol{x}, t) := \frac{1}{T_r}\boldsymbol{q}(\boldsymbol{x}, t) \tag{1}$$

is valid in the linear context. The increment of entropy per unit volume is the opposite of the divergence of the entropy displacement,

$$\eta(\boldsymbol{x}, t) = -\nabla \cdot \boldsymbol{s} = -\frac{\partial s_i}{\partial x_i} \tag{2}$$

where the summation convention is employed for the expression in terms of components, namely the index i that appears twice is implicitly summed over; such an index is called a dummy index. The unit of measure of $\boldsymbol{s}$ is accordingly $[\boldsymbol{s}] = \mathrm{J\ m^{-2}\ K^{-1}}$.

2.2 The extended variational principle

The variational approach has been proven to be the most effective in ensuring the accurate and efficient development of generalized elasticity; thus, we use it to frame a thermo-elastic formulation in a linear context accurately and consistently. In order to obtain the governing equations representing the Euler–Lagrange conditions, we use an extended variational principle adapted for the current scenario. Therefore, for every admissible variation of the kinematic descriptors, $\delta\boldsymbol{u}$ and $\delta\boldsymbol{s}$, the following variational equality holds

$$\int_{t_0}^{t_f} (\delta\mathscr{V} - \delta\mathscr{K})\mathrm{d}t + \int_{t_0}^{t_f} \tilde{\delta}\mathscr{D}\,\mathrm{d}t = \int_{t_0}^{t_f} \delta\mathscr{W}^{\mathrm{ext}}\,\mathrm{d}t \tag{3}$$

where δ is the operator that provides the first variation. Therefore, we can introduce a Lagrangian density:

$$\mathscr{L} = \mathscr{V} - \mathscr{K} \tag{4}$$

which is utilized in the definition of the action functional

$$\mathscr{A} = \int_{t_0}^{t_f} \mathscr{L}\mathrm{d}t \tag{5}$$

in terms of a thermo-elastic potential $\mathscr{V}$ and kinetic energy $\mathscr{K}$ and evaluated between two time instants t_0 and t_f. Thus, the first term of Eq. (3) is the first variation of the action functional, i.e., $\delta\mathscr{A}$.

The second term of Eq. (3) stands for the significant dissipative phenomena and can be associated to the virtual work linked to these effects. Herein, we define a Rayleigh functional $\mathscr{D}$ to describe such phenomena. Since this contribution is not evaluated as the first variation of $\mathscr{D}$, we use a different symbol, i.e., $\tilde{\delta}$, to take into account the dissipation as usually done in a variational framework through the Rayleigh extension of Lagrangian formalism [51, 52].

Finally, the last term of Eq. (3) represents the virtual work related to external generalized forces.

2.3 The thermoelastic potential

By assuming the divergence of the entropy displacement as a measure of thermal 'deformation', that is, a perturbation from a reference state, in a given inertial reference frame, we can postulate in a straightforward way the following thermo-elastic potential:

$$\mathscr{V}[\boldsymbol{u}(\cdot),\boldsymbol{s}(\cdot)] = \int_{\mathcal{B}} \left[\frac{1}{2}\mathbb{C}^{(\eta)}_{ijmn}E_{ij}E_{mn} + \frac{1}{2}k_h^E\left(\frac{\partial s_i}{\partial x_i}\right)^2 + \Gamma_{ij}E_{ij}\frac{\partial s_k}{\partial x_k}\right]\mathrm{d}\boldsymbol{x} \tag{6}$$

where the stiffness tensor $\mathbb{C}^{(\eta)}_{ijmn}$ stands for adiabatic elastic moduli, which furnish the mechanical behavior when no heat exchange is involved in the process, the quadratic term involving the square of the divergence of $\boldsymbol{s}$ is associated with the heat exchanged for conduction, and the last term represents a coupling between the mechanical deformation, $\boldsymbol{E}$, and the heat exchange at the reference temperature T_r given by the divergence of $\boldsymbol{s}$ (see, Eq. (2)). The linearized stain tensor $\boldsymbol{E}$ can be defined by components with the classical expression:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7}$$

for small deformation. The coefficients k_h^E and Γ_{ij} are material parameters. They can be identified as

$$k_h^E = \frac{T_r}{c}, \qquad \Gamma_{ij} = \frac{T_r}{c}\beta_{ij} \tag{8}$$

in which c is the specific heat per unit volume at zero strain, T_r is the reference temperature with respect to which we linearize the problem, and $\boldsymbol{\beta}$ is the thermal dilatation tensor assumed to be symmetric. Based on this identification, we can rewrite the generalized potential energy as

$$\mathscr{V}[\boldsymbol{u}(\cdot),\boldsymbol{s}(\cdot)] = \int_{\mathcal{B}} \left[\frac{1}{2}\mathbb{C}^{(\eta)}_{ijmn}E_{ij}E_{mn} + \frac{1}{2}\frac{T_r}{c}\left(\frac{\partial s_i}{\partial x_i}\right)^2 + \frac{T_r}{c}\beta_{ij}E_{ij}\frac{\partial s_h}{\partial x_h}\right]\mathrm{d}\boldsymbol{x} \tag{9}$$

Therefore, the energy density per unit volume can be defined as follows

$$\psi = \frac{1}{2}\mathbb{C}^{(\eta)}_{ijmn}E_{ij}E_{mn} + \frac{1}{2}\frac{T_r}{c}\left(\frac{\partial s_i}{\partial x_i}\right)^2 + \frac{T_r}{c}\beta_{ij}E_{ij}\frac{\partial s_h}{\partial x_h}. \tag{10}$$

As in the classical approach, the stress tensor is evaluated as the derivative of ψ with respect to $\boldsymbol{E}$,

$$\sigma_{ij} = \frac{\partial \psi}{\partial E_{ij}} = \mathbb{C}^{(\eta)}_{ijmn}E_{mn} + \frac{T_r}{c}\beta_{ij}\frac{\partial s_h}{\partial x_h}\ . \tag{11}$$

Indeed, by computing the first variation of the potential energy Eq. (9), the derivative of ψ with respect to $\boldsymbol{E}$ can immediately be identified as the generalized Lagrangian action that performs work on the virtual strain tensor. Besides, the derivative of ψ with respect to η is, in the same way, defined as

$$\theta = \frac{\partial \psi}{\partial \eta} = -\frac{\partial \psi}{\partial s_{i,i}} = -\frac{T_r}{c}\left(\frac{\partial s_i}{\partial x_i} + \beta_{ij}E_{ij}\right) \tag{12}$$

being the divergence $\frac{\partial s_i}{\partial x_i}$ expressed synthetically with the notation $s_{i,i}$, the absolute temperature T provided by the expression:

$$T(\boldsymbol{x},t) := T_r + \theta(\boldsymbol{x},t) \tag{13}$$

and θ is, therefore, the increment of temperature, namely, the generalized Lagrangian action that does work on the virtual entropy density η. By introducing these dual quantities, we can write the thermo-elastic energy density in a more straightforward way; specifically, we have:

$$\psi = \frac{1}{2}\sigma_{ij}E_{ij} + \frac{1}{2}\theta\eta \tag{14}$$

This expression enables us to analyze the meaning of this potential further. In fact, it can be equivalently reformulated in terms of temperature, as Biot did in [7], with the assumption:

$$\psi = \frac{1}{2}\mathbb{C}^{(\theta)}_{ijmn}E_{ij}E_{mn} + \frac{1}{2}\frac{c}{T_r}\theta^2 \tag{15}$$

where the last term represents a heat potential related to the increment of temperature θ. Naturally, the mechanical stiffness tensor $\mathbb{C}^{(\theta)}_{ijmn}$, in this context, must be specified in terms of isothermal elastic moduli. Therefore, using the kinematical descriptors introduced precisely by Biot, namely, $\boldsymbol{u}$ and $\boldsymbol{s}$, we have

$$\psi = \frac{1}{2}\mathbb{C}^{(\theta)}_{ijmn} E_{ij} E_{mn} + \frac{1}{2}\frac{T_r}{c}\left(\frac{\partial s_i}{\partial x_i} + \beta_{ij} E_{ij}\right)^2 \tag{16}$$

and, hence, the stress with the new isothermal perspective may be evaluated as

$$\begin{aligned}\sigma_{ij} =& \frac{\partial \psi}{\partial E_{ij}} = \mathbb{C}^{(\theta)}_{ijmn} E_{mn} + \beta_{ij}\frac{T_r}{c}\left(\frac{\partial s_h}{\partial x_h} + \beta_{hk} E_{hk}\right) \\ =& \mathbb{C}^{(\theta)}_{ijmn} E_{mn} - \beta_{ij}\theta\end{aligned} \tag{17}$$

The aim of this slightly different point of view resides in the fact that the entropy displacement could be, in the end, replaced by the more common quantity, the temperature, allowing for the retrieval of the classical thermo-elastic formulation that is already accessible.

By comparing the density of the mechanical potential energy, i.e., $1/2\sigma_{ij}E_{ij}$, for the two introduced cases, (11) and (17), we can compute the relationship between the adiabatic elastic moduli and the isothermal elastic moduli, as follows:

$$\mathbb{C}^{(\eta)}_{ijmn} = \mathbb{C}^{(\theta)}_{ijmn} + \frac{T_r}{c}\beta_{ij}\beta_{mn} \tag{18}$$

In this way, we can switch between the adiabatic and the isothermal formulation at our convenience.

2.4 The kinetic energy and the dissipative Rayleigh functional

The kinetic energy, in terms of the kinematical descriptors adopted, can be defined as

$$\mathscr{K}[\dot{\boldsymbol{u}}(\cdot), \dot{\boldsymbol{s}}(\cdot)] = \int_{\mathcal{B}} \frac{1}{2}\left[\varrho\, \frac{\partial u_i}{\partial t}\, \frac{\partial u_i}{\partial t} + \tau_0\, T_r\, \lambda_{ij}\, \frac{\partial s_i}{\partial t}\, \frac{\partial s_j}{\partial t}\right] \mathrm{d}\boldsymbol{x} \tag{19}$$

where the first contribution is the common mechanical term. Then, differently from what was proposed by Biot in his seminal work, we also add a thermal contribution to the kinetic energy [12]. In the ensuing discussion, we aim to elucidate how the supplementary contribution is responsible for the Cattaneo correction in the heat equation. Particularly, in Eq. (19) the parameter ϱ is the mass density, λ is the thermal resistivity tensor assumed to be symmetric, and τ_0 is a thermal relaxation characteristic time.

In order to address the irreversibility of the process, we should also consider some dissipative phenomena. This can be done in a variational framework with a Rayleigh functional $\mathscr{D}$ representing the power dissipated in the process. Specifically, we can assume

$$\begin{aligned}\mathscr{D}[\dot{\boldsymbol{u}}(\cdot), \dot{\boldsymbol{s}}(\cdot)] =& \frac{1}{2}\int_{\mathcal{B}} \mathbb{D}_{ijmn}\frac{\partial E_{ij}}{\partial t}\frac{\partial E_{mn}}{\partial t}\mathrm{d}\boldsymbol{x} \\ &+ \frac{1}{2}\int_{\mathcal{B}} T_r\, \lambda_{ij}\, \frac{\partial s_i}{\partial t}\, \frac{\partial s_j}{\partial t}\, \mathrm{d}\boldsymbol{x}\end{aligned} \tag{20}$$

In Eq. (20), the first contribution, differently from the original approach of Biot, is due to a viscous effect added to generalize the nonconservative source of the problem specifically mechanical, while the second contribution, according to Biot's formulation, is responsible for the dissipation related to thermal effects. Hence, $\mathbb{D}_{ijmn}$ are the components of a viscous tensor. In this context, we can define the Rayleigh potential density as

$$\mathscr{R} = \frac{1}{2}\, \mathbb{D}_{ijmn}\frac{\partial E_{ij}}{\partial t}\frac{\partial E_{mn}}{\partial t} + \frac{1}{2}\, T_r\, \lambda_{ij}\, \frac{\partial s_i}{\partial t}\, \frac{\partial s_j}{\partial t} \tag{21}$$

and, therefore, compute the virtual work done by dissipative generalized forces as

$$\tilde{\delta}\mathscr{D} = \int_{\mathcal{B}} \frac{\partial \mathscr{R}}{\partial \dot{E}_{mn}}\delta E_{mn}\mathrm{d}\boldsymbol{x} + \int_{\mathcal{B}} \frac{\partial \mathscr{R}}{\partial \dot{s}_j}\, \delta s_j\, \mathrm{d}\boldsymbol{x} \tag{22}$$

as it is customary in the context of a variational framework employing the Rayleigh extension of Lagrangian formalism. In Eq. (22), the over-dot symbol represents differentiation with respect to time.

Specifically, the virtual work linked to these dissipative effects becomes

$$\tilde{\delta}\mathscr{D} = \int_{\mathcal{B}} \mathbb{D}_{ijmn}\frac{\partial E_{ij}}{\partial t}\delta E_{mn}\mathrm{d}\boldsymbol{x} + \int_{\mathcal{B}} T_r\, \lambda_{ij}\, \frac{\partial s_i}{\partial t}\, \delta s_j\, \mathrm{d}\boldsymbol{x} \tag{23}$$

2.5 The virtual work done by external generalized forces

The virtual work done by the external generalized forces $\delta\mathscr{W}^{\mathrm{ext}}$ can be assumed, consistently with the hypotheses previously made, as follows:

$$\delta \mathscr{W}^{\text{ext}} = \int_{\mathcal{B}} \left[b_i \, \delta u_i + \vartheta_j \, \delta s_j \right] \mathrm{d}\boldsymbol{x} + \int_{\partial \mathcal{B}} \left(f_i \, \delta u_i + \Theta^{\text{ext}} n_j \, \delta s_j \right) \mathrm{d}A \tag{24}$$

where we can consider two contributions in bulk per unit volume and as many for the boundary of the body $\mathcal{B}$ under consideration, each pair related to the mechanical work and the exchanged heat representing the external interactions for the system. Therefore, b_i are the components of the long-range volumic force acting on $\mathcal{B}$ and f_i are the components of the contact force per unit surface acting on the boundary $\partial\mathcal{B}$ or on a portion of it; ϑ_j may be interpreted as a sort of 'thermo-motive' force, which in analogy with the electro-motive force, is defined by a heat energy density per unit volume of heat flow. The idea here is to consider the term $\vartheta_j \, \delta s_j$ as a source of heat inside the body that can guide some phenomena of heat conduction. The last generalized force, in the Lagrangian sense, namely, Θ^{ext}, is the temperature increment applied on the boundary or at a portion of it.

2.6 Exploitation of the variational principle

In order to apply the variational principle (3) for a thermoelastic continuum, the equations of motion can be obtained by introducing *admissible comparison motions*. In the considered case, the admissible comparison displacements to be considered are

$$\begin{aligned} \tilde{\boldsymbol{u}} &= \boldsymbol{u} + \varepsilon \, \delta \boldsymbol{u} \\ \tilde{\boldsymbol{s}} &= \boldsymbol{s} + \varepsilon \, \delta \boldsymbol{s} \end{aligned} \tag{25}$$

where ε is a real parameter, while $\delta\boldsymbol{u}$ and $\delta\boldsymbol{s}$ are arbitrary and sufficiently smooth vector fields on $\mathcal{B} \times [t_0, t_f]$ subject to the requirements that they vanish where the corresponding kinematic descriptors $\boldsymbol{u}$ and $\boldsymbol{s}$ are assigned on the boundary and for the two time instants. Having introduced these admissible comparison displacements, any functional I can be regarded as a function of ε and its first variation can be computed by taking its derivative with respect to ε and setting $\varepsilon = 0$, as follows

$$\delta I = \left. \frac{\mathrm{d} I(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{s}})}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} \tag{26}$$

Therefore, the first variation of the thermo-elastic potential is

$$\begin{aligned} \delta \mathscr{V} &= \left. \frac{\mathrm{d} \mathscr{V}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{s}})}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} \\ &= \int_{\mathcal{B}} \left[\mathbb{C}^{(\eta)}_{ijmn} E_{ij} \, \delta E_{mn} + \frac{T_r}{c} \frac{\partial s_i}{\partial x_i} \frac{\partial \delta s_h}{\partial x_h} \right. \\ &\quad \left. + \frac{T_r}{c} \beta_{ij} \left(E_{ij} \frac{\partial \delta s_h}{\partial x_h} + \frac{\partial s_h}{\partial x_h} \, \delta E_{ij} \right) \right] \mathrm{d}\boldsymbol{x} \end{aligned} \tag{27}$$

By assuming the symmetry of the tensors $\mathbb{C}^{(\eta)}_{ijmn}$ and β_{ij}, Eq. (27) can be written equivalently as

$$\begin{aligned} \delta \mathscr{V} &= \int_{\mathcal{B}} \left[\mathbb{C}^{(\eta)}_{ijmn} E_{ij} \frac{\partial \delta u_m}{\partial x_n} + \frac{T_r}{c} \frac{\partial s_i}{\partial x_i} \frac{\partial \delta s_h}{\partial x_h} \right. \\ &\left. + \frac{T_r}{c} \beta_{ij} \left(E_{ij} \frac{\partial \delta s_h}{\partial x_h} + \frac{\partial s_h}{\partial x_h} \frac{\partial \delta u_i}{\partial x_j} \right) \right] \mathrm{d}\boldsymbol{x} \end{aligned} \tag{28}$$

where the linearized strain tensor is substituted with the gradient of the displacement to simplify the subsequent calculations.

In fact, the thermo-elastic contribution in the variational principle (3) after an integration by parts becomes:

$$\begin{aligned} \int_{t_0}^{t_f} \delta \mathscr{V} \mathrm{d}t = &- \int_{t_0}^{t_f} \int_{\mathcal{B}} \left\{ \frac{\partial}{\partial x_n} \left(\mathbb{C}^{(\eta)}_{ijmn} E_{ij} + \frac{T_r}{c} \beta_{mn} \frac{\partial s_i}{\partial x_i} \right) \delta u_m \right. \\ &\left. + \frac{\partial}{\partial x_j} \left[\frac{T_r}{c} \left(\frac{\partial s_i}{\partial x_i} + \beta_{ij} E_{ij} \right) \right] \delta s_j \right\} \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &+ \int_{t_0}^{t_f} \int_{\partial \mathcal{B}} \left\{ \left(\mathbb{C}^{(\eta)}_{ijmn} E_{ij} + \frac{T_r}{c} \beta_{mn} \frac{\partial s_i}{\partial x_i} \right) n_n \, \delta u_m \right. \\ &\left. + \left[\frac{T_r}{c} \left(\frac{\partial s_i}{\partial x_i} + \beta_{ih} E_{ih} \right) \right] n_j \, \delta s_j \right\} \mathrm{d}A \, \mathrm{d}t \end{aligned} \tag{29}$$

that is to say

$$\begin{aligned} \int_{t_0}^{t_f} \delta \mathscr{V} \mathrm{d}t = &- \int_{t_0}^{t_f} \int_{\mathcal{B}} \left(\frac{\partial \sigma_{mn}}{\partial x_n} \, \delta u_m - \frac{\partial \theta}{\partial x_j} \, \delta s_j \right) \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \\ &- \int_{t_0}^{t_f} \int_{\partial \mathcal{B}} \left(\sigma_{mn} n_n \, \delta u_m - \theta \, n_j \, \delta s_j \right) \mathrm{d}A \, \mathrm{d}t \end{aligned} \tag{30}$$

if we take into account the constitutive relationships for the stress $\boldsymbol{\sigma}$ and the temperature increment θ.

The inertial contribution linked to the kinetic energy yields:

$$- \int_{t_0}^{t_f} \delta \mathscr{K} \mathrm{d}t = \int_{t_0}^{t_f} \int_{\mathcal{B}} \left[\varrho \, \frac{\partial^2 u_i}{\partial t^2} \, \delta u_i + \tau_0 \, T_r \, \lambda_{ij} \, \frac{\partial^2 s_i}{\partial t^2} \, \delta s_j \right] \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \tag{31}$$

after performing an integration by parts, this time with respect to the temporal variable. In addition, the boundary conditions related to the initial and final time disappear because of the isochronous assumption for the motions. Therefore, as usually done, initial conditions related to $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, $\boldsymbol{s}$, and $\dot{\boldsymbol{s}}$ should be introduced to solve the differential problem obtained.

The virtual work of dissipative phenomena can be evaluated as in (23) or, equivalently, as

$$\tilde{\delta}\mathscr{D} = \int_{\mathcal{B}} \mathbb{D}_{ijmn} \frac{\partial E_{ij}}{\partial t} \frac{\partial \delta u_m}{\partial x_n} \mathrm{d}\boldsymbol{x} + \int_{\mathcal{B}} T_r \, \lambda_{ij} \, \frac{\partial s_i}{\partial t} \, \delta s_j \, \mathrm{d}\boldsymbol{x} \tag{32}$$

for the symmetry of $\mathbb{D}_{ijmn}$. The dissipative contribution in the variational principle (3) can be computed as follows

$$\int_{t_0}^{t_f} \tilde{\delta}\mathscr{D} \, \mathrm{d}t = -\int_{t_0}^{t_f} \int_{\mathcal{B}} \frac{\partial}{\partial x_n} \left(\mathbb{D}_{ijmn} \frac{\partial E_{ij}}{\partial t} \right) \delta u_m \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_{t_0}^{t_f} \int_{\partial\mathcal{B}} \left(\mathbb{D}_{ijmn} \frac{\partial E_{ij}}{\partial t} \right) n_n \delta u_m \mathrm{d}A \, \mathrm{d}t + \int_{t_0}^{t_f} \int_{\mathcal{B}} T_r \, \lambda_{ij} \, \frac{\partial s_i}{\partial t} \, \delta s_j \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \tag{33}$$

with an integration by parts.

2.7 Derivation of the PDEs and BCs of the thermoelastic problem

Finally, by substituting the expressions (29), (31), (33), and (24) into Eq. (3) and applying the fundamental lemma of the calculus of variations (see, e.g., [53] pag. 20), we can deduce the balance of linear momentum as the necessary condition related to the mechanical variable, namely, the displacement $\boldsymbol{u}$, for the adiabatic representation, as follows

$$\varrho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left(\mathbb{C}^{(\eta)}_{ijmn} E_{mn} + \frac{T_r}{c} \beta_{ij} \frac{\partial s_i}{\partial x_i} \right) - \frac{\partial}{\partial x_j} \left(\mathbb{D}_{ijmn} \frac{\partial E_{mn}}{\partial t} \right) = b_i \tag{34}$$

together with the mechanical natural boundary condition

$$\sigma_{ij} n_j + \left(\mathbb{D}_{ijmn} \frac{\partial E_{mn}}{\partial t} \right) n_j = f_i \quad \forall \boldsymbol{x} \in \partial\mathcal{B}_f \text{ and } \forall t \in [t_0, t_f] \tag{35}$$

on the portion of the boundary $\partial\mathcal{B}_f$ where the force per unit area $\boldsymbol{f}$ is applied, or the mechanical essential boundary conditions in which we impose the displacement field $\boldsymbol{u}$

$$u_i(\boldsymbol{x}, t) = g_i(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in \partial\mathcal{B}_u \text{ and } \forall t \in [t_0, t_f] \tag{36}$$

It is worth noting that from this formulation appears a further stress term σ^v_{ij}, namely,

$$\sigma^v_{ij} = \left(\mathbb{D}_{ijmn} \frac{\partial E_{mn}}{\partial t} \right) \tag{37}$$

which represents the contribution of the viscous inner actions introduced here with the viscous part of the Rayleigh functional (20).

In addition, making use of the relationships (18) and (12), the balance of linear momentum can also be expressed for the isothermal representation, as below:

$$\varrho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left(\mathbb{C}^{(\theta)}_{ijmn} E_{mn} - \beta_{ij} \theta \right) - \frac{\partial}{\partial x_j} \left(\mathbb{D}_{ijmn} \frac{\partial E_{mn}}{\partial t} \right) = b_i \tag{38}$$

Once more, by substituting the expressions (29), (31), (33), and (24) into Eq. (3) and applying the fundamental lemma of the calculus of variations, the necessary condition related to the thermal variable, namely, the entropy displacement $\boldsymbol{s}$ provides the Cattaneo law in the following form:

$$-\frac{\partial}{\partial x_j} \left[\frac{T_r}{c} \left(\frac{\partial s_i}{\partial x_i} + \beta_{ij} E_{ij} \right) \right] + T_r \, \lambda_{ij} \left(\tau_0 \, \frac{\partial^2 s_i}{\partial t^2} + \frac{\partial s_i}{\partial t} \right) = \vartheta_j \qquad \forall \boldsymbol{x} \in \mathcal{B} \tag{39}$$

alongside the thermal natural boundary conditions

$$-\theta = \Theta^{\text{ext}} \qquad \forall \boldsymbol{x} \in \partial\mathcal{B}_\theta \text{ and } \forall t \in [t_0, t_f] \tag{40}$$

related to the portion of the boundary where the temperature is applied $\partial\mathcal{B}_\theta$, and, similarly to the mechanical displacement, we can also give the entropy displacement as a thermal essential boundary condition

$$s_i(\boldsymbol{x}, t) = h_i(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in \partial\mathcal{B}_s \text{ and } \forall t \in [t_0, t_f] \, . \tag{41}$$

Equation (35) is significant since it tells us that the dual of the displacement, i.e., f_i, defined in (24), is connected not only with the stress tensor, as in the non-dissipative case, but also to the viscous tensor $\mathscr{D}$ and the strain rate. Equation (24) defines also the dual of the entropy displacement $\Theta^{\text{ext}}\boldsymbol{n}$, that is not only assumed to be parallel to the boundary normal but connected directly with the temperature θ with Eq. (40).

In order to better exploit the fact that the temperature increment is the normal component of the dual of the entropy displacement, we premultiply Eq. (39) by the thermal conductivity $\boldsymbol{\kappa}$, substitute the constitutive relation (12) for θ as well as the expression (1) into Eq. (39) to obtain

$$\kappa_{ij}\frac{\partial \theta}{\partial x_j} + \left(\tau_0 \frac{\partial q_i}{\partial t} + q_i\right) = \kappa_{ij}\vartheta_j \tag{42}$$

being the tensor $\boldsymbol{\kappa} = \boldsymbol{\lambda}^{-1}$.

Finally, we remark that with some manipulations of Eq. (39) and making its divergence, we can obtain the Lord–Shulman model for thermo-elastic anisotropic bodies, completing the thermal field equation

$$\begin{aligned}&\frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial \theta}{\partial x_j}\right) - c\left(\frac{\partial \theta}{\partial t} + \tau_0\frac{\partial^2 \theta}{\partial t^2}\right)\\ &- T_r\beta_{ij}\left(\frac{\partial E_{ij}}{\partial t} + \tau_0\frac{\partial^2 E_{ij}}{\partial t^2}\right) = \frac{\partial}{\partial x_i}\left(\kappa_{ij}\vartheta_j\right)\end{aligned} \tag{43}$$

with the elastic mechanical Eq. (38).

3 A notable case: Isotropic linear thermo-visco-elasticity

In the context of thermoelastic solids that exhibit isotropy, the above-mentioned material tensors are characterized by a specific structure. Specifically, they take the form:

$$\mathbb{C}^{(\eta,\theta)}_{ijmn} = \lambda^{(\eta,\theta)}\delta_{ij}\delta_{mn} + \mu^{(\eta,\theta)}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) \tag{44}$$

$$\beta_{ij} = \beta\delta_{ij} \tag{45}$$

$$\kappa_{ij} = \kappa\,\delta_{ij} = \lambda^{-1}\,\delta_{ij} \tag{46}$$

$$\mathbb{D}_{ijmn} = \left(\kappa^v - \frac{2}{3}\mu^v\right)\delta_{ij}\delta_{mn} + \mu^v\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) \tag{47}$$

where $\lambda^{(\eta)}$ and $\mu^{(\eta)}$ are the Lamé moduli under an adiabatic deformation, β is a thermal dilatation coefficient, κ is the thermal conductivity together with its inverse λ, namely, the thermal resistivity; finally, μ^v and κ^v are the shear and bulk viscosity, respectively. To determine the isothermal Lamé parameters, one may refer to Eq. (18) and (44) and utilize the subsequent relationships

$$\lambda^{(\eta)} = \lambda^{(\theta)} + \frac{T_r}{c}\beta^2, \qquad \mu^{(\eta)} = \mu^{(\theta)} \tag{48}$$

or, alternatively, express them in terms of isothermal Young's modulus, Y, and isothermal Poisson's coefficient, ν, as follows

$$\lambda^{(\theta)} = \frac{Y\nu}{(1+\nu)(1-2\nu)}, \qquad \mu^{(\theta)} = \frac{Y}{2(1+\nu)} \tag{49}$$

For both representations, namely, adiabatic and isotermal, the number of parameters to be identified are, hence: 1) $\lambda^{(\eta,\theta)}$; 2) $\mu^{(\eta,\theta)}$; 3) κ^v; 4) μ^v; 5) κ; 6) c; 7) β; and 8) τ_0. Identifying all these parameters could be a daunting task because of the complexity of the thermo-mechanical responses to consider. It is worth noting that specific techniques that have been utilized for complex generalized materials, such as digital image correlation (DIC), energy methods, and neural networks, might be deemed applicable in this context (see, e.g., [54–58]).

Using the isotropic material symmetry in (44–47), therefore, the constitutive relations for the adiabatic elastic stress in (11) and the temperature increment in (12) are specified by

$$\sigma_{ij} = \left(\lambda^{(\eta)}E_{nn} + \frac{T_r}{c}\beta\frac{\partial s_h}{\partial x_h}\right)\delta_{ij} + 2\mu^{(\eta)}E_{ij} \tag{50}$$

$$\theta = -\frac{T_r}{c}\left(\frac{\partial s_i}{\partial x_i} + \beta E_{hh}\right) \tag{51}$$

By substituting the relations (44) in Eq. (34) and Eq. (39), the PDEs with the adiabatic constants become

$$\begin{aligned}&\varrho\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left[\left(\lambda^{(\eta)}E_{nn} + \frac{T_r}{c}\beta\frac{\partial s_h}{\partial x_h}\right)\delta_{ij} + 2\mu^{(\eta)}E_{ij}\right]\\ &\qquad - \frac{\partial}{\partial x_j}\left[\left(\kappa^v - \frac{2}{3}\mu^v\right)\frac{\partial E_{hh}}{\partial t}\delta_{ij} + 2\,\mu^v\frac{\partial E_{ij}}{\partial t}\right] = b_i\end{aligned} \tag{52}$$

and

$$-\frac{\partial}{\partial x_i}\left[\frac{T_r}{c}\left(\frac{\partial s_n}{\partial x_n} + \beta\,E_{hh}\right)\right] + T_r\,\lambda\left(\tau_0\frac{\partial^2 s_i}{\partial t^2} + \frac{\partial s_i}{\partial t}\right) = \vartheta_i \tag{53}$$

In order to fully solve the set of equations (52) and (53), it is necessary to incorporate specific boundary conditions, which can be essential, if they involve Dirichlet-like boundary conditions in terms of the displacement $\boldsymbol{u}$ and the entropic displacement $\boldsymbol{s}$, or natural if they are Neumann-like boundary conditions in terms of the dual of displacement or entropy displacement. In formulae, we have:

$$\begin{cases} u_i(\boldsymbol{x},t) = g_i(\boldsymbol{x},t) & \forall \boldsymbol{x} \in \partial\mathcal{B}_u \\ \sigma_{ij} n_j + \left[\left(\kappa^v - \frac{2}{3}\mu^v\right)\frac{\partial E_{hh}}{\partial t}\delta_{ij} + 2\,\mu^v \frac{\partial E_{ij}}{\partial t}\right] n_j = f_i & \forall \boldsymbol{x} \in \partial\mathcal{B}_f \\ s_i(\boldsymbol{x},t) = h_i(\boldsymbol{x},t) & \forall \boldsymbol{x} \in \partial\mathcal{B}_s \\ -\theta = \Theta^{\text{ext}} & \forall \boldsymbol{x} \in \partial\mathcal{B}_\theta \end{cases} \tag{54}$$

where $\partial\mathcal{B}_u$, $\partial\mathcal{B}_f$, $\partial\mathcal{B}_s$, and $\partial\mathcal{B}_\theta$ are arbitrary subsets of the entire boundary $\partial\mathcal{B}$ where each condition has been specified.

Furthermore, we must prescribe the necessary initial conditions on the considered domain $\mathcal{B}$ to obtain the desired solution. Specifically, we set:

$$\begin{cases} u_i(\boldsymbol{x},0) = u_i^0(\boldsymbol{x}) \\ \frac{\partial u_i}{\partial t}(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}) \\ s_i(\boldsymbol{x},0) = s_i^0(\boldsymbol{x}) \\ \frac{\partial s_i}{\partial t}(\boldsymbol{x},0) = r_i^0(\boldsymbol{x}) \end{cases} \tag{55}$$

where $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$ are the initial displacement and velocity, respectively. Besides, $\boldsymbol{s}^0$ is the initial entropy displacement and $\boldsymbol{r}^0$ is the initial rate of it. From Eq. (1) $\boldsymbol{r}^0$ is proportional to the initial heat flux. From Eq. (12) $\boldsymbol{s}^0$, in particular its divergence, is related to the initial temperature increment.

Similarly, we can deduce the stress by the insertion of Eq. (51) into Eq. (50) and use of Eq. (48),

$$\sigma_{ij} = \left(\lambda^{(\theta)} E_{nn} - \beta\,\theta\right)\delta_{ij} + 2\mu^{(\theta)} E_{ij} \tag{56}$$

making use of the isothermal moduli.

Consequently, the field equations, with the isothermal parameters, are Eq. (52) with the insertion of Eq. (48) and Eq. (51),

$$\begin{aligned} &\varrho\,\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left[\left(\lambda^{(\theta)} E_{nn} - \beta\,\theta\right)\delta_{ij} + 2\mu^{(\theta)} E_{ij}\right] \\ &-\frac{\partial}{\partial x_j}\left[\left(\kappa^v - \frac{2}{3}\mu^v\right)\frac{\partial E_{hh}}{\partial t}\delta_{ij} + 2\,\mu^v \frac{\partial E_{ij}}{\partial t}\right] = b_i \end{aligned} \tag{57}$$

together with Eq. (53), which remains unchanged, and we repeat here, for the sake of convenience:

$$\frac{\partial \theta}{\partial x_i} + T_r\,\lambda\left(\tau_0\,\frac{\partial^2 s_i}{\partial t^2} + \frac{\partial s_i}{\partial t}\right) = \vartheta_i \tag{58}$$

The differential problem, of course, is completed with the boundary conditions (54) and the initials conditions (55) as well.

The set of equations (57) and (58) are equivalent to the generalized dynamical thermo-elasticity theory developed by H. W. Lord and Y. Shulman [18]. In fact, Eq. (58) can be reformulated, substituting the dependence on the entropy displacement $\boldsymbol{s}$ with the temperature increment θ through Eq. (51). This modification results in a diffusion equation for θ, once one premultiplies by κ and takes the divergence as follows,

$$\begin{aligned} &\frac{\partial}{\partial x_i}\left(\kappa\,\frac{\partial \theta}{\partial x_i}\right) - c\left(\frac{\partial \theta}{\partial t} + \tau_0\,\frac{\partial^2 \theta}{\partial t^2}\right) \\ &- T_r \beta\left(\frac{\partial E_{hh}}{\partial t} + \tau_0 \frac{\partial^2 E_{kk}}{\partial t^2}\right) = \frac{\partial}{\partial x_i}\left(\kappa\,\vartheta_i\right) \end{aligned} \tag{59}$$

which is hyperbolic, and hence it permits fixing the paradox of an infinite velocity of propagation as intended by Lord and Shulman. It is worth to note that the hyperbolicity of the thermal PDE can be recognized also via the original Eq. (53).

The new differential set of Eqs. (57) and (59) should be complemented with the corresponding boundary conditions

$$\begin{cases} u_i(\boldsymbol{x},t) = g_i(\boldsymbol{x},t) & \forall \boldsymbol{x} \in \partial\mathcal{B}_u \\ \sigma_{ij} n_j + \left[\left(\kappa^v - \frac{2}{3}\mu^v\right)\frac{\partial E_{hh}}{\partial t}\delta_{ij} + 2\,\mu^v \frac{\partial E_{ij}}{\partial t}\right] n_j = f_i & \forall \boldsymbol{x} \in \partial\mathcal{B}_f \\ \frac{\partial \theta(\boldsymbol{x},t)}{\partial x_i}\, n_i = \phi(\boldsymbol{x},t) & \forall \boldsymbol{x} \in \partial\mathcal{B}_\phi \\ -\theta = \Theta^{\text{ext}} & \forall \boldsymbol{x} \in \partial\mathcal{B}_\theta \end{cases} \tag{60}$$

and initial conditions

$$\begin{cases} u_i(\boldsymbol{x},0) = u_i^0(\boldsymbol{x}) \\ \frac{\partial u_i}{\partial t}(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}) \\ \theta(\boldsymbol{x},0) = \Theta^0(\boldsymbol{x}) \\ \frac{\partial \theta}{\partial t}(\boldsymbol{x},0) = z^0(\boldsymbol{x}) \end{cases} \tag{61}$$

updated with the newly adopted variables to ensure the uniqueness of the solution. Since we do not have any more access to the variable $\boldsymbol{s}$, we remark that small differences naturally occur in the formulation

Table 1 Summary table of the two formulations in terms of entropy displacement $\boldsymbol{s}$ and increment of temperature θ. The operator 'tr' stands for the trace of a tensor

Variables	Equations	Expressions assigned in BCs
$\boldsymbol{u}, \boldsymbol{s}$	$\varrho\ddot{\boldsymbol{u}} - \nabla\cdot(\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{s}) + \boldsymbol{\sigma}^{v}) = \boldsymbol{b}$	$\boldsymbol{u}, (\boldsymbol{\sigma} + \boldsymbol{\sigma}^{v})\,\boldsymbol{n}$
	$-\nabla(T_r/c\,\nabla\cdot\boldsymbol{s} + \beta\,\mathrm{tr}\boldsymbol{E}) + T_r\lambda(\dot{\boldsymbol{s}} + \tau_0\ddot{\boldsymbol{s}}) = \boldsymbol{\vartheta}$	$\boldsymbol{s}, \theta$
$\boldsymbol{u}, \theta$	$\varrho\ddot{\boldsymbol{u}} - \nabla\cdot(\boldsymbol{\sigma}(\boldsymbol{u},\theta) + \boldsymbol{\sigma}^{v}) = \boldsymbol{b}$	$\boldsymbol{u}, (\boldsymbol{\sigma} + \boldsymbol{\sigma}^{v})\,\boldsymbol{n}$
	$\nabla\cdot(\kappa\nabla\theta) - c\,(\dot{\theta} + \tau_0\ddot{\theta}) - T_r\beta\,(\mathrm{tr}\dot{\boldsymbol{E}} + \tau_0\mathrm{tr}\ddot{\boldsymbol{E}}) = \nabla\cdot(\kappa\boldsymbol{\vartheta})$	$\theta, \nabla\theta\cdot\boldsymbol{n}$

in terms of θ instead of $\boldsymbol{s}$ as it appears from the third condition of (60), which is a standard Neumann condition for θ, while the third condition of (54), which is a standard Dirichlet condition for $\boldsymbol{s}$. Besides, the last two equations of (61) are due to the second order time-differentiation of the thermal diffusion Eq. (59) and are connected with $(55)_{3,4}$ by divergence operation. These two approaches are very similar, but slight differences can still be found. We remark that the boundary condition $(54)_3$ is vectorial in nature and, therefore, gives us the possibility to provide more information about the system on the boundary, while the $(60)_3$ is of scalar type. For the same reasons, the initial conditions $(55)_{3,4}$ appear more comprehensive than the standard $(61)_{3,4}$ being scalars.

Finally, to summarize the two possibilities that come from the proposed formulation, in Table 1, we give a synoptic picture of the main variables, the governing equations, and the expressions involved in the boundary conditions for the sake of comparing the novel formulation in terms of entropy displacement and the one, already available, in terms of temperature increment.

4 Conclusion

In the paper, we discuss with a variational formulation the development of a mathematical model that combines mechanical and thermal aspects to study how a three-dimensional continuum medium responds to external forces and temperature changes. The model extends existing theories, incorporates concepts like thermal inertia associated with Cattaneo's law for heat conduction, and ultimately leads to the derivation of a specific version of the Lord–Shulman model. Our starting point is Biot's theory, which explains the behavior of classical thermoelastic linear materials; namely, heat conduction is based on Fourier's law. Biot's approach uses a Lagrangian variational perspective to take into account both the mechanical and thermal behavior of the material and simultaneously the irreversibility of the chosen processes. Herein, we extend the model of Biot, including the effects of thermal inertia, which means considering that the heat conduction is based on Cattaneo's law instead. We remark that Cattaneo's law is straightforwardly derived as a consequence of the hypotheses assumed from the adopted variational principle in terms of a generalization of the form of the kinetic energy. With this improvement, we are able to obtain a description involving a finite propagation speed for heat. In this variational framework, therefore, we deduce the Lord–Shulman model for the general case of thermoelastic anisotropic materials. The isotropic case is also discussed in detail, together with the boundary and initial conditions needed to complete the differential problem. The primary contribution of the paper lies in its use of the variational formulation. Our aim is to showcase that the well-known Lord-Shulman model can be derived directly from this new approach, thus allowing for the straightforward generalization of the thermo-mechanical problem with the most advanced models that are otherwise challenging to obtain using the standard approach. This new approach eliminates the need for a plethora of potentially incoherent hypotheses, which must be justified a posteriori, thus providing a more efficient way of addressing this problem. Additionally, it can be argued that the variational viewpoint presents a better physical ground compared to the conventional approach. It is noteworthy that Cattaneo's correction can be interpreted as an inertial contribution here, as it appears as a conservative term in the Hamiltonian formulation rather than a dissipative phenomenon simply because it is related to the rate of the heat flux.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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