



A priori estimates for solutions of FitzHugh–Rinzel system

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Abstract The FitzHugh–Rinzel system is able to describe some biophysical phenomena, such as bursting oscillations, and the study of its solutions can help to better understand several behaviours of the complex dynamics of biological systems. We express the solutions by means of an integral equation involving the fundamental solution $H(x, t)$ related to a non linear integro-differential equation. Properties of $H(x, t)$ allow us to obtain a priori estimates for solutions determined in the whole space, showing both the influence of the initial data and the source term.

Keywords FitzHugh–Rinzel model · Fundamental solution · Laplace transform

1 Introduction

Mathematical biophysics models, such as the FitzHugh–Nagumo system (FHN), play an important role in studying the nervous system, as they can help describe biophysical phenomena that are relevant to neuronal excitability.

The FHN consists of two differential equations that model several engineering applications and there

exist many scientific results and an extensive bibliography in regard [1–7]. However, it has been noted that only using a reset or adding noise, it is possible to evaluate bursting phenomena. This phenomenon occurs in a number of different cell types and it consists of a behaviour characterized by brief bursts of oscillatory activity alternating with periods of quiescence during which the membrane potential changes only slowly [8].

Bursting phenomena are becoming more and more important and their studies are increasing in many scientific fields (see, f.i. [9] and references therein). For example, in the restoration of synaptic connections, it appears that some nanoscale memristor devices have the potential to reproduce the behavior of a biological synapse [10, 11]. This will lead in the future, especially in case of traumatic injuries, to the introduction of electronic synapses to directly connect neurons.

A model that seems to be more mathematically appropriate for incorporating nerve cell bursting phenomena is the FitzHugh–Rinzel model (FHR). It derives from FitzHugh–Nagumo and, differently from FHN, consists of three equations just to insert slow modulation of the current [1, 12–15]. Indeed, bursting oscillations can be characterized by a system variable that periodically changes from an active phase of rapid spike oscillations to a silent phase.

As for the FHR model, the following system is considered:

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$$\begin{cases} \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - w + y + f(u) \\ \frac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u) \\ \frac{\partial y}{\partial t} = \delta(-u + h - dy) \end{cases} \quad (1)$$

where

$$f(u) = u(a - u)(u - 1) \quad (0 < a < 1), \quad (2)$$

and terms β , c , d , h , ε , δ are positive constants that characterize the model's kinetics. The second order term with $D > 0$ can be associated to the axial current in the axon, and it derives from the Hodgkin-Huxley theory for nerve membranes. Indeed, if b represents the axon diameter and r_i is the resistivity, the spatial variation in the potential V gives the term $(b/4r_i)V_{,xx}$ from which term $Du_{,xx}$ derives (see f.i. [16]), and in [9] an analysis on contribution due to this term has been developed. Furthermore, when the fast variable u simulates the membrane potential of a nerve cell, while the slow variable w and the super-slow variable y determine the corresponding number densities of ions, the model (1) simulates the propagation of impulses from one neuron to another, and studies on solutions can help in testing the responses of the various models in neuroscience.

Several methods have been developed to find exact solutions related to partial differential equations and an extensive bibliography on the study of analytical behaviors exists (see, f.i. [17–23]). The aim of this paper is to determine a priori estimates for the FHR solution by means of suitable properties of the fundamental solution $H(x, t)$, showing how the effects due to the initial perturbation are vanishing when t tends to infinity, and simultaneously, as time increases, the effect of the nonlinear source remains bounded.

The paper is organized as follows: in Sect. 2 we define the mathematical problem and report some of the results already proved in [9], as well as other results well known in literature. In Sect. 3, some properties related to the fundamental solution $H(x, t)$ are obtained and, in a subsection, some relationships on convolutions which characterize the explicit solution, are highlighted. In Sect. 4, estimates on convolution are proved and in Sect. 5, the solution is expressed

by means of these particular convolution integrals. Finally, in Sect. 6, a priori estimates are showed.

2 Mathematical considerations

Indicating by T an arbitrary positive constant, let us consider the set:

$$\Omega_T = \{(x, t) : x \in \mathfrak{R}, 0 < t \leq T\}.$$

Moreover, if

$$\begin{aligned} u(x, 0) &= u_0, w(x, 0) \\ &= w_0 y(x, 0) \\ &= y_0, (x \in \mathfrak{R}) \end{aligned} \quad (3)$$

represent the initial values, then from (1)_{2,3}, one deduces:

$$\begin{cases} w = w_0 e^{-\varepsilon\beta t} + \frac{c}{\beta}(1 - e^{-\varepsilon\beta t}) + \varepsilon \int_0^t e^{-\varepsilon\beta(t-\tau)} u(x, \tau) d\tau \\ y = y_0 e^{-\delta dt} + \frac{h}{d}(1 - e^{-\delta dt}) - \delta \int_0^t e^{-\delta d(t-\tau)} u(x, \tau) d\tau. \end{cases} \quad (4)$$

Besides, letting

$$\begin{aligned} f(u) &= -au + \varphi(u) \\ \text{with } \varphi(u) &= u^2(a + 1 - u) \quad 0 < a < 1, \end{aligned} \quad (5)$$

system (1) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - au - w + y + \varphi(u) \\ \frac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u) \\ \frac{\partial y}{\partial t} = \delta(-u + h - dy), \end{cases} \quad (6)$$

and hence, when

$$\begin{aligned} F(x, t, u) &= \varphi(u) - w_0(x)e^{-\varepsilon\beta t} + y_0(x)e^{-\delta dt} \\ &\quad - \frac{c}{\beta}(1 - e^{-\varepsilon\beta t}) + \frac{h}{d}(1 - e^{-\delta dt}) \end{aligned} \quad (7)$$

denotes the source term, problem (6) with initial conditions (3), can be modified into the following initial value problem \mathcal{P} :

$$\begin{cases} u_t - Du_{xx} + au + \int_0^t [\epsilon e^{-\epsilon\beta(t-\tau)} + \delta e^{-\delta d(t-\tau)}]u(x, \tau)d\tau = F(x, t, u) \\ u(x, 0) = u_0(x) \quad x \in \mathfrak{R}. \end{cases} \tag{8}$$

In order to determine the solution of problem (8), let us consider the following functions:

$$H_1(x, t) = \frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi Dt}} e^{-at} - \frac{1}{2} \int_0^t \frac{e^{-\frac{x^2}{4Dy} - ay}}{\sqrt{t-y}} \frac{\sqrt{\epsilon} e^{-\beta\epsilon(t-y)}}{\sqrt{\pi D}} J_1(2\sqrt{\epsilon y(t-y)}) dy, \tag{9}$$

$$H_2 = \int_0^t H_1(x, y) e^{-\delta d(t-y)} \sqrt{\frac{\delta y}{t-y}} J_1(2\sqrt{\delta y(t-y)}) dy \tag{10}$$

where $J_1(z)$ is the Bessel function of first kind and order 1.

In [9] it has been verified that function $H(x, t)$:

$$H = H_1 - H_2 \tag{11}$$

represents the fundamental solution of the parabolic operator

$$Lu \equiv u_t - Du_{xx} + au + \int_0^t [\epsilon e^{-\epsilon\beta(t-\tau)} + \delta e^{-\delta d(t-\tau)}]u(x, \tau) d\tau, \tag{12}$$

and the following theorem has been proved:

Theorem 1 *In the half-plane $\mathfrak{R}es > \max(-a, -\beta\epsilon, -\delta d)$ the Laplace integral $\mathcal{L}_t H$ converges absolutely for all $x > 0$, and it results:*

$$\mathcal{L}_t H = \frac{1}{\sqrt{D}} \frac{e^{-\frac{|x|}{\sqrt{D}}\sigma}}{2\sigma} \tag{13}$$

where

$$\sigma^2 = s + a + \frac{\delta}{s + \delta d} + \frac{\epsilon}{s + \beta\epsilon}. \tag{14}$$

Moreover, function $H(x,t)$ satisfies some properties typical of the fundamental solution of heat equation, such as:

(a) $H(x, t) \in C^\infty, t > 0, x \in \mathfrak{R},$

- (b) for fixed $t > 0, H$ and its derivatives are vanishing exponentially fast as $|x|$ tends to infinity.
- (c) In addition, it results $\lim_{t \rightarrow 0} H(x, t) = 0,$ for any fixed $\eta > 0,$ uniformly for all $|x| \geq \eta.$ □

To obtain results of existence and uniqueness for the problem (8), the theorem of fixed point can be applied

and therefore, also according to [24], for initial term and source function we shall admit:

Assumption A Initial data u_0 is continuously differentiable and bounded together with its first derivative. The source term $F(x, t, u)$ is defined and continuous on the following set:

$$Z = \{(x, t, u) : (x, t) \in \Omega_T, -\infty < u < \infty\}. \tag{15}$$

Besides, for each $K > 0$ and $|u| < K, F(x, t, u)$ is uniformly Lipschitz

continuous in (x, t) for each compact set of Ω_T and it is bounded for bounded $u.$

Then, for all $(u_1, u_2),$ there exists a positive constant W_F such that:

$$|F(x, t, u_1) - F(x, t, u_2)| \leq W_F |u_1 - u_2|. \tag{16}$$

□

As a consequence, when the fundamental solution $H(x, t)$ and source function $F(x, t, u)$ satisfy theorem 1 and Assumption A, respectively, indicating by $u(x, t)$ a solution of problem $\mathcal{P},$ then u assumes this form:

$$u(x, t) = \int_{\mathfrak{R}} H(x - \xi, t) u_0(\xi) d\xi + \int_0^t d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] d\xi. \tag{17}$$

On the other hand, if $u(x, t)$ is a continuous and bounded solution of (17), it is possible to prove that u satisfies (8).

Consequently, it is possible to conclude that

Theorem 2 *Initial value problem (8) admits a unique solution only if (17) admits a unique continuous and bounded solution.* □

Besides, by means of fixed point theorem,(and extensive proofs can be found, f.i., in [24–29]), it is possible to prove the following theorem:

Theorem 3 *When Assumption A is satisfied, then the initial value problem (8) admits a unique regular solution $u(x, t)$ in Ω_T .* □

In this case, taking into account the source term $F(x, t)$ defined in (7), solution (17) assumes the following form:

$$\begin{aligned}
 u(x, t) = & \int_0^t d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) \varphi[\xi, \tau, u(\xi, \tau)] d\xi \\
 & + \left(\frac{h}{d} - \frac{c}{\beta} \right) \int_0^t d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) d\xi + \frac{c}{\beta} \int_0^t e^{-\beta\epsilon\tau} d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) d\xi \\
 & - \int_0^t e^{-\beta\epsilon\tau} d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) w_0(\xi) d\xi - \frac{h}{d} \int_0^t e^{-\delta d\tau} d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) d\xi \\
 & + \int_0^t e^{-\delta d\tau} d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) y_0(\xi) d\xi + \int_{\mathfrak{R}} H(x - \xi, t) u_0(\xi) d\xi
 \end{aligned} \tag{18}$$

and this formula, together with relations (4), allows us to determine also $v(x, t)$ and $y(x, t)$ in terms of the data.

3 Some properties related to $H(x,t)$

In order to obtain a priori estimates and asymptotic effects, some properties related to the fundamental solution H need to be evaluated.

More precisely, formula (18) shows the need to evaluate the convolution of the fundamental solution H with respect to time and space.

Consequently, this section will include a first part where two theorems involving some properties related to $H(x, t)$ are showed, and a subsection where some premises allowing to prove properties related to convolution integrals, will be stated.

Let us start indicating by

$$A(t) = \frac{e^{-\beta\epsilon t} - e^{-at}}{a - \beta\epsilon}; \quad B(t) = \frac{e^{-\delta d t} - e^{-at}}{a - \delta d}; \quad C(t) = \frac{e^{-\delta d t} - e^{-\beta\epsilon t}}{\beta\epsilon - \delta d} \tag{19}$$

three positive functions, then the following theorem holds:

Theorem 4 *The solution function H defined in (11) satisfies the following estimate:*

$$|H| \leq \frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi Dt}} \left[e^{-at} + t\epsilon A(t) + \delta t \left(1 + \frac{\epsilon t}{|a - \beta\epsilon|} \right) B(t) + \frac{\epsilon t}{|a - \epsilon\beta|} C(t) \right] \tag{20}$$

Proof Since

$$|J_1(2\sqrt{\epsilon y(t-y)})| \leq \sqrt{\epsilon y(t-y)} \quad (y \leq t) \tag{21}$$

from (9) it results:

$$|H_1(x, t)| \leq \frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi Dt}} \left[e^{-at} + \epsilon t \int_0^t e^{-ay} e^{-\beta\epsilon(t-y)} dy \right]$$

and hence:

$$|H_1(x, t)| \leq \frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi Dt}} \left[e^{-at} + \epsilon t \frac{e^{-\beta\epsilon t} - e^{-at}}{a - \epsilon\beta} \right]. \tag{22}$$

Moreover, from (10) and by means of (22), it results:

$$|H_2| \leq \int_0^t \frac{e^{-\frac{x^2}{4Dy}}}{2\sqrt{\pi Dy}} \left[e^{-ay} + \epsilon y \frac{e^{-\beta\epsilon y} - e^{-ay}}{a - \epsilon\beta} \right] e^{-\delta d(t-y)} \delta y dy.$$

Consequently one obtains:

$$|H_2| \leq \frac{\delta t e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi Dt}} \left[\frac{e^{-\delta t} - e^{-at}}{a - \delta d} \left(1 + \frac{\epsilon t}{|a - \epsilon \beta|} \right) + \frac{\epsilon t}{|a - \epsilon \beta|} \frac{e^{-\delta t} - e^{-\beta \epsilon t}}{\beta \epsilon - \delta d} \right] \tag{23}$$

Hence, according to (11), for (22) and (23), theorem holds. \square

Now, let us introduce as I_0 the modified Bessel function of the first kind and order 0, and let

$$l = \min(a, \beta \epsilon), \quad q = \min\{a, \beta \epsilon, \delta d\}, \tag{24}$$

$$\lambda(t) \equiv 1 + \pi t(\sqrt{\epsilon} + \sqrt{\delta} + \pi t\sqrt{\delta \epsilon}). \tag{25}$$

The following theorem holds:

Theorem 5 *The fundamental solution $H(x, t)$ defined in (11) satisfies the following estimates:*

$$\int_{\mathfrak{R}} |H(x - \xi, t)| d\xi \leq e^{-at} + \sqrt{\epsilon} \pi t e^{-\frac{\beta \epsilon + a}{2} t} I_0\left(\frac{\beta \epsilon - a}{2} t\right) + \sqrt{\delta} \pi t \left[e^{-\frac{\delta d + a}{2} t} I_0\left(\frac{\delta d - a}{2} t\right) + \sqrt{\epsilon} \pi t e^{-\frac{\delta d + l}{2} t} I_0\left(\frac{\delta d - l}{2} t\right) \right]; \tag{26}$$

$$\int_{\mathfrak{R}} |H(x - \xi, t)| d\xi \leq \lambda(t) e^{-qt} \tag{27}$$

Besides, indicating by

$$S = 1/a + \sqrt{\epsilon} \pi \frac{a + \beta \epsilon}{2(a\beta \epsilon)^{3/2}} + \sqrt{\delta} \pi \left[\frac{\delta d + a}{(a\delta d)^{3/2}} + 3\pi \sqrt{\epsilon} \frac{\delta^2 d^2 + l^2}{4(l\delta d)^{5/2}} \right], \tag{28}$$

one has:

$$\int_0^t d\tau \int_{\mathfrak{R}} |H(x - \xi, t - \tau)| d\xi \leq S. \tag{29}$$

Proof Considering that

$$H = H_1 - H_2, \tag{30}$$

we will firstly focus on the integral involving H_1 , and then on that involving H_2 .

Since it results:

$$\int_{\mathfrak{R}} e^{-\frac{x^2}{4Dt}} dx = 2\sqrt{\pi Dt}; \quad |J_1(z)| \leq 1, \tag{31}$$

from (9) one obtains:

$$\int_{\mathfrak{R}} |H_1(x, t)| dx \leq e^{-at} + \sqrt{\epsilon} \int_0^t e^{-\beta \epsilon(t-y)} e^{-ay} \frac{\sqrt{y}}{\sqrt{t-y}} dy \tag{32}$$

with

$$\begin{aligned} & \int_0^t e^{-\beta \epsilon(t-y)} e^{-ay} \sqrt{\frac{y}{t-y}} dy \\ &= - \int_0^t e^{-\beta \epsilon(t-y)} e^{-ay} (t/2 - y) \frac{dy}{\sqrt{y(t-y)}} + \\ &+ \int_0^t e^{-\beta \epsilon(t-y)} e^{-ay} \frac{t/2 dy}{\sqrt{y(t-y)}}. \end{aligned} \tag{33}$$

Now, taking into account that

$$\int_0^{2b} e^{-sy} (b - y) \frac{1}{\sqrt{2by - y^2}} dy = \pi b e^{-sb} I_1(sb) \tag{34}$$

and

$$\int_0^{2b} e^{-sy} \frac{1}{\sqrt{2by - y^2}} dy = \pi e^{-sb} I_0(sb) \tag{35}$$

for $b = t/2$ and $s = a - \beta \epsilon$, one has:

$$\begin{aligned} & \int_0^t e^{-\beta \epsilon(t-y)} e^{-ay} \sqrt{\frac{y}{t-y}} dy \\ &= \frac{\pi t}{2} \left[e^{-\frac{a - \beta \epsilon}{2} t} \left(I_0\left(\frac{a - \beta \epsilon}{2} t\right) - I_1\left(\frac{a - \beta \epsilon}{2} t\right) \right) \right]. \end{aligned}$$

Consequently, as for

$$I_1(-z) = -I_1(z) \quad I_0(z) = I_0(-z) \quad I_1(|z|) \leq I_0(|z|),$$

it results:

$$\int_{\mathfrak{R}} |H_1(x - \xi, t)| d\xi \leq e^{-at} + \sqrt{\epsilon} \pi t e^{-\frac{a + \beta \epsilon}{2} t} I_0\left(\frac{\beta \epsilon - a}{2} t\right). \tag{36}$$

Now, being $I_0(|z|) < e^{|z|}$, from (36) one deduces that

$$\int_{\mathfrak{R}} |H_1(x - \xi, t)| d\xi \leq e^{-at} + \sqrt{\epsilon} \pi t e^{-lt} \tag{37}$$

where l is defined in (24)₁.

As for function H_2 , taking into account that $|J_1| \leq 1$, from (10) and by means of (37), it results:

$$\int_{\mathfrak{R}} |H_2(x - \xi, t)| d\xi \leq \sqrt{\delta} \int_0^t (e^{-ay} + \sqrt{\varepsilon} \pi y e^{-ly}) e^{-\delta d(t-y)} \sqrt{\frac{y}{t-y}} dy.$$

Hence, returning to the previous reasoning, one obtains:

$$\int_{\mathfrak{R}} |H_2| \leq \sqrt{\delta} \pi t \left[e^{-\frac{\delta d+a}{2} t} I_0\left(\frac{\delta d-a}{2} t\right) + \sqrt{\varepsilon} \pi t e^{-\frac{\delta d+l}{2} t} I_0\left(\frac{\delta d-l}{2} t\right) \right] \tag{38}$$

from which, along with (36), (26) is proved.

Moreover, from (38), an inequality analogous to (37) can be obtained. In this way, according to (30), (27) follows, too.

Lastly, since it results

$$\int_0^\infty e^{-pt} t I_0(bt) dt = p(\sqrt{p^2 - b^2})^{-3} \quad \text{Re } p > |\text{Re } b| \tag{39}$$

$$\int_0^\infty e^{-pt} t^2 I_0(bt) dt = (\sqrt{p^2 - b^2})^{-3/2} \left(\frac{3p^2}{p^2 - b^2} - 1 \right) \quad \text{Re } p > |\text{Re } b|, \tag{40}$$

from (36) and (38), property (29) can be proved. \square

3.1 Premises on convolution integrals referring to the solution

In order to determine the estimates related to the solution, it is necessary to highlight every convolution integrals that characterize the solution itself. Therefore, in this subsection convolutions K_δ and H_δ will be introduced and, by means of them, solution $u(x, t)$ will be expressed (Formula 52).

Hence, let us consider

$$K_\delta(x, t) \equiv \int_0^t e^{-\delta d(t-y)} H_1(x, y) J_0(2\sqrt{\delta y(t-y)}) dy \tag{41}$$

and let

$$g_1(x, t) * g_2(x, t) = \int_0^t g_1(x, t - \tau) g_2(x, \tau) d\tau \tag{42}$$

be the convolution with respect to t .

In [9] it has been proved that:

$$e^{-\delta dt} * H = K_\delta \tag{43}$$

and

$$e^{-\varepsilon \beta t} * H = K_\delta + (\delta d - \varepsilon \beta) e^{-\beta \varepsilon t} * K_\delta. \tag{44}$$

Now, denoting by

$$H_\delta = \int_0^t e^{-\varepsilon \beta(t-\tau)} d\tau \int_0^\tau H_1(x, y) e^{-\delta d(\tau-y)} J_0(2\sqrt{\delta y(\tau-y)}) dy \tag{45}$$

it results:

$$e^{-\beta \varepsilon t} * K_\delta = H_\delta, \tag{46}$$

and as a consequence, from (44), one one:

$$e^{-\varepsilon \beta t} * H = K_\delta + (\delta d - \varepsilon \beta) H_\delta. \tag{47}$$

Moreover, let us denote by

$$g_1(x, t) \diamond g_2(x, t) = \int_{\mathfrak{R}} f_1(\xi, t) g_2(x - \xi, t) d\xi \tag{48}$$

the convolution with respect to the space, and

$$H \otimes F = \int_0^t d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] d\xi. \tag{49}$$

Since (43) and (47), it results:

$$\begin{cases} H \otimes e^{-\delta dt} = \int_{\mathfrak{R}} K_\delta(\xi, t) d\xi, \\ H \otimes e^{-\beta \varepsilon t} = \int_{\mathfrak{R}} [K_\delta + (\delta d - \varepsilon \beta) H_\delta] d\xi \end{cases} \tag{50}$$

and

$$\begin{cases} H \otimes (y_0(x) e^{-\delta dt}) = y_0 \diamond K_\delta \\ H \otimes (w_0(x) e^{-\beta \varepsilon t}) = w_0 \diamond [K_\delta + (\delta d - \varepsilon \beta) H_\delta]. \end{cases} \tag{51}$$

Consequently, given (18), we get:

$$\begin{aligned}
 u(x, t) &= H \diamond u_0(x) + K_\delta \diamond (y_0(x) - w_0(x)) + H \otimes \varphi(u) \\
 &+ (\varepsilon\beta - \delta d) H_\delta \diamond w_0(x) + \frac{c}{\beta} H_\delta \diamond (\delta d - \varepsilon\beta) \\
 &+ H \otimes \left(\frac{h}{d} - \frac{c}{\beta} \right) + K_\delta \diamond \left(\frac{c}{\beta} - \frac{h}{d} \right)
 \end{aligned} \tag{52}$$

and this formula explicitly shows all the convolutions involved in the solution $u(x, t)$.

4 On convolutions involving functions K_δ and H_δ

Formula (52) shows that an analysis of the solution directly implies estimates on both $H(x, t)$ and on functions K_δ, H_δ , defined in (41) and (45).

For this, let us consider $A(t), B(t), C(t), \lambda(t)$ defined in (19) and (25), respectively. Moreover, let

$$E(t) = \frac{e^{-qt} - e^{-\delta dt}}{\delta d - q} \quad L(t) = \frac{e^{-qt} - e^{-\beta \varepsilon t}}{\beta \varepsilon - q} \tag{53}$$

with q defined by (24)₂.

In addition,

$$M = \frac{1}{|\delta d - q| \delta q d} \left[q + \delta d + \pi(\sqrt{\varepsilon} + \sqrt{\delta}) \left(\frac{q^2 + \delta^2 d^2}{\delta d q} \right) + 2\pi^2 \sqrt{\delta \varepsilon} \left(\frac{q^3 + \delta^3 d^3}{(q \delta d)^2} \right) \right] \tag{54}$$

$$N = \frac{1}{|\beta \varepsilon - q| q \beta \varepsilon} \left[q + \beta \varepsilon + \pi(\sqrt{\varepsilon} + \sqrt{\delta}) \left(\frac{q^2 + \beta^2 \varepsilon^2}{\beta \varepsilon q} \right) + 2\pi^2 \sqrt{\delta \varepsilon} \left(\frac{q^3 + \beta^3 \varepsilon^3}{(q \beta d)^2} \right) \right] \tag{55}$$

$$g(t) = \frac{\lambda(t)}{|\beta \varepsilon - \delta d|} [E(t) + L(t)] \tag{56}$$

$$\int_{\mathfrak{R}} |H_\delta(x, t)| dx \leq g(t) \tag{63}$$

$$h(t) = \frac{\lambda(t)}{(\varepsilon\beta - \delta d)^2} [L(t) + (1 + t(\delta d - \varepsilon\beta))E(t)]. \tag{57}$$

$$\int_0^t d\tau \int_{\mathfrak{R}} |H_\delta(x, t - \tau)| dx \leq \frac{M + N}{|\beta \varepsilon - \delta d|} \tag{64}$$

The following theorems hold:

Theorem 6 *Function $K_\delta(x, t)$ defined in (41) satisfies the following estimates:*

$$\int_{\mathfrak{R}} |K_\delta(x, t)| \leq \lambda(t) E(t); \tag{58}$$

$$\int_0^t e^{-\delta d \tau} d\tau \int_{\mathfrak{R}} |H_\delta(x, t - \tau)| dx \leq h(t). \tag{65}$$

$$\int_0^t d\tau \int_{\mathfrak{R}} |K_\delta(x, \tau)| dx \leq M. \tag{59}$$

$$\int_0^t e^{-\beta \varepsilon \tau} d\tau \int_{\mathfrak{R}} |H_\delta(x, t - \tau)| dx \leq \frac{t \lambda(t)}{|\delta d - q|} [C(t) + L(t)] \tag{66}$$

Proof According to (46), one has:

$$\int_0^t e^{-\delta d \tau} d\tau \int_{\mathfrak{R}} |K_\delta(x, t - \tau)| dx \leq t \lambda(t) E(t) \tag{60}$$

Proof By means of (43) and property (27) on $\int_{\mathfrak{R}} |H(\xi, t)| d\xi$, inequality (58) follows.

By this estimate, according to (25), and taking into account that

$$\begin{aligned}
 \int_0^t y e^{-\alpha y} &\leq 1/\alpha^2; & \int_0^t y^2 e^{-\alpha y} &\leq 2/\alpha^3 \\
 (t > 0, \quad \alpha > 0), & & &
 \end{aligned} \tag{61}$$

(59) holds, too.

Moreover, because of (43), it results

$$e^{-\delta dt} * K_\delta = e^{-\delta dt} * H * e^{-\delta dt} = (t e^{-\delta dt}) * H \tag{62}$$

and inequality (60) follows. □

Theorem 7 *Referring to (45), function $H_\delta(x, t)$ satisfies the inequalities below:*

$$\int_{\mathfrak{R}} |H_{\delta}(x, t)| dx = \int_0^t e^{-\beta\epsilon\tau} d\tau \int_{\mathfrak{R}} |K_{\delta}(x, t - \tau)| dx \quad (67)$$

with, since (43), it results:

$$e^{-\beta\epsilon t} * K_{\delta} = e^{-\beta\epsilon t} * H * e^{-\delta dt} = C(t) * H(x, t) \quad (68)$$

where $C(t)$ is defined in (19)₃. Hence, since (27), inequality (63) holds.

Consequently, also (64) follows.

Estimate (65) is proved by means of

$$e^{-\delta dt} * H_{\delta} = e^{-\delta dt} * K_{\delta} * e^{-\beta\epsilon t} = (t e^{-\delta dt}) * e^{-\beta\epsilon t} * H. \quad (69)$$

Finally, taking into account that

$$e^{-\beta\epsilon t} * H_{\delta} = e^{-\beta\epsilon t} * K_{\delta} * e^{-\beta\epsilon t} = (t e^{-\beta\epsilon t}) * K_{\delta}, \quad (70)$$

from (58), (66) is proved, too. \square

5 Analysis of solution

In order to analyse functions $u(x, t)$, $w(x, t)$, and $y(x, t)$, it appears necessary to make explicit the integrals of convolutions involving functions H_{δ} and K_{δ} whose estimates have been established in the previous section.

Therefore, since (52), by means of convolution properties, we get:

$$\begin{aligned} u(x, t) &= \int_0^t d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) \varphi[\xi, \tau, u(\xi, \tau)] d\xi \\ &+ \left(\frac{h}{d} - \frac{c}{\beta} \right) \left[\int_0^t d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) d\xi - \int_{\mathfrak{R}} K_{\delta}(x - \xi, t) d\xi \right] \\ &+ \int_{\mathfrak{R}} K_{\delta}(x - \xi, t) [y_0(\xi) - w_0(\xi)] d\xi - (\delta d - \epsilon\beta) \int_{\mathfrak{R}} H_{\delta}(x - \xi, t) w_0(\xi) d\xi \\ &+ \frac{c}{\beta} (\delta d - \epsilon\beta) \int_{\mathfrak{R}} H_{\delta}(x - \xi, t) d\xi + \int_{\mathfrak{R}} H(x - \xi, t) u_0(\xi) d\xi. \end{aligned} \quad (71)$$

Moreover, as for functions $w(x, t)$ and $y(x, t)$ defined in (4), according to (43), (46) and (47), since (71), the following integrals must be considered:

$$\begin{aligned} \int_0^t e^{-\beta\epsilon(t-\tau)} u(x, \tau) d\tau &= \int_{\mathfrak{R}} K_{\delta}(x - \xi, t) u_0(\xi) d\xi \\ &+ \int_0^t d\tau \int_{\mathfrak{R}} K_{\delta}(x - \xi, t - \tau) \left[\varphi[\xi, \tau, u(\xi, \tau)] + \frac{h}{d} - \frac{c}{\beta} \right] d\xi \\ &+ (\delta d - \epsilon\beta) \int_0^t d\tau \int_{\mathfrak{R}} H_{\delta}(x - \xi, t - \tau) \left[\varphi[\xi, \tau, u(\xi, \tau)] + \frac{h}{d} - \frac{c}{\beta} \right] d\xi \\ &+ (\delta d - \epsilon\beta) \int_0^t e^{-\beta\epsilon(t-\tau)} d\tau \int_{\mathfrak{R}} H_{\delta}(x - \xi, \tau) \left[\frac{c}{\beta} - w_0(\xi) \right] d\xi \\ &+ \int_{\mathfrak{R}} H_{\delta}(x - \xi, t) [y_0(\xi) - w_0(\xi) - \frac{h}{d} + \frac{c}{\beta} + (\delta d - \epsilon\beta) u_0(\xi)] d\xi \end{aligned} \quad (72)$$

and

$$\begin{aligned}
 \int_0^t e^{-\delta d(t-\tau)} u(x, \tau) d\tau &= \int_{\mathfrak{R}} K_\delta(x - \xi, t) u_0(\xi) d\xi \\
 &+ \int_0^t d\tau \int_{\mathfrak{R}} K_\delta(x - \xi, t - \tau) \left[\varphi[\xi, \tau, u(\xi, \tau)] + \frac{h}{d} - \frac{c}{\beta} \right] d\xi \\
 &+ \int_0^t e^{-\delta d(t-\tau)} d\tau \int_{\mathfrak{R}} K_\delta(x - \xi, \tau) [y_0(\xi) - w_0(\xi) - \frac{h}{d} + \frac{c}{\beta}] d\xi \\
 &+ (\delta d - \varepsilon \beta) \int_0^t e^{-\delta d(t-\tau)} d\tau \int_{\mathfrak{R}} H_\delta(x - \xi, \tau) \left[\frac{c}{\beta} - w_0(\xi) \right] d\xi.
 \end{aligned} \tag{73}$$

6 Estimates of solution

As for the analysis of solutions of the non linear reaction diffusion model, there exists a large bibliography. In particular in [30, 31] the existence of bounded solutions is proved.

Therefore, in the class of bounded solutions, let us assume initial data and function $\varphi(x, t, u)$ satisfy Assumption A, and let

$$\begin{aligned}
 \|u_0\| &= \sup_{\mathfrak{R}} |u_0(x)|, & \|w_0\| &= \sup_{\mathfrak{R}} |w_0(x)|, \\
 \|y_0\| &= \sup_{\mathfrak{R}} |y_0(x)|, \\
 \|u\| &= \sup_{\Omega_T} |u(x, t)| & \|\varphi\| &= \sup_Z |\varphi(x, t, u)|
 \end{aligned}$$

with φ defined in (5) and Z defined in (15).

In order to give a priori estimates of the solution of FHR system, the following theorem is proved:

Theorem 8 *If function $\varphi(x, t, u)$ and initial data $u_0(x), w_0(x), y_0(x)$ are compatible with Assumption A, then the problem (1)–(3) satisfies the following estimates:*

$$\begin{aligned}
 |u(x, t)| &\leq \|u_0(x)\| \lambda(t) e^{-qt} + \left(\|\varphi\| + \left| \frac{h}{d} - \frac{c}{\beta} \right| \right) S \\
 &+ \left(\|y_0\| + \|w_0\| + \left| \frac{h}{d} - \frac{c}{\beta} \right| \right) \lambda(t) E(t) \\
 &+ \left(\|w_0\| + \frac{c}{\beta} \right) (\delta d - \varepsilon \beta) g(t);
 \end{aligned} \tag{74}$$

$$\begin{aligned}
 |w(x, t)| &\leq \|w_0\| e^{-\beta \varepsilon t} + \frac{c}{\beta} + \varepsilon \|u_0\| \lambda(t) E(t) \\
 &+ \varepsilon \left(\|\varphi\| + \left| \frac{h}{d} - \frac{c}{\beta} \right| \right) (2M + N) + \\
 &+ \varepsilon \frac{|\delta d - \varepsilon \beta|}{|\delta d - q|} \left[\frac{c}{\beta} + \|w_0(x)\| \right] t \lambda(t) [C(t) + L(t)] \\
 &+ \varepsilon \left[\|y_0\| + \|w_0\| + \left| \frac{c}{\beta} - \frac{h}{d} \right| + |\delta d - \varepsilon \beta| \|u_0\| \right] g(t);
 \end{aligned} \tag{75}$$

$$\begin{aligned}
 |y(x, t)| &\leq \|y_0\| e^{-\delta dt} + \frac{h}{d} + \delta \|u_0\| \lambda(t) E(t) + \\
 &+ \delta \left[\|y_0\| + \|w_0\| + \left| \frac{c}{\beta} - \frac{h}{d} \right| \right] t \lambda(t) E(t) + \\
 &\delta \left[\|\varphi\| + \left| \frac{h}{d} - \frac{c}{\beta} \right| \right] M + (\delta d - \varepsilon \beta) \left(\|w_0\| + \frac{c}{\beta} \right) h(t)
 \end{aligned} \tag{76}$$

where constants q, S, M, N are introduced in (24)₂, (28), (54), and (55), respectively.

Besides, functions $C(t), \lambda(t), E(t), L(t), g(t), h(t)$ are defined in (19)₃ (25), (53)_{1,2}, (56), and (57).

Proof According to (71) and by means of inequalities (27), (29), (58), and (63), estimate (74) follows.

As for inequalities (75) and (76), functions defined in (4) have to be considered.

More precisely, from (4)₁ and (72), taking into account inequalities (58), (59), (63), (64) and (66), estimate (75) is proved.

Analogously, from (4)₂ and (73), for (58)–(60) and (65), also (76) holds.

□

Remark These estimates show that the solution of the FitzHugh–Rinzel system is bounded for all t . Besides, when t tends to infinity, the effect of the non linear term $\varphi(x, t)$ is bounded, while the effects of initial perturbances $u_0(x), w_0(x), y_0(x)$ are vanishing.

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Declarations

Conflict of interest The author declares that she has no conflict of interest.

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