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Analyzing the angular acceleration vector of a moving rigid body

J. Jesús Cervantes-Sánchez · José M. Rico-Martínez · Mario A. García-Murillo · Marco A. Gómez-Álvarez

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Abstract This paper presents a novel and systematic approach for obtaining the angular acceleration vector of a moving rigid body. The novelty of the proposed method lies in the particular form of writing the pose of the moving rigid body, as well as in the procedure to compute its time derivatives. The derivation process goes directly to the very foundations of rotational motion and exploits the phenomenological connection between orientation, angular velocity, angular acceleration, and spatial motion of a rigid body. Hence, as a remarkable result, a symbolic expression for the angular acceleration vector arises naturally without the need to solve the inverse acceleration problem. The novel and general expression of the angular acceleration vector involves relationships between the position, velocity, and acceleration vectors of three non-collinear points of the body, which can be easily understood and physically interpreted without particular knowledge of specialized techniques or advanced mathematical tools. Due to its vector nature, the expression for the angular acceleration vector proposed in this paper is relatively simple, as well as, it is very robust against computational singularities. Two fully detailed case studies demonstrate the robustness of the proposed angular acceleration vector compared with other expressions appearing in the literature.

Keywords Angular acceleration vector · Spatial motion · Rigid body · Non-collinear points

1 Introduction

There is a phenomenological connection between orientation, angular velocity, and angular acceleration, which are strongly associated with the spatial motion of a rigid body. The formal study of rigid body rotations may be traced to 1775 when Euler published his seminal work [1], which was rediscovered independently by Rodrigues [2] in 1840. Since then, any number of discoveries [3–6] have flowed continuously from one author to the next, all of them adding something new to the results obtained by their predecessors.

A systematic formulation of the dynamic model of a mechanical system is needed to predict and understand its behavior. The effectiveness in formulating the equations of motion depends primarily on the ability to construct simple and correct mathematical

J. J. Cervantes-Sánchez (✉) · J. M. Rico-Martínez · M. A. García-Murillo · M. A. Gómez-Álvarez
Departamento de Ingeniería Mecánica, Universidad de Guanajuato, DICIS, 36885 Salamanca, Guanajuato, Mexico
e-mail: jecer@ugto.mx

J. M. Rico-Martínez
e-mail: jrico@ugto.mx

M. A. García-Murillo
e-mail: garcia.mario@ugto.mx

M. A. Gómez-Álvarez
e-mail: marco.gomez@ugto.mx

expressions for kinematic quantities such as angular velocities and accelerations of rigid bodies, as well as, velocities and accelerations of points of moving rigid bodies [7]. For example, Euler's second law requires a proper formulation of the angular velocity vector and angular acceleration vector of each rigid body composing a spatial mechanical system moving in a three-dimensional space [8]. One difficulty with the study of the angular velocity vector, and the angular acceleration vector, is that it becomes increasingly difficult to formulate as the complexity of the motion of the rigid body increases, such as occurs with the links of spatial parallel manipulators [9], flight simulators [10], and complex machines, e.g., the turbula machine [11], or the human head [12].

A fundamental problem in rigid body kinematics is the *inverse acceleration problem* [13]. This is a very challenging problem for spatial motion that consists of the determination of the angular acceleration vector in terms of the position, velocity, and acceleration vectors of three non-collinear points of a moving rigid body. As far as we know, very few attempts have been reported to solve this problem. Among these investigations, regarding vector-based approaches, it is fair to highlight the contributions of Condurache and Matcovschi [13], Field and Ziwet [14], Soutas-Little and Inman [15], and Wittenburg [16, 17]. On the other hand, Angeles [18–20], and Condurache and Matcovschi [21] address the same problem, but with a matrix-based approach.

All the derivations [13–21] for the angular acceleration vector have much in common and, to a greater or lesser extent, all of them follow a general pattern: (a) Start from classical and well-known equations related to the acceleration state of a rigid body, (b) It is required to solve the inverse acceleration problem, (c) They do not provide further details about the intrinsic nature of the angular acceleration, (d) Involve the angular velocity, which, in turn, comes from the velocity state of a moving rigid body, (e) The denominators of the resulting expressions are prone to computational singularities, and (f) Suffer from a rather heavy computational burden associated with all the required matrix computations, which significantly obscures the geometrical nature of the angular acceleration vector. Hence the motivation to devise an alternative and more comprehensive approach that overcomes the shortcomings (a)-(f) mentioned above. Furthermore, the objective is to exploit the fact that angular acceleration is a kinematic property related to the acceleration state of a moving rigid body, which arises naturally when a proper description of the spatial motion of a rigid body is carried out.

2 Description of the spatial motion of a moving rigid body

To start with, it is reasonable to think that to obtain a good mathematical model of the angular acceleration, one must first make a careful description and analysis of both, the orientation and the angular velocity of the body. On the other hand, a moving rigid body may be translating and rotating simultaneously in a general spatial motion. Hence, it is important to correlate the angular motion of the body with the translational motion of any point of the moving body. In this regard, we take some ideas from previous investigations [22], which deal with the angular velocity of a rigid body in motion.

2.1 The pose of a moving rigid body

Consider the rigid body shown in Fig. 1, which may be moving in any manner with respect to a fixed reference frame *XYZ*. The body has three arbitrary and non-collinear points, namely, points 1, 2, and 3, whose location with respect to the origin *O* of the fixed frame *XYZ* is given by position vectors, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, respectively.

The *pose* of the rigid body shown in Fig. 1 can be described in terms of the *location* and *orientation* of the moving frame *UVW*, fixed to the body, with respect to the fixed frame *XYZ*. On one hand, the location of the origin of the moving frame *UVW* may be defined by the position vector of point 1, namely, $\mathbf{p}_1$. On the other hand, the orientation of the body is completely determined once the set of coordinate axes *UVW* has been oriented relative to the fixed reference frame *XYZ*. Both requirements may be stated in terms of the position vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, of the three non-collinear points 1, 2, and 3, respectively, which are shown in Fig. 1. To this end, we define the following unit vectors:

$$\mathbf{u} \equiv \frac{\mathbf{p}_2 - \mathbf{p}_1}{\sqrt{(\mathbf{p}_2 - \mathbf{p}_1) \cdot (\mathbf{p}_2 - \mathbf{p}_1)}} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{L_{12}} \tag{1}$$

$$\mathbf{m} \equiv \frac{\mathbf{p}_3 - \mathbf{p}_1}{\sqrt{(\mathbf{p}_3 - \mathbf{p}_1) \cdot (\mathbf{p}_3 - \mathbf{p}_1)}} = \frac{\mathbf{p}_3 - \mathbf{p}_1}{L_{13}} \tag{2}$$

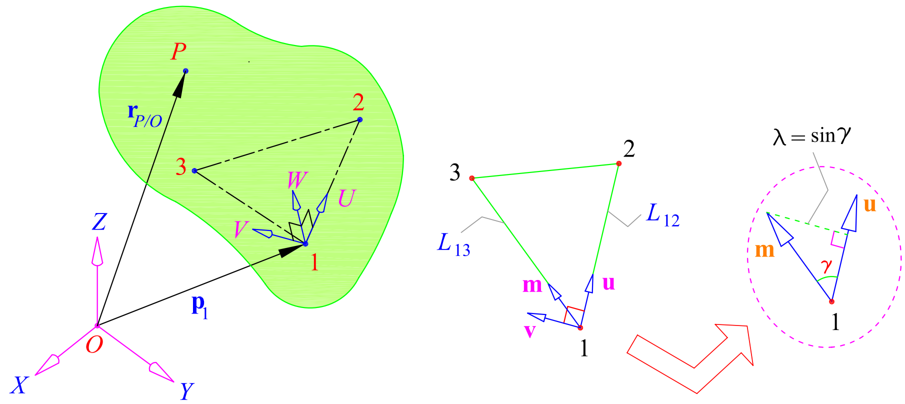

Fig. 1 Fixed (*XYZ*) and moving (*UVW*) frames used for describing the pose of a rigid body

$$\mathbf{v} \equiv \frac{\mathbf{m} - (\mathbf{m}\cdot\mathbf{u})\mathbf{u}}{\lambda} = k_1\,\mathbf{m} - k_2\,\mathbf{u}, \quad k_1 \equiv \frac{1}{\lambda}, \quad k_2 \equiv \frac{\mathbf{m}\cdot\mathbf{u}}{\lambda}. \tag{3}$$

$$\mathbf{w} \equiv \mathbf{u} \times \mathbf{v} \tag{4}$$

where the scalar parameter:

$$\lambda \equiv \sqrt{\{\mathbf{m} - (\mathbf{m}\cdot\mathbf{u})\mathbf{u}\}\cdot\{\mathbf{m} - (\mathbf{m}\cdot\mathbf{u})\mathbf{u}\}} \tag{5}$$

is graphically depicted in Fig. 1.

It is important to remark that unit vectors **u**, **v**, and **w** are directed along the axes *U*, *V*, and *W*, respectively, and they are used to describe the relative orientation between frames *UVW* and *XYZ*. Furthermore, unit vectors **u**, and **m** are directly related to the the position vectors, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, whereas unit vectors **v**, and **w** can be computed in terms of unit vectors **u**, and **m**.

We may interpret the pose of a body as a way to know about the position of all the points of the body in the space. In this regard, consider a typical point *P*, fixed in the body, which is shown in Figs. 1 and 2. Thus, one may state that the position of point *P*, as seen from *XYZ*, equals the position of point 1, as seen from *XYZ*, plus the position of point *P* relative to point 1, that is:

$$\mathbf{r}_{P/O} = \mathbf{p}_1 + \mathbf{r} \tag{6}$$

where position vector **r** is used to represent the position of an arbitrary point *P* of the moving body with respect to the origin of the moving frame *UVW*. Thus, position vector **r** maintains a constant magnitude and orientation in the *UVW* frame, that is, its coordinates *u*, *v*, and *w* remain fixed even if the body rotates. However, its Cartesian coordinates (measured in the fixed frame *XYZ*), namely, *x*, *y*, and *z*, are continuously changing, whereas rotational motion occurs. In this way, the position vector **r** can be expressed with respect to the moving frame as follows:

$$\mathbf{r} = u\,\mathbf{u} + v\,\mathbf{v} + w\,\mathbf{w} \tag{7}$$

Finally, substitution of Eq. (7) into Eq. (6) yields the following result:

$$\mathbf{r}_{P/O} = \mathbf{p}_1 + u\,\mathbf{u} + v\,\mathbf{v} + w\,\mathbf{w}. \tag{8}$$

which is a vector equation[1] related to an arbitrary pose of the moving rigid body depicted in Figs. 1, and 2. This equation clearly shows that position vector $\mathbf{p}_1$, and unit vectors **u**, **v**, and **w** can be used to locate all the points of the body in space. Moreover, the Eq. (8) is the key equation to correlate the angular motion of the body, which is represented by rotating unit vectors **u**, **v**, and **w**, with the translational motion of any point of the moving body. Furthermore, as will be seen later, this equation will lead to a clear, simple, and systematic way to obtain the velocity and acceleration state of the rigid body in motion.

2.2 The first time derivative of body pose

Since the body-fixed *UVW* frame translates and rotates relative to the *XYZ* frame, the position vector $\mathbf{p}_1$, as well as unit vectors **u**, **v** and **w** will change

[1] It should be noted that Eq. (8) can be equivalently written as $\mathbf{r}_{P/O} = \mathbf{p}_1 + \mathbf{R}\,\mathbf{r}$, where the (3×3) matrix $\mathbf{R} \equiv [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ describes the orientation of the rigid body with respect to reference frame *XYZ*, and $\mathbf{r} \equiv (u, v, w)^T$.

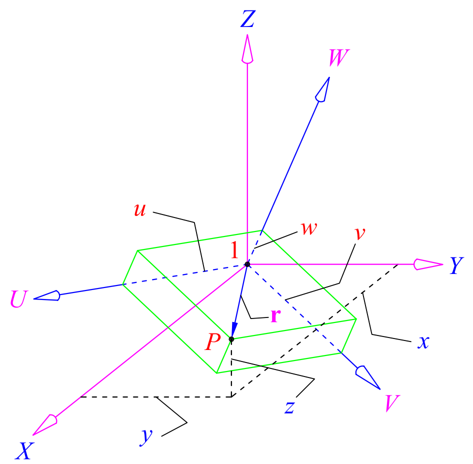

Fig. 2 Orientation of moving frame *UVW* with respect to fixed frame *XYZ*

through time, and they may be considered as functions of time, denoted by the symbol t. Therefore, this section starts by taking the first time derivative of the vector Eq. (6), which is given by:

$$\left(\frac{d\,\mathbf{r}_{P/O}}{d\,t}\right)_{XYZ} = \left(\frac{d\,\mathbf{p}_1}{d\,t}\right)_{XYZ} + \left(\frac{d\,\mathbf{r}}{d\,t}\right)_{XYZ} \tag{9}$$

where $(d/dt)_{XYZ}$ is used to denote the time derivative as seen from *XYZ*.

On the one hand, the first two terms of Eq. (9) have a direct physical interpretation. The time rate change of $\mathbf{r}_{P/O}$, as seen from *XYZ*, represents the velocity vector of point P with respect to fixed point O, namely, $\mathbf{v}_{P/O}$, whereas the time rate change of $\mathbf{p}_1$ with respect to fixed point O, as seen from *XYZ*, represents the velocity vector of point 1, namely, $\mathbf{v}_{1/O}$. Thus, we can express the foregoing statements as follows:

$$\mathbf{v}_{P/O} \equiv \left(\frac{d\,\mathbf{r}_{P/O}}{d\,t}\right)_{XYZ}, \quad \mathbf{v}_{1/O} \equiv \left(\frac{d\,\mathbf{p}_1}{d\,t}\right)_{XYZ}. \tag{10}$$

On the other hand, the second term appearing on the left-hand side of the Eq. (9) deserves special treatment. To this end, and recalling Eq. (7), we have that:

$$\left(\frac{d\,\mathbf{r}}{d\,t}\right)_{XYZ} = u\,\dot{\mathbf{u}} + v\,\dot{\mathbf{v}} + w\,\dot{\mathbf{w}} \tag{11}$$

since the components u, v and w do not change through the time. Moreover, when it is clear from the discussion what frame is involved for a time derivative, we will use a dot over a parameter to indicate a time derivative of that parameter, e.g., $d\,a/d\,t \equiv \dot{a}$.

Next, by projecting each time derivative of the involved unit vectors onto the axes of the *UVW* frame, we may write:

$$\begin{aligned}
\dot{\mathbf{u}} &= (\dot{\mathbf{u}}\cdot\mathbf{u})\mathbf{u} + (\dot{\mathbf{u}}\cdot\mathbf{v})\mathbf{v} + (\dot{\mathbf{u}}\cdot\mathbf{w})\mathbf{w}\\
\dot{\mathbf{v}} &= (\dot{\mathbf{v}}\cdot\mathbf{u})\,\mathbf{u} + (\dot{\mathbf{v}}\cdot\mathbf{v})\,\mathbf{v} + (\dot{\mathbf{v}}\cdot\mathbf{w})\,\mathbf{w}\\
\dot{\mathbf{w}} &= (\dot{\mathbf{w}}\cdot\mathbf{u})\mathbf{u} + (\dot{\mathbf{w}}\cdot\mathbf{v})\mathbf{v} + (\dot{\mathbf{w}}\cdot\mathbf{w})\mathbf{w}
\end{aligned} \tag{12}$$

Furthermore, unit vectors must satisfy the following relationships:

$$\mathbf{u}\cdot\mathbf{u} = 1, \quad \mathbf{v}\cdot\mathbf{v} = 1, \quad \mathbf{w}\cdot\mathbf{w} = 1. \tag{13}$$

as well as:

$$\mathbf{u}\cdot\mathbf{v} = 0, \quad \mathbf{u}\cdot\mathbf{w} = 0, \quad \mathbf{v}\cdot\mathbf{w} = 0. \tag{14}$$

Then, time differentiation of Eqs. (13) and (14) leads to:

$$\dot{\mathbf{u}}\cdot\mathbf{u} = 0, \quad \dot{\mathbf{v}}\cdot\mathbf{v} = 0, \quad \dot{\mathbf{w}}\cdot\mathbf{w} = 0, \tag{15}$$

$$\dot{\mathbf{u}}\cdot\mathbf{v} = -\mathbf{u}\cdot\dot{\mathbf{v}}, \quad \dot{\mathbf{u}}\cdot\mathbf{w} = -\mathbf{u}\cdot\dot{\mathbf{w}}, \quad \dot{\mathbf{v}}\cdot\mathbf{w} = -\mathbf{v}\cdot\dot{\mathbf{w}}. \tag{16}$$

Thus, Eq. (12) become:

$$\begin{aligned}
\dot{\mathbf{u}} &= -(\mathbf{u}\cdot\dot{\mathbf{v}})\mathbf{v} + (\dot{\mathbf{u}}\cdot\mathbf{w})\mathbf{w}\\
\dot{\mathbf{v}} &= +(\dot{\mathbf{v}}\cdot\mathbf{u})\,\mathbf{u} - (\mathbf{v}\cdot\dot{\mathbf{w}})\,\mathbf{w}\\
\dot{\mathbf{w}} &= -(\dot{\mathbf{u}}\cdot\mathbf{w})\mathbf{u} + (\mathbf{v}\cdot\dot{\mathbf{w}})\mathbf{v}
\end{aligned} \tag{17}$$

Substituting Eq. (17) into Eq. (11) we find that:

$$\begin{aligned}
\dot{\mathbf{r}} = \{&v(\dot{\mathbf{v}}\cdot\mathbf{u}) - w(\dot{\mathbf{u}}\cdot\mathbf{w})\}\mathbf{u} + \{w(\mathbf{v}\cdot\dot{\mathbf{w}}) - u(\mathbf{u}\cdot\dot{\mathbf{v}})\}\mathbf{v}\\
&+\{u(\dot{\mathbf{u}}\cdot\mathbf{w}) - v(\mathbf{v}\cdot\dot{\mathbf{w}})\}\mathbf{w}
\end{aligned} \tag{18}$$

By resorting to the definition of the cross product between two vectors, one may notice that Eq. (18) can be written as follows:

$$\dot{\mathbf{r}} = -\{(\mathbf{v}\cdot\dot{\mathbf{w}})\mathbf{u} + (\dot{\mathbf{u}}\cdot\mathbf{w})\mathbf{v} + (\mathbf{u}\cdot\dot{\mathbf{v}})\mathbf{w}\} \times \{u\,\mathbf{u} + v\,\mathbf{v} + w\,\mathbf{w}\} \tag{19}$$

Thus, from Eqs. (7) and (16), Eq. (19) becomes:

$$\dot{\mathbf{r}} = \{(\dot{\mathbf{v}}\cdot\mathbf{w})\mathbf{u} - (\dot{\mathbf{u}}\cdot\mathbf{w})\mathbf{v} + (\dot{\mathbf{u}}\cdot\mathbf{v})\mathbf{w}\} \times \mathbf{r} \tag{20}$$

where first time derivatives of unit vectors, $\dot{\mathbf{u}}$, $\dot{\mathbf{v}}$ and $\dot{\mathbf{w}}$, can be computed by taking the first time derivative of Eq. (1)–(4), respectively, thus yielding:

$$\dot{\mathbf{u}} = \frac{\dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_1}{L_{12}} \tag{21}$$

$$\dot{\mathbf{m}} = \frac{\dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_1}{L_{13}} \tag{22}$$

$$\dot{\mathbf{v}} = k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}} \tag{23}$$

$$\dot{\mathbf{w}} = \dot{\mathbf{u}} \times \mathbf{v} + \mathbf{u} \times \dot{\mathbf{v}} \tag{24}$$

where it is important to remark that unit vectors $\dot{\mathbf{u}}$, and $\dot{\mathbf{m}}$ are directly related to the velocity vectors of points 1, 2, and 3, namely, $\dot{\mathbf{p}}_1$, $\dot{\mathbf{p}}_2$, and $\dot{\mathbf{p}}_3$, whereas vectors $\dot{\mathbf{v}}$, and $\dot{\mathbf{w}}$ can be computed in terms of vectors $\dot{\mathbf{u}}$, and $\dot{\mathbf{m}}$.

Finally, from Eqs. (10), and (20), Eq. (9) becomes:

$$\mathbf{v}_{P/O} = \mathbf{v}_{1/O} + \{(\dot{\mathbf{v}}\cdot\mathbf{w})\,\mathbf{u} - (\dot{\mathbf{u}}\cdot\mathbf{w})\,\mathbf{v} + (\dot{\mathbf{u}}\cdot\mathbf{v})\,\mathbf{w}\} \times \mathbf{r}. \tag{25}$$

which may be considered as the vector equation representing the first-time derivative of the body pose illustrated in Fig. 1. This equation is closely related to the *velocity state* [23] of the rigid body since it provides enough information to find the velocity of any point of the moving rigid body.

2.3 The second time derivative of body pose

Both sides of the vector Eq. (25) may be differentiated with respect to time to obtain the acceleration equation, which is given by:

$$\mathbf{a}_{P/O} = \mathbf{a}_{1/O} + \mathbf{a}_{P/1} \tag{26}$$

where

$$\mathbf{a}_{P/O} \equiv \left(\frac{d\,\mathbf{v}_{P/O}}{d\,t}\right)_{XYZ}, \quad \mathbf{a}_{1/O} \equiv \left(\frac{d\,\mathbf{v}_{1/O}}{d\,t}\right)_{XYZ}. \tag{27}$$

represents the acceleration vector of point P with respect to fixed point O, and the acceleration vector of point 1 with respect to fixed point O, respectively, whereas the second term of the left-hand side of Eq. (26) deserves special treatment, that is:

$$\mathbf{a}_{P/1} \equiv \left(\frac{d\,\dot{\mathbf{r}}}{d\,t}\right)_{XYZ} = \dot{\mathbf{n}} \times \mathbf{r} + \mathbf{n} \times \dot{\mathbf{r}}. \tag{28}$$

where, we have that:

$$\mathbf{n} \equiv (\dot{\mathbf{v}}\cdot\mathbf{w})\mathbf{u} - (\dot{\mathbf{u}}\cdot\mathbf{w})\mathbf{v} + (\dot{\mathbf{u}}\cdot\mathbf{v})\mathbf{w}. \tag{29}$$

$$\begin{aligned}\dot{\mathbf{n}} &= (\ddot{\mathbf{v}}\cdot\mathbf{w})\mathbf{u} + (\dot{\mathbf{v}}\cdot\dot{\mathbf{w}})\mathbf{u} + (\dot{\mathbf{v}}\cdot\mathbf{w})\dot{\mathbf{u}} - (\ddot{\mathbf{u}}\cdot\mathbf{w})\mathbf{v} - (\dot{\mathbf{u}}\cdot\dot{\mathbf{w}})\mathbf{v} - (\dot{\mathbf{u}}\cdot\mathbf{w})\dot{\mathbf{v}} \\ &\quad + (\ddot{\mathbf{u}}\cdot\mathbf{v})\mathbf{w} + (\dot{\mathbf{u}}\cdot\dot{\mathbf{v}})\mathbf{w} + (\dot{\mathbf{u}}\cdot\mathbf{v})\dot{\mathbf{w}}. \\ \dot{\mathbf{n}} &\equiv \dot{\mathbf{n}}_1 + \dot{\mathbf{n}}_2 + \dot{\mathbf{n}}_3 - \dot{\mathbf{n}}_4 - \dot{\mathbf{n}}_5 - \dot{\mathbf{n}}_6 + \dot{\mathbf{n}}_7 + \dot{\mathbf{n}}_8 + \dot{\mathbf{n}}_9.\end{aligned} \tag{30}$$

and the second time derivative of unit vectors, namely, $\ddot{\mathbf{u}}$, $\ddot{\mathbf{v}}$, and $\ddot{\mathbf{w}}$, can be computed by taking the first time derivative of Eqs. (21)–(24), respectively, thus yielding:

$$\ddot{\mathbf{u}} = \frac{\ddot{\mathbf{p}}_2 - \ddot{\mathbf{p}}_1}{L_{12}} \tag{31}$$

$$\ddot{\mathbf{m}} = \frac{\ddot{\mathbf{p}}_3 - \ddot{\mathbf{p}}_1}{L_{13}} \tag{32}$$

$$\ddot{\mathbf{v}} = k_1\ddot{\mathbf{m}} - k_2\ddot{\mathbf{u}} \tag{33}$$

$$\ddot{\mathbf{w}} = \ddot{\mathbf{u}} \times \mathbf{v} + 2\,\dot{\mathbf{u}} \times \dot{\mathbf{v}} + \mathbf{u} \times \ddot{\mathbf{v}}. \tag{34}$$

where it is important to remark that unit vectors $\ddot{\mathbf{u}}$, and $\ddot{\mathbf{m}}$ are directly related to the acceleration vectors of points 1, 2, and 3, namely, $\ddot{\mathbf{p}}_1$, $\ddot{\mathbf{p}}_2$, and $\ddot{\mathbf{p}}_3$, whereas vectors $\ddot{\mathbf{v}}$, and $\ddot{\mathbf{w}}$ can be computed in terms of vectors $\ddot{\mathbf{u}}$, and $\ddot{\mathbf{m}}$.

Equation (30) involves unit vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, as well as, their first, and second time derivatives, and it is composed of nine vector terms denoted as $\dot{\mathbf{n}}_1, \dot{\mathbf{n}}_2, \cdots \dot{\mathbf{n}}_9$. In order to improve the readability of the article, a detailed computation of each term is presented in Appendix 1. As shown in that appendix, the general idea is to include only $\mathbf{u}$, $\dot{\mathbf{u}}$, $\ddot{\mathbf{u}}$, $\mathbf{m}$, $\dot{\mathbf{m}}$, and $\ddot{\mathbf{m}}$, since these vectors are directly related to the position, the velocity, and the acceleration of the three non-collinear points of the moving rigid body under analysis.

3 The angular acceleration vector

We are now in a position to derive the expression for the angular acceleration vector. To this end, the general Eq. (26) is combined with Eqs. (20), (28), and (29), thus leading to:

$$\mathbf{a}_{P/O} = \mathbf{a}_{1/O} + \dot{\mathbf{n}} \times \mathbf{r} + \mathbf{n} \times (\mathbf{n} \times \mathbf{r}). \tag{35}$$

which is an equation closely related to the *acceleration state* [23] of the moving rigid body since it provides enough information to find the acceleration of any point of the moving rigid body.

A dimensional analysis reveals that Eq. (35) has the dimensions of acceleration, and vector $\dot{\mathbf{n}}$ has the same units as the angular acceleration. Moreover, vector $\dot{\mathbf{n}}$ is indeed the angular acceleration vector, whereas vector $\mathbf{n}$ is the angular velocity vector. Furthermore, the angular acceleration vector is usually denoted by the bold Greek symbol $\boldsymbol{\alpha}$. Hence, from this point, $\dot{\mathbf{n}} \equiv \boldsymbol{\alpha}$. Thus, a careful collection of all the nine terms shown in Eq. (30) yields the following result:

$$\boldsymbol{\alpha} = k_1^2 \{\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_4 + \boldsymbol{\kappa}_5 + \boldsymbol{\kappa}_6 + \boldsymbol{\kappa}_7\} - k_1 k_2 \{\boldsymbol{\kappa}_8 + \boldsymbol{\kappa}_9\}. \tag{36}$$

where we have the following scalar parameters:

$$k_1 \equiv \left(\frac{1}{\lambda}\right), \quad k_2 \equiv \left(\frac{\mathbf{m}\cdot\mathbf{u}}{\lambda}\right), \tag{37}$$
$$\lambda \equiv \sqrt{\{\mathbf{m} - (\mathbf{m}\cdot\mathbf{u})\mathbf{u}\}\cdot\{\mathbf{m} - (\mathbf{m}\cdot\mathbf{u})\mathbf{u}\}}.$$

as well as the following vectors:

$$\begin{aligned}
\boldsymbol{\kappa}_1 &\equiv (\ddot{\mathbf{m}}\times\mathbf{u})\times(\mathbf{u}\times\mathbf{m})\\
\boldsymbol{\kappa}_2 &\equiv \ddot{\mathbf{u}}\times\{(\mathbf{u}\times\mathbf{m})\times\mathbf{m}\}\\
\boldsymbol{\kappa}_3 &\equiv \{\dot{\mathbf{m}}\cdot(\dot{\mathbf{u}}\times\mathbf{m})\}\mathbf{u}\\
\boldsymbol{\kappa}_4 &\equiv \{\dot{\mathbf{m}}\cdot(\mathbf{u}\times\mathbf{m})\}\dot{\mathbf{u}}\\
\boldsymbol{\kappa}_5 &\equiv \dot{\mathbf{u}}\times\left\{(\mathbf{u}\times\mathbf{m})\times\dot{\mathbf{m}}\right\}\\
\boldsymbol{\kappa}_6 &\equiv \dot{\mathbf{u}}\times\left\{(\dot{\mathbf{u}}\times\mathbf{m})\times\mathbf{m}\right\}\\
\boldsymbol{\kappa}_7 &\equiv \dot{\mathbf{u}}\times\left\{(\mathbf{u}\times\dot{\mathbf{m}})\times\mathbf{m}\right\}\\
\boldsymbol{\kappa}_8 &\equiv (\ddot{\mathbf{u}}\cdot\mathbf{u})(\mathbf{u}\times\mathbf{m})\\
\boldsymbol{\kappa}_9 &\equiv (\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})(\mathbf{u}\times\mathbf{m})
\end{aligned} \tag{38}$$

whose detailed derivations are presented in Appendix 1.

It is important to remark that Eq. (36) is a novel and general expression for computing the angular acceleration vector of a rigid body moving in space. As far as we know, this equation and its detailed derivation have not been reported previously in the literature. This equation has remarkable features, such as:

(a) This is a result that arose naturally, without the need to solve the inverse acceleration problem.
(b) It involves only the position, velocity, and acceleration of three non-collinear points of a moving rigid body.
(c) It does not require the computation of the angular velocity vector of the moving body. All the expressions for the angular acceleration reported in [14, 15, 17, 20], require a previous computation of the corresponding angular velocity vector of the moving body.
(d) Its denominator is given by a simple scalar parameter, namely, λ^2, see Eqs. (3), and (5). Moreover, it can be proved that $\lambda = \sin\gamma$, where γ is the angle formed by unit vectors $\mathbf{u}$, and $\mathbf{m}$, see Fig. 1. Hence, the parameter λ only vanishes when the three given points, 1, 2, and 3, are collinear, i.e., $\gamma = 0°$, or $\gamma = 180°$, which is not the general case treated in this paper. Therefore, it is very robust against computational singularities.
(e) It has a simple mathematical structure since only basic principles of vector calculus were used in its formulation.

4 Representations of the angular acceleration vector

In reviewing the literature we found five typical representations for the angular acceleration vector, which are different from the one given by the Eq. (36). All those representations are required to solve the inverse acceleration problem in rigid body kinematics [13], and they are briefly presented next for completeness purposes.

4.1 First representation of the angular acceleration vector

A first representation of the angular acceleration vector is due to Field, and Ziwet [14], which is given by:

$$\begin{aligned}
\boldsymbol{\alpha}_{FW} =& \left(\frac{\boldsymbol{\omega}_{FW}\cdot\ddot{\mathbf{q}}}{\boldsymbol{\omega}_{FW}\cdot(\mathbf{p}\times\mathbf{q})}\right)\mathbf{p} - \left(\frac{\boldsymbol{\omega}_{FW}\cdot\ddot{\mathbf{p}}}{\boldsymbol{\omega}_{FW}\cdot(\mathbf{p}\times\mathbf{q})}\right)\mathbf{q}\\
&+ \left(\frac{(\boldsymbol{\omega}_{FW}\cdot\mathbf{p})(\boldsymbol{\omega}_{FW}\cdot\mathbf{q})}{\boldsymbol{\omega}_{FW}\cdot(\mathbf{p}\times\mathbf{q})}\right)\boldsymbol{\omega}_{FW} -\\
&-\left(\frac{(\boldsymbol{\omega}_{FW}\cdot\boldsymbol{\omega}_{FW})(\mathbf{p}\cdot\mathbf{q})}{\boldsymbol{\omega}_{FW}\cdot(\mathbf{p}\times\mathbf{q})}\right)\boldsymbol{\omega}_{FW} - \left(\frac{\mathbf{p}\cdot\ddot{\mathbf{q}}}{\boldsymbol{\omega}_{FW}\cdot(\mathbf{p}\times\mathbf{q})}\right)\boldsymbol{\omega}_{FW}.\\
\boldsymbol{\alpha}_{FW} \equiv& \ \mathbf{s}_1 - \mathbf{s}_2 + \mathbf{s}_3 - \mathbf{s}_4 - \mathbf{s}_5.
\end{aligned} \tag{39}$$

where

$$\boldsymbol{\omega}_{FW} = \frac{\dot{\mathbf{p}} \times \dot{\mathbf{q}}}{\dot{\mathbf{p}} \cdot \mathbf{q}} \tag{40}$$

is the angular velocity vector of the moving rigid body, and $\mathbf{p} \equiv \mathbf{p}_2 - \mathbf{p}_1$, $\mathbf{q} \equiv \mathbf{p}_3 - \mathbf{p}_1$, $\dot{\mathbf{p}} \equiv \dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_1$, $\dot{\mathbf{q}} \equiv \dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_1$, $\ddot{\mathbf{p}} \equiv \ddot{\mathbf{p}}_2 - \ddot{\mathbf{p}}_1$, and $\ddot{\mathbf{q}} \equiv \ddot{\mathbf{p}}_3 - \ddot{\mathbf{p}}_1$.

4.2 Second representation of the angular acceleration vector

A second representation for the angular acceleration vector has been proposed by Soutas-Little and Inman [15]. The corresponding formula is as follows:

$$\boldsymbol{\alpha}_{SLI} = \frac{\mathbf{q}_{B/A} \times \mathbf{q}_{C/A}}{\mathbf{q}_{B/A} \cdot \mathbf{r}_{C/A}}, \quad \text{for} \quad \mathbf{q}_{B/A} \cdot \mathbf{r}_{C/A} \neq 0. \tag{41}$$

where

$$\mathbf{q}_{B/A} \equiv \ddot{\mathbf{p}}_2 - \ddot{\mathbf{p}}_1 - \boldsymbol{\omega}_{SLI} \times \{\boldsymbol{\omega}_{SLI} \times (\mathbf{p}_2 - \mathbf{p}_1)\}. \tag{42}$$

$$\mathbf{q}_{C/A} \equiv \ddot{\mathbf{p}}_3 - \ddot{\mathbf{p}}_1 - \boldsymbol{\omega}_{SLI} \times \{\boldsymbol{\omega}_{SLI} \times (\mathbf{p}_3 - \mathbf{p}_1)\}. \tag{43}$$

$$\mathbf{r}_{C/A} \equiv \mathbf{p}_3 - \mathbf{p}_1. \tag{44}$$

and $\boldsymbol{\omega}_{SLI}$ denotes the angular velocity vector, which is given by:

$$\boldsymbol{\omega}_{SLI} = \frac{\mathbf{v}_{B/A} \times \mathbf{v}_{C/A}}{\mathbf{v}_{B/A} \cdot \mathbf{r}_{C/A}}, \quad \text{for} \quad \mathbf{v}_{B/A} \cdot \mathbf{r}_{C/A} \neq 0. \tag{45}$$

being:

$$\mathbf{v}_{B/A} \equiv \dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_1, \quad \mathbf{v}_{C/A} \equiv \dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_1. \tag{46}$$

4.3 Third representation of the angular acceleration vector

Professor Wittenburg proposed a third representation of the angular acceleration vector [17]. The corresponding formula is as follows:

$$\boldsymbol{\alpha}_{W1} = \frac{\mathbf{q}_{1/3} \times \mathbf{q}_{2/3}}{\mathbf{r}_{3/1} \cdot \mathbf{q}_{2/3}}, \quad \text{for} \quad \mathbf{r}_{3/1} \cdot \mathbf{q}_{2/3} \neq 0. \tag{47}$$

where

$$\mathbf{q}_{1/3} \equiv \ddot{\mathbf{p}}_1 - \ddot{\mathbf{p}}_3 - \boldsymbol{\omega}_W \times (\dot{\mathbf{p}}_1 - \dot{\mathbf{p}}_3). \tag{48}$$

$$\mathbf{q}_{2/3} \equiv \ddot{\mathbf{p}}_2 - \ddot{\mathbf{p}}_3 - \boldsymbol{\omega}_W \times (\dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_3). \tag{49}$$

$$\mathbf{r}_{3/1} \equiv \mathbf{p}_3 - \mathbf{p}_1. \tag{50}$$

However, when the denominator of Eq. (47) equals zero, in [17] it is proposed the following alternative formula:

$$\boldsymbol{\alpha}_{W2} = \mu_1(\mathbf{p}_1 - \mathbf{p}_2) + \mu_2(\mathbf{p}_2 - \mathbf{p}_3), \quad \text{for} \quad \mathbf{r}_{3/1} \cdot \mathbf{q}_{2/3} = 0. \tag{51}$$

where

$$\mu_1 \equiv \left(\frac{1}{\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}}\right) \{ \boldsymbol{\varepsilon} \cdot [\, \ddot{\mathbf{p}}_3 - \ddot{\mathbf{p}}_2 - \boldsymbol{\omega}_W \times (\dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_2) \,] \}. \tag{52}$$

$$\mu_2 \equiv \left(\frac{1}{\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}}\right) \{ \boldsymbol{\varepsilon} \cdot [\, \ddot{\mathbf{p}}_1 - \ddot{\mathbf{p}}_2 - \boldsymbol{\omega}_W \times (\dot{\mathbf{p}}_1 - \dot{\mathbf{p}}_2) \,] \}. \tag{53}$$

$$\boldsymbol{\varepsilon} \equiv (\mathbf{p}_1 - \mathbf{p}_2) \times (\mathbf{p}_3 - \mathbf{p}_2). \tag{54}$$

The corresponding angular velocity vector, namely, $\boldsymbol{\omega}_W$, is given by:

$$\boldsymbol{\omega}_{W1} = \frac{(\dot{\mathbf{p}}_1 - \dot{\mathbf{p}}_3) \times (\dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_3)}{(\mathbf{p}_3 - \mathbf{p}_1) \cdot (\dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_3)}, \quad \text{for} \quad (\mathbf{p}_3 - \mathbf{p}_1) \cdot (\dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_3) \neq 0, \tag{55}$$

$$\boldsymbol{\omega}_{W2} = \mu_3(\mathbf{p}_1 - \mathbf{p}_2) + \mu_4(\mathbf{p}_2 - \mathbf{p}_3), \quad \text{for} \quad (\mathbf{p}_3 - \mathbf{p}_1) \cdot (\dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_3) = 0. \tag{56}$$

where

$$\mu_3 \equiv \left(\frac{1}{\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}}\right) \{ \boldsymbol{\varepsilon} \cdot (\dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_2) \}, \quad \mu_4 \equiv \left(\frac{1}{\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}}\right) \{ \boldsymbol{\varepsilon} \cdot (\dot{\mathbf{p}}_1 - \dot{\mathbf{p}}_2) \}. \tag{57}$$

4.4 Fourth representation of the angular acceleration vector

A fourth representation of the angular acceleration vector was developed by Angeles [20], and the corresponding formulas are given by:

$$\boldsymbol{\alpha}_{A1} = \mathbf{D}^{-1} \operatorname{vec} (\ddot{\mathbf{P}} - \boldsymbol{\Omega}_1^2 \mathbf{P}), \quad \text{for} \quad \operatorname{tr}(\mathbf{P}) \neq 0. \tag{58}$$

where

$$\mathbf{D}^{-1} = \alpha_1\,\mathbf{I} - \beta\,\mathbf{P}^2, \quad \alpha_1 \equiv \frac{2}{\mathrm{tr}(\mathbf{P})}, \quad \beta \equiv \frac{4}{\mathrm{tr}(\mathbf{P})(\mathrm{tr}(\mathbf{P}^2) - \mathrm{tr}^2(\mathbf{P}))},$$
$$\mathbf{P} \equiv \left[\mathbf{p}_1 - \mathbf{c}\;\; \mathbf{p}_2 - \mathbf{c}\;\; \mathbf{p}_3 - \mathbf{c}\right],$$
$$\dot{\mathbf{P}} \equiv [\dot{p}_{ij}] = \left[\dot{\mathbf{p}}_1 - \dot{\mathbf{c}}\;\; \dot{\mathbf{p}}_2 - \dot{\mathbf{c}}\;\; \dot{\mathbf{p}}_3 - \dot{\mathbf{c}}\right], \quad \ddot{\mathbf{P}} \equiv \left[\ddot{\mathbf{p}}_1 - \ddot{\mathbf{c}}\;\; \ddot{\mathbf{p}}_2 - \ddot{\mathbf{c}}\;\; \ddot{\mathbf{p}}_3 - \ddot{\mathbf{c}}\right],$$
$$\mathbf{c} \equiv \frac{\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3}{3}, \quad \dot{\mathbf{c}} \equiv \frac{\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 + \dot{\mathbf{p}}_3}{3}, \quad \ddot{\mathbf{c}} \equiv \frac{\ddot{\mathbf{p}}_1 + \ddot{\mathbf{p}}_2 + \ddot{\mathbf{p}}_3}{3},$$
$$\mathrm{vec}(\dot{\mathbf{P}}) \equiv \frac{1}{2}\begin{bmatrix} \dot{p}_{32} - \dot{p}_{23} \\ \dot{p}_{13} - \dot{p}_{31} \\ \dot{p}_{21} - \dot{p}_{12} \end{bmatrix}, \quad \boldsymbol{\omega}_{A1} \equiv \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \mathbf{D}^{-1}\,\mathrm{vec}(\dot{\mathbf{P}}),$$
$$\boldsymbol{\Omega}_1 \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

and $\mathrm{tr}(\mathbf{P})$ denotes the trace of a (3×3) matrix $\mathbf{P}$, and $\mathbf{I}$ is the (3×3) identity matrix.

On the other hand, when the denominator of Eq. (58) equals zero, the author of [20] proposes an alternative formula:

$$\boldsymbol{\alpha}_{A2} = 2\,\mathbf{J}^{-1}\,\mathrm{vec}\,(\ddot{\mathbf{P}}\mathbf{P}^T - \boldsymbol{\Omega}_2^2\mathbf{R}), \quad \text{for} \quad \mathrm{tr}(\mathbf{P}) = 0. \tag{59}$$

where

$$\mathbf{J} = \mathrm{tr}(\mathbf{R})\,\mathbf{I} - \mathbf{R}, \quad \mathbf{R} \equiv \mathbf{P}\mathbf{P}^T, \quad \dot{\mathbf{P}}\mathbf{P}^T \equiv [\dot{q}_{ij}].$$
$$\mathrm{vec}(\dot{\mathbf{P}}\mathbf{P}^T) \equiv \frac{1}{2}\begin{bmatrix} \dot{q}_{32} - \dot{q}_{23} \\ \dot{q}_{13} - \dot{q}_{31} \\ \dot{q}_{21} - \dot{q}_{12} \end{bmatrix}, \quad \boldsymbol{\omega}_{A2} \equiv \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 2\,\mathbf{J}^{-1}\,\mathrm{vec}(\dot{\mathbf{P}}\mathbf{P}^T), \quad \boldsymbol{\Omega}_2 \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

and $\boldsymbol{\Omega}_2^2 \equiv \boldsymbol{\Omega}_2\boldsymbol{\Omega}_2$.

4.5 Fifth representation of the angular acceleration vector

A fifth representation of the angular acceleration vector has been reported by Condurache and Matcovschi, [13, 21], and the corresponding formulation is given by:

$$\boldsymbol{\alpha}_{CM} = \left(\frac{1}{2}\right)\left\{\mathbf{r}_1^* \times (\ddot{\mathbf{p}}_1 - \ddot{\mathbf{p}}_Q) + \mathbf{r}_2^* \times (\ddot{\mathbf{p}}_2 - \ddot{\mathbf{p}}_Q) + \mathbf{r}_3^* \times (\ddot{\mathbf{p}}_3 - \ddot{\mathbf{p}}_Q)\right\} \tag{60}$$

where

$$\begin{aligned}\ddot{\mathbf{p}}_Q = {} & [\,\ddot{\mathbf{p}}_1\cdot(\mathbf{p}_1-\mathbf{p}_Q) + (\dot{\mathbf{p}}_1-\dot{\mathbf{p}}_Q)\cdot(\dot{\mathbf{p}}_1 - \dot{\mathbf{p}}_Q)\,]\,\mathbf{r}_1^* \\ & + [\,\ddot{\mathbf{p}}_2\cdot(\mathbf{p}_2-\mathbf{p}_Q) + (\dot{\mathbf{p}}_2-\dot{\mathbf{p}}_Q)\cdot(\dot{\mathbf{p}}_2-\dot{\mathbf{p}}_Q)\,]\,\mathbf{r}_2^* + \\ & + [\,\ddot{\mathbf{p}}_3\cdot(\mathbf{p}_3-\mathbf{p}_Q) + (\dot{\mathbf{p}}_3-\dot{\mathbf{p}}_Q)\cdot(\dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_Q)\,]\,\mathbf{r}_3^*.\end{aligned} \tag{61}$$

$$\begin{aligned}\dot{\mathbf{p}}_Q = {} & [\,\dot{\mathbf{p}}_1\cdot(\mathbf{p}_1-\mathbf{p}_Q)\,]\,\mathbf{r}_1^* + [\,\dot{\mathbf{p}}_2\cdot(\mathbf{p}_2-\mathbf{p}_Q)\,]\,\mathbf{r}_2^* \\ & + [\,\dot{\mathbf{p}}_3\cdot(\mathbf{p}_3-\mathbf{p}_Q)\,]\,\mathbf{r}_3^*\end{aligned} \tag{62}$$

$$\mathbf{r}_1^* = \frac{(\mathbf{p}_2-\mathbf{p}_Q)\times(\mathbf{p}_3-\mathbf{p}_Q)}{\sigma}, \quad \mathbf{r}_2^* = \frac{(\mathbf{p}_3-\mathbf{p}_Q)\times(\mathbf{p}_1-\mathbf{p}_Q)}{\sigma}, \quad \mathbf{r}_3^* = \frac{(\mathbf{p}_1-\mathbf{p}_Q)\times(\mathbf{p}_2-\mathbf{p}_Q)}{\sigma}. \tag{63}$$

$$\sigma \equiv (\mathbf{p}_1-\mathbf{p}_Q)\cdot[(\mathbf{p}_2-\mathbf{p}_Q)\times(\mathbf{p}_3-\mathbf{p}_Q)]. \tag{64}$$

and position vector $\mathbf{p}_Q$ denotes the location of an arbitrary point Q (a fourth point) of the moving rigid body which must be non-coplanar with points 1, 2, and 3. Hence, we arbitrarily defined it as follows:

$$\mathbf{p}_Q \equiv \left(\frac{1}{3}\right)\{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3\} + \frac{(\mathbf{p}_2-\mathbf{p}_1)\times(\mathbf{p}_3-\mathbf{p}_1)}{\chi},$$

$$\chi \equiv \sqrt{\{(\mathbf{p}_2-\mathbf{p}_1)\times(\mathbf{p}_3-\mathbf{p}_1)\}\cdot\{(\mathbf{p}_2-\mathbf{p}_1)\times(\mathbf{p}_3-\mathbf{p}_1)\}}, \quad \forall \quad \sigma \neq 0. \tag{65}$$

Here it is important to note that the formulation of Condurache and Matcovschi, [13, 21], is the only one that explicitly involves a fourth point of the rigid body, which is not the centroid of the set of points 1, 2, and 3.

5 First case study

The objective of this section is to show the application details of the different approaches to obtain the angular acceleration vector. To this end, consider a representative example taken from [20]. This example provides the position vectors, the velocity vectors, and the acceleration vectors of three non-collinear points of a moving rigid body:

$$\mathbf{p}_1 = \begin{bmatrix} 1/2 \\ -\sqrt{3}/6 \\ 0 \end{bmatrix}, \ \mathbf{p}_2 = \begin{bmatrix} 0 \\ \sqrt{3}/3 \\ 0 \end{bmatrix}, \ \mathbf{p}_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/6 \\ 0 \end{bmatrix}, \tag{66}$$

$$\dot{\mathbf{p}}_1 = \frac{4-\sqrt{2}}{4}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \dot{\mathbf{p}}_2 = \frac{4-\sqrt{3}}{4}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \dot{\mathbf{p}}_3 = \frac{4+\sqrt{2}}{4}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{67}$$

$$\dot{\mathbf{p}} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ \sqrt{2}-\sqrt{3} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \dot{\mathbf{p}}\cdot\mathbf{q} = 0. \tag{69}$$

and therefore, Eq. (40) fails to compute the angular velocity vector. However, this vector is required to compute the corresponding angular acceleration vector. Hence, Eq. (39) is not valid for the particular numerical data of this first case study.

5.2 Angular acceleration of the second representation

Equation (41) represents the angular acceleration vector for the second representation. However, the computation fails from the beginning for the numerical data of this first case study since the denominator of the corresponding angular velocity vector (45) vanishes, that is:

$$\mathbf{v}_{B/A} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ \sqrt{2}-\sqrt{3} \end{bmatrix}, \quad \mathbf{r}_{C/A} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_{B/A}\cdot\mathbf{r}_{C/A} = 0. \tag{70}$$

and therefore, Eq. (45) fails to compute the angular velocity vector. However, this vector is required to compute the corresponding angular acceleration vector. Hence, Eq. (41) is not valid for the particular numerical data of this first case study.

$$\ddot{\mathbf{p}}_1 = \frac{1}{24}\begin{bmatrix} -6+4\sqrt{3} \\ 12-3\sqrt{2} \\ 0 \end{bmatrix}, \ \ddot{\mathbf{p}}_2 = -\frac{1}{24}\begin{bmatrix} 8\sqrt{3}+3\sqrt{6} \\ 3\sqrt{3} \\ 0 \end{bmatrix}, \ \ddot{\mathbf{p}}_3 = \frac{1}{24}\begin{bmatrix} 6+4\sqrt{3} \\ -12+3\sqrt{2} \\ 0 \end{bmatrix}. \tag{68}$$

The goal of this first case study is to obtain the angular acceleration vector of the corresponding rigid body using those approaches shown previously.

5.1 Angular acceleration of the first representation

Equation (39) is the angular acceleration vector of the first representation. The computation of the corresponding angular velocity vector (40) for the numerical data of this first case study fails from the beginning since the denominator vanishes, that is:

5.3 Angular acceleration of the third representation

For the angular acceleration vector of the third representation, we have two choices, namely, Eqs. (47) or (51). However, we need to compute the angular velocity vector first. To this end, we observe that the denominator of the angular velocity vector $\boldsymbol{\omega}_{W1}$, Eq. (55), vanishes for the numerical data of this first case study. Hence, we resort to the alternative angular velocity vector $\boldsymbol{\omega}_{W2}$, given by Eq. (56), thus yielding:

$$\boldsymbol{\omega}_{W2} = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{2} \\ 0 \end{bmatrix}. \tag{71}$$

Now, since $\mathbf{r}_{3/1} \cdot \mathbf{q}_{2/3} \neq 0$, the angular acceleration vector can be computed by the formula (47), which yields the following numerical results:

$$\mathbf{q}_{1/3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{q}_{2/3} = \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_{3/1} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_{3/1} \cdot \mathbf{q}_{2/3} = \frac{\sqrt{3}}{2}. \tag{72}$$

therefore, we see that the final result is given by:

$$\boldsymbol{\alpha}_{W1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{rad/s}^2. \tag{73}$$

5.4 Angular acceleration of the fourth representation

The corresponding angular acceleration vector is given by Eq. (58). We readily obtain the following numerical results for the particular data of the first case study:

$$\mathbf{P} = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ -\sqrt{3} & 2\sqrt{3} & -\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{tr}(\mathbf{P}) = \frac{1}{2} + \frac{\sqrt{3}}{3}, \quad \dot{\mathbf{P}} = \frac{1}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{3}-3\sqrt{2} & -2\sqrt{3} & \sqrt{3}+3 \end{bmatrix},$$

$$\boldsymbol{\omega}_{A1} = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{2} \\ 0 \end{bmatrix}, \quad \ddot{\mathbf{P}} = \frac{1}{24}\begin{bmatrix} -6+4\sqrt{3}+\sqrt{6} & -8\sqrt{3}-2\sqrt{6} & 6+4\sqrt{3}+\sqrt{6} \\ 12-3\sqrt{2}+\sqrt{3} & -2\sqrt{3} & -12+3\sqrt{2}+\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{6(\sqrt{3}+2)}{3+2\sqrt{3}} & 0 & -\frac{6\sqrt{3}}{3+2\sqrt{3}} \\ -\frac{2\sqrt{3}(\sqrt{3}+2)}{3+2\sqrt{3}} & 4 & \frac{2\sqrt{3}(\sqrt{3}-2)}{3+2\sqrt{3}} \\ 0 & 0 & \frac{12}{3+2\sqrt{3}} \end{bmatrix}, \quad \alpha_1 = \frac{12}{3+2\sqrt{3}}, \quad \beta = -\frac{24\sqrt{3}}{3+2\sqrt{3}}.$$

thus resulting the following angular acceleration vector:

$$\boldsymbol{\alpha}_{A1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{rad/s}^2. \tag{74}$$

5.5 Angular acceleration of the fifth representation

The corresponding angular acceleration vector is given by Eq. (60). In this case there are obtained the following numerical results:

$$\mathbf{p}_Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \sigma = -\frac{\sqrt{3}}{2}, \quad \mathbf{r}_1^* = \begin{bmatrix} 1 \\ -\frac{\sqrt{3}}{3} \\ -\frac{1}{3} \end{bmatrix},$$

$$\mathbf{r}_2^* = \begin{bmatrix} 0 \\ \frac{2\sqrt{3}}{3} \\ -\frac{1}{3} \end{bmatrix}, \quad \mathbf{r}_3^* = \begin{bmatrix} -1 \\ -\frac{\sqrt{3}}{3} \\ -\frac{1}{3} \end{bmatrix},$$

$$\dot{\mathbf{p}}_Q = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2} \\ 1-\frac{\sqrt{3}}{12} \end{bmatrix}, \quad \ddot{\mathbf{p}}_Q = \begin{bmatrix} -\frac{\sqrt{6}}{24} \\ -\frac{\sqrt{3}}{24} \\ -\frac{3}{4} \end{bmatrix}.$$

which produce the following angular acceleration vector:

$$\boldsymbol{\alpha}_{CM} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{rad/s}^2. \tag{75}$$

5.6 Angular acceleration of the formula proposed in this paper

Equation (36) represents the angular acceleration vector proposed in this paper. The corresponding terms for the numerical data of this first case study are the following:

(b) The basic equations reported in [14] and [15] failed to compute the angular acceleration vector corresponding to the numerical data of this first case study.

$$L_{12}=1,\quad L_{13}=1,\quad \mathbf{u}=\frac{1}{2}\begin{bmatrix}-1\\ \sqrt{3}\\ 0\end{bmatrix},\quad \mathbf{m}=\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix},\quad \lambda=\frac{\sqrt{3}}{2},\quad k_1=\frac{2\sqrt{3}}{3},\quad k_2=\frac{\sqrt{3}}{3},$$

$$\dot{\mathbf{u}}=\frac{1}{4}\begin{bmatrix}0\\ 0\\ \sqrt{2}-\sqrt{3}\end{bmatrix},\quad \dot{\mathbf{m}}=\begin{bmatrix}0\\ 0\\ \sqrt{2}/2\end{bmatrix},\quad \ddot{\mathbf{u}}=\frac{1}{8}\begin{bmatrix}2-\sqrt{6}-4\sqrt{3}\\ -4+\sqrt{2}-\sqrt{3}\\ 0\end{bmatrix},\quad \ddot{\mathbf{m}}=\frac{1}{4}\begin{bmatrix}2\\ \sqrt{2}-4\\ 0\end{bmatrix},$$

$$\boldsymbol{\kappa}_1=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},\quad \boldsymbol{\kappa}_2=\frac{\sqrt{3}}{16}\begin{bmatrix}0\\ 0\\ \sqrt{6}+4\sqrt{3}-2\end{bmatrix},\quad \boldsymbol{\kappa}_3=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},\quad \boldsymbol{\kappa}_4=\frac{\sqrt{6}}{16}\begin{bmatrix}0\\ 0\\ \sqrt{2}-\sqrt{3}\end{bmatrix},$$

$$\boldsymbol{\kappa}_5=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},\ \boldsymbol{\kappa}_6=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},\ \boldsymbol{\kappa}_7=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},\ \boldsymbol{\kappa}_8=\frac{\sqrt{3}}{32}\begin{bmatrix}0\\ 0\\ 2\sqrt{6}-5\end{bmatrix},\ \boldsymbol{\kappa}_9=\frac{\sqrt{3}}{32}\begin{bmatrix}0\\ 0\\ (\sqrt{3}-\sqrt{2})^2\end{bmatrix}.$$

which produce the following angular acceleration vector:

$$\boldsymbol{\alpha}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\text{rad/s}^2. \tag{76}$$

5.7 Remarks on the first case study

A careful analysis of the numerical results related to previous computations shows that:

(a) The formula proposed in this paper and the formulas that appear in [13, 17], and [20] were able to compute the correct numerical value of the angular acceleration vector corresponding to the numerical data of this first case study. All the formulas produced the same numerical result.

6 Second case study

The second case study considers a very representative example taken from [15]. This example provides the position vectors, the velocity vectors, and the acceleration vectors of three non-collinear points of a moving rigid body, namely:

$$\mathbf{p}_1=\begin{bmatrix}100\\ 100\\ 0\end{bmatrix},\ \mathbf{p}_2=\begin{bmatrix}300\\ 300\\ 0\end{bmatrix},\ \mathbf{p}_3=\begin{bmatrix}220\\ 180\\ 0\end{bmatrix}, \tag{77}$$

$$\dot{\mathbf{p}}_1=\begin{bmatrix}600\\ -400\\ 100\end{bmatrix},\ \dot{\mathbf{p}}_2=\begin{bmatrix}200\\ 0\\ 0\end{bmatrix},\ \dot{\mathbf{p}}_3=\begin{bmatrix}440\\ -160\\ 40\end{bmatrix}, \tag{78}$$

$$\ddot{\mathbf{p}}_1=\begin{bmatrix}850\\ 1200\\ -240\end{bmatrix},\ \ddot{\mathbf{p}}_2=\begin{bmatrix}200\\ 200\\ 0\end{bmatrix},\ \ddot{\mathbf{p}}_3=\begin{bmatrix}420\\ 760\\ -140\end{bmatrix}. \tag{79}$$

which are given in millimeters, millimeters per second, and millimeters per second squared, respectively.

The objective of this second case study is to obtain the angular acceleration vector of the moving rigid body using all five formulas shown previously.

6.1 Angular acceleration of the first representation

Equation (39) represents the angular acceleration vector of the first representation. This formula requires the following numerical computations:

$$\mathbf{p} = \begin{bmatrix} 200 \\ 200 \\ 0 \end{bmatrix}, \ \mathbf{q} = \begin{bmatrix} 120 \\ 80 \\ 0 \end{bmatrix}, \ \dot{\mathbf{p}} = \begin{bmatrix} -400 \\ 400 \\ -100 \end{bmatrix}, \ \dot{\mathbf{q}} = \begin{bmatrix} -160 \\ 240 \\ -60 \end{bmatrix}, \ \ddot{\mathbf{p}} = \begin{bmatrix} -650 \\ -1000 \\ 240 \end{bmatrix},$$
$$\ddot{\mathbf{q}} = \begin{bmatrix} -430 \\ -440 \\ 100 \end{bmatrix}, \ \boldsymbol{\omega}_{FZ} = \begin{bmatrix} 0 \\ 1/2 \\ 2 \end{bmatrix}, \ \mathbf{s}_1 = \begin{bmatrix} 1/4 \\ 1/4 \\ 0 \end{bmatrix}, \ \mathbf{s}_2 = \begin{bmatrix} 3/20 \\ 1/10 \\ 0 \end{bmatrix},$$
$$\mathbf{s}_3 = \begin{bmatrix} 0 \\ -1/8 \\ -1/2 \end{bmatrix}, \ \mathbf{s}_4 = \begin{bmatrix} 0 \\ -85/16 \\ -85/4 \end{bmatrix}, \ \mathbf{s}_5 = \begin{bmatrix} 0 \\ 87/16 \\ 87/4 \end{bmatrix}.$$

which produce the following numerical result for the angular acceleration vector:

$$\boldsymbol{\alpha}_{FZ} = \begin{bmatrix} 1/10 \\ -1/10 \\ -1 \end{bmatrix} \text{rad/s}^2. \tag{80}$$

6.2 Angular acceleration of the second representation

The corresponding angular acceleration vector is given by Eq. (41). The numerical values associated with this formula are the following:

$$\mathbf{v}_{B/A} = \begin{bmatrix} -400 \\ 400 \\ \sqrt{2} - 100 \end{bmatrix}, \quad \mathbf{v}_{C/A} = \begin{bmatrix} -160 \\ 240 \\ -60 \end{bmatrix}, \quad \mathbf{r}_{C/A} = \begin{bmatrix} 120 \\ 80 \\ 0 \end{bmatrix},$$
$$\boldsymbol{\omega}_{SLI} = \begin{bmatrix} 0 \\ 1/2 \\ 2 \end{bmatrix}, \quad \mathbf{q}_{B/A} = \begin{bmatrix} 200 \\ -200 \\ 40 \end{bmatrix}, \quad \mathbf{q}_{C/A} = \begin{bmatrix} 80 \\ -120 \\ 20 \end{bmatrix}.$$

and we get the following angular acceleration vector:

$$\boldsymbol{\alpha}_{SLI} = \begin{bmatrix} 1/10 \\ -1/10 \\ -1 \end{bmatrix} \text{rad/s}^2. \tag{81}$$

6.3 Angular acceleration of the third representation

The angular acceleration vector for the third representation is given by Eq. (47). The numerical values associated with this formula are the following:

$$\boldsymbol{\omega}_{W1} = \begin{bmatrix} 0 \\ 1/2 \\ 2 \end{bmatrix}. \tag{82}$$

Now, since $\mathbf{r}_{3/1} \cdot \mathbf{q}_{2/3} \neq 0$, the angular acceleration vector can be computed by the formula (47), which yields the following numerical results:

$$\mathbf{q}_{1/3} = \begin{bmatrix} -80 \\ 120 \\ -20 \end{bmatrix}, \quad \mathbf{q}_{2/3} = \begin{bmatrix} 120 \\ -80 \\ 20 \end{bmatrix},$$
$$\mathbf{r}_{3/1} = \begin{bmatrix} 120 \\ 80 \\ 0 \end{bmatrix}, \quad \mathbf{r}_{3/1} \cdot \mathbf{q}_{2/3} = 8000. \tag{83}$$

thus, we see that the final result is given by:

$$\boldsymbol{\alpha}_{W1} = \begin{bmatrix} 1/10 \\ -1/10 \\ -1 \end{bmatrix} \text{rad/s}^2. \tag{84}$$

6.4 Angular acceleration of the fourth representation

Equation (58) is the basic formula for the angular acceleration vector of the fourth representation. A quick calculation of the matrix **P** for the particular data of the second case study reveals that:

$$\mathbf{P} = \frac{1}{3}\begin{bmatrix} -320 & 280 & 40 \\ -280 & 320 & -40 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{tr}\,(\mathbf{P}) = 0. \tag{85}$$

but the term tr($\mathbf{P}$) is part of the denominator of the parameters α_1, and β, which are related to the angular acceleration vector given by Eq. (58). Therefore, we have to resort to the alternative formula of the angular acceleration vector (59), which produces the following numerical results:

which produce the following angular acceleration vector:

$$\boldsymbol{\alpha}_{CM} = \begin{bmatrix} 1/10 \\ -1/10 \\ -1 \end{bmatrix} \mathrm{rad/s}^2. \tag{87}$$

$$\mathbf{R} = \frac{1}{3}\begin{bmatrix} 60800 & 59200 & 0 \\ 59200 & 60800 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{tr}\,(\mathbf{R}) = \frac{121600}{3}, \quad \mathbf{J}^{-1} = \begin{bmatrix} \frac{19}{20000} & \frac{37}{40000} & 0 \\ \frac{37}{40000} & \frac{19}{20000} & 0 \\ 0 & 0 & \frac{3}{121600} \end{bmatrix},$$

$$\dot{\mathbf{P}}\mathbf{P}^T = \frac{1}{3}\begin{bmatrix} -118400 & -121600 & 0 \\ 121600 & 118400 & 0 \\ -30400 & -29600 & 0 \end{bmatrix}, \quad \boldsymbol{\omega}_{A2} = \begin{bmatrix} 0 \\ 1/2 \\ 2 \end{bmatrix}, \quad \boldsymbol{\Omega}_2 = \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

and, finally, it is obtained the following angular acceleration vector:

$$\boldsymbol{\alpha}_{A2} = \begin{bmatrix} 1/10 \\ -1/10 \\ -1 \end{bmatrix} \mathrm{rad/s}^2. \tag{86}$$

6.5 Angular acceleration of the fifth representation

The corresponding angular acceleration vector is given by Eq. (60). In this particular case, the following results are obtained:

6.6 Angular acceleration of the formula proposed in this paper

Equation (36) represents the angular acceleration vector proposed in this paper. The corresponding computations for the numerical data of the second case study are the following:

$$\mathbf{p}_Q = \begin{bmatrix} \frac{620}{3} \\ \frac{580}{3} \\ -1 \end{bmatrix}, \quad \sigma = -8000, \quad \mathbf{r}_1^* = \begin{bmatrix} -\frac{3}{200} \\ \frac{1}{100} \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{r}_2^* = \begin{bmatrix} -\frac{1}{100} \\ \frac{3}{200} \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{r}_3^* = \begin{bmatrix} \frac{1}{40} \\ -\frac{1}{40} \\ \frac{1}{3} \end{bmatrix},$$

$$\dot{\mathbf{p}}_Q = \begin{bmatrix} \frac{2477}{6} \\ -\frac{560}{3} \\ \frac{140}{3} \end{bmatrix}, \quad \ddot{\mathbf{p}}_Q = \begin{bmatrix} \frac{4901}{10} \\ \frac{7191}{10} \\ -\frac{1517}{12} \end{bmatrix}.$$

$$L_{12} = 200\sqrt{2}\,\text{mm}, \quad L_{13} = 40\sqrt{13}\,\text{mm}, \quad \mathbf{u} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{m} = \frac{\sqrt{13}}{13}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix},$$

$$\lambda = \frac{\sqrt{26}}{26}, \; k_1 = \sqrt{26}, \quad k_2 = 5, \; \dot{\mathbf{u}} = \frac{\sqrt{2}}{4}\begin{bmatrix} -4 \\ 4 \\ -1 \end{bmatrix}, \quad \dot{\mathbf{m}} = \frac{\sqrt{13}}{26}\begin{bmatrix} -8 \\ 12 \\ -3 \end{bmatrix},$$

$$\ddot{\mathbf{u}} = \frac{\sqrt{2}}{40}\begin{bmatrix} -65 \\ -100 \\ 24 \end{bmatrix}, \quad \ddot{\mathbf{m}} = \frac{\sqrt{13}}{52}\begin{bmatrix} -43 \\ -44 \\ 10 \end{bmatrix}, \quad \boldsymbol{\kappa}_1 = \frac{1}{52}\begin{bmatrix} -5 \\ -5 \\ 0 \end{bmatrix},$$

$$\boldsymbol{\kappa}_2 = \begin{bmatrix} \frac{9}{65} \\ \frac{6}{65} \\ \frac{79}{104} \end{bmatrix}, \quad \boldsymbol{\kappa}_3 = \frac{1}{13}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\kappa}_4 = \frac{1}{104}\begin{bmatrix} -12 \\ 12 \\ -3 \end{bmatrix},$$

$$\boldsymbol{\kappa}_5 = \frac{1}{26}\begin{bmatrix} 2 \\ -3 \\ -20 \end{bmatrix}, \quad \boldsymbol{\kappa}_6 = \frac{1}{26}\begin{bmatrix} -2 \\ 3 \\ 20 \end{bmatrix}, \quad \boldsymbol{\kappa}_7 = \frac{1}{26}\begin{bmatrix} 0 \\ -5 \\ -20 \end{bmatrix},$$

$$\boldsymbol{\kappa}_8 = \frac{33\sqrt{26}}{208}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{\kappa}_9 = \frac{33\sqrt{26}}{208}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

which produce the following angular acceleration vector:

$$\boldsymbol{\alpha} = \begin{bmatrix} 1/10 \\ -1/10 \\ -1 \end{bmatrix} \text{rad/s}^2. \tag{88}$$

6.7 Remarks on the second case study

A detailed analysis of all the numerical results related to the previous computations shows that:

(a) The equations reported in [13–15, 17], and the formula proposed in this paper were able to compute the angular acceleration vector corresponding to the numerical data of the second case study. All the formulas produced the same numerical result.
(b) Only the basic formula proposed in [20] failed to compute the angular acceleration vector corresponding to the numerical data related to the second case study. However, the alternative formula shown in [20] produces the correct numerical value of the angular acceleration vector.

7 Conclusions

This paper presented a systematic and detailed approach for the computation of the angular acceleration vector of a rigid body moving in space. Moreover, the proposed approach systematically groups some desirable features, which leads to a simpler and better way of conceiving and understanding the correlation that exists between the rotational motion of a rigid body and the velocity and acceleration of an arbitrary point of the body. Some advantages of the proposed approach are the following:

(1) The novelty of the proposed method lies in the particular form of writing the pose of the moving rigid body, as well as in the procedure to compute its time derivatives. As a result, the method leads naturally to obtaining a novel expression of the angular acceleration vector, where it is not required to solve the inverse acceleration problem in rigid body kinematics [13].
(2) The integrated nature of the proposed approach may help to visualize the physical connection between rotation, angular velocity, and angular

acceleration, which are present during the spatial motion of a rigid body. On the one hand, the proposed method treats the rotational phenomenon in a way that agrees directly with physical insight. On the other hand, the approach allows us to correlate the angular motion of the body with the translational motion of any point of the moving body.

(3) The approach presented in this paper does not require the computation of the angular velocity vector of the moving body. However, all the representations of the angular acceleration vector reported in [14, 15, 17], and [20] requires the computation of the angular velocity vector. This issue may represent a serious drawback since obtaining the angular velocity vector may lead to computational singularities.

(4) A singularity occurs when the denominator of the angular acceleration vector equals zero. The denominators of the angular acceleration vectors reported in [14, 15, 17], and [20] involves several vectors, in consequence, they are sensitive to singularities. That is because these results come from an intensive vector and matrix algebra manipulation of preconceived equations of classical kinematics. On the other hand, the denominator of the expression for the angular acceleration vector proposed in this paper is very simple, namely, the scalar parameter λ^2, and therefore, it is very robust against singularities. It may produce a singularity if, and only if, the three points of the body are collinear.

Last, the authors hope that the present contribution may help to a better understanding of the acceleration analysis of complex multibody systems, such as machines, mechanisms, parallel robots, and the human body.

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Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

Appendix 1

The objective of this appendix is to present a detailed derivation of the nine vector terms $\dot{\mathbf{n}}_1, \dot{\mathbf{n}}_2, \cdots \dot{\mathbf{n}}_9$ involved into Eq. (30). To this end, the general idea is to include only $\mathbf{u}$, $\dot{\mathbf{u}}$, $\ddot{\mathbf{u}}$, $\mathbf{m}$, $\dot{\mathbf{m}}$, and $\ddot{\mathbf{m}}$, since these vectors are directly related to the position, the velocity, and the acceleration of the three non-collinear points of the moving rigid body under analysis.

(1) Computation of the first term, $\dot{\mathbf{n}}_1$.

The first term, namely, $\dot{\mathbf{n}}_1$, has been previously defined in Eq. (30), and it may be handled as follows:

$$\begin{aligned}\dot{\mathbf{n}}_1 &\equiv (\ddot{\mathbf{v}}\cdot\mathbf{w})\mathbf{u} = \{(k_1\ddot{\mathbf{m}} - k_2\ddot{\mathbf{u}})\cdot(\mathbf{u}\times\mathbf{v})\}\mathbf{u} \\ &= k_1\{\ddot{\mathbf{m}}\cdot(\mathbf{u}\times\mathbf{v})\}\mathbf{u} - k_2\{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{v})\}\mathbf{u}.\end{aligned} \tag{89}$$

By using vector product identities, we have that:

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\ddot{\mathbf{m}}) &= \{\ddot{\mathbf{m}}\cdot(\mathbf{u}\times\mathbf{v})\}\mathbf{u} - \{\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})\}\ddot{\mathbf{m}} \\ &= \{\ddot{\mathbf{m}}\cdot(\mathbf{u}\times\mathbf{v})\}\mathbf{u}, \quad \mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0.\end{aligned} \tag{90}$$

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\ddot{\mathbf{u}}) &= \{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{v})\}\mathbf{u} - \{\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})\}\ddot{\mathbf{u}} \\ &= \{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{v})\}\mathbf{u}, \quad \mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0.\end{aligned} \tag{91}$$

where

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \mathbf{u}\times(k_1\mathbf{m} - k_2\mathbf{u}) = k_1(\mathbf{u}\times\mathbf{m}) \\ &- k_2(\mathbf{u}\times\mathbf{u}) = k_1(\mathbf{u}\times\mathbf{m}), \quad \mathbf{u}\times\mathbf{u} = \mathbf{0}.\end{aligned} \tag{92}$$

In this way, according to the rules of cross vector products, Eq. (89) may be written as follows:

$$\dot{\mathbf{n}}_1 = k_1^2\,(\ddot{\mathbf{m}}\times\mathbf{u})\times(\mathbf{u}\times\mathbf{m}) - k_1k_2\,(\ddot{\mathbf{u}}\times\mathbf{u})\times(\mathbf{u}\times\mathbf{m}). \tag{93}$$

(2) Computation of the second term, $\dot{\mathbf{n}}_2$.

The algebraic handling of the second term, namely, $\dot{\mathbf{n}}_2$, is described below:

$$\dot{\mathbf{n}}_2 \equiv (\dot{\mathbf{v}}\cdot\dot{\mathbf{w}})\mathbf{u} = \{(k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}})\cdot(\dot{\mathbf{u}}\times\mathbf{v} + \mathbf{u}\times\dot{\mathbf{v}})\}\mathbf{u} \tag{94}$$

where

$$\dot{\mathbf{u}}\times\mathbf{v} = \dot{\mathbf{u}}\times(k_1\mathbf{m} - k_2\mathbf{u}) = k_1(\dot{\mathbf{u}}\times\mathbf{m}) - k_2(\dot{\mathbf{u}}\times\mathbf{u}) \tag{95}$$

$$\mathbf{u}\times\dot{\mathbf{v}} = \mathbf{u}\times(k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}}) = k_1(\mathbf{u}\times\dot{\mathbf{m}}) - k_2(\mathbf{u}\times\dot{\mathbf{u}}) \tag{96}$$

Thus, vector $\dot{\mathbf{n}}_2$ becomes:

$$\dot{\mathbf{n}}_2 = k_1^2 \{\dot{\mathbf{m}} \cdot (\dot{\mathbf{u}} \times \mathbf{m})\}\mathbf{u} - k_1 k_2 \{\dot{\mathbf{u}} \cdot (\mathbf{u} \times \dot{\mathbf{m}})\}\mathbf{u} \quad (97)$$

Next, by resorting to the following vector product identity, it is found that:

$$\begin{aligned}(\dot{\mathbf{u}}\times\mathbf{u})\times(\mathbf{u}\times\dot{\mathbf{m}}) &= \{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\}\mathbf{u} - \{\mathbf{u}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\}\dot{\mathbf{u}} \\ &= \{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\}\mathbf{u},\ \ \mathbf{u}\cdot(\mathbf{u}\times\dot{\mathbf{m}}) = 0.\end{aligned} \quad (98)$$

Therefore, it is finally obtained that:

$$\dot{\mathbf{n}}_2 = k_1^2 \{\dot{\mathbf{m}} \cdot (\dot{\mathbf{u}} \times \mathbf{m})\}\mathbf{u} - k_1 k_2 (\dot{\mathbf{u}}\times\mathbf{u})\times(\mathbf{u}\times\dot{\mathbf{m}}). \quad (99)$$

(3) Computation of the third term, $\dot{\mathbf{n}}_3$.

This section shows the computation of the third term, namely, $\dot{\mathbf{n}}_3$. The process begins with the following expression:

$$\dot{\mathbf{n}}_3 \equiv (\dot{\mathbf{v}}\cdot\mathbf{w})\dot{\mathbf{u}} = \{(k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}}) \cdot (\mathbf{u} \times \mathbf{v})\}\dot{\mathbf{u}} \quad (100)$$

where

$$\begin{aligned}\mathbf{u}\times\mathbf{v} = \mathbf{u}\times(k_1\mathbf{m} - k_2\mathbf{u}) &= k_1(\mathbf{u}\times\mathbf{m}) \\ - k_2(\mathbf{u}\times\mathbf{u}) &= k_1(\mathbf{u}\times\mathbf{m}),\ \ \mathbf{u}\times\mathbf{u} = \mathbf{0}.\end{aligned} \quad (101)$$

Then, the third term can be expressed as:

$$\dot{\mathbf{n}}_3 = k_1^2 \{\dot{\mathbf{m}} \cdot (\mathbf{u} \times \mathbf{m})\}\dot{\mathbf{u}} - k_1 k_2 \{\dot{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{m})\}\dot{\mathbf{u}}. \quad (102)$$

(4) Computation of the fourth term, $\dot{\mathbf{n}}_4$.

The algebraic manipulation of the fourth term is as follows:

$$\begin{aligned}\dot{\mathbf{n}}_4 &\equiv (\ddot{\mathbf{u}}\cdot\mathbf{w})\mathbf{v} = \{\ddot{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v})\}\mathbf{v} = \{\ddot{\mathbf{u}} \cdot (k_1\mathbf{u} \times \mathbf{m})\}(k_1\mathbf{m} - k_2\mathbf{u}) \\ \dot{\mathbf{n}}_4 &= k_1^2 \{\ddot{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{m})\}\mathbf{m} - k_1 k_2 \{\ddot{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{m})\}\mathbf{u}.\end{aligned} \quad (103)$$

Recalling the following vector product identities:

$$(\ddot{\mathbf{u}}\times\mathbf{m})\times(\mathbf{u}\times\mathbf{m}) = \{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\mathbf{m} - \{\mathbf{m}\cdot(\mathbf{u}\times\mathbf{m})\}\ddot{\mathbf{u}} = \{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\mathbf{m},\ \ \mathbf{m}\cdot(\mathbf{u}\times\mathbf{m}) = 0. \quad (104)$$

$$(\ddot{\mathbf{u}}\times\mathbf{u})\times(\mathbf{u}\times\mathbf{m}) = \{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\mathbf{u} - \{\mathbf{u}\cdot(\mathbf{u}\times\mathbf{m})\}\ddot{\mathbf{u}} = \{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\mathbf{u},\ \ \mathbf{u}\cdot(\mathbf{u}\times\mathbf{m}) = 0. \quad (105)$$

Having laid the necessary groundwork, we get the following result:

$$\dot{\mathbf{n}}_4 = k_1^2 (\ddot{\mathbf{u}}\times\mathbf{m})\times(\mathbf{u}\times\mathbf{m}) - k_1 k_2 (\ddot{\mathbf{u}}\times\mathbf{u})\times(\mathbf{u}\times\mathbf{m}). \quad (106)$$

(5) Computation of the fifth term, $\dot{\mathbf{n}}_5$.

The fifth term can be formulated in a way that yields a convenient vector expression, which starts with the relation:

$$\begin{aligned}\dot{\mathbf{n}}_5 &\equiv (\dot{\mathbf{u}}\cdot\dot{\mathbf{w}})\mathbf{v} = \{\dot{\mathbf{u}} \cdot (\dot{\mathbf{u}} \times \mathbf{v} + \mathbf{u} \times \dot{\mathbf{v}})\} \\ \mathbf{v} &= \{\dot{\mathbf{u}} \cdot (\dot{\mathbf{u}} \times \mathbf{v}) + \dot{\mathbf{u}} \cdot (\mathbf{u} \times \dot{\mathbf{v}})\}\mathbf{v} \\ \dot{\mathbf{n}}_5 &= \{\dot{\mathbf{u}} \cdot (\mathbf{u} \times \dot{\mathbf{v}})\}\mathbf{v},\ \ \dot{\mathbf{u}} \cdot (\dot{\mathbf{u}} \times \mathbf{v}) = 0.\end{aligned} \quad (107)$$

where

$$\mathbf{u} \times \dot{\mathbf{v}} = \mathbf{u} \times (k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}}) = k_1(\mathbf{u} \times \dot{\mathbf{m}}) - k_2(\mathbf{u} \times \dot{\mathbf{u}}) \quad (108)$$

$$\begin{aligned}\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{v}}) &= k_1\{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\} - k_2\{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{u}})\} \\ &= k_1\{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\},\ \ \dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{u}}) = 0.\end{aligned} \quad (109)$$

Then Eq. (107) becomes:

$$\begin{aligned}\dot{\mathbf{n}}_5 &= k_1 \{\dot{\mathbf{u}} \cdot (\mathbf{u}\times\dot{\mathbf{m}})\}(k_1\mathbf{m} - k_2\mathbf{u}) \\ &= k_1^2 \{\dot{\mathbf{u}} \cdot (\mathbf{u} \times \dot{\mathbf{m}})\}\mathbf{m} - k_1 k_2 \{\dot{\mathbf{u}} \cdot (\mathbf{u} \times \dot{\mathbf{m}})\}\mathbf{u}.\end{aligned} \quad (110)$$

To complete the reduction process, we now use the well-known vector product identity:

$$\begin{aligned}(\dot{\mathbf{u}}\times\mathbf{u})\times(\mathbf{u}\times\dot{\mathbf{m}}) &= \{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\}\mathbf{u} - \{\mathbf{u}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\} \\ \dot{\mathbf{u}} &= \{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\}\mathbf{u},\ \ \mathbf{u}\cdot(\mathbf{u}\times\dot{\mathbf{m}}) = 0.\end{aligned} \quad (111)$$

Thus, the sought expression is, therefore:

$$\dot{\mathbf{n}}_5 = k_1^2 \{\dot{\mathbf{u}} \cdot (\mathbf{u} \times \dot{\mathbf{m}})\}\mathbf{m} - k_1 k_2 (\dot{\mathbf{u}}\times\mathbf{u})\times(\mathbf{u}\times\dot{\mathbf{m}}). \quad (112)$$

(6) Computation of the sixth term, $\dot{\mathbf{n}}_6$.

In this section, we examine another means of expressing the so-called sixth term. To this end, in the first instance, we have the following equation:

$$\dot{\mathbf{n}}_6 \equiv (\dot{\mathbf{u}}\cdot\mathbf{w})\dot{\mathbf{v}} = \{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{v})\}\dot{\mathbf{v}} = \{\dot{\mathbf{u}}\cdot(k_1\mathbf{u}\times\mathbf{m})\}\dot{\mathbf{v}} = \{\dot{\mathbf{u}}\cdot(k_1\mathbf{u}\times\mathbf{m})\}(k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}}) \tag{113}$$

and, after some vector algebra, we obtain the final result given by:

$$\dot{\mathbf{n}}_6 = k_1^2\,\{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\dot{\mathbf{m}} - k_1k_2\,\{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\dot{\mathbf{u}}. \tag{114}$$

(7) Computation of the seventh term, $\dot{\mathbf{n}}_7$.

In this section, we analyze the mathematical form associated with the seventh term described in Eq. (30), namely:

$$\begin{aligned}\dot{\mathbf{n}}_7 &\equiv (\ddot{\mathbf{u}}\cdot\mathbf{v})\mathbf{w} = \{\ddot{\mathbf{u}}\cdot(k_1\mathbf{m} - k_2\mathbf{u})\}\mathbf{w} = \{\ddot{\mathbf{u}}\cdot(k_1\mathbf{m} - k_2\mathbf{u})\}(\mathbf{u}\times\mathbf{v})\\ \dot{\mathbf{n}}_7 &= \{\ddot{\mathbf{u}}\cdot(k_1\mathbf{m} - k_2\mathbf{u})\}(k_1\mathbf{u}\times\mathbf{m}) = k_1^2\,(\ddot{\mathbf{u}}\cdot\mathbf{m})(\mathbf{u}\times\mathbf{m}) - k_1k_2\,(\ddot{\mathbf{u}}\cdot\mathbf{u})(\mathbf{u}\times\mathbf{m})\end{aligned} \tag{115}$$

The first term of the above equation may be conveniently transformed by the following vector identity:

$$\ddot{\mathbf{u}}\times\{(\mathbf{u}\times\mathbf{m})\times\mathbf{m}\} = (\ddot{\mathbf{u}}\cdot\mathbf{m})(\mathbf{u}\times\mathbf{m}) - \{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\mathbf{m} \tag{116}$$

Thus, the seventh term is given by:

$$\dot{\mathbf{n}}_7 = k_1^2\,\ddot{\mathbf{u}}\times\{(\mathbf{u}\times\mathbf{m})\times\mathbf{m}\} + k_1^2\,\{\ddot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\mathbf{m} - k_1k_2\,(\ddot{\mathbf{u}}\cdot\mathbf{u})(\mathbf{u}\times\mathbf{m}). \tag{117}$$

(8) Computation of the eighth term, $\dot{\mathbf{n}}_8$.

We now consider an alternative derivation of the formula for the eighth term that was previously defined. The procedure is as follows:

$$\begin{aligned}\dot{\mathbf{n}}_8 &\equiv (\dot{\mathbf{u}}\cdot\dot{\mathbf{v}})\mathbf{w} = \{\dot{\mathbf{u}}\cdot(k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}})\}\mathbf{w} = \{\dot{\mathbf{u}}\cdot(k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}})\}(\mathbf{u}\times\mathbf{v})\\ \dot{\mathbf{n}}_8 &= \{\dot{\mathbf{u}}\cdot(k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}})\}(k_1\mathbf{u}\times\mathbf{m}) = k_1^2(\dot{\mathbf{u}}\cdot\dot{\mathbf{m}})(\mathbf{u}\times\mathbf{m}) - k_1k_2(\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})(\mathbf{u}\times\mathbf{m})\end{aligned} \tag{118}$$

To get a more convenient form of the first term of the above equation, we resort to the following vector identity:

$$\dot{\mathbf{u}}\times\{(\mathbf{u}\times\mathbf{m})\times\dot{\mathbf{m}}\} = (\dot{\mathbf{u}}\cdot\dot{\mathbf{m}})(\mathbf{u}\times\mathbf{m}) - \{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\dot{\mathbf{m}} \tag{119}$$

Therefore, the eighth term gets the following form:

$$\dot{\mathbf{n}}_8 = k_1^2\,\dot{\mathbf{u}}\times\{(\mathbf{u}\times\mathbf{m})\times\dot{\mathbf{m}}\} + k_1^2\,\{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\mathbf{m})\}\dot{\mathbf{m}} - k_1k_2\,(\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})(\mathbf{u}\times\mathbf{m}). \tag{120}$$

(9) Computation of the ninth term, $\dot{\mathbf{n}}_9$.

This section is dedicated to finding an alternative formula for the ninth term defined in Eq. (30). This term can be expressed as:

$$\dot{\mathbf{n}}_9 \equiv (\dot{\mathbf{u}}\cdot\mathbf{v})\dot{\mathbf{w}} = \{\dot{\mathbf{u}}\cdot(k_1\mathbf{m} - k_2\mathbf{u})\}\dot{\mathbf{w}} = \{k_1(\dot{\mathbf{u}}\cdot\mathbf{m}) - k_2(\dot{\mathbf{u}}\cdot\mathbf{u})\}(\dot{\mathbf{u}}\times\mathbf{v} + \mathbf{u}\times\dot{\mathbf{v}}) \tag{121}$$

where

$$\dot{\mathbf{u}}\times\mathbf{v} = \dot{\mathbf{u}}\times(k_1\mathbf{m} - k_2\mathbf{u}) = k_1(\dot{\mathbf{u}}\times\mathbf{m}) - k_2(\dot{\mathbf{u}}\times\mathbf{u}) \tag{122}$$

$$\mathbf{u}\times\dot{\mathbf{v}} = \mathbf{u}\times(k_1\dot{\mathbf{m}} - k_2\dot{\mathbf{u}}) = k_1(\mathbf{u}\times\dot{\mathbf{m}}) - k_2(\mathbf{u}\times\dot{\mathbf{u}}) \tag{123}$$

$$\dot{\mathbf{u}}\cdot\mathbf{u} = 0, \quad \dot{\mathbf{u}}\times\mathbf{u} = -\mathbf{u}\times\dot{\mathbf{u}}. \tag{124}$$

From these relationships, we have that:

$$\dot{\mathbf{n}}_9 = k_1^2\,(\dot{\mathbf{u}}\cdot\mathbf{m})(\dot{\mathbf{u}}\times\mathbf{m}) + k_1^2\,(\dot{\mathbf{u}}\cdot\mathbf{m})(\mathbf{u}\times\dot{\mathbf{m}}). \tag{125}$$

The algebraic process continues using the following two vector identities:

$$\dot{\mathbf{u}}\times\{(\dot{\mathbf{u}}\times\mathbf{m})\times\mathbf{m}\} = (\dot{\mathbf{u}}\cdot\mathbf{m})(\dot{\mathbf{u}}\times\mathbf{m}) - \{\dot{\mathbf{u}}\cdot(\dot{\mathbf{u}}\times\mathbf{m})\}\mathbf{m} = (\dot{\mathbf{u}}\cdot\mathbf{m})(\dot{\mathbf{u}}\times\mathbf{m}),\ \dot{\mathbf{u}}\cdot(\dot{\mathbf{u}}\times\mathbf{m}) = 0. \tag{126}$$

$$\dot{\mathbf{u}}\times\{(\mathbf{u}\times\dot{\mathbf{m}})\times\mathbf{m}\} = (\dot{\mathbf{u}}\cdot\mathbf{m})(\mathbf{u}\times\dot{\mathbf{m}}) - \{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\}\mathbf{m}. \tag{127}$$

By using the foregoing identities it is obtained the final result given by:

$$\dot{\mathbf{n}}_9 = k_1^2\,\dot{\mathbf{u}}\times\{(\dot{\mathbf{u}}\times\mathbf{m})\times\mathbf{m}\} + k_1^2\,\dot{\mathbf{u}}\times\{(\mathbf{u}\times\dot{\mathbf{m}})\times\mathbf{m}\} + k_1^2\,\{\dot{\mathbf{u}}\cdot(\mathbf{u}\times\dot{\mathbf{m}})\}\mathbf{m}. \tag{128}$$

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