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BRIEF NOTES AND DISCUSSIONS

Elementary scales and the lack of Fourier paradox for Fourier fluids

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Abstract Focusing on the Fourier fluids in the liquid state, which are characterized by linear thermal constitutive equation and low compressibility, this short note proposes a discrete approach based on the elementary scales, which allows removing the so-called Fourier paradox in classical continuum thermomechanics. As a corollary, the adopted line of reasoning allows highlighting some features on the elementary scales.

Keywords Elementary scales · Fourier fluids · Fourier heat equation · Fourier paradox · Local thermodynamic equilibrium condition

1 Introduction

The phenomenological description of fluid flows is founded on the spatio-temporal continuity of the motion [1, 2]. In agreement with this hypothesis, the generic field functions, such as density ρ, velocity $\boldsymbol{v}$, temperature T, pressure p, are regular functions of both space $\boldsymbol{r}$ and time t. For the sake of clarity, this note is restricted to homogeneous, isotropic, electrically neutral, chemically inert fluids. The continuity in the space involves the continuum hypothesis: the material system is considered as a continuum system of fluid particles; the fluid particles are, virtually, in one-to-one correspondence with the points of the Euclidean space. The size of the fluid particles coincides with the Representative Elementary Volume (REV) $\mathrm{d}V$. The REV, which has been first introduced for modeling the transport phenomena in porous media [3], is required for any continuous macroscopic representation of the material systems [4]. In line with the method of homogenization, the REV is *large* compared to the molecular size and *small* compared to the size of the flow domain [4]. Consequently, the continuity in space implies that $\frac{\updownarrow}{|\mathrm{d}\boldsymbol{r}|} \ll 1$, where $\updownarrow$ is a characteristic length of the material system, $|\mathrm{d}\boldsymbol{r}|$ the elementary spatial scale (i.e. the REV scale). In classical continuum thermomechanics, the continuity in time requires that $\frac{t_r}{|\mathrm{d}t|} \ll 1$, where t_r is the relaxation time, i.e. the interval time required to restore a Local Thermodynamic Equilibrium (LTE) condition in place of a local thermodynamic non-equilibrium condition, $|\mathrm{d}t|$ the elementary time scale. According to this formulation:

- the fluid is in LTE condition (also referred to as hydrodynamic regime, [5]);
- the generic field function b is well defined in both $\boldsymbol{r}$ and t;

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- $b(\boldsymbol{r} \pm \delta\boldsymbol{r}) = b(\boldsymbol{r})$ for $|\delta\boldsymbol{r}| < |\mathrm{d}\boldsymbol{r}|$ and $b(t \pm \delta t) = b(t)$ for $|\delta t| < |\mathrm{d}t|$;
- if $|\mathrm{d}b|$ is the elementary scale of b, then $b + \delta b = b$ for $|\delta b| < |\mathrm{d}b|$.

In Sect. 2 some features of $|\mathrm{d}\boldsymbol{r}|$ and $|\mathrm{d}t|$ are highlighted and the elementary scales of density $|\mathrm{d}\rho|$, pressure $|\mathrm{d}p|$, and temperature $|\mathrm{d}T|$, are introduced. Focusing on the Fourier fluids in the liquid state, which are characterized by linear thermal constitutive equation and low compressibility, Sect. 3 defines the constraints to be imposed to elementary scales. In Sect. 4, a discrete approach based on the elementary scales allows removing the heat conduction paradox of the infinite speed of signal diffusion in Fourier theory. The conjunction of the Fourier constitutive equation and heat balance equation gives rise to the parabolic heat equation which in turn leads to the Fourier paradox [6, 7]. The conclusions close the paper.

2 Elementary scales

From a mathematical point of view, the relationship:

$$\boldsymbol{r}^* = \boldsymbol{r} + \mathrm{d}\boldsymbol{r} = \boldsymbol{r} + \boldsymbol{v}(\boldsymbol{r}, t)\mathrm{d}t \tag{1}$$

where $\boldsymbol{r}$ identifies the position of a fluid particle at the time t, can be thought of as a coordinate transformation [8], where the Jacobian tensor $\underline{\boldsymbol{J}}$ is given as:

$$\underline{\boldsymbol{J}} = \frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{r}} = \nabla \boldsymbol{r}^* = \underline{\boldsymbol{I}} + \nabla \boldsymbol{v}\mathrm{d}t \tag{2}$$

By defining $\mathrm{d}V$ as the REV at the position $\boldsymbol{r}$ at the time t, $\mathrm{d}V = \mathrm{d}V(\boldsymbol{r}, t)$, and $\mathrm{d}V^* = \mathrm{d}V(\boldsymbol{r}^*, t^*) = \mathrm{d}V(\boldsymbol{r} + \mathrm{d}\boldsymbol{r}, t + \mathrm{d}t)$, with $t^* = t + \mathrm{d}t$, the connection between $\mathrm{d}V$ and $\mathrm{d}V^*$ is expressed by:

$$\mathrm{d}V^* = J\mathrm{d}V \tag{3}$$

where J is the Jacobian determinant:

$$J = \det\underline{\boldsymbol{J}} = \det\left(\underline{\boldsymbol{I}} + \nabla \boldsymbol{v}\mathrm{d}t\right) \tag{4}$$

By identifying $\mathrm{d}V$ as a material volume (with constant mass), the principle of mass conservation can be expressed as:

$$\rho\mathrm{d}V = \rho^*\mathrm{d}V^* \tag{5}$$

where:

$$\rho = \rho(\boldsymbol{r}, t) \tag{6}$$

$$\begin{aligned} \rho^*(r^*, t^*) &= \rho(r + dr, t + dt) \\ &= \rho(r, t) + \frac{\partial}{\partial t}\rho(r, t)dt + \frac{\partial}{\partial r}\rho(r, t)\, dr \\ &= \rho + \frac{\partial \rho}{\partial t}dt + \nabla\rho\, dr \\ &= \rho + \frac{\partial \rho}{\partial t}dt + \nabla\rho\, vdt \end{aligned} \tag{7}$$

If $\|\nabla \boldsymbol{v}\mathrm{d}t\| \ll 1$, Eq. (4) can be approximated as [8]:

$$J = 1 + \nabla \cdot \boldsymbol{v}\mathrm{d}t \tag{8}$$

and Eq. (5) reads as:

$$\rho\mathrm{d}V = \left(\rho + \frac{\partial \rho}{\partial t}\mathrm{d}t + \nabla\rho \cdot \boldsymbol{v}\mathrm{d}t\right)(1 + \nabla \cdot \boldsymbol{v}\mathrm{d}t)\mathrm{d}V \tag{9}$$

Equation (9) reduces to the well-known continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{10}$$

when the higher order infinitesimals are neglected. Equation (10) is valid for both Linear Non-Equilibrium Regime (LNER, which corresponds to the viscous flow regime) and Non-Linear Non-Equilibrium Regime (NLNER, which corresponds to the turbulent flow regime). The LNER, which is stable and regular, is characterized by *small* values of $\|\nabla \boldsymbol{v}\|$ and $|\nabla T|$: a perturbation of the mechanical and thermodynamic state regresses during the evolution of the motion. The NLNER, which is unstable and chaotic, is characterized by *large* values of $\|\nabla \boldsymbol{v}\|$ and/or $|\nabla T|$: a perturbation amplifies and has systematic effects on the motion features.

Formally, the obtained result implies that $\forall \nabla \boldsymbol{v} \exists \mathrm{d}\boldsymbol{t} : \frac{t_r}{\mathrm{d}t} \ll \boldsymbol{I}$ and $\|\boldsymbol{\nabla v}\mathrm{d}\boldsymbol{t}\| \ll \boldsymbol{I}$. If $\|\nabla \boldsymbol{v}\mathrm{d}t\| \ll 1$ also $|\nabla \cdot \boldsymbol{v}\mathrm{d}t| \ll 1$ (if $\|\nabla \boldsymbol{v}\mathrm{d}t\| \ll 1$, $\det\left(\underline{\boldsymbol{I}} + \nabla \boldsymbol{v}\mathrm{d}t\right) = 1 + \nabla \cdot \boldsymbol{v}\mathrm{d}t$ is very close to 1 and the following approximation holds $|\nabla \cdot \boldsymbol{v}\mathrm{d}t| \ll 1$).

In equivalent forms, the continuity Eq. (10) can be written as:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla \cdot \boldsymbol{v} = 0 \tag{11}$$

or as:

$$d\rho + \rho\nabla \cdot \boldsymbol{v}dt = 0 \tag{12}$$

where $\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \nabla b \cdot \boldsymbol{v}$ is the material derivative of b, $db = \frac{\partial b}{\partial t}dt + \nabla b \cdot d\boldsymbol{r}$. Equation (12) allows to define the elementary density scale as $|d\rho| = |\rho\nabla \cdot \boldsymbol{v}dt|$. The comparison between Eq. (12) and the equation of state $\rho = \rho(p, T)$, written in the differential form as [9]:

$$-\frac{1}{\rho}d\rho = -\frac{1}{\varepsilon}dp + \alpha dT \tag{13}$$

- with $\varepsilon = \varepsilon(p, T)$ the bulk modulus of elasticity, and $\alpha = \alpha(p, T)$ the thermal expansion coefficient– yields to:

$$\alpha dT = \nabla \cdot \boldsymbol{v}dt + \frac{1}{\varepsilon}dp \tag{14}$$

Equation (13), which is valid for both LNER and NLNER, shows the influence of $d\rho$ and dp on dT. According to Eq. (13), $|dp|$ is the elementary scale of p and $|dT|$ the elementary scale of T.

3 Fourier fluids

Fourier Fluids (FF) are that for which the conduction part of the heat flux vector is linearly related to the temperature gradient [10–12]. For this kind of fluids, the thermal constitutive equation is given as:

$$\boldsymbol{q} = -k_T\nabla T \tag{15}$$

In Eq. (15), $\boldsymbol{q}$ is the heat flux vector due to thermal conduction, $k_T = k_T(p, T)$ is the thermal conductivity.

Whitin the framework of FF, the attention is paid to the Fourier Fluids in the Liquid State (FFLS). The FFLS are characterized by low compressibility: the bulk modulus of elasticity ε is the order of $10^9 Pa$. In agreement with Eq. (14), putting $\varepsilon = O(10^9 Pa) \gg 1$, it follows that $\frac{1}{\varepsilon}|dp| \ll 1$, $|\nabla \cdot \boldsymbol{v}dt| \ll 1$, $\alpha|dT| \ll 1$. Observing that $dT = \frac{\partial T}{\partial t}dt + \nabla T \cdot d\boldsymbol{r}$, the relationship $\alpha|dT| \ll 1$, formally, implies that $\forall\nabla T \exists d\boldsymbol{r} : \frac{}{|d\boldsymbol{r}|} \ll \boldsymbol{1}$, $\alpha|\nabla T \cdot d\boldsymbol{r}| \ll 1$.

4 The so-called Fourier paradox

For FFLS at rest ($\boldsymbol{v} = 0$ in the chosen inertial reference frame), the heat balance equation [9] reads as:

$$\rho c\frac{\partial T}{\partial t} = -\nabla \cdot \boldsymbol{q} \tag{16}$$

where c is the liquid heat capacity. In conjunction with the Fourier constitutive Eq. (15) it provides the heat equation:

$$\rho c\frac{\partial T}{\partial t} = \nabla(k_T\nabla T) \tag{17}$$

Setting $k_T \cong constant$ (as usual in many classical continuum thermomechanics problems [12]), Eq. (17) reduces to the well-known parabolic heat equation:

$$\rho c\frac{\partial T}{\partial t} = k_T\nabla^2 T \tag{18}$$

For one-dimensional heat flow in infinite domain, Eq. (18) reduces to:

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \tag{19}$$

where $D = \frac{k_T}{\rho c}$ is the thermal diffusivity. Assuming that, in a given system of units, at the initial instant $t = 0$ the temperature takes the value 1 for $x = 0$ and 0 elsewhere:

$$T(x, 0) = \delta(x) \tag{20}$$

where $\delta(x)$ is the Dirac distribution, then the solution of Eq. (19) decays exponentially [7]:

$$T(x, t) = \frac{1}{2\sqrt{\pi Dt}}e^{-\frac{x^2}{4Dt}} \tag{21}$$

It should be stressed that the condition $T = 0$ at infinity, for an infinite medium stimulated at $x = 0$, is not part of the Fourier model but it is a given boundary condition used for solving Eq. (19).

According to Eq. (21), for $t > 0$ and for any x, it results that $T > 0$. Such a property, which involves the infinitesimal heat diffusion process, is considered by many authors to be paradoxical [13, 14]. To overcome the paradox, several generalizations of the classical heat conduction theory have been proposed (a review can be found in [7]). On the other hand, some theoretical works against the existence of the paradox have been suggested (a review can be found in [6]).

In line with the formulation proposed in this note, if $|\delta T| < |\mathrm{d}T|$, $T + \delta T = T$ and, consequently, the paradoxical nature of Fourier theory is only apparent: for $t < \mathrm{d}t$, $T = 0$; for $t \geq \mathrm{d}t \exists \bar{x}$: for$x > \bar{x}, T < \mathrm{d}T$. The infinitesimal heat diffusion process involves the elementary temperature scale which, in turn, involves the discrete aspect of classical continuum thermomechanics.

5 Conclusions

Within the framework of classical continuum thermomechanics, a discrete approach, based on the elementary scales, has been introduced. The spatio-temporal continuity of the motion allows defining the elementary spatial scale, $|\mathrm{d}\boldsymbol{r}|$, and the elementary time scale, $|\mathrm{d}t|$. Next to $|\mathrm{d}\boldsymbol{r}|$ and $|\mathrm{d}t|$, the elementary scales for the density, $|\mathrm{d}\rho|$, temperature, $|\mathrm{d}T|$, and pressure, $|\mathrm{d}p|$, have been defined. The link between $|\mathrm{d}\rho|$, $|\mathrm{d}T|$ and $|\mathrm{d}p|$ has been deduced using the continuity equation and the equation of state. In agreement with the spatio-temporal continuity of the motion, the elementary scales are regular functions of both space $\boldsymbol{r}$ and time t. The classical continuum thermomechanics imposes some constraints to $|\mathrm{d}\boldsymbol{r}|$ and $|\mathrm{d}t|$, which in turn involve some dimensionless numbers, such as $\|\nabla \boldsymbol{v} \mathrm{d}t\|$, $|\nabla \cdot \boldsymbol{v} \mathrm{d}t|$, $\alpha|\mathrm{d}T|$, $\alpha|\nabla T \cdot \mathrm{d}\boldsymbol{r}|$. These obtained results indicate that $|\mathrm{d}\boldsymbol{r}|$ and $|\mathrm{d}t|$, which are related to the Local Thermodynamic Equilibrium (LTE) condition and, in turn, to the $\|\nabla \boldsymbol{v}\|$ and $|\nabla T|$, are frame indifferent.

The proposed approach, based on the elementary scales, has been employed to remove the heat conduction paradox concerning the infinite speed of signal diffusion in the Fourier theory.

For the sake of clarity, this paper is focused on the Fourier fluids in the liquid state. The line of reasoning can be extended to any continuum system which is in hydrodynamic regime (in LTE condition). The hydrodynamic regime is characterized by long wavelengths and low frequencies. Beyond these limits, the classical continuum thermomechanics fails in providing an adequate description of the physical phenomena: the phenomena which involve very high frequencies and short wavelengths require a formulation that extends the classical continuum thermomechanics [14].

5.1 Conflict of interst

The authors declare that they have no conflict of interest.

Declarations

Conflicts of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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