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# A generalized integro-differential theory of nonlocal elasticity of *n*-Helmholtz type—part II: boundary-value problems in the one-dimensional case

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**Abstract** This paper is the second in a series of two that deal with a generalized theory of nonlocal elasticity of $n$-Helmholtz type. This terminology is motivated by the fact that the attenuation function (kernel) of the integral type nonlocal constitutive equation is the Green function associated with a generalized Helmholtz differential operator of order $n$. In the first paper, the governing equations have been derived and supported by suitable thermodynamic arguments. In this second paper, the proposed nonlocal model is specialized for the one-dimensional case to solve boundary-value problems. First, the relevant higher-order nonstandard boundary conditions in the differential (or, more precisely, integro-differential) version of the theory are derived. These boundary conditions are consistent with the particular family of attenuation functions adopted in the integral formulation. Then, some simple applications to statics and dynamics problems are presented. In particular, the theory is used to capture the static response and to perform free vibration analysis of a discrete lattice model with periodic microstructure (mass-and-spring chain) featured by nearest neighbor and next nearest neighbor particle interactions. In the latter case, boundary effects arise at the two lattice ends that are well captured by the proposed nonlocal continuum formulation. The nonlocal material parameters are identified a priori by matching the dispersion curve of the discrete lattice model, and a comparison in terms of attenuation function is also presented.

**Keywords** Nonlocal elasticity · Gradient elasticity · Internal length scale · Enriched continua · Helmholtz equation · Wave dispersion · Discrete lattice · Size effects

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## 1 Introduction

The motivations of this study (comprising two companion Part I and II papers) have been stated in the Introduction of the Part I paper. Here, we limit ourselves to provide a summary of the main aspects and basic formulae of the previous Part I paper for completeness, and an outline of the most important contents of the present Part II paper.

### 1.1 Summary of the part I paper

Eringen's 1983 theory of nonlocal elasticity [1] is governed by an integral constitutive relation between the nonlocal stress and the local strain through the convolution of an attenuation function (or kernel) and the local Hookean stress field. This theory can also be cast as a two-phase local/nonlocal integral model [2–4] as follows

$$t_{ij}(\mathbf{x}) = \xi_0 C_{ijkl}\varepsilon_{kl}(\mathbf{x}) + \xi_1 \int_V \alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;\ell) C_{ijkl}\varepsilon_{kl}(\mathbf{y})\mathrm{d}V_{\mathbf{y}} \tag{1}$$

in which one phase (of volume fraction $\xi_0$) has local elastic behavior and the complementary phase (of volume fraction $\xi_1$, with $\xi_0 + \xi_1 = 1$) complies with nonlocal elasticity. The superscript (1) in the kernel $\alpha$ entering the integral of the nonlocal phase in (1) denotes nonlocal elasticity of first order—Eringen type. In other words, Eq. (1) can be re-written as

$$t_{ij}(\mathbf{x}) = \xi_0 C_{ijkl}\varepsilon_{kl}(\mathbf{x}) + \xi_1 C_{ijkl}\bar{\varepsilon}^{(1)}_{kl}(\mathbf{x}) \tag{2}$$

with $\bar{\varepsilon}^{(1)}_{kl}(\mathbf{x})$ representing the so-called first order nonlocal strain tensor, obtained by applying an integral, *first order regularization operator* $\mathscr{R}^{(1)}_\ell$ to the local strain field as follows

$$\bar{\varepsilon}^{(1)}_{kl}(\mathbf{x}) = \mathscr{R}^{(1)}_\ell[\varepsilon_{kl}(\mathbf{x})] = \int_V \alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;\ell)\varepsilon_{kl}(\mathbf{y})\mathrm{d}V_{\mathbf{y}} \tag{3}$$

and the attenuation function $\alpha^{(1)}(|\mathbf{x}-\mathbf{y}|,\ell)$ is termed *first order kernel* accordingly.

Among the mathematical properties of the first order (Eringen type) kernel $\alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;\ell)$, it was demonstrated that it is the Green function of a linear differential operator, for instance the *first order Helmholtz differential operator* $\mathscr{L}^{(1)}_\ell = 1 - \ell^2\nabla^2$

$$\mathscr{L}^{(1)}_\ell \alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;\ell) = \delta(\mathbf{x}-\mathbf{y}) \,. \tag{4}$$

Therefore, this is also called nonlocal elasticity theory of Helmholtz type. In the literature, nonlocal elasticity theory of bi-Helmholtz type was proposed as extension of the Helmholtz type one. According to the nomenclature of this paper, the bi-Helmholtz theory is governed by a *second order nonlocal kernel* $\alpha^{(2)}$ and by a *second order Helmholtz differential operator* $\mathscr{L}^{(2)}_{c_1,c_2}$ that can be defined as follows, respectively

$$\begin{aligned}
\alpha^{(2)}(|\mathbf{x}-\mathbf{y}|;c_1,c_2) &= \alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;c_1) * \alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;c_2) \\
&= \int_{V_\infty} \alpha^{(1)}(|\mathbf{x}-\mathbf{y}_1|;c_1)\,\alpha^{(1)}(|\mathbf{y}_1-\mathbf{y}|;c_2)\mathrm{d}V_{\mathbf{y}_1}, \\
\mathscr{L}^{(2)}_{c_1,c_2} &= \mathscr{L}^{(1)}_{c_1}\mathscr{L}^{(1)}_{c_2} = \left(1 - c_1^2\nabla^2\right)\left(1 - c_2^2\nabla^2\right) \\
&= 1 - \ell_1^2\nabla^2 + \ell_2^4\nabla^4
\end{aligned} \tag{5}$$

where the symbol $*$ denotes the convolution product. In other words, the second order kernel $\alpha^{(2)}$ can be viewed as the convolution product of two first order (Eringen type) kernels $\alpha^{(1)}$, while the second order Helmholtz differential operator $\mathscr{L}^{(2)}$ is the product of two first order Helmholtz differential operators $\mathscr{L}^{(1)}$, each of which with its own length scale parameter $c_i\,(i = 1, 2)$. As such, the second order kernel is the Green function of the second order Helmholtz differential operator, that is

$$\mathscr{L}^{(2)}_{c_1,c_2}\alpha^{(2)}(|\mathbf{x}-\mathbf{y}|;c_1,c_2) = \delta(\mathbf{x}-\mathbf{y}) \,. \tag{6}$$

The bi-Helmholtz type nonlocal elasticity theory was envisaged by Eringen in a preliminary format [5, 6], and then subsequently (and more extensively) studied by Lazar et al. [7], who also provided the nonlocal kernels of bi-Helmholtz type in one, two and three dimensions along with the physical admissibility conditions between the length scale parameters $\ell_1, \ell_2$ and the length scale coefficients $c_1, c_2$.

Building on these insights and, above all, on these particular interpretations of the bi-Helmholtz type nonlocal elasticity theory via the aforementioned second order kernels and second order Helmholtz differential operator, we have extended this concept to develop a generalized nonlocal elasticity of $n$-Helmholtz type. This generalized theory is governed by a so-called *nth order kernel* $\alpha^{(n)}$ and *nth order Helmholtz differential operator* $\mathscr{L}^{(n)}$ defined as, respectively

$$\alpha^{(n)}(|\mathbf{x}-\mathbf{y}|;c_1,c_2,\ldots,c_n)=\alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;c_1)* \alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;c_2)*\cdots(n \text{ times})\cdots*\alpha^{(1)}(|\mathbf{x}-\mathbf{y}|;c_n) = \int_{V_\infty}\cdots(n-1)\text{ times}\int_{V_\infty}\alpha^{(1)}(|\mathbf{x}-\mathbf{y}_1|;c_1)\,\alpha^{(1)} \Big(|\mathbf{y}_1-\mathbf{y}_2|;c_2)\cdots\alpha^{(1)}(|\mathbf{y}_{n-1}-\mathbf{y}|;c_n)\mathrm{d}V_{\mathbf{y}_1}\mathrm{d}V_{\mathbf{y}_2} \cdots\mathrm{d}V_{\mathbf{y}_{n-1}} \mathscr{L}^{(n)}_{c_1,c_2,\ldots,c_n}=\mathscr{L}^{(1)}_{c_1}\mathscr{L}^{(1)}_{c_2}\cdots(n\text{ times}) \cdots\mathscr{L}^{(1)}_{c_n}=\prod_{k=1}^{n}\mathscr{L}^{(1)}_{c_k}=\sum_{k=0}^{n}(-1)^k\ell_k^{2k}\,\nabla^{2k} \tag{7}$$

which represent the convolution product of $n$ first order kernels and the product of $n$ first order Helmholtz differential operator $\mathscr{L}^{(1)}$, each of which with its own length scale parameters $c_k\,(k=1,2,\ldots,n)$. The relations between the length scale parameters $\ell_k(k=1,\ldots,n)$ appearing on the right-hand-side of $\mathscr{L}^{(n)}$ in $(7)_2$, and the length scale coefficients $c_k\;(k=1,2,\ldots,n)$ of the originating first order nonlocal kernels were reported in Appendix A of the Part I paper. In the spirit of Eq. (2) representing the two-phase local/nonlocal integral model for nonlocal elasticity of first order, the corresponding constitutive equation for the nonlocal elasticity theory of order $n$ is generalized as follows

$$t_{ij}(\mathbf{x})=C_{ijkl}\sum_{j=0}^{n}\xi_j\mathscr{R}^{(j)}_{c_1,c_2,\ldots,c_j}[\varepsilon_{kl}(\mathbf{x})] = C_{ijkl}\sum_{j=0}^{n}\xi_j\bar{\varepsilon}^{(j)}_{kl}(\mathbf{x}) \tag{8}$$

with $\mathscr{R}^{(j)}_{c_1,c_2,\ldots,c_j}[\cdot]=\mathscr{R}^{(1)}_{c_1}[\mathscr{R}^{(1)}_{c_2}[\ldots\mathscr{R}^{(1)}_{c_j}[\cdot]]]$ denoting the $j$th order regularization operator and $\bar{\varepsilon}^{(j)}_{kl}(\mathbf{x})$ the corresponding $j$th order nonlocal strain tensor that are related to each other through the $j$th order kernel $\alpha^{(j)}$ (the latter defined in $(7)_1$ for $j$ in place of $n$)

$$\bar{\varepsilon}^{(j)}_{kl}(\mathbf{x})=\mathscr{R}^{(j)}_{c_1,c_2,\ldots,c_j}[\varepsilon_{kl}(\mathbf{x})] =\int_V\alpha^{(j)}(|\mathbf{x}-\mathbf{y}|;c_1,c_2,\ldots,c_j)\varepsilon_{kl}(\mathbf{y})\mathrm{d}V_{\mathbf{y}} \tag{9}$$

while the zero order regularization operator $\mathscr{R}^{(0)}[\cdot]$ corresponds to a kernel coinciding with the Dirac delta function, i.e. $\alpha^{(0)}(|\mathbf{x}-\mathbf{y}|)=\delta(\mathbf{x}-\mathbf{y})$, and is such that the related nonlocal and local strain tensor of order zero coincide

$$\bar{\varepsilon}^{(0)}_{kl}(\mathbf{x})=\mathscr{R}^{(0)}[\varepsilon_{kl}(\mathbf{x})] =\int_V\delta(\mathbf{x}-\mathbf{y})\varepsilon_{kl}(\mathbf{y})\mathrm{d}V_{\mathbf{y}}=\varepsilon_{kl}(\mathbf{x}). \tag{10}$$

Expression (8) represents a generalized constitutive relation for an $(n+1)$-phase material, in which one phase (of volume fraction $\xi_0$) has local elastic behavior, and the remaining $n$ phases (of volume fractions $\xi_i$, $i=1,\ldots,n$ and such that $\xi_0+\xi_1+\cdots+\xi_n=1$) comply with nonlocal elasticity of higher order.

In Appendix B of the Part I paper, it was demonstrated that the $n$th order kernel $\alpha^{(n)}$ is the Green function of the $n$th order Helmholtz differential operator $\mathscr{L}^{(n)}$, that is

$$\mathscr{L}^{(n)}_{c_1,c_2,\ldots,c_n}\alpha^{(n)}(|\mathbf{x}-\mathbf{y}|;c_1,c_2,\ldots,c_n)=\delta(\mathbf{x}-\mathbf{y}). \tag{11}$$

Furthermore, a set of other useful properties of the $\mathscr{L}$ operator in relationship to the nonlocal strain tensor have been introduced and demonstrated in the Part I paper, which are here recalled for completeness (see Appendix B of the Part I paper for more details)

$$\mathscr{L}^{(n)}_{c_1,c_2,\ldots,c_n}\bar{\varepsilon}^{(n)}_{kl}(\mathbf{x})=\varepsilon_{kl}(\mathbf{x})$$
$$\begin{cases}\mathscr{L}^{(n)}_{c_1,c_2,\ldots,c_n}\bar{\varepsilon}^{(j)}_{kl}(\mathbf{x})=\mathscr{L}^{(n-j)}_{c_{j+1},c_{j+2},\ldots,c_n}\varepsilon_{kl}(\mathbf{x}) & \text{for } n>j\\ \bar{\varepsilon}^{(j)}_{kl}(\mathbf{x})=\mathscr{L}^{(n-j)}_{c_{j+1},c_{j+2},\ldots,c_n}\bar{\varepsilon}^{(n)}_{kl}(\mathbf{x}) & \text{for } n>j\end{cases} \tag{12}$$

where $\mathscr{L}^{(n-j)}_{c_{j+1},c_{j+2},\ldots,c_n}$ represents the truncated $(n-j)$th order Helmholtz operator (with $n>j$) defined as

$$\mathscr{L}^{(n-j)}_{c_{j+1},c_{j+2},\ldots,c_n}=\mathscr{L}^{(1)}_{c_{j+1}}\mathscr{L}^{(1)}_{c_{j+2}}\cdots((n-j)\text{ times})\cdots\mathscr{L}^{(1)}_{c_{j+n}} =\prod_{k=1}^{n-j}\mathscr{L}^{(1)}_{c_{j+k}}=\sum_{k=0}^{n-j}(-1)^k\ell_{j,k}^{2k}\,\nabla^{2k} \tag{13}$$

for $n>j$.

The relationships between the length scale parameters $\ell^{2k}_{j,k}(k=1,\ldots,n-j)$ of the truncated $(n-j)$th order Helmholtz operator and the length scale coefficients $c_{j+1},c_{j+2},\ldots,c_n$ of the originating first order Helmholtz operators are derived in Appendix A of the Part I paper.

It was demonstrated that Eq. (8) represents a thermodynamically consistent nonlocal elasticity model that can be derived from suitable thermodynamics arguments. Additionally, it was demonstrated that the generalized nonlocal elasticity model in the integral form (8) can be converted into equivalent differential and integro-differential forms as follows, respectively

$$\mathscr{L}^{(n)} t_{ij}(\mathbf{x}) = C_{ijkl} \sum_{j=0}^{n} \xi_j \mathscr{L}^{(n-j)}_{c_{j+1},c_{j+2},\ldots,c_n} \varepsilon_{kl}(\mathbf{x})$$
$$t_{ij}(\mathbf{x}) = C_{ijkl} \sum_{j=0}^{n} \xi_j \mathscr{L}^{(n-j)}_{c_{j+1},c_{j+2},\ldots,c_n} \bar{\varepsilon}^{(n)}_{kl}(\mathbf{x}) \tag{14}$$

which, considering the definitions in (7) and (13), can be more explicitly written as, respectively

$$\sum_{k=0}^{n} (-1)^k \ell_k^{2k} \nabla^{2k} t_{ij}(\mathbf{x}) = C_{ijkl} \sum_{k=0}^{n} (-1)^k \ell_{\xi,k}^{2k} \nabla^{2k} \varepsilon_{kl}(\mathbf{x})$$
$$t_{ij}(\mathbf{x}) = C_{ijkl} \sum_{k=0}^{n} (-1)^k \ell_{\xi,k}^{2k} \nabla^{2k} \bar{\varepsilon}^{(n)}_{kl}(\mathbf{x}) \tag{15}$$

where an additional set of $n$ length scale parameters $\ell_{\xi,k}^{2k}$ appears on the right-hand-side of both the expressions, which are simple linear functions of the material phase parameters $\xi_j$ multiplied by the corresponding $\ell_{j,k}^{2k}$ length scale parameter of the truncated $(n-j)$th order Helmholtz operator, namely

$$\ell_{\xi,k}^{2k} = \sum_{j=0}^{n-k} \xi_j \ell_{j,k}^{2k} \tag{16}$$

with $\ell_{\xi,0} = 1$ and $\ell_{\xi,n}^{2n} = \xi_0 \ell_{0,n}^{2n} \equiv [1 - (\xi_1 + \xi_2 + \cdots + \xi_n)]\ell_n^{2n}$ because $\xi_0 + \xi_1 + \xi_2 + \cdots + \xi_n = 1$. As a result, the differential and integro-differential constitutive relations (15) are characterized by $2n$ free parameters, namely the $n$ length scale parameters $\ell_k (k = 1, \ldots, n)$ for the nonlocal stress (which are functions of the $n$ length scale coefficients $c_k (k = 1, \ldots, n)$ of the originating kernels) and other $n$ length scale parameters $\ell_{\xi,k}^{2k} (k = 1, \ldots, n)$ for the nonlocal strain (which are functions of the previous $n$ length scale coefficients $c_k (k = 1, \ldots, n)$ and of $n$ independent material phase parameters $\xi_j (j = 1, \ldots, n)$, as $\xi_0$ can be expressed in terms of the remaining $n$, $\xi_0 = 1 - (\xi_1 + \xi_2 + \cdots + \xi_n)$). When the $n$ Helmholtz operators in $(7)_2$ and (13) are assumed with the same length scale coefficient $c_1 = c_2 = \cdots = c_n = \ell$, the length scale parameters $\ell_k (k = 1, \ldots, n)$ and $\ell_{\xi,k}^{2k} (k = 1, \ldots, n)$ of the differential and intergro-differential constitutive equations (15) are linear functions of the unique length scale coefficient $\ell$ and assume the following simplified format

$$\ell_k^{2k} = \beta_{k,n} \ell^{2k}$$
$$\ell_{\xi,k}^{2k} = \sum_{j=0}^{n-k} \xi_j \beta_{k,n-j} \ell^{2k} \tag{17}$$

where the constants $\beta_{k,n}$ have the expression

$$\beta_{k,n} = \frac{n!}{k!\,(n-k)!}\,. \tag{18}$$

In this case, the number of free nonlocal material parameter reduces from $2n$ to $n+1$, namely $n$ independent material phase parameters $\xi_j (j = 1, \ldots, n)$ and one length scale parameter $\ell$.

### 1.2 Outline of the present part II paper

Based on the previous expressions of the generalized nonlocal elasticity theory of $n$-Helmholtz type in the multi-dimensional space, the present Part II paper is concerned with the particularization for the *one-dimensional case*. In passing from the integral to the differential formulation, a set of accompanying non-standard boundary conditions must be introduced to solve the boundary-value problem [8]. These nonstandard boundary conditions are here derived in explicit form in the one-dimensional case and are consistent with the particular family of attenuation functions adopted in the integral formulation. This allows the identification of closed-form solutions for simple benchmark problems that are useful for the validation of the proposed approach. Then, a few simple application of the theory in statics and dynamics are presented. In particular, the nonlocal theory is used to capture the response of a discrete lattice system with nearest neighbor (NN) and next-nearest neighbor (NNN) particle interactions. The nonlocal material parameters are identified by matching the dispersion curve of the corresponding discrete lattice model. Boundary effects arise for the NNN interactions that are well described by the proposed generalized theory of nonlocal elasticity. The generalized attenuation function for different order of the nonlocal elasticity theory is compared to the exact attenuation function of the discrete lattice derived through the Fourier transform method. Free vibration analysis is also performed to compare a set of eigenfunctions (modal shapes) obtained by the higher-order nonlocal continuum with the natural modes of vibration of the corresponding discrete lattice model.

## 2 Generalized theory of nonlocal elasticity in the one-dimensional case

Let us consider a one-dimensional bar element of length $L$ made of nonlocal elastic material. In the one-dimensional case, the generalized nonlocal elasticity model relates the nonlocal stress component $t \equiv t_x$ to the local strain component $\varepsilon \equiv \varepsilon_x$ through the following integral, differential and integro-differential relationships, respectively

$$\begin{aligned} t(x) &= E\sum_{j=0}^{n} \xi_j \mathscr{R}^{(j)}[\varepsilon(x)] = E\sum_{j=0}^{n} \xi_j \bar{\varepsilon}^{(j)}(x) \\ \sum_{k=0}^{n}(-1)^k \ell_k^{2k}\frac{d^{2k}t(x)}{dx^{2k}} &= E\sum_{k=0}^{n}(-1)^k \ell_{\xi,k}^{2k}\frac{d^{2k}\varepsilon(x)}{dx^{2k}} \\ t(x) &= E\sum_{k=0}^{n}(-1)^k \ell_{\xi,k}^{2k}\frac{d^{2k}\bar{\varepsilon}^{(n)}(x)}{dx^{2k}} \end{aligned} \tag{19}$$

with $E$ denoting the Young's modulus and $x$ denoting the reference axis of the one-dimensional bar. It is instructive to particularize expressions (19) in the following cases:

- for $n = 1$

$$\begin{aligned} t(x) &= E\xi_0\varepsilon(x) + E\xi_1\bar{\varepsilon}^{(1)}(x) \\ t(x) - \ell_1^2 t''(x) &= E\left(\varepsilon(x) - \ell_{\xi,1}^2\,\varepsilon''(x)\right) \\ t(x) &= E\left(\bar{\varepsilon}^{(1)}(x) - \ell_{\xi,1}^2\,\bar{\varepsilon}^{(1)''}(x)\right) \end{aligned} \tag{20}$$

  where primes denote derivatives with respect to the spatial coordinate $x$;
- for $n = 2$

$$\begin{aligned} t(x) &= E\xi_0\varepsilon(x) + E\xi_1\bar{\varepsilon}^{(1)}(x) + E\xi_2\bar{\varepsilon}^{(2)}(x) \\ t(x) - \ell_1^2 t''(x) + \ell_2^4 t''''(x) &= E\left(\varepsilon(x) - \ell_{\xi,1}^2\,\varepsilon''(x) + \ell_{\xi,2}^4\,\varepsilon''''(x)\right) \\ t(x) &= E\left(\bar{\varepsilon}^{(2)}(x) - \ell_{\xi,1}^2\,\bar{\varepsilon}^{(2)''}(x) + \ell_{\xi,2}^4\,\bar{\varepsilon}^{(2)''''}(x)\right). \end{aligned} \tag{21}$$

### 2.1 Derivation of higher-order nonlocal kernels for the one-dimensional case

In the one-dimensional case, the first order kernel $\alpha^{(1)}$, which is the Green function of the first order Helmholtz operator $\mathscr{L}_{c_1}^{(1)} = \mathscr{L}_H(c_1) = 1 - c_1^2\frac{d^2}{dx^2}$, is the bi-exponential function

$$\alpha^{(1)}(r\,;c_1) = \frac{1}{2c_1}\exp\left(-\frac{r}{c_1}\right) \tag{22}$$

where $r = |x - y|$. The $n$th order kernel $\alpha^{(n)}$ can be evaluated through the convolution product of $\alpha^{(1)}$ $n$ times as per Eq. $(7)_1$ (each nonlocal kernel with its own length scale coefficient $c_k\,(k = 1, 2, \ldots, n)$), which for $n = 2, 3, 4$ gives

$$\begin{aligned} \alpha^{(2)}(r\,;c_1,c_2) &= \frac{c_1\exp(-r/c_1)}{2(c_1^2-c_2^2)} + \frac{c_2\exp(-r/c_2)}{2(c_2^2-c_1^2)} \\ \alpha^{(3)}(r\,;c_1,c_2,c_3) &= \frac{c_1^3\exp(-r/c_1)}{2(c_1^2-c_2^2)(c_1^2-c_3^2)} \\ &\quad + \frac{c_2^3\exp(-r/c_2)}{2(c_2^2-c_1^2)(c_2^2-c_3^2)} \\ &\quad + \frac{c_3^3\exp(-r/c_3)}{2(c_3^2-c_1^2)(c_3^2-c_2^2)} \\ \alpha^{(4)}(r\,;c_1,c_2,c_3,c_4) &= \frac{c_1^5\exp(-r/c_1)}{2(c_1^2-c_2^2)(c_1^2-c_3^2)(c_1^2-c_4^2)} \\ &\quad + \frac{c_2^5\exp(-r/c_2)}{2(c_2^2-c_1^2)(c_2^2-c_3^2)(c_2^2-c_4^2)} \\ &\quad + \frac{c_3^5\exp(-r/c_3)}{2(c_3^2-c_1^2)(c_3^2-c_2^2)(c_3^2-c_4^2)} \\ &\quad + \frac{c_4^5\exp(-r/c_4)}{2(c_4^2-c_1^2)(c_4^2-c_2^2)(c_4^2-c_3^2)} \end{aligned} \tag{23}$$

showing a certain sequential mathematical structure that can replicated for higher $n$ values

$$\alpha^{(n)}(r;c_1,c_2,\ldots,c_n) = \frac{c_1^{2n-3}\exp(-r/c_1)}{2(c_1^2-c_2^2)\cdots(c_1^2-c_n^2)} + \frac{c_2^{2n-3}\exp(-r/c_2)}{2(c_2^2-c_1^2)\cdots(c_2^2-c_n^2)} \cdots + \frac{c_n^{2n-3}\exp(-r/c_n)}{2(c_n^2-c_1^2)\cdots(c_n^2-c_{n-1}^2)} = \frac{1}{2}\sum_{i=1}^{n}\frac{c_i^{2n-3}\exp(-r/c_i)}{\prod_{k=1\wedge k\neq i}^{n}(c_i^2-c_k^2)} \quad \text{for } n>1 \tag{24}$$

with $\wedge$ denoting the logical AND operator. In the limit case of nonlocal kernels having the same length scale coefficient, i.e., $c_1 = c_2 = \cdots = c_n = \ell$, expressions (24) up to sixth order ($n = 6$) simplify into

$$\begin{aligned}
\alpha^{(2)}(r;\ell) &= \frac{1}{4\ell^2}(\ell + r)\exp\left(-\frac{r}{\ell}\right)\\
\alpha^{(3)}(r;\ell) &= \frac{1}{16\ell^3}(3\ell^2 + 3\ell r + r^2)\exp\left(-\frac{r}{\ell}\right)\\
\alpha^{(4)}(r;\ell) &= \frac{1}{96\ell^4}(15\ell^3 + 15\ell^2 r + 6\ell r^2 + r^3)\exp\left(-\frac{r}{\mu}\right)\\
\alpha^{(5)}(r;\ell) &= \frac{1}{768\ell^5}\\
&\quad (105\ell^4 + 105\ell^3 r + 45\ell^2 r^2 + 10\ell r^3 + r^4)\exp\left(-\frac{r}{\mu}\right)\\
\alpha^{(6)}(r;\ell) &= \frac{1}{7680\ell^6}(945\ell^5 + 945\ell^4 r + 420\ell^3 r^2\\
&\quad + 105\ell^2 r^3 + 15\ell r^4 + r^5)\exp\left(-\frac{r}{\mu}\right)
\end{aligned} \tag{25}$$

which are displayed in Fig. 1 along with the first-order kernel (22).

As can be observed, the higher the order of the kernel, the larger the influence distance within which the nonlocal effects take place. It is worth noting that the second order kernels $\alpha^{(2)}$ reported in (23) and (25) (for different and equal length scales, respectively) coincide with the nonlocal kernel of bi-Helmholtz type (Green function of the bi-Helmholtz differential operator) provided by Lazar et al. [7] for the one-dimensional case.

Following Eringen [6] and Lazar et al. [7], an alternative and more elegant derivation of the higher-order nonlocal kernels can be obtained through the Fourier transform method. We use the following notation for the Fourier transform and its inverse

$$\mathscr{F}[f(r)] = \tilde{f}(k) = \int_{-\infty}^{\infty} f(r)\exp(\mathrm{i}kr)dr,$$
$$\mathscr{F}^{-1}\left[\tilde{f}(k)\right] = f(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)\exp(-\mathrm{i}kr)dk. \tag{26}$$

The Green function associated to the $n$-Helmholtz differential operator $(7)_2$ can be obtained by applying the Fourier transform to both sides of Eq. (11), which leads to

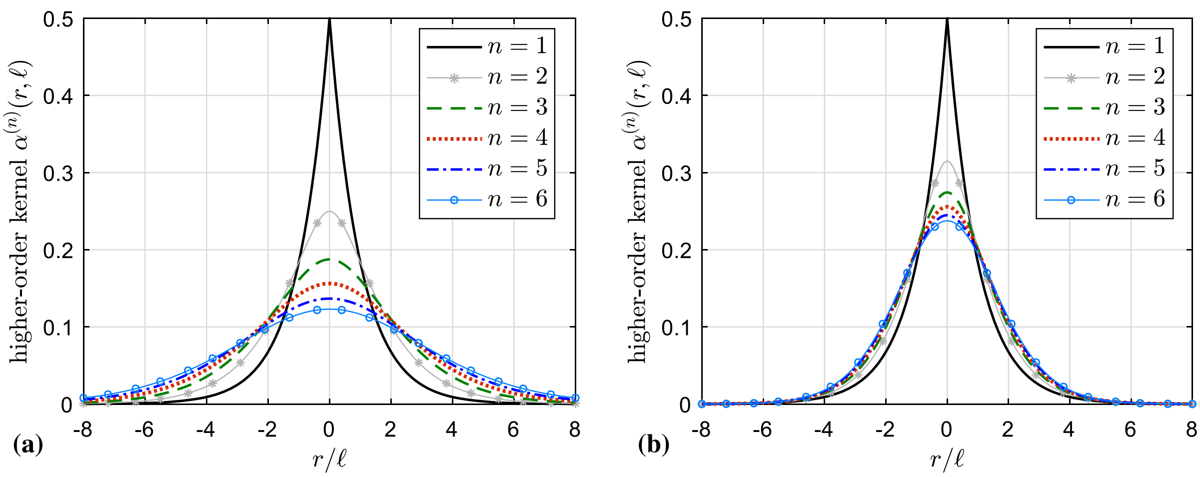



**Fig. 1** Higher-order nonlocal kernels (up to sixth order) as per Eq. (25): **a** for the same length scale parameter $\ell = 1$; **b** for the same influence zone across the distance $|r| = 6\ell$

$$\mathscr{F}\left[\mathscr{L}^{(n)}_{c_1,c_2,\ldots,c_n}\alpha^{(n)}(r\,;c_1,c_2,\ldots,c_n)\right] = \int_{-\infty}^{\infty}\mathscr{L}^{(n)}_{c_1,c_2,\ldots,c_n}\alpha^{(n)}(r\,;c_1,c_2,\ldots,c_n)\exp(\mathrm{i}kr)dr = \int_{-\infty}^{\infty}\delta(r)\exp(\mathrm{i}kr)dr = 1 \tag{27}$$

where, as before, $r = |x-y|$. Exploiting the properties of the Fourier transform, we obtain the Fourier transform of the Green function of the $n$-Helmholtz differential operator $(7)_2$ as

$$\mathscr{F}\left[\alpha^{(n)}(r\,;c_1,c_2,\ldots,c_n)\right] = \tilde{\alpha}^{(n)}(k\,;c_1,c_2,\ldots,c_n) = \frac{1}{\left(1+k^2c_1^2\right)\left(1+k^2c_2^2\right)\cdots\left(1+k^2c_n^2\right)} = \prod_{i=1}^{n}\frac{1}{(1+k^2c_i^2)} \tag{28}$$

which is the factorized as the product of $n$ terms $\left(1+k^2c_i^2\right)^{-1}$, each one being affected by a distinct length scale parameters $c_i\,(i=1,\ldots,n)$. We recall that the inverse Fourier transform of the first-order kernel is

$$\mathscr{F}^{-1}\left[\tilde{\alpha}^{(1)}(k\,;c_1)\right] = \alpha^{(1)}(r\,;c_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\alpha}^{(1)}(k\,;c_1)\exp(-\mathrm{i}kr)dk = \frac{1}{2c_1}\exp\left(-\frac{r}{c_1}\right) \tag{29}$$

which, in fact, coincides with the first order kernel in (22). Considering now the factorized structure of Eq. (28), by using the convolution theorem, the inverse Fourier transform applied to (28) straightforwardly leads to

$$\mathscr{F}^{-1}\left[\tilde{\alpha}^{(n)}(k\,;c_1,c_2,\ldots,c_n)\right] = \alpha^{(n)}(r\,;c_1,c_2,\ldots,c_n) = \alpha^{(1)}(r\,;c_1)*\alpha^{(1)}(r\,;c_2)*\cdots(n\text{ times})\cdots*\alpha^{(1)}(r\,;c_n) \tag{30}$$

which, in fact, coincides with the definition of the $n$th order nonlocal kernel in (7). In the limit case of nonlocal kernels having the same length scale coefficient, i.e., $c_1 = c_2 = \cdots = c_n = \ell$, the Fourier transform (28) reduces to

$$\mathscr{F}\left[\alpha^{(n)}(r\,;\ell)\right] = \tilde{\alpha}^{(n)}(k\,;\ell) = \frac{1}{(1+k^2\ell^2)(1+k^2\ell^2)\cdots(1+k^2\ell^2)} = \frac{1}{(1+k^2\ell^2)^n}. \tag{31}$$

### 2.2 Nonstandard boundary conditions consistent with the attenuation functions

The equivalence between integral $(19)_1$ and differential representations $(19)_{2,3}$ is accomplished through a set of nonstandard boundary conditions. In this paper, we derive the appropriate boundary conditions that are consistent with the particular choice of the family of attenuation functions reported in $(7)_1$ and explicitly written in (24) for a generic order $n$. It will be demonstrated that the integro-differential format of the Eq. $(19)_3$ is more convenient than the differential form $(19)_2$ for the derivation of the boundary conditions.

To this end, let us introduce the notion of left and right (one-dimensional) Helmholtz differential operators, $\mathscr{L}_\ell^{(-)}$ and $\mathscr{L}_\ell^{(+)}$, respectively, which arise from factorizing the first order Helmholtz operator as follows

$$\mathscr{L}_\ell^{(-)} = 1-\ell\frac{d}{dx};\quad \mathscr{L}_\ell^{(+)} = 1+\ell\frac{d}{dx};\quad \mathscr{L}_\ell^{(1)} = \mathscr{L}_\ell^{(-)}\mathscr{L}_\ell^{(+)} \tag{32}$$

Bearing in mind the one-dimensional domain of the bar $[0,L]$, we exploit the particular choice of the attenuation function (22) in the definition of the first order nonlocal strain (3)

$$\bar{\varepsilon}^{(1)}(x) = \int_0^L \alpha^{(1)}(|x-y|;c_1)\varepsilon(y)\mathrm{d}y = \frac{1}{2c_1}\int_0^L \exp\left(-\frac{|x-y|}{c_1}\right)\varepsilon(y)\mathrm{d}y. \tag{33}$$

The integral can be split into two parts in order to avoid the absolute value of the integrand, which also allows evaluating its first spatial derivative easily

$$\bar{\varepsilon}^{(1)}(x) = \frac{1}{2c_1}\left\{\int_0^x \exp[-(x-y)/c_1]\varepsilon(y)\mathrm{d}y + \int_x^L \exp[(x-y)/c_1]\varepsilon(y)\mathrm{d}y\right\},$$
$$\frac{d}{dx}\bar{\varepsilon}^{(1)}(x) = \frac{1}{2c_1^2}\left\{\int_0^x -\exp[-(x-y)/c_1]\varepsilon(y)\mathrm{d}y + \int_x^L \exp[(x-y)/c_1]\varepsilon(y)\mathrm{d}y\right\}. \tag{34}$$

Combining Eq. (34) with the left and right Helmholtz operators (32) gives

$$\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}^{(1)}(x)\big|_{x=0} = \left[\left(1 - c_1\frac{d}{dx}\right)\bar{\varepsilon}^{(1)}(x)\right]\Big|_{x=0} = 0$$
$$\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}^{(1)}(x)\big|_{x=L} = \left[\left(1 + c_1\frac{d}{dx}\right)\bar{\varepsilon}^{(1)}(x)\right]\Big|_{x=L} = 0 \tag{35}$$

which represent the nonstandard boundary conditions for the nonlocal elasticity of Helmholtz type ($n = 1$). Thus, for nonlocal elasticity of first order (or of Helmholtz type, $n = 1$) there are two boundary conditions (one in 0 and one in $L$). These boundary conditions are consistent with the attenuation function (22).

*Remark 1* It is worth noting that $\bar{\varepsilon}^{(1)}(x)$ is an integral functional of the local strain $\varepsilon(x)$, where the spatial variable $x$ appears in the two extremes of the definite integrals. To evaluate its first spatial derivative with respect to $x$, i.e., Equation $(34)_2$, the Leibniz integral rule has been applied, which is recalled here for completeness—for more details, see e.g. [9]

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y\right) = f(x,b(x))\frac{db(x)}{dx} - f(x,a(x))\frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\mathrm{d}y \tag{36}$$

where $a(x)$ and $b(x)$ are, alternatively, equal to a constant or equal to $x$ in the two definite integrals in $(34)_1$. As such, the first two terms on the right-hand-side (RHS) of (36) cancel out ($\varepsilon(x) - \varepsilon(x) = 0$) and only the last term with the partial derivative inside the integral is nonzero, which leads to the final expression reported in $(34)_2$.

As a next step, let us consider the nonlocal elasticity theory of bi-Helmholtz type ($n = 2$). We exploit the second order kernel $\alpha^{(2)}$ reported in (23) to calculate the second order nonlocal strain and its spatial derivatives (exploiting again the Leibniz rule (36) multiple times)

$$\begin{aligned}\bar{\varepsilon}^{(2)}(x) &= \int_0^L \left\{\frac{c_1\exp[-|x-y|/c_1]}{2(c_1^2-c_2^2)} + \frac{c_2\exp[-|x-y|/c_2]}{2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y \\ &= \int_0^x \left\{\frac{c_1\exp[-(x-y)/c_1]}{2(c_1^2-c_2^2)} + \frac{c_2\exp[-(x-y)/c_2]}{2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y \\ &\quad + \int_x^L \left\{\frac{c_1\exp[(x-y)/c_1]}{2(c_1^2-c_2^2)} + \frac{c_2\exp[(x-y)/c_2]}{2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y\end{aligned} \tag{37}$$

$$\frac{d}{dx}\bar{\varepsilon}^{(2)}(x) = \int_0^x \left\{-\frac{\exp[-(x-y)/c_1]}{2(c_1^2-c_2^2)} - \frac{\exp[-(x-y)/c_2]}{2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y + \int_x^L \left\{\frac{\exp[(x-y)/c_1]}{2(c_1^2-c_2^2)} + \frac{\exp[(x-y)/c_2]}{2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y \tag{38}$$

$$\frac{d^2}{dx^2}\bar{\varepsilon}^{(2)}(x) = \int_0^x \left\{\frac{\exp[-(x-y)/c_1]}{2c_1(c_1^2-c_2^2)} + \frac{\exp[-(x-y)/c_2]}{2c_2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y + \int_x^L \left\{\frac{\exp[(x-y)/c_1]}{2c_1(c_1^2-c_2^2)} + \frac{\exp[(x-y)/c_2]}{2c_2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y \tag{39}$$

$$\frac{d^3}{dx^3}\bar{\varepsilon}^{(2)}(x) = \int_0^x \left\{-\frac{\exp[-(x-y)/c_1]}{2c_1^2(c_1^2-c_2^2)} - \frac{\exp[-(x-y)/c_2]}{2c_2^2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y + \int_x^L \left\{\frac{\exp[(x-y)/c_1]}{2c_1^2(c_1^2-c_2^2)} + \frac{\exp[(x-y)/c_2]}{2c_2^2(c_2^2-c_1^2)}\right\}\varepsilon(y)\mathrm{d}y \tag{40}$$

Combining Eqs. (37)–(40) and considering the left and right Helmholtz operators in (32) with two independent length scales $c_1, c_2$, it follows that

$$
\begin{aligned}
\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}^{(2)}(x)\big|_{x=0} &= \left[\left(1-(c_1+c_2)\frac{d}{dx}+c_1c_2\frac{d^2}{dx^2}\right)\bar{\varepsilon}^{(2)}(x)\right]\Bigg|_{x=0}=0\\
\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}^{(2)}(x)\big|_{x=L} &= \left[\left(1+(c_1+c_2)\frac{d}{dx}+c_1c_2\frac{d^2}{dx^2}\right)\bar{\varepsilon}^{(2)}(x)\right]\Bigg|_{x=L}=0
\end{aligned}
\tag{41}
$$

which represent two nonstandard boundary conditions for the second order nonlocal strain in the nonlocal elasticity of bi-Helmholtz type ($n=2$). These boundary conditions are consistent with the attenuation function $\alpha^{(2)}$ in (23). Moreover, an additional set of two nonstandard boundary conditions can be derived as follows

$$
\begin{aligned}
\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(2)}(x)\big|_{x=0} &= \left[\left(1-c_2\frac{d}{dx}-c_1^2\frac{d^2}{dx^2}+c_1^2c_2\frac{d^3}{dx^3}\right)\bar{\varepsilon}^{(2)}(x)\right]\Bigg|_{x=0}=0,\\
\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(2)}(x)\big|_{x=L} &= \left[\left(1+c_2\frac{d}{dx}-c_1^2\frac{d^2}{dx^2}-c_1^2c_2\frac{d^3}{dx^3}\right)\bar{\varepsilon}^{(2)}(x)\right]\Bigg|_{x=L}=0.
\end{aligned}
\tag{42}
$$

Thus, combining (41) and (42), for nonlocal elasticity of second order (or of bi-Helmholtz type, $n=2$) there are four boundary conditions, two at $x=0$ and two at $x=L$.

We can extend this result further by considering nonlocal elasticity of third order ($n=3$). The third order nonlocal strain is governed by the third order nonlocal kernel $\alpha^{(3)}$ in (23), i.e.

$$
\begin{aligned}
\bar{\varepsilon}^{(3)}(x) &= \int_0^L \left[\frac{c_1^3\exp(-|x-y|/c_1)}{2(c_1^2-c_2^2)(c_1^2-c_3^2)}+\frac{c_2^3\exp(-|x-y|/c_2)}{2(c_2^2-c_1^2)(c_2^2-c_3^2)}\right.\\
&\quad\left.+\frac{c_3^3\exp(-|x-y|/c_3)}{2(c_3^2-c_1^2)(c_3^2-c_2^2)}\right]\varepsilon(y)\mathrm{d}y\\
&= \int_0^x \left\{\frac{c_1^3\exp[-(x-y)/c_1]}{2(c_1^2-c_2^2)(c_1^2-c_3^2)}+\frac{c_2^3\exp[-(x-y)/c_2]}{2(c_2^2-c_1^2)(c_2^2-c_3^2)}\right.\\
&\quad\left.+\frac{c_3^3\exp[-(x-y)/c_3]}{2(c_3^2-c_1^2)(c_3^2-c_2^2)}\right\}\varepsilon(y)\mathrm{d}y\\
&\quad+\int_x^L \left\{\frac{c_1^3\exp[(x-y)/c_1]}{2(c_1^2-c_2^2)(c_1^2-c_3^2)}+\frac{c_2^3\exp[(x-y)/c_2]}{2(c_2^2-c_1^2)(c_2^2-c_3^2)}\right.\\
&\quad\left.+\frac{c_3^3\exp[(x-y)/c_3]}{2(c_3^2-c_1^2)(c_3^2-c_2^2)}\right\}\varepsilon(y)\mathrm{d}y.
\end{aligned}
\tag{43}
$$

Computing the spatial derivatives of Eq. (43) in a similar manner to what already done for $n=2$ and $n=1$, it can be easily proved that the following three nonstandard boundary conditions consistent with the attenuation function $\alpha^{(3)}$ are obtained at $x=0$

$$
\begin{aligned}
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}^{(3)}(x)\big|_{x=0}\\
&\quad=\left[\left(1-(c_1+c_2+c_3)\frac{d}{dx}+(c_1c_2+c_1c_3\right.\right.\\
&\left.\left.+c_2c_3)\frac{d^2}{dx^2}-c_1c_2c_3\frac{d^3}{dx^3}\right)\bar{\varepsilon}^{(3)}(x)\right]\Bigg|_{x=0}=0\\
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(3)}(x)\big|_{x=0}\\
&\quad=\left[\left(1-(c_2+c_3)\frac{d}{dx}+(c_2c_3-c_1^2)\frac{d^2}{dx^2}\right.\right.\\
&\left.\left.+(c_1^2c_2+c_1^2c_3)\frac{d^3}{dx^3}-c_1^2c_2c_3\frac{d^4}{dx^4}\right)\bar{\varepsilon}^{(3)}(x)\right]\Bigg|_{x=0}=0\\
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(3)}(x)\big|_{x=0}=\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}^{(3)}(x)\big|_{x=0}\\
&\quad=\left[\left(1-c_3\frac{d}{dx}-(c_1^2+c_2^2)\frac{d^2}{dx^2}+(c_1^2c_3+c_2^2c_3)\frac{d^3}{dx^3}\right.\right.\\
&\quad\left.\left.+c_1^2c_2^2\frac{d^4}{dx^4}-c_1^2c_2^2c_3\frac{d^5}{dx^5}\right)\bar{\varepsilon}^{(3)}(x)\right]\Bigg|_{x=0}=0
\end{aligned}
\tag{44}
$$

wherein use has been made of the previous relations (41) and (35). Moreover, the following three nonstandard boundary conditions consistent with the attenuation function $\alpha^{(3)}$ are obtained at $x=L$

$$
\begin{aligned}
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}^{(3)}(x)\big|_{x=L}\\
&\quad=\left[\left(1+(c_1+c_2+c_3)\frac{d}{dx}+(c_1c_2+c_1c_3\right.\right.\\
&\left.\left.+c_2c_3)\frac{d^2}{dx^2}+c_1c_2c_3\frac{d^3}{dx^3}\right)\bar{\varepsilon}^{(3)}(x)\right]\Bigg|_{x=L}=0,\\
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(3)}(x)\big|_{x=L}\\
&\quad=\left[\left(1+(c_2+c_3)\frac{d}{dx}+(c_2c_3-c_1^2)\frac{d^2}{dx^2}\right.\right.\\
&\left.\left.-(c_1^2c_2+c_1^2c_3)\frac{d^3}{dx^3}-c_1^2c_2c_3\frac{d^4}{dx^4}\right)\bar{\varepsilon}^{(3)}(x)\right]\Bigg|_{x=L}=0,\\
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(3)}(x)\big|_{x=L}=\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}^{(3)}(x)\big|_{x=L}\\
&\quad=\left[\left(1+c_3\frac{d}{dx}-(c_1^2+c_2^2)\frac{d^2}{dx^2}-(c_1^2c_3+c_2^2c_3)\frac{d^3}{dx^3}\right.\right.\\
&\quad\left.\left.+c_1^2c_2^2\frac{d^4}{dx^4}+c_1^2c_2^2c_3\frac{d^5}{dx^5}\right)\bar{\varepsilon}^{(3)}(x)\right]\Bigg|_{x=L}=0.
\end{aligned}
\tag{45}
$$

Thus, combining (44) and (45), for nonlocal elasticity of third order, the differential equations must be supplemented with six boundary conditions, three at $x=0$ and three at $x=L$ in order for the differential form to be equivalent to the integral one.

Following the same rationale, it can be demonstrated that for nonlocal elasticity of fourth order ($n = 4$) the 4th order nonlocal strain must satisfy the following eight nonstandard boundary conditions, four at $x = 0$ and four at $x = L$, viz.

$$\begin{aligned}
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}^{(4)}(x)\big|_{x=0}= 0,\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(4)}(x)\big|_{x=0}= 0,\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(4)}(x)\big|_{x=0}\\
&\quad = \mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}^{(4)}(x)\big|_{x=0}= 0,\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(1)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(4)}(x)\big|_{x=0}\\
&\quad = \mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_1,c_2,c_3}^{(3)}\bar{\varepsilon}^{(4)}(x)\big|_{x=0}= 0,\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}^{(4)}(x)\big|_{x=L}= 0,\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(4)}(x)\big|_{x=L}= 0,\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(4)}(x)\big|_{x=L}\\
&\quad = \mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}^{(4)}(x)\big|_{x=L}= 0,\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(1)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}^{(4)}(x)\big|_{x=L}\\
&\quad = \mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_1,c_2,c_3}^{(3)}\bar{\varepsilon}^{(4)}(x)\big|_{x=L}= 0.
\end{aligned}\tag{46}$$

By extension of the above results, it can be demonstrated that for a generalized nonlocal elasticity model of $n$-Helmholtz type there are a total of $2n$ nonstandard boundary conditions that are consistent with the attenuation function $\alpha^{(n)}$, in particular $n$ conditions at $x = 0$ and the remaining $n$ at $x = L$, which can be expressed in the following compact form

$$\begin{cases}\mathscr{L}_{c_n,c_{n-1},\ldots,c_j}^{(p-)}\mathscr{L}_{c_1,c_2,\ldots,c_{j-1}}^{(j-1)}\bar{\varepsilon}^{(n)}(x)\big|_{x=0}= 0\\ \mathscr{L}_{c_n,c_{n-1},\ldots,c_j}^{(p+)}\mathscr{L}_{c_1,c_2,\ldots,c_{j-1}}^{(j-1)}\bar{\varepsilon}^{(n)}(x)\big|_{x=L}= 0\end{cases}\quad \left(\begin{array}{l}\text{for } j = 1,2,\ldots,n;\\ p = n-j+1; \quad n \geq j\end{array}\right)\tag{47}$$

where $\mathscr{L}^{(0)} = 1$ and the following definitions have been introduced for compact notation

$$\begin{aligned}
\mathscr{L}_{c_n,c_{n-1},\ldots,c_j}^{(p-)} &= \mathscr{L}_{c_n}^{(-)}\mathscr{L}_{c_{n-1}}^{(-)}\cdots(p\text{ times})\cdots\mathscr{L}_{c_j}^{(-)}\\
&= \prod_{k=0}^{n-j}\mathscr{L}_{c_{n-k}}^{(-)},\\
\mathscr{L}_{c_n,c_{n-1},\ldots,c_j}^{(p+)} &= \mathscr{L}_{c_n}^{(+)}\mathscr{L}_{c_{n-1}}^{(+)}\cdots(p\text{ times})\cdots\mathscr{L}_{c_j}^{(+)}\\
&= \prod_{k=0}^{n-j}\mathscr{L}_{c_{n-k}}^{(+)}.
\end{aligned}\tag{48}$$

It is worth noting that the nonstandard boundary conditions (47) are such that the number of $\mathscr{L}$ differential operators is equal to the order of the nonlocal strain field $\bar{\varepsilon}^{(n)}(x)$ to which they are applied (ascribed to the regularization integral operator of order $n$). A set of supplemental nonstandard boundary conditions of order $n$, i.e., involving $n$ $\mathscr{L}$ differential operators, applied to nonlocal strain field of order less than $n$, i.e., $\bar{\varepsilon}^{(k)}(x)$ $(k = 1, 2\ldots, n-1)$, is presented in Appendix A. These supplemental boundary conditions are useful for the next derivations of the eigenfunctions in the free vibration analysis in subsection 4.2. We emphasize that, to keep the analytical treatment as general as possible, the derivation of the boundary conditions has been presented for length scale parameters being different from each other ($c_1 \neq c_2 \neq c_3 \neq \cdots \neq c_n$). When $c_1 = c_2 = c_3 = \cdots = c_n = \ell$ all the previous relations still hold but the resulting boundary conditions can be compacted considerably.

In conclusion, the integro-differential equations $(19)_3$, which express the nonlocal stress $t(x)$ in terms of the $n$th order nonlocal strain $\bar{\varepsilon}^{(n)}(x)$, supplemented by the nonstandard boundary conditions (47) in terms of the $n$th order nonlocal strain $\bar{\varepsilon}^{(n)}(x)$ fully define the constitutive equations of the generalized theory of nonlocal elasticity of order $n$ (or of $n$-Helmholtz type). For the particular family of attenuation functions defined in (22) and (24), this set of integro-differential equations and boundary conditions are equivalent to the nonlocal elasticity model in integral form expressed by Eq. $(19)_1$. Indeed, the nonstandard boundary conditions (47) are consistent with such choice of nonlocal kernels in the integral formulation. It is noted that the integro-differential equations $(19)_3$ are more convenient than the differential equations $(19)_2$ since the boundary conditions are more easily derived in terms of higher-order nonlocal strains $\bar{\varepsilon}^{(n)}(x)$ rather than in terms of local strains $\varepsilon(x)$.

## 3 Governing constitutive equations in the one-dimensional case

In this section, we present the governing constitutive equations for the generalized nonlocal elasticity theory in the one-dimensional case. Let us introduce the compatibility equation and the cross-section equilibrium equation as

$$\varepsilon(x,t)=\frac{\partial u(x,t)}{\partial x}=u'(x,t)\quad\text{(compatibility equation)}$$
$$N(x,t)=\int_A t(x,t)\,\mathrm{d}A\qquad\text{(cross-section equilibrium equation)}\tag{49}$$

where $u(x,t)$ denotes the axial displacement of the bar at abscissa $x$ and at time $t$, $N(x,t)$ is the corresponding axial force, and $A$ the cross sectional area. The dynamic equilibrium equation of the bar is

$$N'(x,t)+q(x,t)=\rho A\,\ddot{u}(x,t)\quad\text{(dynamic equilibrium equation)}\tag{50}$$

where dots indicate derivatives with respect to time, $\rho$ is the mass density per unit volume, and $q\equiv q_x$ the axial distributed load per unit length. Substituting the compatibility Eq. $(49)_1$ into (19), integrating both sides over the area $A$ and taking into account Eq. $(49)_2$ leads to the following integral, differential and integro-differential generalized nonlocal elasticity models

$$N(x,t)=EA\sum_{j=0}^{n}\xi_j\mathscr{R}^{(j)}_{c_1,c_2,\ldots,c_j}[\varepsilon(x,t)]=EA\sum_{j=0}^{n}\xi_j\bar{\varepsilon}^{(j)}(x,t)$$
$$\sum_{k=0}^{n}(-1)^k\ell_k^{2k}\frac{\partial^{2k}N(x,t)}{\partial x^{2k}}=EA\sum_{k=0}^{n}(-1)^k\ell_{\xi,k}^{2k}\frac{\partial^{2k}\varepsilon(x,t)}{\partial x^{2k}}$$
$$N(x,t)=EA\sum_{k=0}^{n}(-1)^k\ell_{\xi,k}^{2k}\frac{\partial^{2k}\bar{\varepsilon}^{(n)}(x,t)}{\partial x^{2k}}\tag{51}$$

where $\bar{\varepsilon}^{(j)}(x,t)=\mathscr{R}^{(j)}_{c_1,c_2,\ldots,c_j}[\varepsilon(x,t)]$ is the $j$th order generalized nonlocal strain, and $\ell_{\xi,k}^{2k}$ are simple linear functions of the material phase parameters $\xi_j$ (such that $\sum_{j=0}^{n}\xi_j=1$) multiplied by the corresponding $\ell_{j,k}^{2k}$ length scale parameter of the truncated $(n-j)$th order Helmholtz operator, namely

$$\ell_{\xi,k}^{2k}=\sum_{j=0}^{n-k}\xi_j\ell_{j,k}^{2k}.\tag{52}$$

It is instructive to particularize expressions (51) in the following cases:

- for $n=1$

$$N(x,t)=EA\left[\xi_0\,\varepsilon(x,t)+\xi_1\,\bar{\varepsilon}^{(1)}(x,t)\right]$$
$$N(x,t)-\ell_1^2N''(x,t)=EA\left[\varepsilon(x,t)-\ell_{\xi,1}^2\varepsilon''(x,t)\right]$$
$$N(x,t)=EA\left[\bar{\varepsilon}^{(1)}(x,t)-\ell_{\xi,1}^2\bar{\varepsilon}^{(1)''}(x,t)\right]\tag{53}$$

Considering that $\xi_0=1-\xi_1$, expressions (53) underlie a two-parameter nonlocal elasticity model (independent parameters $\ell_1,\xi_1$);

- for $n=2$

$$N(x,t)=EA\left[\xi_0\,\varepsilon(x,t)+\xi_1\,\bar{\varepsilon}^{(1)}(x,t)+\xi_2\,\bar{\varepsilon}^{(2)}(x,t)\right]$$
$$N(x,t)-\ell_1^2N''(x,t)+\ell_2^4N''''(x,t)=EA\left[\varepsilon(x,t)-\ell_{\xi,1}^2\varepsilon''(x,t)+\ell_{\xi,2}^4\varepsilon''''(x,t)\right]$$
$$N(x,t)=EA\left[\bar{\varepsilon}^{(2)}(x,t)-\ell_{\xi,1}^2\bar{\varepsilon}^{(2)''}(x,t)+\ell_{\xi,2}^4\bar{\varepsilon}^{(2)''''}(x,t)\right]\tag{54}$$

Considering that $\xi_0=1-\xi_1-\xi_2$, expressions (54) underlie a four-parameter nonlocal elasticity model (independent parameters $\ell_1,\ell_2,\xi_1,\xi_2$). By extension of the above results, it can be seen that for a generic order $n$, there are $2n$ independent parameters, namely the $n$ length scale parameters $\ell_k(k=1,\ldots,n)$ and the $n$ independent material phase parameters $\xi_k(k=1,\ldots,n)$. When the $n$ Helmholtz operators are assumed with the same length scale coefficient $c_1=c_2=\cdots=c_n=\ell$, the number of free nonlocal material parameter reduces from $2n$ to $n+1$, namely $n$ independent material phase parameters $\xi_j(j=1,\ldots,n)$ and one length scale parameter $\ell$. All this is in line with the generalized nonlocal continuum formulation presented in the Part I paper.

Since Eqs. (51) are formally similar to (19), the corresponding nonstandard boundary conditions that are consistent with the attenuation functions resemble the expressions (47), namely

$$\begin{cases}\mathscr{L}^{(p-)}_{c_n,c_{n-1},\ldots,c_j}\mathscr{L}^{(j-1)}_{c_1,c_2,\ldots,c_{j-1}}\bar{\varepsilon}^{(n)}(x)\big|_{x=0}=0\\ \mathscr{L}^{(p+)}_{c_n,c_{n-1},\ldots,c_j}\mathscr{L}^{(j-1)}_{c_1,c_2,\ldots,c_{j-1}}\bar{\varepsilon}^{(n)}(x)\big|_{x=L}=0\\ \left(\begin{array}{l}\text{for } j=1,2,\ldots,n;\\ p=n-j+1;\quad n\geq j\end{array}\right)\end{cases}\tag{55}$$

which represent $2n$ boundary conditions for the $n$th order nonlocal strain. In conclusion, the integro-differential equations $(51)_3$, which express the nonlocal axial force $N(x,t)$ in terms of the $n$th order nonlocal strain $\bar{\varepsilon}^{(n)}(x,t)$, supplemented by the nonstandard boundary conditions (55), in terms of $\bar{\varepsilon}^{(n)}(x,t)$, fully define the nonlocal generalized constitutive equations of order $n$ (or of $n$-Helmholtz type) in the one-dimensional case. For the particular family of attenuation functions defined in (22) and (24), this set of integro-differential equations and boundary conditions are equivalent to the nonlocal elasticity model in integral form expressed by Eq. $(51)_1$. Indeed, the nonstandard boundary conditions (55) are consistent with such choice of nonlocal kernels in the integral formulation.

## 4 The boundary value problem for the generalized nonlocal elasticity theory in the one-dimensional case

In this subsection, we present the governing equations of the initial-boundary value problem for the generalized nonlocal elasticity theory in the one-dimensional case. The following derivations are based on the previous expressions reported for the generic order $n$, and then particularized for $n = 1, 2$ representing the Helmholtz and bi-Helmholtz type nonlocal elasticity theory.

Based on the previous derivations, the governing equations are

$$\begin{aligned} &N'(x,t) + q(x,t) = \rho A\,\ddot{u}(x,t) \quad \text{(dynamic equilibrium equation)} \\ &\mathscr{L}^{(n)}\bar{\varepsilon}^{(n)}(x,t) = \varepsilon(x,t) = \frac{\partial u(x,t)}{\partial x} \quad \text{(compatibility equation)} \\ &N(x,t) = EA\sum_{k=0}^{n}(-1)^k \ell_{\xi,k}^{2k}\frac{\partial^{2k}\bar{\varepsilon}^{(n)}(x,t)}{\partial x^{2k}} \quad \text{(constitutive equation)} \end{aligned} \tag{56}$$

Deriving the constitutive Eq. $(56)_3$ and substituting into the dynamic equilibrium Eq. $(56)_1$ we obtain the integro-differential equation (in terms of $\bar{\varepsilon}^{(n)}(x,t)$)

$$EA\left[\sum_{k=0}^{n}(-1)^k \ell_{\xi,k}^{2k}\frac{\partial^{2k+1}\bar{\varepsilon}^{(n)}(x,t)}{\partial x^{2k+1}}\right] = \rho A\,\ddot{u}(x,t) - q(x,t) \tag{57}$$

which can be solved in terms of the nonlocal elongation $\bar{\varepsilon}^{(n)}(x,t)$ by supplementing the relevant boundary conditions (55) along with some initial conditions for the displacement $u(x,t)$ and its derivatives. By applying the differential operator $\mathscr{L}^{(n)}_{c_1,c_2,\ldots,c_n}$ to Eq. (57) and taking into account the compatibility Eq. $(56)_2$ we obtain the following differential equation in terms of the displacement $u(x,t)$ and its spatial derivatives up to order $2n+2$, the acceleration $\ddot{u}(x,t)$ and its spatial derivatives up to order $2n$

$$\sum_{k=0}^{n}(-1)^k \ell_{k,\varepsilon}^{2k}\frac{\partial^{2k}[\rho A\,\ddot{u}(x,t) - q(x,t)]}{\partial x^{2k}} = EA\left[\sum_{k=0}^{n}(-1)^k \ell_{\xi,k,\varepsilon}^{2k}\frac{\partial^{2k+2}u(x,t)}{\partial x^{2k+2}}\right]. \tag{58}$$

### 4.1 Solutions in statics

Assuming, by hypothesis, that the distributed load $q(x,t)$ is applied in a quasi-static manner, we can ignore inertia effects in Eq. (57) and the time dependence of all the variables

$$EA\left[\sum_{k=0}^{n}(-1)^k \ell_{\xi,k}^{2k}\frac{d^{2k+1}\bar{\varepsilon}^{(n)}(x)}{dx^{2k+1}}\right] = -q(x). \tag{59}$$

The solution of Eq. (59) can be obtained as the sum of a homogeneous solution and a particular solution, that is $\bar{\varepsilon}^{(n)}(x) = \bar{\varepsilon}^{(n)}_{\text{hom}}(x) + \bar{\varepsilon}^{(n)}_{\text{part}}(x)$. The homogeneous solution can be assumed as a general exponential function $\bar{\varepsilon}^{(n)}_{\text{hom}}(x) = \exp(\kappa x)$, which leads to the following characteristic equation in $\kappa$

$$\kappa\sum_{k=0}^{n}(-1)^k \ell_{\xi,k}^{2k}\kappa^{2k} = 0 \tag{60}$$

whose $2n+1$ solutions depend on the numerical values of the $n$ length scale coefficients $\ell_{\xi,k}^{2k}(k = 1,\ldots,n)$, which in turn depend on the $n$ independent material phase parameters $\xi_k(k = 1,\ldots,n)$ and are related to the length scale parameters $\ell_k^{2k}(k = 1,\ldots,n)$. The general solution $\bar{\varepsilon}^{(n)}(x)$ is obtained by imposing a set of $2n+2$ boundary conditions, two of which of standard type (either essential or natural) and the remaining $2n$ of nonstandard type according to (47). It is instructive to particularize the solution in the following cases:

- for $n = 1$

the governing Eq. (57) in the quasi-static case reads

$$EA\left[\bar{\varepsilon}^{(1)'}(x) - \ell^2_{\xi,1}\bar{\varepsilon}^{(1)'''}(x)\right] = -q(x) \tag{61}$$

and the corresponding characteristic Eq. (60) reduces to

$$\kappa\left(1 - \ell^2_{\xi,1}\kappa^2\right) = 0 \tag{62}$$

which leads to three real solutions in $\kappa$, namely

$$\kappa_{1,2} = \pm\frac{1}{\sqrt{\ell^2_{\xi,1}}}; \quad \kappa_3 = 0. \tag{63}$$

These solutions form the homogeneous solution so that the general solution can be expressed in the following form

$$\bar{\varepsilon}^{(1)}(x) = C_1\exp(\kappa_1 x) + C_2\exp(\kappa_2 x) + C_3 + \bar{\varepsilon}^{(1)}_{\text{part}}(x). \tag{64}$$

Once the solution in terms of $\bar{\varepsilon}^{(1)}(x)$ is known, the solution in terms of local strain field can be obtained exploiting the compatibility Eq. $(56)_2$, that is

$$\varepsilon(x) = \mathscr{L}^{(1)}_{c_1}\bar{\varepsilon}^{(1)}(x) = \bar{\varepsilon}^{(1)}(x) - c_1^2\bar{\varepsilon}^{(1)''}(x) \tag{65}$$

and, finally, the solution in terms of displacements is obtained by integrating the local strain field (65) with an additional constant of integration as follows

$$u(x) = \int \varepsilon(x)\mathrm{d}x + C_4. \tag{66}$$

Thus, there are four constants of integration $C_1, C_2, C_3, C_4$ to be determined. Consequently, the general solution is obtained by imposing four boundary conditions, two of which of standard type that can be either essential boundary conditions in terms of

$$\begin{aligned}[u(x)]\big|_{x=0} &= \bar{u}_0\\ [u(x)]\big|_{x=L} &= \bar{u}_L\end{aligned} \tag{67}$$

or natural boundary conditions arising from the constitutive equations $(56)_3$

$$\begin{aligned}EA\left[\bar{\varepsilon}^{(1)}(x) - \ell^2_{\xi,1}\bar{\varepsilon}^{(1)''}(x)\right]\Big|_{x=0} &= \bar{N}_0\\ EA\left[\bar{\varepsilon}^{(1)}(x) - \ell^2_{\xi,1}\bar{\varepsilon}^{(1)''}(x)\right]\Big|_{x=L} &= \bar{N}_L\end{aligned} \tag{68}$$

and the remaining two of nonstandard type according to (35), namely

$$\begin{aligned}\mathscr{L}^{(-)}_{c_1}\bar{\varepsilon}^{(1)}(x)\big|_{x=0} &= \left[\left(1 - c_1\frac{d}{dx}\right)\bar{\varepsilon}^{(1)}(x)\right]\Big|_{x=0} = 0\\ \mathscr{L}^{(+)}_{c_1}\bar{\varepsilon}^{(1)}(x)\big|_{x=L} &= \left[\left(1 + c_1\frac{d}{dx}\right)\bar{\varepsilon}^{(1)}(x)\right]\Big|_{x=L} = 0\end{aligned} \tag{69}$$

- for $n = 2$

the governing Eq. (57) in the quasi-static case reads

$$EA\left[\bar{\varepsilon}^{(2)'}(x) - \ell^2_{\xi,1}\bar{\varepsilon}^{(2)'''}(x) + \ell^4_{\xi,2}\bar{\varepsilon}^{(2)'''''}(x)\right] = -q(x) \tag{70}$$

and the corresponding characteristic Eq. (60) reduces to

$$\kappa\left(1 - \ell^2_{\xi,1}\kappa^2 + \ell^4_{\xi,2}\kappa^4\right) = 0 \tag{71}$$

which leads to five real solutions in $\kappa$, namely

$$\kappa_{1,2,3,4} = \pm\frac{1}{\sqrt{2}}\sqrt{\frac{\ell^2_{\xi,1}}{\ell^4_{\xi,2}} \pm \frac{\sqrt{\ell^4_{\xi,1} - 4\ell^4_{\xi,2}}}{\ell^4_{\xi,2}}}; \quad \kappa_5 = 0. \tag{72}$$

These solutions form the homogeneous solution so that the general solution can be expressed in the following form

$$\begin{aligned}\bar{\varepsilon}^{(2)}(x) &= C_1\exp(\kappa_1 x) + C_2\exp(\kappa_2 x) + C_3\exp(\kappa_2 x)\\ &\quad + C_4\exp(\kappa_2 x) + C_5 + \bar{\varepsilon}^{(2)}_{\text{part}}(x).\end{aligned} \tag{73}$$

Once the solution in terms of $\bar{\varepsilon}^{(2)}(x)$ is known, the solution in terms of local strain field can be obtained exploiting the compatibility Eq. $(56)_2$, namely

$$\begin{aligned}\varepsilon(x) &= \mathscr{L}^{(2)}_{c_1,c_2}\bar{\varepsilon}^{(2)}(x)\\ &= \bar{\varepsilon}^{(2)}(x) - \ell_1^2\bar{\varepsilon}^{(2)''}(x) + \ell_2^4\bar{\varepsilon}^{(2)''''}(x)\end{aligned} \tag{74}$$

and, then, the displacement field can be calculated by integration as explained before for $n = 1$, with an additional constant of integration $C_6$. Thus, there are six constants of integration $C_1 - C_6$ to be determined. Consequently, the general solution is obtained by imposing six boundary conditions, two of which of standard type that can be either essential boundary conditions in terms of

$$[u(x)]|_{x=0} = \bar{u}_0$$
$$[u(x)]|_{x=L} = \bar{u}_L \tag{75}$$

or natural boundary conditions arising from the constitutive equations $(56)_3$

$$EA\left[\bar{\varepsilon}^{(2)}(x) - \ell^2_{\xi,1}\bar{\varepsilon}^{(2)''}(x) + \ell^4_{\xi,2}\bar{\varepsilon}^{(2)''''}(x)\right]\Big|_{x=0} = \bar{N}_0$$
$$EA\left[\bar{\varepsilon}^{(2)}(x) - \ell^2_{\xi,1}\bar{\varepsilon}^{(2)''}(x) + \ell^4_{\xi,2}\bar{\varepsilon}^{(2)''''}(x)\right]\Big|_{x=L} = \bar{N}_L \tag{76}$$

and the remaining four of nonstandard type according to (41) and (42), namely

$$\mathscr{L}^{(-)}_{c_2}\mathscr{L}^{(-)}_{c_1}\bar{\varepsilon}^{(2)}(x)|_{x=0} = \left[\left(1-(c_1+c_2)\frac{d}{dx}+c_1c_2\frac{d^2}{dx^2}\right)\bar{\varepsilon}^{(2)}(x)\right]\Big|_{x=0} = 0$$
$$\mathscr{L}^{(-)}_{c_2}\mathscr{L}^{(1)}_{c_1}\bar{\varepsilon}^{(2)}(x)|_{x=0} = \left[\left(1-c_1\frac{d}{dx}-c_2^2\frac{d^2}{dx^2}+c_1c_2^2\frac{d^3}{dx^3}\right)\bar{\varepsilon}^{(2)}(x)\right]\Big|_{x=0} = 0$$
$$\mathscr{L}^{(+)}_{c_2}\mathscr{L}^{(+)}_{c_1}\bar{\varepsilon}^{(2)}(x)|_{x=L} = \left[\left(1+(c_1+c_2)\frac{d}{dx}+c_1c_2\frac{d^2}{dx^2}\right)\bar{\varepsilon}^{(2)}(x)\right]\Big|_{x=L} = 0$$
$$\mathscr{L}^{(+)}_{c_2}\mathscr{L}^{(1)}_{c_1}\bar{\varepsilon}^{(2)}(x)|_{x=L} = \left[\left(1+c_2\frac{d}{dx}-c_1^2\frac{d^2}{dx^2}-c_1^2c_2\frac{d^3}{dx^3}\right)\bar{\varepsilon}^{(2)}(x)\right]\Big|_{x=L} = 0 \tag{77}$$

## 4.2 Free vibration analysis and dispersion relations

Derivation of the dispersion relation results from the differential Eq. (58), by setting the distributed load equal to zero for free vibration analysis, i.e., $q(x,t) = 0$, which leads to

$$\rho\left[\sum_{k=0}^{n}(-1)^k\ell_k^{2k}\frac{\partial^{2k}\ddot{u}(x,t)}{\partial x^{2k}}\right] = E\left[\sum_{k=0}^{n}(-1)^k\ell_{\xi,k}^{2k}\frac{\partial^{2k+2}u(x,t)}{\partial x^{2k+2}}\right]. \tag{78}$$

In Eq. (58) $E$ and $\rho$ have been implicitly assumed as constant coefficients. Therefore, Eq. (78) implies a relation between the acceleration field along with its gradients up to order $2n$, and the displacement field along with its gradients up to order $2n+2$. Since $E, \rho$ are constant coefficients, Eq. (78) admits solutions given by a general harmonic function

$$u(x,t) = U\exp[\mathrm{i}(kx-\omega t)] = U\exp[\mathrm{i}(x-ct)] \tag{79}$$

where $U$ is the amplitude, i the imaginary unit, $k$ the wave number, $\omega$ the angular frequency and $c = \omega/k$ the phase velocity. Substituting the trial solution (79) into Eq. (78) yields

$$\omega^2\left(1+\sum_{j=1}^{2n}k^{2j}\ell^{2j}_{j,\varepsilon}\right) = c_e^2k^2\left(1+\sum_{j=1}^{2n}k^{2j}\ell^{2j}_{\xi,j,\varepsilon}\right) \tag{80}$$

with $c_e = \sqrt{E/\rho}$ being the one-dimensional bar velocity of classical elasticity. Expression (80) represents the dispersion relation of the generalized nonlocal elasticity model of $n$-Helmholtz type. It can be re-written in dimensionless form in terms of normalized angular frequency and normalized phase velocity, respectively

$$\frac{\omega}{\omega_0} = \bar{k}\sqrt{\frac{1+\sum_{j=1}^{2n}\bar{k}^{2j}\,\bar{\ell}^{2j}_{\xi,j,\varepsilon}}{1+\sum_{j=1}^{2n}\bar{k}^{2j}\,\bar{\ell}^{2j}_{j,\varepsilon}}} \quad\rightarrow\quad \frac{c}{c_e} = \sqrt{\frac{1+\sum_{j=1}^{2n}\bar{k}^{2j}\,\bar{\ell}^{2j}_{\xi,j,\varepsilon}}{1+\sum_{j=1}^{2n}\bar{k}^{2j}\,\bar{\ell}^{2j}_{j,\varepsilon}}} \tag{81}$$

where $\omega_0 = c_e/\ell$ represents the natural circular frequency with $\ell$ being an internal characteristic length scale (for instance, a lattice constant), $\bar{k} = k\ell$ is the dimensionless wave number, $\bar{\ell}_{\xi,j} = \ell_{\xi,j}/\ell$, $\bar{\ell}_j = \ell_j/\ell$ are dimensionless coefficients that "adjust" the relative magnitudes between the various length scales appearing in the axial displacement and acceleration gradients, respectively. As such, these coefficients are to be calibrated according to the problem being studied.

For the determination of the eigenfunctions, the general harmonic solution can also be written as

$$u(x,t) = \phi(x)\exp(-\mathrm{i}\omega t) = \exp(\mathrm{i}\kappa x)\exp(-\mathrm{i}\omega t) \tag{82}$$

where $\phi(x) = \exp(\mathrm{i}\kappa x)$ is the sought eigenfunction and $\kappa$ the wave number. Substituting this expression into (78) leads to the following algebraic equation in terms of $\kappa$ rather than in terms of $\omega$

$$E\sum_{j=0}^{n}\ell_{\xi,j}^{2j}\kappa^{2j+2}-\rho\omega^2\sum_{j=0}^{n}\ell_j^{2j}\kappa^{2j}=0\quad\forall t \tag{83}$$

which leads to $2n+2$ solutions in $\kappa$, namely $\kappa_1,\kappa_2,\ldots,\kappa_{2n+2}$, that are affected by the numerical values of the $2n$ length scale coefficients $\ell_j^{2j},\ell_{\xi,j}^{2j}(j=1,\ldots,n)$. Therefore, the resulting eigenfunction has the following shape

$$\phi(x)=C_1\exp(\mathrm{i}\kappa_1 x)+C_2\exp(\mathrm{i}\kappa_2 x)+\cdots+C_{2n+2}\exp(\mathrm{i}\kappa_{2n+2}x) \tag{84}$$

and the constants of integration $C_1,C_2,\ldots,C_{2n+2}$ are determined by imposing $2n+2$ boundary conditions, two of which of standard type that can be either essential boundary conditions in terms of axial displacement—related to the value of the eigenfunction $\phi(x)$

$$\begin{cases}[u(x,t)]|_{x=0}=\bar{u}_0\\ [u(x,t)]|_{x=L}=\bar{u}_L\end{cases}\quad\forall t\quad\rightarrow\quad\begin{cases}[\phi(x)]|_{x=0}=\bar{u}_0\\ [\phi(x)]|_{x=L}=\bar{u}_L\end{cases} \tag{85}$$

or natural boundary conditions in terms of axial force—related to the higher order spatial derivatives of the eigenfunction $\phi(x)$. To determine the latter relationship between $N(x)$ and the higher order spatial derivatives of $\phi(x)$, let us apply the $\mathscr{L}^{(n)}$ differential operator to the constitutive Eq. $(56)_3$ while considering the compatibility Eq. $(56)_2$

$$\sum_{k=0}^{n}(-1)^k\ell_k^{2k}\frac{\partial^{2k}N(x,t)}{\partial x^{2k}}=EA\sum_{k=0}^{n}(-1)^k\ell_{\xi,k}^{2k}\frac{\partial^{2k+1}u(x,t)}{\partial x^{2k}} \tag{86}$$

which, combined with the dynamic equilibrium Eq. $(56)_1$ for $q_x(x,t)=0$, leads to

$$N(x,t)=-\rho A\sum_{k=1}^{n}(-1)^k\ell_k^{2k}\frac{\partial^{2k-1}\ddot{u}(x,t)}{\partial x^{2k-1}}+EA\sum_{k=0}^{n}(-1)^k\ell_{\xi,k}^{2k}\frac{\partial^{2k+1}u(x,t)}{\partial x^{2k}}. \tag{87}$$

Substituting the expression of the displacement $u(x,t)$ in terms of $\phi(x)$ as given in (82) into Eq. (87) leads to the sought relationship between $N(x)$ and the higher order spatial derivatives of $\phi(x)$

$$N(x,t)=\left\{\rho A\omega^2\left[\sum_{k=1}^{n}(-1)^k\ell_k^{2k}\frac{d^{2k-1}\phi(x)}{dx^{2k-1}}\right]+EA\left[\sum_{k=0}^{n}(-1)^k\ell_{\xi,k}^{2k}\frac{d^{2k+1}\phi(x)}{dx^{2k}}\right]\right\}\exp(-\mathrm{i}\omega t). \tag{88}$$

Based on Eq. (88) the natural boundary conditions in terms of axial force are written as

$$\begin{cases}\left\{\rho A\omega^2\left[\sum_{k=1}^{n}(-1)^k\ell_k^{2k}\frac{d^{2k-1}\phi(x)}{dx^{2k-1}}\right]+EA\left[\sum_{k=0}^{n}(-1)^k\ell_{\xi,k}^{2k}\frac{d^{2k+1}\phi(x)}{dx^{2k}}\right]\right\}\Bigg|_{x=0}=\bar{N}_0=0\\ \left\{\rho A\omega^2\left[\sum_{k=1}^{n}(-1)^k\ell_k^{2k}\frac{d^{2k-1}\phi(x)}{dx^{2k-1}}\right]+EA\left[\sum_{k=0}^{n}(-1)^k\ell_{\xi,k}^{2k}\frac{d^{2k+1}\phi(x)}{dx^{2k}}\right]\right\}\Bigg|_{x=L}=\bar{N}_L=0\end{cases}\quad\forall t. \tag{89}$$

Finally, the remaining $2n$ nonstandard boundary conditions relate the axial force $N(x,t)$ and its spatial derivatives to the eigenfunction $\phi(x)$ and its spatial derivatives. To determine such relationships, the starting point is the constitutive equation in integral form $(51)_1$, that is

$$N(x,t)=EA\sum_{j=0}^{n}\xi_j\bar{\varepsilon}^{(j)}(x,t)=EA\left[\xi_0\varepsilon(x,t)+\xi_1\bar{\varepsilon}^{(1)}(x,t)+\cdots+\xi_n\bar{\varepsilon}^{(n)}(x,t)\right]. \tag{90}$$

The sought $2n$ nonstandard boundary conditions arise from applying the differential operators $\mathscr{L}^{(p-)}_{c_n,c_{n-1},\ldots,c_j}\mathscr{L}^{(j-1)}_{c_1,c_2,\ldots,c_{j-1}}$ and $\mathscr{L}^{(p+)}_{c_n,c_{n-1},\ldots,c_j}\mathscr{L}^{(j-1)}_{c_1,c_2,\ldots,c_{j-1}}$ (of order $n$), previously defined in (48), to either side of Eq. (90), accounting for relations (47) (when applied to the nonlocal strain of order $n$, $\bar{\varepsilon}^{(n)}(x)$) and for relations (133) (when applied to the nonlocal strain of order $k$ $\bar{\varepsilon}^{(k)}(x)$ $(k=1,\ldots,n-1)$) and converting the derivatives of the axial force in terms of higher order inertia, as per the dynamic equilibrium Eq. $(56)_1$ for $q_x(x,t)=0$.

It is instructive to particularize Eqs. (78), (81) and the determination of the eigenfunctions $\phi(x)$ in the following cases:

- for $n=1$

$$\rho\left[\ddot{u}(x,t)-\ell_1^2\ddot{u}''(x,t)\right]=E\left[u''(x,t)-\ell_{\xi,1}^2u''''(x,t)\right] \tag{91}$$

which leads to the following dimensionless dispersion relation

$$\frac{\omega}{\omega_0} = \bar{k}\sqrt{\frac{1+\bar{k}^2\bar{\ell}_{\xi,1}^2}{1+\bar{k}^2\bar{\ell}_1^2}} \quad \rightarrow \quad \frac{c}{c_e} = \sqrt{\frac{1+\bar{k}^2\bar{\ell}_{\xi,1}^2}{1+\bar{k}^2\bar{\ell}_1^2}}. \tag{92}$$

This represents the so-called *dynamically consistent model* presented by Askes and Aifantis [10–12]. It may easily be demonstrated that the corresponding dispersion curve shows a diagonal asymptote, whose slope is governed by the ratio $\ell_{\xi,1}/\ell_1$: for $\ell_{\xi,1} > \ell_1$ ($\ell_{\xi,1} < \ell_1$) the higher wave numbers travel faster (slower) than the lower wave numbers, whereas for $\ell_{\xi,1} = \ell_1$ a non-dispersive behavior is obtained, similar to classical elasticity. For the determination of the eigenfunctions $\phi(x)$, Eq. (83) reduces to

$$E\ell_{\xi,1}^2\kappa^4 + \left(E - \rho\omega^2\ell_1^2\right)\kappa^2 - \rho\omega^2 = 0 \quad \forall t \tag{93}$$

which leads to four solutions in $\kappa$, which are of the kind $\kappa_{1,2} = \pm i s_1$ and $\kappa_{3,4} = \pm r_1$, so that, exploiting Euler's formula, the solution can be expressed in trigonometric form as

$$\phi(x) = C_1\cosh(s_1 x) + C_2\sinh(s_1 x) + C_3\cos(r_1 x) + C_4\sin(r_1 x). \tag{94}$$

The four constants of integration $C_1, C_2, C_3, C_4$ are determined by imposing 4 boundary conditions, two of which of standard type that can be either essential boundary conditions in terms of axial displacement

$$\begin{cases} [u(x,t)]|_{x=0} = 0 \\ [u(x,t)]|_{x=L} = 0 \end{cases} \quad \forall t \quad \rightarrow \quad \begin{cases} [\phi(x)]|_{x=0} = 0 \\ [\phi(x)]|_{x=L} = 0 \end{cases} \tag{95}$$

or natural boundary conditions in terms of axial force according to (89)

$$\begin{cases} \left\{-\rho A\omega^2\ell_1^2\phi'(x) + EA\left[\phi'(x) - \ell_{\xi,1}^2\phi'''(x)\right]\right\}\Big|_{x=0} = \bar{N}_0 = 0 \\ \left\{-\rho A\omega^2\ell_1^2\phi'(x) + EA\left[\phi'(x) - \ell_{\xi,1}^2\phi'''(x)\right]\right\}\Big|_{x=L} = \bar{N}_L = 0 \end{cases} \quad \forall t. \tag{96}$$

To determine the two remaining nonstandard boundary conditions, let us consider the expression

$$N(x,t) = EA\left[\xi_0\varepsilon(x,t) + \xi_1\bar{\varepsilon}^{(1)}(x,t)\right]. \tag{97}$$

According to Eq. (35), let us apply $\mathscr{L}_{c_1}^{(-)}$ and $\mathscr{L}_{c_1}^{(+)}$ to either side of (97)

$$\begin{aligned} \mathscr{L}_{c_1}^{(-)}N(x,t)\big|_{x=0} &= \mathscr{L}_{c_1}^{(-)}EA\left[\xi_0\varepsilon(x,t) + \xi_1\bar{\varepsilon}^{(1)}(x,t)\right]\Big|_{x=0} \\ \mathscr{L}_{c_1}^{(+)}N(x,t)\big|_{x=L} &= \mathscr{L}_{c_1}^{(+)}EA\left[\xi_0\varepsilon(x,t) + \xi_1\bar{\varepsilon}^{(1)}(x,t)\right]\Big|_{x=L} \end{aligned} \tag{98}$$

which may be expanded in terms of eigenfunction $\phi(x)$ and its spatial derivatives considering the dynamic equilibrium Eq. $(56)_1$ as follows

$$\begin{cases} \left[N(x) - c_1N'(x)\right]\Big|_{x=0} = \left[N(x) + \rho A\omega^2 c_1\phi(x)\right]\big|_{x=0} = EA\left[\xi_0\phi'(x) - \xi_0 c_1\phi''(x)\right]\Big|_{x=0} \\ \left[N(x) + c_1N'(x)\right]\Big|_{x=L} = \left[N(x) - \rho A\omega^2 c_1\phi(x)\right]\big|_{x=L} = EA\left[\xi_0\phi'(x) + \xi_0 c_1\phi''(x)\right]\Big|_{x=L} \end{cases} \quad \forall t \tag{99}$$

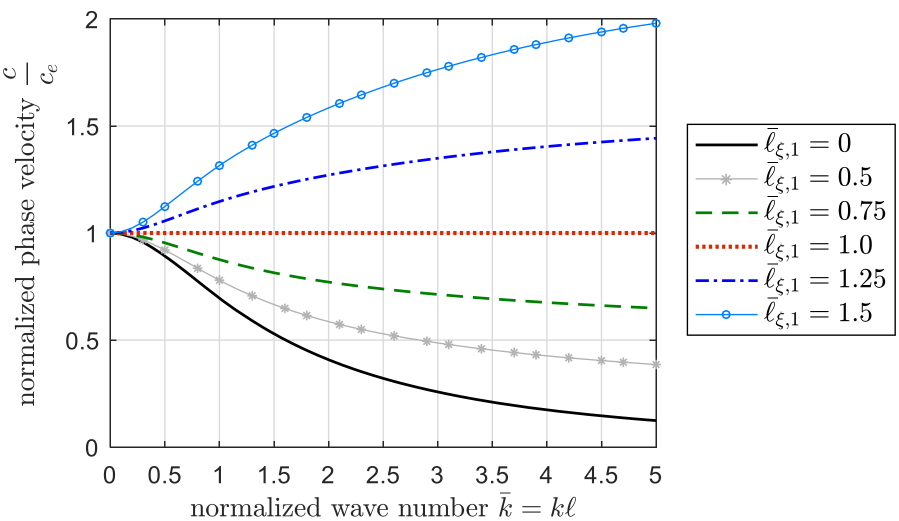



**Fig. 2** Dispersion curves of 2nd order nonlocal elasticity model (second strain gradient with second velocity gradient)

where account has been taken of the relations (35). Inserting the expression (94) in (95) or (96) and in (99) and setting the determinant of the coefficients equal to zero allows the evaluation of all the natural frequencies of the nonlocal continuum system $\omega_i (i = 1, 2, \ldots)$ for the particular set of boundary conditions considered. All these frequencies should lie on the dispersion curve (92). Given the natural frequencies of the nonlocal continuum, the corresponding wave numbers can be determined for each natural frequency $\omega_i (i = 1, 2, \ldots)$, viz. $\kappa_j^i (j = 1, 2, 3, 4)$, and the four constants of integrations $C_j^i (j = 1, 2, 3, 4)$ are obtained by imposing the four boundary conditions for the given $\omega_i$ accordingly. The eigenfunction $\phi(x)$ can thus be plotted for the specific $i$th natural mode of vibration and the procedure can be repeated again for any $i = 1, 2, \ldots$.

- for $n = 2$

$$\rho\left[\ddot{u}(x,t) - \ell_1^2 \ddot{u}''(x,t) + \ell_2^4 \ddot{u}''''(x,t)\right] = E\left[u''(x,t) - \ell_{\xi,1}^2 u''''(x,t) + \ell_{\xi,2}^4 u''''''(x,t)\right] \quad (100)$$

which leads to the following dimensionless dispersion relation

$$\frac{\omega}{\omega_0} = \bar{k}\sqrt{\frac{1 + \bar{k}^2\bar{\ell}_{\xi,1}^2 + \bar{k}^4\bar{\ell}_{\xi,2}^4}{1 + \bar{k}^2\bar{\ell}_1^2 + \bar{k}^4\bar{\ell}_2^4}} \quad \rightarrow \quad \frac{c}{c_e} = \sqrt{\frac{1 + \bar{k}^2\bar{\ell}_{\xi,1}^2 + \bar{k}^4\bar{\ell}_{\xi,2}^4}{1 + \bar{k}^2\bar{\ell}_1^2 + \bar{k}^4\bar{\ell}_2^4}}. \quad (101)$$

This represents the second strain gradient with second velocity gradient inertia model proposed by Polizzotto [13]. A similar model was recently proposed by De Domenico and Askes [14, 15] (the latter for $\ell_{\xi,2} = 0$) and applied to capture wave dispersion characteristics of a range of materials with microstructure [16–19]. As noted by Polizzotto [13], any set of length scale ratios complying with the conditions

$$\frac{\bar{\ell}_{\xi,1}}{\bar{\ell}_1} < 1; \quad \frac{\bar{\ell}_2}{\bar{\ell}_1} > \frac{\bar{\ell}_{\xi,2}}{\bar{\ell}_1}; \quad \bar{\ell}_1 \neq 0; \quad \frac{\bar{\ell}_2}{\bar{\ell}_1} \neq 0 \quad (102)$$

ensures that the phase velocity curve is entirely below the horizontal line $c/c_e$. In the limit as $\bar{k} \rightarrow \infty$ the $c/c_e$ ratio tends to the diagonal asymptote whose slope is governed by the ratio $(\bar{\ell}_{\xi,2}/\bar{\ell}_2)^2$. This four-length-scale model is versatile and it is able to accommodate wave dispersion characteristics of a wide variety of real materials.

As an example, in Fig. 2 the normalized phase velocity $c/c_e$ from Eq. (101) is plotted as a function of the normalized wave number $\bar{k}$ for different values of the $\bar{\ell}_{\xi,1}$ ratio, assuming $\bar{\ell}_{\xi,2} = \bar{\ell}_{\xi,1}/2$, $\bar{\ell}_1 = 1$, $\bar{\ell}_2 = 0.5$. The $c/c_e$ exceeds the non-dispersive asymptote $c/c_e = 1$ only for $\bar{\ell}_{\xi,1} > \bar{\ell}_1 = 1$. The normalized group velocity of the model in (101) is obtained through

$$\frac{c_g}{c_e} = \bar{c}_g = \frac{d\bar{\omega}}{d\bar{k}} = \frac{1 + \bar{k}^2(2 + \bar{k}^2\bar{\ell}_1^2)\bar{\ell}_{\xi,1}^2 + \bar{k}^4(3 + 2\bar{k}^2\bar{\ell}_1^2 + \bar{k}^4\bar{\ell}_2^4)\bar{\ell}_{\xi,2}^4 + \bar{k}^4\bar{\ell}_2^4}{\sqrt{\left(1 + \bar{k}^2\bar{\ell}_{\xi,1}^2 + \bar{k}^4\bar{\ell}_{\xi,2}^4\right)\left(1 + \bar{k}^2\bar{\ell}_1^2 + \bar{k}^4\bar{\ell}_2^4\right)^3}} \quad (103)$$

which is equal to 1 for $k \rightarrow 0$. The group velocity for the model with $n = 1$ is simply obtained by setting $\bar{\ell}_2 = \bar{\ell}_{\xi,2} = 0$ in (103). For the determination of the eigenfunctions $\phi(x)$, Eq. (83) reduces to

$$E\ell_{\xi,2}^4\kappa^6 + \left(E\ell_{\xi,1}^2 - \rho\omega^2\ell_2^4\right)\kappa^4 + \left(E - \rho\omega^2\ell_1^2\right)\kappa^2 - \rho\omega^2 = 0 \quad \forall t \quad (104)$$

which leads to six solutions in $\bar{\kappa}$, $\kappa$, which are of the kind $\kappa_{1,2,3,4} = \pm r_1 \pm \mathrm{i}\, s_1$ and $\kappa_{5,6} = \pm p_1$, so that, exploiting Euler's formula, the solution can be expressed in trigonometric form as

$$\phi(x) = C_1\cosh(s_1x)cos(r_1x) + C_2\cosh(s_1x)\sin(r_1x) + C_3\sinh(s_1x)\sin(r_1x) + C_4\sinh(s_1x)cos(r_1x) + C_5\cos(p_1x) + C_6\sin(p_1x). \quad (105)$$

The six constants of integration $C_1 - C_6$ are determined by imposing 6 boundary conditions, two of which of standard type that can be either essential boundary conditions in terms of axial displacement

$$\begin{cases} [u(x,t)]|_{x=0} = \bar{u}_0 \\ [u(x,t)]|_{x=L} = \bar{u}_L \end{cases} \forall t \quad \rightarrow \quad \begin{cases} [\phi(x)]|_{x=0} = \bar{u}_0 \\ [\phi(x)]|_{x=L} = \bar{u}_L \end{cases} \quad (106)$$

or natural boundary conditions in terms of axial force according to (89)

$$\begin{cases} \left\{\rho A\omega^2\left[-\ell_1^2\phi^{'}(x)+\ell_2^4\phi^{'''}(x)\right]+EA\left[\phi^{'}(x)-\ell_{\xi,1}^2\phi^{'''}(x)+\ell_{\xi,2}^4\phi^{'''''}(x)\right]\right\}\Big|_{x=0}=\bar{N}_0=0 \\ \left\{\rho A\omega^2\left[-\ell_1^2\phi^{'}(x)+\ell_2^4\phi^{'''}(x)\right]+EA\left[\phi^{'}(x)-\ell_{\xi,1}^2\phi^{'''}(x)+\ell_{\xi,2}^4\phi^{'''''}(x)\right]\right\}\Big|_{x=L}=\bar{N}_L=0 \end{cases} \quad \forall t. \tag{107}$$

To determine the two remaining nonstandard boundary conditions, let us consider the expression

$$N(x,t)=EA\left[\xi_0\varepsilon(x,t)+\xi_1\bar{\varepsilon}^{(1)}(x,t)+\xi_2\bar{\varepsilon}^{(2)}(x,t)\right]. \tag{108}$$

According to Eqs. (41) and (42), let us apply $\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}, \mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}, \mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}, \mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}$ to either side of (108)

$$\begin{aligned} &\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}N(x,t)\big|_{x=0} \\ &\quad=\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}EA\left[\xi_0\varepsilon(x,t)+\xi_1\bar{\varepsilon}^{(1)}(x,t)+\xi_2\bar{\varepsilon}^{(2)}(x,t)\right]\Big|_{x=0}, \\ &\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}N(x,t)\big|_{x=0} \\ &\quad=\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}EA\left[\xi_0\varepsilon(x,t)+\xi_1\bar{\varepsilon}^{(1)}(x,t)+\xi_2\bar{\varepsilon}^{(2)}(x,t)\right]\Big|_{x=0}, \\ &\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}N(x,t)\big|_{x=L} \\ &\quad=\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}EA\left[\xi_0\varepsilon(x,t)+\xi_1\bar{\varepsilon}^{(1)}(x,t)+\xi_2\bar{\varepsilon}^{(2)}(x,t)\right]\Big|_{x=L}, \\ &\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}N(x,t)\big|_{x=L} \\ &\quad=\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}EA\left[\xi_0\varepsilon(x,t)+\xi_1\bar{\varepsilon}^{(1)}(x,t)+\xi_2\bar{\varepsilon}^{(2)}(x,t)\right]\Big|_{x=L}. \end{aligned} \tag{109}$$

which may be expanded in terms of eigenfunction $\phi(x)$ and its spatial derivatives considering the dynamic equilibrium Eq. $(56)_1$ as follows

$$\begin{cases} \left[N(x)-(c_1+c_2)N^{'}(x)+c_1c_2N^{''}(x)\right]\Big|_{x=0}=\left\{N(x)-\rho A\omega^2\left[-(c_1+c_2)\phi(x)+c_1c_2\phi^{'}(x)\right]\right\}\Big|_{x=0} \\ \quad=EA\left\{\xi_0\left[\phi^{'}(x)-(c_1+c_2)\phi^{''}(x)+c_1c_2\phi^{'''}(x)\right]-\xi_1\left[\dfrac{c_2}{c_1}\phi^{'}(x)\right]\right\}\Big|_{x=0} \\ \left[N(x)-c_2N^{'}(x)-c_1^2N^{''}(x)+c_1^2c_2N^{'''}(x)\right]\Big|_{x=0}=\left\{N(x)-\rho A\omega^2\left[-c_2\phi(x)-c_1^2\phi^{'}(x)+c_1^2c_2\phi^{''}(x)\right]\right\}\Big|_{x=0} \\ \quad=EA\left\{\xi_0\left[\phi^{'}(x)-c_2\phi^{''}(x)-c_1^2\phi^{'''}(x)+c_1^2c_2\phi^{''''}(x)\right]+\xi_1\left[\phi^{'}(x)-c_2\phi^{''}(x)\right]\right\}\Big|_{x=0} \\ \left[N(x)+(c_1+c_2)N^{'}(x)+c_1c_2N^{''}(x)\right]\Big|_{x=L}=\left\{N(x)-\rho A\omega^2\left[(c_1+c_2)\phi(x)+c_1c_2\phi^{'}(x)\right]\right\}\Big|_{x=L} \\ \quad=EA\left\{\xi_0\left[\phi^{'}(x)+(c_1+c_2)\phi^{''}(x)+c_1c_2\phi^{'''}(x)\right]-\xi_1\left[\dfrac{c_2}{c_1}\phi^{'}(x)\right]\right\}\Big|_{x=L} \\ \left[N(x)+c_2N^{'}(x)-c_1^2N^{''}(x)-c_1^2c_2N^{'''}(x)\right]\Big|_{x=L}=\left\{N(x)-\rho A\omega^2\left[c_2\phi(x)-c_1^2\phi^{'}(x)-c_1^2c_2\phi^{''}(x)\right]\right\}\Big|_{x=L} \\ \quad=EA\left\{\xi_0\left[\phi^{'}(x)+c_2\phi^{''}(x)-c_1^2\phi^{'''}(x)-c_1^2c_2\phi^{''''}(x)\right]+\xi_1\left[\phi^{'}(x)+c_2\phi^{''}(x)\right]\right\}\Big|_{x=L} \end{cases} \tag{110}$$

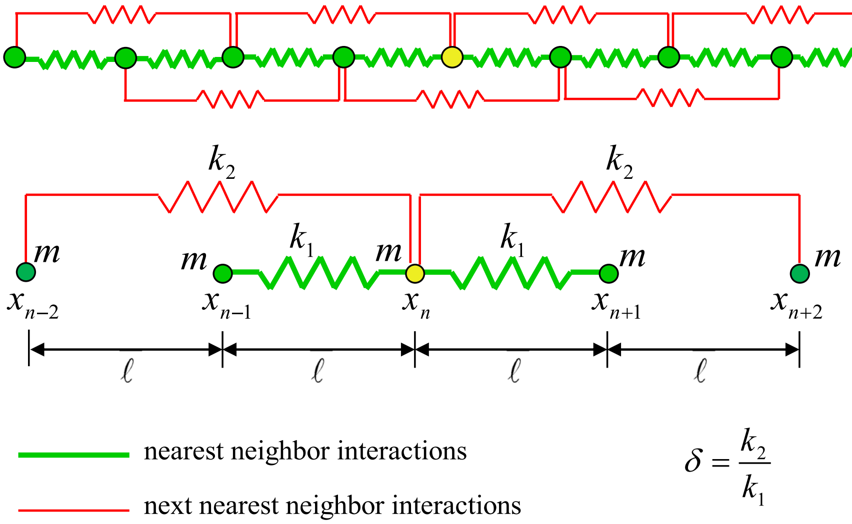



**Fig. 3** Lattice model with nearest neighbor (NN) and next nearest neighbor (NNN) interactions

valid $\forall t$, where account has been taken of the identities (41), (42) as well as of the relations (127) reported in Appendix A. Inserting the expression (105) in (106) or (107) and in (110) and setting the determinant of the coefficients equal to zero allows the evaluation of all the natural frequencies of the nonlocal continuum system $\omega_i (i = 1, 2, \ldots)$ for the particular set of boundary conditions considered. All these frequencies should lie on the dispersion curve (101). Given the natural frequencies of the nonlocal continuum, the corresponding wave numbers can be determined for each natural frequency $\omega_i (i = 1, 2, \ldots)$, viz. $\kappa_j^i (j = 1, \ldots, 6)$, and the six constants of integrations $C_j^i (j = 1, \ldots, 6)$ are obtained by imposing the six boundary conditions for the given $\bar{\omega}_i$ accordingly. The eigenfunction $\phi(x)$ can thus be plotted for the specific $i$th natural mode of vibration and the procedure can be repeated again for any $i = 1, 2, \ldots$.

## 5 Lattice model with nearest neighbor (NN) and next nearest neighbor (NNN) interactions

The simplest discrete medium is the one-dimensional lattice model formed by a mass-spring chain, as per the sketch in Fig. 3. Each atom is represented by a lumped mass $m$, which interacts with neighboring atoms in the lattice model. The discrete masses are located at uniform particle spacing $\ell$ that may be interpreted as the unit cell of a heterogeneous material (lattice constant). The interacting forces between atoms depends upon the Morse potential. Adopting a harmonic approximation of this potential leads to spring-like interatomic forces. These forces can be represented by a set of springs of constants $k_1$ and $k_2$ if the nearest neighbor (NN) and next nearest neighbor (NNN) particles are considered, respectively, with $\delta = k_2/k_1$ a dimensionless parameter that measures the second-order interaction contribution.

The equation of motion for the $n$th mass particle reads

$$m\ddot{u}_n(t) = k_1(u_{n-1}(t) - 2u_n(t) + u_{n+1}(t)) + k_2(u_{n-2}(t) - 2u_n(t) + u_{n+2}(t)) \tag{111}$$

where $u_n(t)$ denotes the displacement of the mass particle $n$ initially located at $x_n$, and $u_{n\pm p}(t)$ that of the particle $n \pm p$ located at $x_{n\pm j} = x_n \pm p\ell$, $p = 1, 2$. In the static case, the inertia effects on the left-hand-side (LHS) of Eq. (111) can be ignored. In this case, the static response of the periodic chain displayed in Fig. 3 subject to a given loading condition is obtained by solving a simple equation $\mathbf{Ku} = \mathbf{F}$ where $\mathbf{K}$ is the stiffness matrix of the lattice (calculated by generalizing (111)), $\mathbf{F}$ is the load vector that depends on the loading conditions, and $\mathbf{u}$ is the vector collecting the unknown axial displacements of the mass particles.

The dispersion curve of the discrete lattice can be obtained by considering a general harmonic function for the displacement of the $n$th mass particle, namely

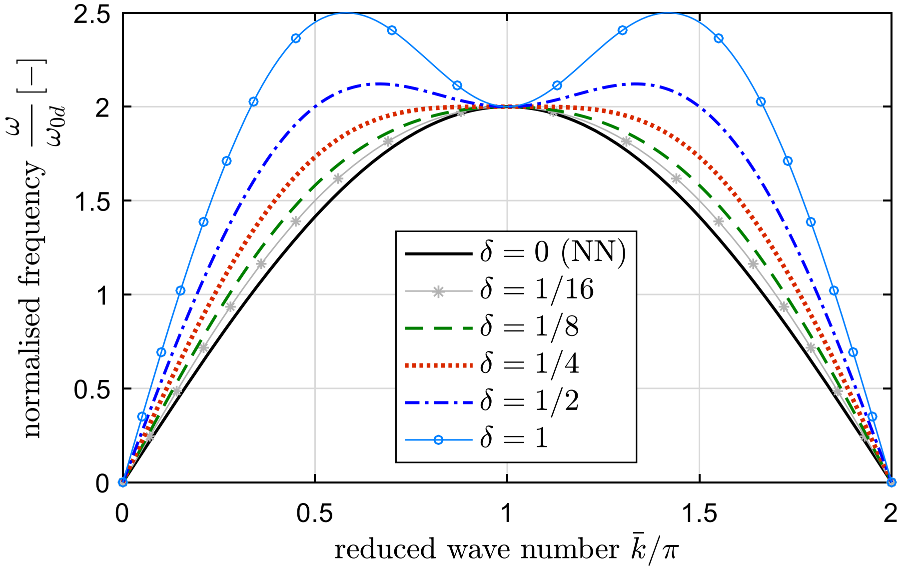



**Fig. 4** Dispersion curves of lattice model (discrete chain of masses and springs) with NNN interactions for different values of the $\delta = k_2/k_1$ second-order interaction ratio

$$u_n(x,t) = U\exp[\mathrm{i}(kx_n - \omega t)] \tag{112}$$

and that of the $(n \pm p)$th particle is

$$\begin{aligned} u_{n\pm p}(x,t) &= U\exp\left[\mathrm{i}\left(kx_{n\pm p} - \omega t\right)\right] \\ &= U\exp[\mathrm{i}(k(x_n \pm p\ell) - \omega t)] \\ &= u_n(x,t)\exp[\pm \mathrm{i}kp\ell] \qquad (\text{with } p = 1,2). \end{aligned} \tag{113}$$

Substituting (112) and (113) into the equation of motion of the $n$th mass particle (111), after some straightforward algebra, yields

$$\frac{\omega}{\omega_{0d}} = \bar{\omega} = 2\sqrt{\sin^2\left(\frac{\bar{k}}{2}\right) + \delta\sin^2(\bar{k})} \quad \text{(lattice model, NNN interactions)} \tag{114}$$

where the subscript $d$ stands for "discrete", and $\omega_{0d} = \sqrt{k_1/m}$ is the natural circular frequency of the discrete lattice. For $\delta = 0$, Eq. (114) reduces to the well-known Born–von Kármán model with nearest neighbor interactions only [6]

$$\frac{\omega}{\omega_{0d}} = \bar{\omega} = 2\sin\left(\frac{\bar{k}}{2}\right) \quad \text{(lattice model, NN interactions).} \tag{115}$$

Expression (114) is plotted in Fig. 4 for different values of the $\delta$ ratio.

Finally, the eigenvectors (modal shapes) $\mathbf{\Phi}$ of the discrete lattice model can be obtained from Eq. (111), which leads to

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\mathbf{\Phi} = 0 \tag{116}$$

where, assuming lumped mass according to Fig. 3, the mass matrix is $\mathbf{M} = \mathrm{diag}\{\ m\}$. Setting $\det[\mathbf{K} - \omega^2\mathbf{M}] = 0$ gives the natural frequencies of the lattice that, when substituted back into (116), provide the corresponding modal shapes (arbitrary within a multiplicative factor).

## 6 Identification of nonlocal material parameters

### 6.1 Dispersion law derived from Fourier transform of global attenuation function

Based on Remark 1 of the Part I paper, a modified (or global) attenuation function associated with the generalized nonlocal elasticity theory of order $n$ can be introduced as

$$\begin{aligned}\mathscr{A}(r\,;c_1,c_2,\ldots,c_n)\\ &= \xi_0\delta(r) + \xi_1\alpha^{(1)}(r\,;c_1) + \xi_2\alpha^{(2)}(r\,;c_1,c_2)\\ &\quad + \cdots + \xi_n\alpha^{(n)}(r\,;c_1,c_2,\ldots,c_n)\\ &= \sum_{j=0}^{n}\xi_j\alpha^{(j)}(r\,;c_1,c_2,\ldots,c_j)\end{aligned} \tag{117}$$

where account has been taken of the fact that $\alpha^{(0)}(r) = \delta(r)$. Considering Eq. (117), the generalized nonlocal constitutive equation in integral form $(51)_1$ can be re-written as follows

$$N(x,t) = EA\int_0^L \mathscr{A}(r\,;c_1,c_2,\ldots,c_n)\,\varepsilon(y)\mathrm{d}y. \tag{118}$$

Following Eringen [1, 6], the dispersion relation can be derived through

$$\begin{aligned}\omega^2 &= \omega_0^2 k^2\tilde{\mathscr{A}}(k\,;c_1,c_2,\ldots,c_n)\\ &= \omega_0^2 k^2\left[\sum_{j=0}^{n}\left(\xi_j\prod_{i=0}^{j}\left(1+k^2c_i^2\right)^{-1}\right)\right]\end{aligned} \tag{119}$$

where $\tilde{\mathscr{A}}(k\,;c_1,c_2,\ldots,c_n) = \mathscr{F}[\mathscr{A}(r\,;c_1,c_2,\ldots,c_n)]$, and the properties of the Fourier transform of $\alpha^{(n)}(r\,;c_1,c_2,\ldots,c_n)$ as per Eq. (28) have been exploited, with $c_0 = 0$. Expression (119) can be re-written in dimensionless form as follows

$$\begin{aligned}\frac{\omega}{\omega_0} &= \bar{k}\sqrt{\tilde{\mathscr{A}}(\bar{k}\,;c_1,c_2,\ldots,c_n)}\\ &= \bar{k}\sqrt{\sum_{j=0}^{n}\left(\xi_j\prod_{i=0}^{j}\left(1+\bar{k}^2\bar{c}_i^2\right)^{-1}\right)}\end{aligned} \tag{120}$$

where $\bar{c}_i = c_i/\ell\,(i = 1,\ldots,n)$ are normalized length scale coefficients for each nonlocal kernel, and $\bar{c}_0 = c_0/\ell = 0$. By recalling from Appendix A of the Part I paper the relations between the length scale coefficients $c_k\,(k = 1,2,\ldots,n)$ and the length scale parameters $\ell_{\xi,k}^{2k}$ and $\ell_k(k = 1,\ldots,n)$ of the Helmholtz operators, it can be demonstrated that Eq. (120) is perfectly identical to Eq. (81) derived above from free vibration analysis. In order to identify the $n$ length scale coefficients $\bar{c}_i\,(i = 1,\ldots,n)$ and the $n$ material phase parameters $\xi_j\,(j = 1,\ldots,n)$ (with $\xi_0 = 1 - \sum_{j=1}^{n}\xi_j$) a common strategy adopted in the literature [1, 6, 14, 20, 21] is to match the dispersion relation (120) with the relevant lattice model (discussed in the previous section). To this end, it is preliminarily useful to calculate the natural circular frequency of the higher order nonlocal continuum $\omega_0 = c_e/\ell$ and compare it to the discrete counterpart $\omega_{0d} = \sqrt{k_1/m}$ in (114). For NNN interactions being included in the lattice model, the equivalent stiffness of the lattice model is $k_{\mathrm{eq}} = k_1 + 4k_2 = k_1(1+4\delta)$. Thus, the Young's modulus of the nonlocal bar is $E = k_{\mathrm{eq}}\ell/A = k_1(1+4\delta)\ell/A$ and the mass density (per unit volume) is $\rho = m/(A\ell)$, so that $c_e = \ell\sqrt{k_1/m}\sqrt{(1+4\delta)}$. Therefore, $\omega_0$ is related to $\omega_{0d}$ through the following formula: $\omega_0 = \omega_{0d}\sqrt{1+4\delta}$. Substituting this value into (120), the dispersion relation is generalized to include NNN interactions as follows

$$\begin{aligned}\frac{\omega}{\omega_{0d}} &= \bar{\omega} = \bar{k}\sqrt{1+4\delta}\sqrt{\tilde{\mathscr{A}}(\bar{k}\,;c_1,c_2,\ldots,c_n)}\\ &= \bar{k}\sqrt{1+4\delta}\sqrt{\sum_{j=0}^{n}\left(\xi_j\prod_{i=0}^{j}\left(1+\bar{k}^2\bar{c}_i^2\right)^{-1}\right)}.\end{aligned} \tag{121}$$

In this format, the LHS of both Eqs. (121) and (114) is the same. Hence, comparing the RHS of Eqs. (121) and (114) leads to a physically substantiated identification procedure of the $2n$ nonlocal material parameters of the higher-order continuum model, namely the $n$ length scale coefficients $\bar{c}_i\,(i = 1,\ldots,n)$ and the $n$ material phase parameters $\xi_j\,(j = 1,\ldots,n)$ characterizing the generalized theory of nonlocal elasticity of $n$-Helmholtz type.

*Remark 2* It is worth noting that the relation $\omega_0 = \omega_{0d}\sqrt{1+4\delta}$ in (121) ensures that the dispersion curve $\bar{\omega}(\bar{k})\big|_{\bar{k}=0}$, and the group velocity $\bar{c}_g = d\bar{\omega}/d\bar{k}\big|_{\bar{k}=0}$ in the higher order nonlocal continuum and in the discrete lattice have the same value. Thus, the supplemental conditions must be enforced for other wave numbers, e.g. at the end of the first Brillouin zone (FBZ) for $\bar{k} = \pi$ in terms of $\bar{\omega}$, $\bar{c}_g = \mathrm{d}\bar{\omega}/\mathrm{d}\bar{k}$ and/or higher order derivatives of $\bar{c}_g$.

*Remark 3* The following limit values of the group velocity hold

$$\lim_{\bar{k}\to 0}\bar{c}_g = 1\,; \quad \lim_{\bar{k}\to\infty}\bar{c}_g = \sqrt{\xi_0}. \tag{122}$$

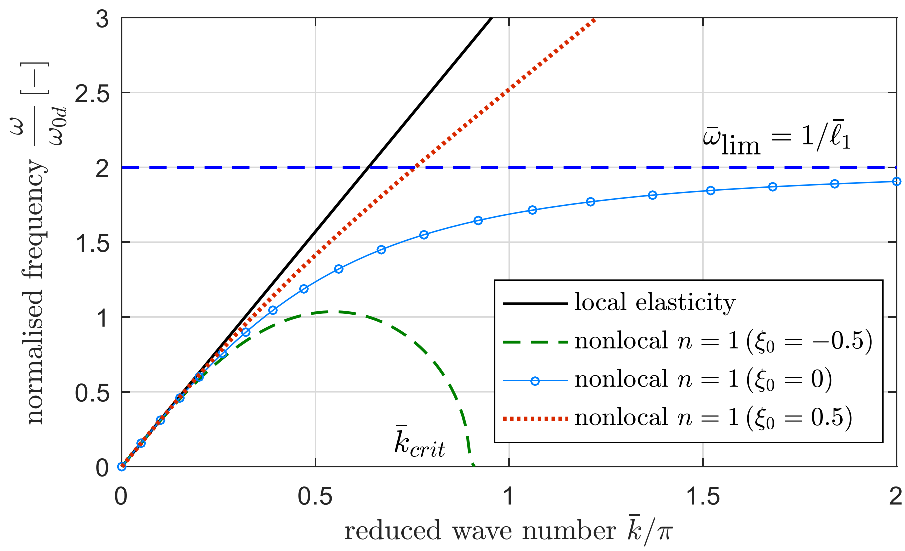



**Fig. 5** Dispersion curves for nonlocal continuum model of first order ($n = 1$) compared to local elasticity, for fixed $\bar{\ell}_1 = 0.5$ and different $\xi_0$ material phase parameter

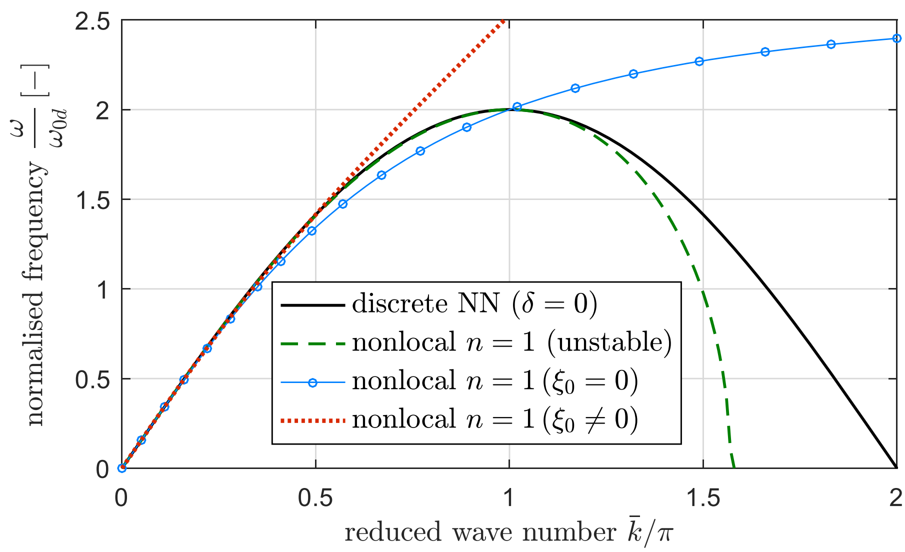



**Fig. 6** Dispersion curves for discrete lattice model with NN interactions against nonlocal continuum model of first order ($n = 1$)

Equation (122) allows to highlight some specific features of the nonlocal model at low and high frequency. In fact, at low frequency the nonlocal model approaches the classical local one, for all values of the model parameters. At high-frequency the behaviour is ruled by the parameter $\xi_0$ as the dispersion curve exhibits an asymptote $\bar{\omega} = \bar{k}\sqrt{\xi_0}$ that, for $0 < \xi_0 < 1$ or $\xi_0 > 1$, is located below or above the dispersion curve $\bar{\omega} = \bar{k}$ of the classical local model, respectively. This implies that the high frequency waves transmit the energy slower or faster than the classical compressional waves, respectively.

Moreover, some restrictions on the admissible values of parameter $\xi_0$ can be inferred: in order to guarantee stability, $\xi_0$ must be nonnegative; indeed, if $\xi_0 < 0$ at high frequencies the nonlocal model becomes unstable. As an example, for the first order nonlocal model ($n = 1$), if we assume $\xi_0 < 0$, the waves having wave number $\bar{k} > \bar{k}_{crit} = \left(\bar{\ell}_1\sqrt{|\xi_0|}\right)^{-1}$ are unstable, cf. Fig. 5. If $\xi_0 = 0$ and $\xi_1 = 0$ (Eringen model), the first order nonlocal model is stable, the group velocity vanish for $k \to \infty$ and the dispersion curve tends to a horizontal asymptote at $\bar{\omega}_{\rm lim} = 1/\bar{\ell}_1$,, cf. again Fig. 5. A gradient model with a negative value of $\xi_0$ was

**Table 1** Nonlocal material parameters of the generalized nonlocal elasticity theory for $n = 1$ and $n = 2$ identified by matching the dispersion curve of the lattice model with NN interactions

| Order $n$ | Nonlocal material parameters | | | | Length scale parameters | | | |
|---|---|---|---|---|---|---|---|---|
| | $\xi_1$ | $\xi_2$ | $\bar{c}_1$ | $\bar{c}_2$ | $\bar{\ell}^2_{\xi,1}$ | $\bar{\ell}^4_{\xi,2}$ | $\bar{\ell}^2_1$ | $\bar{\ell}^4_2$ |
| 1 | 1.867 | – | 0.218 | – | −0.041 | – | 0.047 | – |
| 1† | 1.00* | – | 0.386 | – | 0.000 | – | 0.149 | – |
| 1‡ | 0.49 | – | 0.501 | – | 0.127 | – | 0.251 | – |
| 2 | 1.83 | −0.23 | 0.277 | 0.499 | 0.261 | −0.011 | 0.325 | 0.019 |
| 2* | −18.86 | 12.95 | 0.109 | 0.109 | −0.060 | $9.7 \times 10^{-4}$ | 0.024 | $1.4 \times 10^{-4}$ |

†This is a particular subcase of the proposed generalized theory of nonlocal elasticity by setting $\xi_0 = 0$ and $\xi_1 = 1.0$ so that the length scale parameter $\bar{\ell}_{\xi,1} = 0$; ‡this set of parameters is obtained by setting the value $\bar{\omega}(\bar{k})\big|_{\bar{k}=\pi/2}$ and $d^2\bar{\omega}/d^2\bar{k}\big|_{\bar{k}=\pi/4}$; *this represents the special case of Helmholtz operators with the same lengths scale coefficient $c_1 = c_2 = c = \varepsilon_0 \ell$

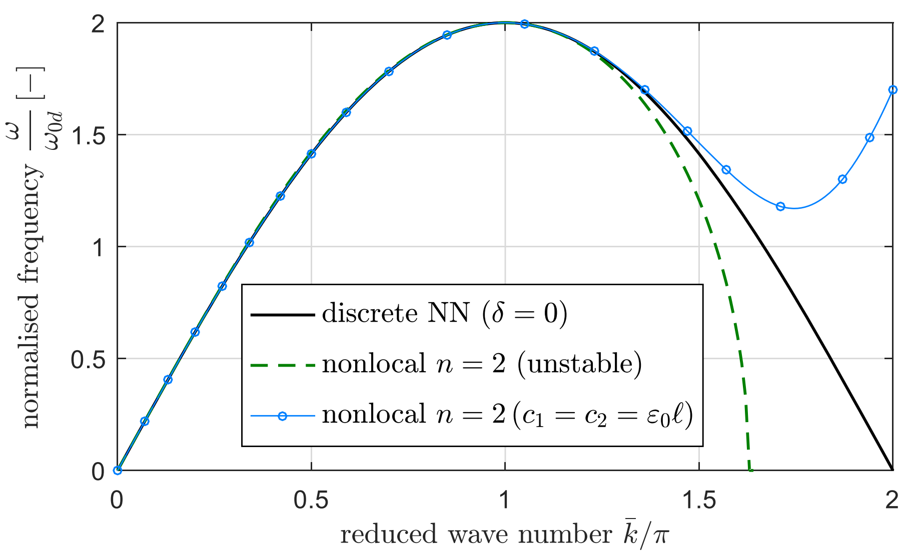



**Fig. 7** Dispersion curves for discrete lattice model with NN interactions against nonlocal elasticity of second order ($n = 2$)

proposed by Challamel et al. [22] to achieve an excellent matching of the dispersive curve of the Born–von Kármán model of lattice dynamics in the FBZ. Unfortunately, this model is unstable for higher wave numbers beyond the FBZ.

### 6.2 Matching the dispersion curve of the discrete lattice model

In order to match the dispersion curve of the nonlocal continuum with that of the discrete lattice model, a set of conditions are enforced, depending on the order of the nonlocal elasticity theory. As a first step, we approximate the discrete lattice with NN interactions ($\delta = 0$). For $n = 1$ there are two free nonlocal material parameters $c_1, \xi_1$, thus only two conditions can be imposed. As a first choice, we enforce the value $\bar{\omega}(\bar{k})\big|_{\bar{k}=\pi}$ and the value of the group velocity $\bar{c}_g\big|_{\bar{k}=\pi}$ at the end of the FBZ—relevant results are displayed in Fig. 6, while the identified parameters are listed in Table 1.

As can be seen, the resulting dispersion curve shows imaginary frequencies beyond a certain wave number, which denotes destabilizing behavior. Since $\xi_1 = 1.867$, the resulting $\xi_0 = 1 - \xi_1 = -0.867$ is negative, and this denotes instability as explained in the previous Remark 3. This is also confirmed by the negative value of the length scale parameter $\bar{\ell}^2_{\xi,1}$ reported in Table 1. The values $\xi_0 = -0.867$ and $\bar{c}_1 = 0.218$ are perfectly in line with the work by Challamel

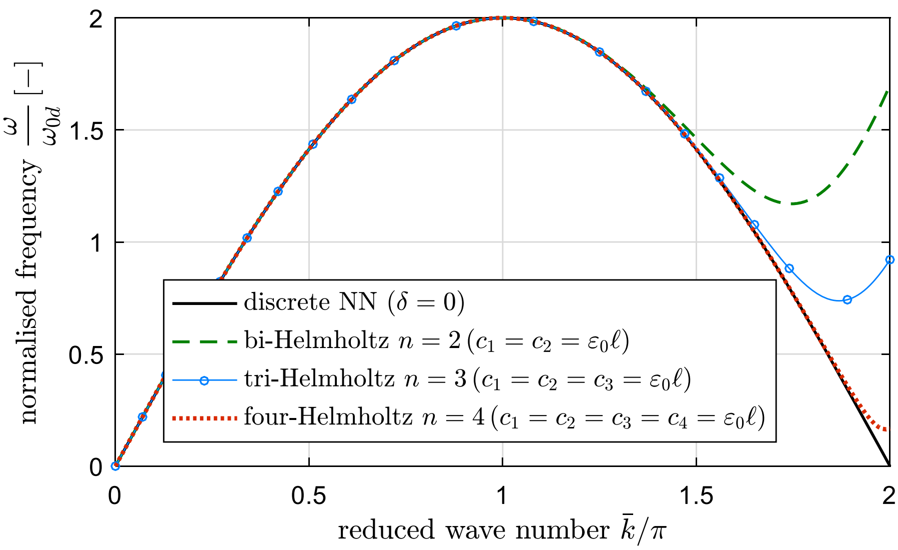



**Fig. 8** Dispersion curves for discrete lattice model with NN interactions against higher order nonlocal elasticity ($n = 2, 3, 4$)

et al. [22]. As an alternative, we could eliminate one of the two free parameters by setting $\xi_0 = 0$, so that $\xi_1 = 1$ accordingly (Eringen model). This also implies that the length scale parameter $\bar{\ell}^2_{\xi,1} = 0$. It is worth noting that this combination of parameters leads to the Helmholtz type nonlocal elasticity theory discussed in Lazar et al. [7] with the differential operator applied only to the LHS of the equations of motion (whereas in the proposed nonlocal elasticity theory the differential operator is applied to either side). The well-known value $\bar{c}_1 = 0.386$ coincides with the value $\varepsilon_0 = 0.39$ reported by Lazar et al. [7] and by Challamel et al. [22] for nonlocal elasticity of Helmholtz type. However, this eliminates the local phase and reverts to the Eringen's 1983 nonlocal (stress gradient) theory. The corresponding integral-type constitutive relation is a Fredholm integral equation of first type, and it is not invertible (see the remarks in the part I paper). To keep the local phase and, correspondingly, a Fredholm integral equation of second type (which is invertible) we consider an additional solution with $\xi_0 \neq 0$, but in order to avoid destabilizing behavior we impose a set of different conditions, namely the value in $\bar{\omega}(\bar{k})\big|_{\bar{k}=\pi/2}$ and the value of $d\bar{c}_g/d\bar{k}\big|_{\bar{k}=\pi/4} = d^2\bar{\omega}/d^2\bar{k}\big|_{\bar{k}=\pi/4}$. The relevant dispersion curve is plotted in Fig. 6 as well, and the corresponding material parameters reported in Table 1.

As second step, the generalized nonlocal elasticity theory of second order ($n = 2$) is considered. In general, for $n = 2$ there are four independent nonlocal material parameters $c_1, c_2, \xi_1, \xi_2$, thus four conditions should be imposed accordingly. Therefore, there are two further conditions in addition to the ones adopted for $n = 1$, which are preliminarily chosen as: $\bar{c}_g\big|_{\bar{k}=\pi/2}$ and $d\bar{c}_g/d\bar{k}\big|_{\bar{k}=\pi}$. This results in the dispersion curve shown in Fig. 7, which exhibits a destabilizing behavior beyond certain wave numbers. To avoid this, we resort to the simplification of the bi-Helmholtz nonlocal elasticity model with the same length scale coefficient for the two Helmholtz operators, namely $c_1 = c_2 = c = \varepsilon_0\ell$ [7] where $\varepsilon_0$ is a coefficient that multiplies the characteristic length scale $\ell$ (i.e., a lattice constant). This model has now 3 free parameters, namely $\varepsilon_0, \xi_1, \xi_2$. Imposing this time $\bar{\omega}(\bar{k})\big|_{\bar{k}=\pi/2}$ leads to a stable behavior, as shown in Fig. 7 and reported in Table 1 with $\bar{c}_1 = \bar{c}_2 = \varepsilon_0 = 0.109$. Despite the negative value of the $\bar{\ell}^4_{\xi,2}$ parameter, the model does not manifest imaginary frequencies because $1 + \bar{\ell}^2_{\xi,1}\bar{k}^2 + \bar{\ell}^4_{\xi,2}\bar{k}^4 > 0$ throughout the range of wave numbers. This is also confirmed by the positive $\xi_0$ value obtained as $\xi_0 = 1 - \xi_1 - \xi_2$ with $\xi_1$ and $\xi_2$ being reported in Table 1.

Obviously, increasing the order of the nonlocal elasticity theory improves the matching of the lattice model further. To prove this, in Fig. 8 we report the bi-, tri- and four-Helmholtz nonlocal elasticity models with the same length scale coefficient $c_i = \varepsilon_0\ell \ \forall i$. Besides the above conditions $\bar{\omega}(\bar{k})\big|_{\bar{k}=\pi}$ and $\bar{c}_g\big|_{\bar{k}=\pi}$, the length scale parameters reported in Table 2 are

**Table 2** Nonlocal material parameters of the generalized nonlocal elasticity theory for $n=3$ and $n=4$ identified by matching the dispersion curve of the lattice model with NN interactions

| $n$ | Nonlocal material parameters | | | | | Length scale parameters | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\xi_1$ | $\xi_2$ | $\xi_3$ | $\xi_4$ | $\varepsilon_0$ | $\ell^2_{\xi,1}$ | $\ell^4_{\xi,2}$ | $\ell^6_{\xi,3}$ | $\ell^8_{\xi,4}$ | $\ell^2_1$ | $\ell^4_2$ | $\ell^6_3$ | $\ell^8_4$ |
| 3 | − 11.59 | 13.45 | − 3.81 | – | 0.147 | − 0.02 | − 1.3 × $10^{-3}$ | 2.98 × $10^{-5}$ | – | 0.06 | 1.4 × $10^{-3}$ | 1.0 × $10^{-5}$ | – |
| 4 | − 4181 | 7405 | − 5836 | 1726 | 0.056 | − 0.07 | 1.8 × $10^{-3}$ | − 1.95 × $10^{-5}$ | − 8.6 × $10^{-8}$ | 0.01 | 5.9 × $10^{-5}$ | − 1.2 × $10^{-7}$ | − 9.7 × $10^{-11}$ |

identified by adding supplemental conditions in the form of $d\bar{c}_g/d\bar{k}\big|_{\bar{k}=\pi}$ and $d^3\bar{c}_g/d\bar{k}^3\big|_{\bar{k}=\pi}$ for $n=3$, and in the form $d\bar{c}_g/d\bar{k}\big|_{\bar{k}=\pi}$, $d^3\bar{c}_g/d\bar{k}^3\big|_{\bar{k}=\pi}$ and $d^5\bar{c}_g/d\bar{k}^5\big|_{\bar{k}=\pi}$ for $n=4$. This implies capturing the inflexion of the lattice model beyond the FBZ more accurately. It can be demonstrated that both the identified models with $n=3$ and $n=4$ exhibit a stable behavior, although some of the length scale parameters appear with negative sign, cf. Table 2.

As a second step, we repeat the length scale identification procedure by matching the dispersion curve of the lattice model with NNN interactions for $\delta = 1/4$. Relevant results are shown in Fig. 9, and the corresponding values are listed in Table 3. Despite the mixed positive and negative signs of the length scale parameters, all the identified models are stable, as confirmed by trends in Fig. 9.

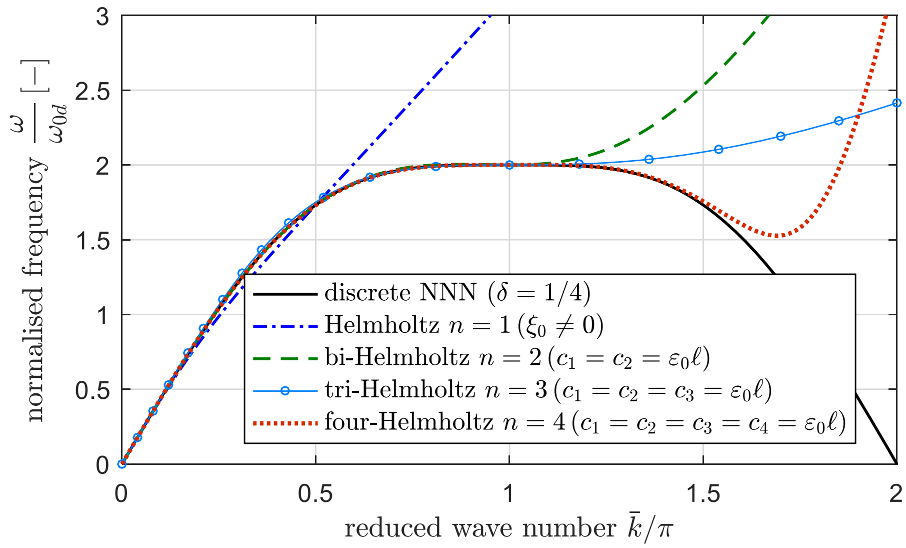
## 7 Nonlocal continuum versus discrete lattice model

### 7.1 Comparison in terms of global attenuation function

Once the nonlocal material parameters are calibrated by matching the dispersion curve of the lattice model, comparison in terms of the resulting global attenuation function $\mathscr{A}(r\,; c_1, c_2, \ldots, c_n)$ reported in (117) is straightforward. Such comparison enables a visual and conceptual assessment of the description of the nonlocal diffusion processes taking place in the underlying microstructure that simulate the long-range interactions occurring in the lattice model. To this end, comparing Eq. (121) of the nonlocal continuum with the corresponding Eq. (114) of the lattice model leads to the determination of the Fourier transform of the nonlocal continuum model having the same dispersion curve of the reference lattice model, that is

$$\tilde{\mathscr{A}}_{\text{lattice}}(\bar{k}, \delta) = \frac{\bar{\omega}^2}{\bar{k}^2(1+4\delta)} = \frac{4\left[\sin^2\left(\frac{\bar{k}}{2}\right) + \delta\sin^2(\bar{k})\right]}{\bar{k}^2(1+4\delta)}. \tag{123}$$

**Fig. 9** Dispersion curves for discrete lattice model with NNN interactions for $\delta = 1/4$ against higher order nonlocal elasticity ($n = 1, 2, 3, 4$)




Through the inverse Fourier transform of (123) the sought attenuation function of the lattice model is evaluated as follows

$$\mathscr{A}_{\text{lattice}}(\bar{r},\delta) = \mathscr{F}^{-1}\left[\tilde{\mathscr{A}}_{\text{lattice}}(\bar{k},\delta)\right] = \frac{1}{2(1+4\delta)}\left[-(1+\delta)f(\bar{r},0) + f(\bar{r},1) + \delta f(\bar{r},2)\right] \tag{124}$$

where $\bar{r} = r/\ell$ and $f(\bar{r},p) = |\bar{r}+p| + |\bar{r}-p|$. The plots of $\tilde{\mathscr{A}}_{\text{lattice}}(\bar{k},\delta)$ and $\mathscr{A}_{\text{lattice}}(\bar{r},\delta)$ are reported in Fig. 10 for different values of the second order interaction ratio $\delta$.

Thus, Eq. (124) can be compared to the global attenuation function of the generalized nonlocal continuum $\mathscr{A}_{\text{nonlocal}}(\bar{r},\varepsilon_0\ell)$ reported in (117), expressed in terms of a single length scale parameter $\varepsilon_0\ell$ and in terms of dimensionless $\bar{r} = r/\ell$ parameter

$$\begin{aligned}\mathscr{A}_{\text{nonlocal}}(\bar{r},\varepsilon_0\ell) &= \xi_0\delta(\bar{r}) + \xi_1\alpha^{(1)}(\bar{r},\varepsilon_0\ell) \\ &\quad + \xi_2\alpha^{(2)}(\bar{r},\varepsilon_0\ell) + \cdots \\ &\quad + \xi_n\alpha^{(n)}(\bar{r},\varepsilon_0\ell) \\ &= \sum_{j=0}^{n}\xi_j\alpha^{(j)}(\bar{r},\varepsilon_0\ell)\end{aligned} \tag{125}$$

where $\varepsilon_0$ and $\xi_j (j = 0,\ldots,n)$ have calibrated in the previous subsection by matching the dispersion curve of the lattice model. Comparison of $\mathscr{A}_{\text{lattice}}(\bar{r},\delta)$ for $\delta = 1/4$ and the corresponding $\mathscr{A}_{\text{nonlocal}}(\bar{r},\varepsilon_0\ell)$ with nonlocal material parameters listed in Table 3 is shown in Fig. 11. As can be seen, in the central zone for $\bar{r} = 0$ the local contribution, characterized by the term $\xi_0\delta(\bar{r})$ in Eq. (125), is dominant. On the contrary, in the zones far from the origin, i.e. for $|\bar{r}| > 0$, size effects induced by the discrete microstructure take place and the nonlocal long-range interactions become manifest, as described by the attenuation function. These long-range interactions are well captured by the nonlocal continuum model: indeed, the attenuation function $\mathscr{A}_{\text{nonlocal}}(\bar{r},\varepsilon_0\ell)$ is quite close to $\mathscr{A}_{\text{lattice}}(\bar{r},\delta)$ for $|\bar{r}| > 0$, with accuracy that is higher with increasing order of nonlocal elasticity theory. For $n \geq 2$ the approximation of $\mathscr{A}_{\text{lattice}}(\bar{r},\delta)$ is satisfactory for a wide range of normalized distances $\bar{r}$.

### 7.2 Capturing the static and dynamic response of the discrete lattice

The static and dynamic response of the lattice model sketched in Fig. 3 is simulated through the nonlocal elasticity model with the material parameters identified in the previous subsections. We consider a discrete lattice comprising 21 mass particles, so that the total length of the corresponding nonlocal continuum bar is $L = 20\ell$. We consider NNN interactions ($\delta = 1/4$). The input data of the lattice are: $\ell = 0.1421$ nm (distance between two carbon atoms), $k_1 = 305$ nN/nm (first order stiffness), $k_2 = \delta k_1$, and $A = 1\,\text{nm}^2$.

As a first example, the static response of the discrete lattice subject to two concentrated axial forces $P = 0.01$ nN at the two free ends is considered. The

**Table 3** Nonlocal material parameters of the generalized nonlocal elasticity theory for $n = 1, 2, 3, 4$ identified by matching the dispersion curve of the lattice model with NNN interactions for $\delta = 1/4$

| $n$ | Nonlocal material parameters | | | | | Length scale parameters | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\bar{\xi}_1$ | $\bar{\xi}_2$ | $\bar{\xi}_3$ | $\bar{\xi}_4$ | $\bar{\varepsilon}_0$ | $\bar{\ell}^2_{\xi,1}$ | $\bar{\ell}^4_{\xi,2}$ | $\bar{\ell}^6_{\xi,3}$ | $\bar{\ell}^8_{\xi,4}$ | $\bar{\ell}^2_1$ | $\bar{\ell}^4_2$ | $\bar{\ell}^6_3$ | $\bar{\ell}^8_4$ |
| 1† | 1 | – | – | – | 0.631 | 0 | – | – | – | 0.399 | – | – | – |
| 1‡ | 0.55 | – | – | – | 0.997 | 0.445 | – | – | – | 0.994 | – | – | – |
| 2 | −2.40 | 2.74 | – | – | 0.264 | −0.075 | $3.2 \times 10^{-3}$ | – | – | 0.139 | $4.9 \times 10^{-3}$ | – | – |
| 3 | −0.19 | 2.86 | −1.73 | – | 0.508 | 0.69 | $9.3 \times 10^{-4}$ | $1.20 \times 10^{-3}$ | – | 0.78 | 0.20 | 0.017 | – |
| 4 | −407 | 797 | −692 | 226 | 0.129 | −0.14 | 0.0118 | $-4.49 \times 10^{-4}$ | $6.1 \times 10^{-6}$ | 0.07 | $1.7 \times 10^{-3}$ | $1.9 \times 10^{-5}$ | $7.8 \times 10^{-8}$ |

†This is a particular subcase of the proposed generalized theory of nonlocal elasticity by setting $\bar{\xi}_0 = 0$ and $\bar{\xi}_1 = 1.0$ so that the length scale parameter $\bar{\ell}_{\xi,1} = 0$; ‡this set of parameters is obtained by setting the value $\bar{\omega}(\bar{k})\big|_{\bar{k}=\pi/2}$ and $d^3\bar{\omega}/d^3\bar{k}\big|_{\bar{k}=\pi/2}$

stiffness matrix $\mathbf{K}$ is easily constructed considering the generic equation of motion of the $n$th mass particle (111). Ignoring inertia effects (as if the load were applied in a quasi-static manner), the governing equation in statics $\mathbf{K}\mathbf{u} = \mathscr{F}$ is solved to determine the displacements of the mass particles $\mathbf{u}$ and, then, the corresponding strain field. In the case of NN interactions ($\delta = 0$), the strain field would be constant throughout the lattice and equal to $\varepsilon = P/(EA)$ with $E = k_1\ell/A$. However, in the case of NNN interactions the strain field is a bit different. While in the central zone of the lattice the strain field is constant and equal to $\varepsilon = P/(E_{\mathrm{eq}}A)$ with $E_{\mathrm{eq}} = k_{\mathrm{eq}}\ell/A = k_1(1+4\delta)\ell/A$, the boundary masses are affected by a reduction of stiffness near the lattice ends: indeed, the second order stiffness contributions $k_2$ reduce in the two boundary atoms in comparison with the other ones along the lattice length. This stiffness variation in the second order interactions causes an increase of axial strain in the two boundary masses and some fluctuations in the neighboring atoms that cannot be captured with a local continuum model. Instead, these fluctuations can be approximated with a nonlocal continuum model, as demonstrated in Fig. 12. In addition to the continuous profile of the axial strain computed through the nonlocal continuum model, a piecewise plot is also reported for facilitating the comparison with the discretized profile of the lattice model. This piecewise profile is computed at each midpoint $x_0$ of the particle spacing $\ell$ as weighted average of the continuous function within such a lattice spacing, i.e.,

$$\varepsilon_{\mathrm{average}}(x_0) = \frac{1}{\ell}\int_{x_0-\ell/2}^{x_0+\ell/2} \varepsilon(y)\,\mathrm{d}y. \tag{126}$$

As can be seen from Fig. 12, the nonlocal continuum model of second order ($n = 2$) is able to describe the actual boundary fluctuations of the static response of the lattice model with good accuracy, much better than the first order model ($n = 1$).

As a second example, the eigenvectors of the nonlocal continuum bar (i.e., the eigenfunctions $\phi_1(x), \phi_2(x), \ldots$) are calculated and compared to the natural modes of vibration of the discrete lattice model obtained by solving the eigenvalue problem (116). The mass of each atom is set as $m = 1.66 \cdot 10^{-18}$ nKg so that the mass per unit volume is $\rho = m/(A\ell) = 1.1682 \times 10^{-17}$nKg/nm$^3$. The other

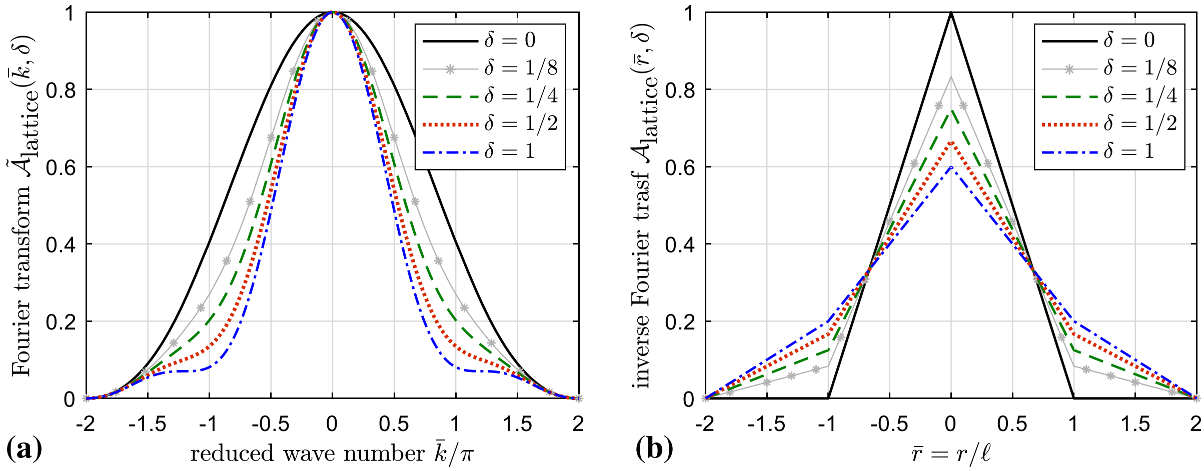
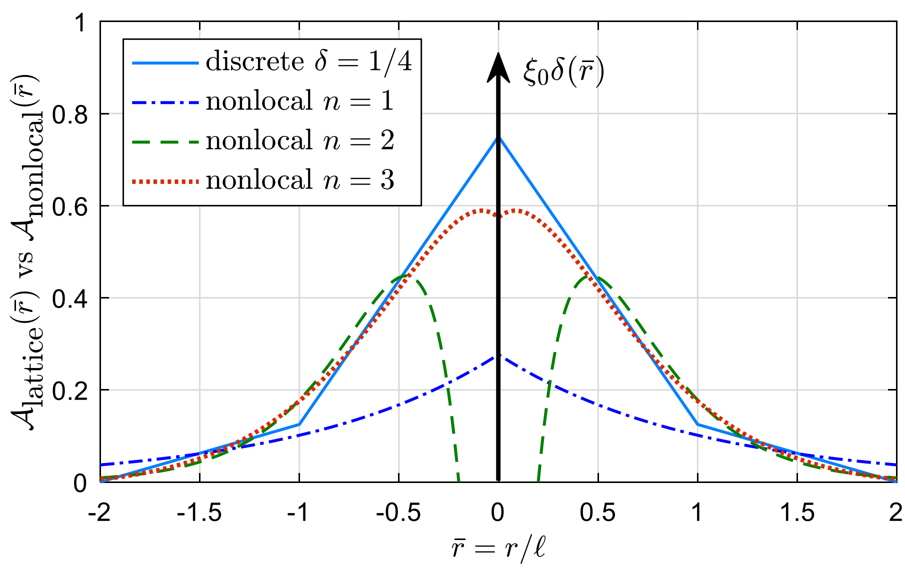



**Fig. 10** Fourier transform $\tilde{\mathscr{A}}_{\text{lattice}}(\bar{k},\delta)$ and inverse Fourier $\mathscr{A}_{\text{lattice}}(\bar{r},\delta)$ of the attenuation function of the discrete lattice model for different values of the second order interaction ratio $\delta$

**Fig. 11** Comparison of global attenuation function of the discrete lattice model and of the higher-order continuum model for NNN interactions with $\delta = 1/4$




parameters are kept the same as defined above. In order to have the same total mass $m_{\text{tot}} = 21m$ in the discrete lattice and in the nonlocal continuum model, the length of the nonlocal bar is set as $L = 21\ell$ in this case (so that $\rho L = m_{\text{tot}} = 21m$).

Comparisons relevant to the second order nonlocal elasticity model ($n = 2$) are depicted in Figs. 13 and 14 for the first 12 natural modes of vibration of the lattice model with NN and NNN interactions, respectively. It is seen that the nonlocal material parameters identified in the previous section (cf. Tables 1 and 3 for NN and NNN interactions, respectively) provide an excellent simulation of the free vibration response of the discrete lattice model, even for relatively high frequencies. The corresponding $\bar{\varpi} - \bar{k}$ couples of the discrete lattice model and of the nonlocal continuum bar lie on the corresponding dispersion curves (of the lattice model, Eq. (114), and of the nonlocal model, Eq. (101)). These $\bar{\varpi} - \bar{k}$ couples are reported in Fig. 15 for NN and NNN interactions. As can be observed, for NN interactions the $\bar{\varpi} - \bar{k}$ couples of the lattice model are superimposed to the ones of the nonlocal bar, and the description of the dispersion curve is perfect within the FBZ; for NNN interactions,

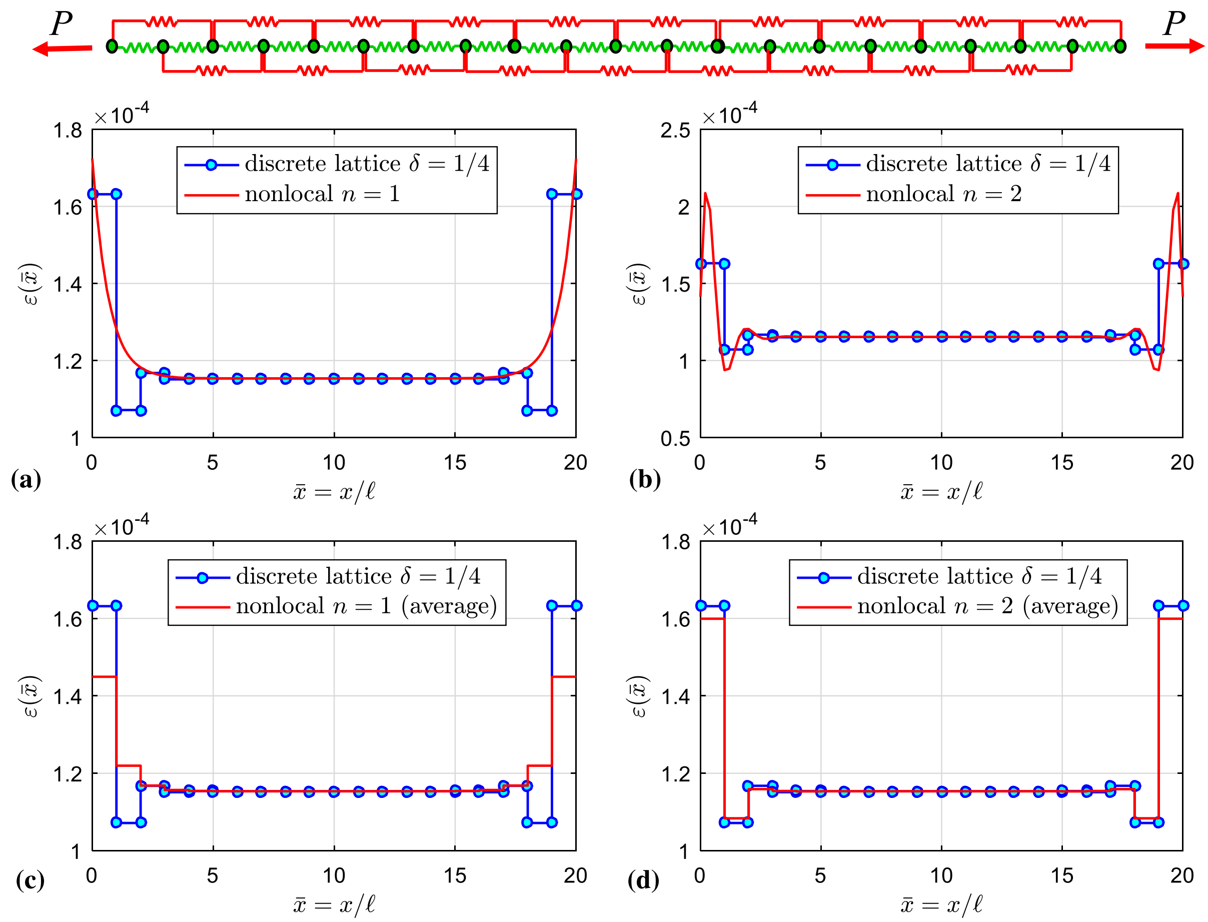



**Fig. 12** Simulation of the static response, in terms of axial strain, of discrete lattice with 21 mass particles and NNN interactions as sketched in Fig. 3 through nonlocal continuum model of first and second order

the description is excellent up to reduced wave numbers $\bar{k} \approx 0.4\pi$ and then some modest discrepancies arise for higher wave numbers.

It can be concluded that the proposed nonlocal elasticity model of second order is capable of describing size effects and long-range interactions observed in the static and dynamic response of the discrete lattice model with NN and NNN particle interactions.

## 8 Conclusions

The generalized theory of nonlocal elasticity of $n$-Helmholtz type presented in the Part I paper has been particularized here for the one-dimensional case. The starting point has been to assume the Eringen nonlocal (bi-exponential) kernel to define the first order nonlocal strain field. By convolution product, it has been possible to extend the derivations up to the definition of a generic $n$th nonlocal strain field. These derivations have also been replicated through the Fourier transform method for completeness. From this particular family of nonlocal kernels, the corresponding nonstandard boundary conditions that are consistent with the attenuation functions have been derived. These nonstandard boundary conditions are useful to pass from the integral formulation to the differential (or, also, integro-differential) format of the proposed generalized nonlocal elasticity theory in a consistent manner.

Combining the generalized constitutive equations with the compatibility and dynamic equilibrium equations, the expressions governing the boundary

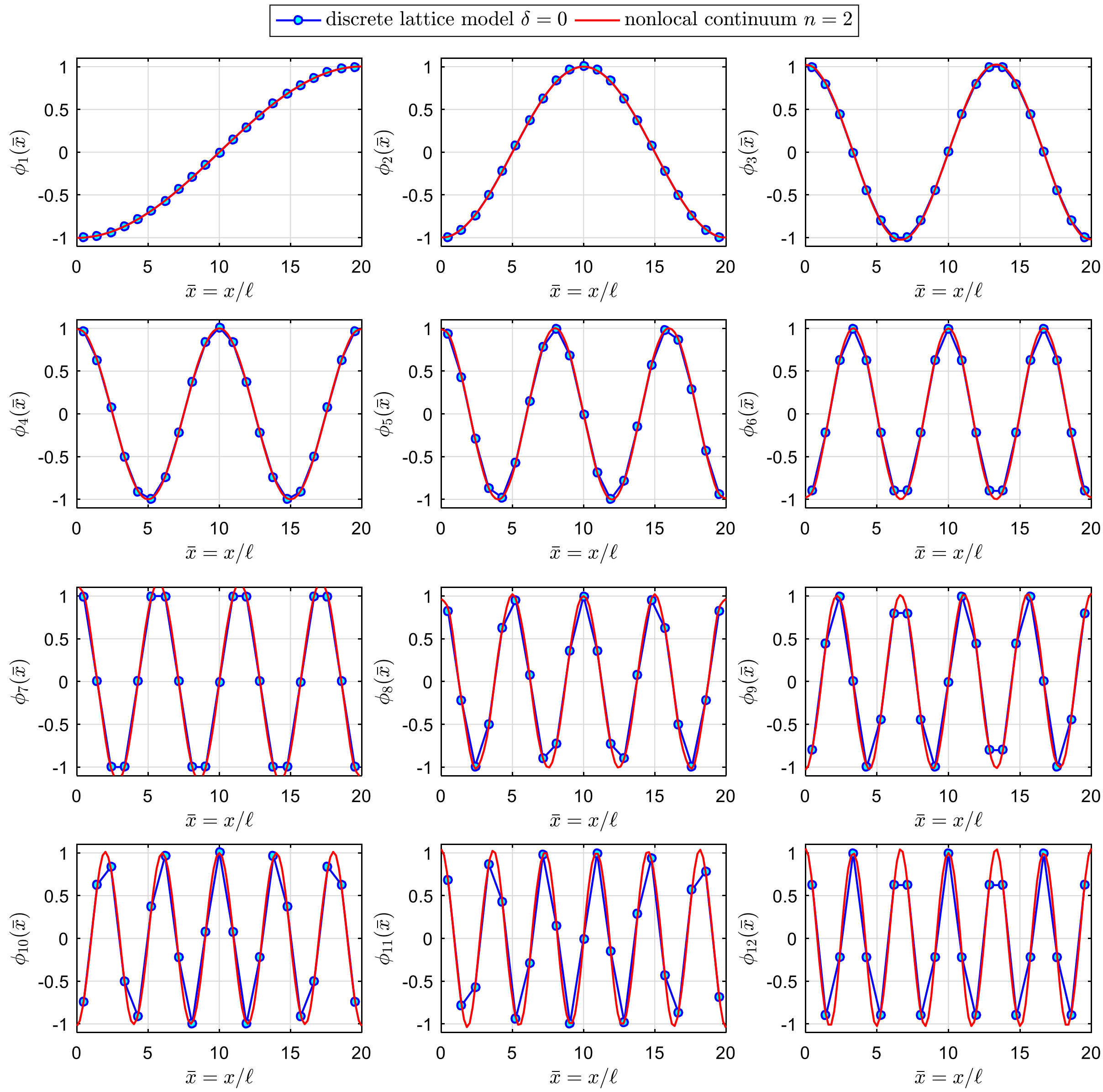



**Fig. 13** Simulation of the first 12 natural modes of vibration of discrete lattice with 21 mass particles and NN interactions ($\delta = 0$) as sketched in Fig. 3 through second order nonlocal continuum model

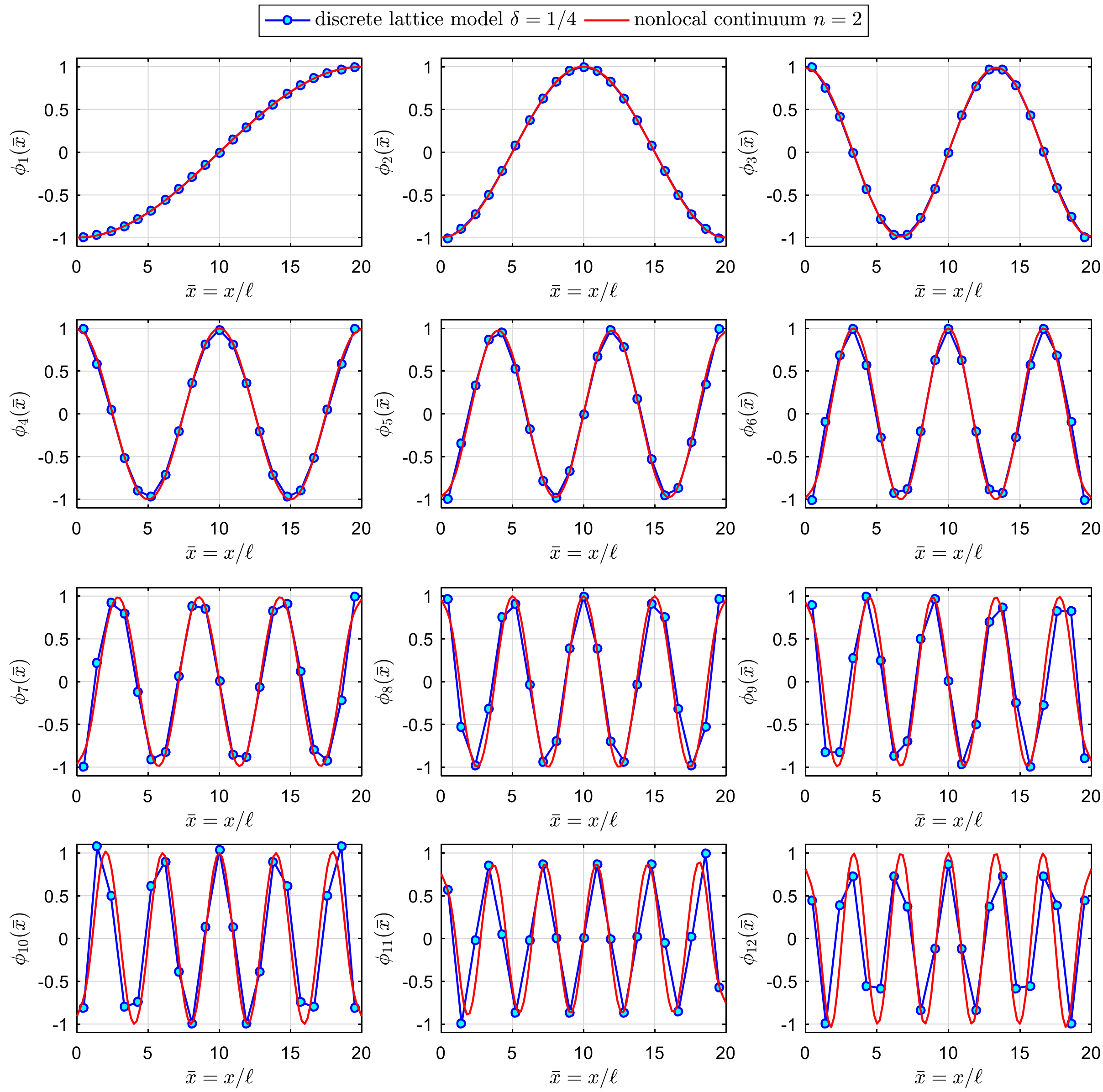



**Fig. 14** Simulation of the first 12 natural modes of vibration of discrete lattice with 21 mass particles and NNN interactions ($\delta = 1/4$) as sketched in Fig. 3 through second order nonlocal continuum model

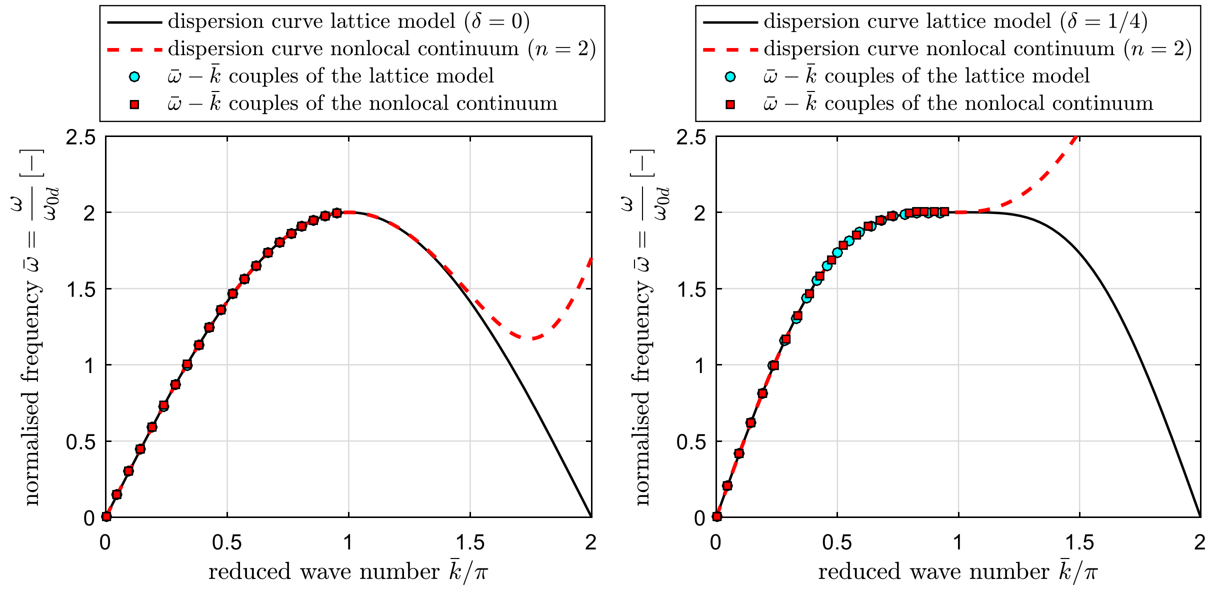



**Fig. 15** $\bar{\omega} - \bar{k}$ couples of the lattice model and of the nonlocal bar for NN interactions (left) and NNN interactions (right)

value problem for the one-dimensional case have been presented. These expressions have been used to simulate the static and dynamic response of a discrete lattice model with long-range interactions. In line with other studies from the relevant literature, the nonlocal material parameters have been identified by matching the dispersion curve of the discrete lattice. Using the identified nonlocal material parameters, a comparison in terms of global attenuation functions has also been presented, by exploiting the Fourier transform method to determine the exact attenuation function of the lattice model with NNN interactions. Some simple examples in statics and dynamics highlight the accuracy of the nonlocal elasticity theory of second order to capture size effects and long-range interactions occurring in the lattice model.

It is emphasized here that this particular format of differential and integro-differential model is perfectly consistent with the integral nonlocal formulation. Thus, once the calibration of the nonlocal material parameters is carried out with reference to the differential (weakly nonlocal) theory, further analyses can be performed through the integral (strongly nonlocal) theory by exploiting the resulting nonlocal kernels that are consistent with the identified length scale parameters.

In conclusion, two final remarks are made here regarding both the Part I and Part II papers. Increasing the order $n$ in the generalized theory of nonlocal elasticity entails the following consequences:

(1) the addition of further higher-order terms in the corresponding governing differential Eq. (58) (including derivatives of the acceleration field up to order $2n$ and derivatives of the displacement field up to order $2n + 2$);
(2) the addition of further terms in the global attenuation function (125) that is, therefore, better able to describe nonlocal long-range interactions;
(3) the addition of further terms in the dispersion curve of the nonlocal continuum (121) that is,

therefore, better able to describe the dispersion curve of lattice models;

(4) the implication of additional higher-order nonstandard boundary conditions to be enforced at the two bar ends, whose number is strictly related to the order $n$ (in particular, $2n$ nonstandard boundary conditions).

While extension to multi-dimensional problems (e.g. plates and shells) is quite challenging as it requires additional effort due to the implied higher-order 2D nonstandard boundary conditions that are consistent with the 2D attenuation functions, application of the proposed theory to beam models [23, 24] (e.g., Euler–Bernoulli, or Timoshenko beam theories) seems straightforward.

**Acknowledgements** This research work is the result of fruitful scientific discussions of the authors with Prof. Elias C. Aifantis. This article is dedicated to his academic career and achievements on the occasion of his 70th birthday.

**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

## Appendix A: Nonstandard boundary conditions of order *n* applied to nonlocal strain of order less than *n*

In this Appendix, we present some supplemental nonstandard boundary conditions that useful for the determination of the eigenfunctions $\phi(x)$ in the free vibration analysis. These supplemental relationships arise from applying the nonstandard boundary conditions of order $n$, i.e. involving $n$ $\mathscr{L}$ differential operators, to nonlocal strain of order of order $k$ less than $n$, i.e., $k = 1, 2, \ldots, n-1$. Let us consider the expression of $\bar{\varepsilon}_x^{(1)}(x)$ given in (33). By exploiting the Leibniz integral rule (36), the nonstandard boundary conditions for $n = 2$ as per (41) and (42) applied to $\bar{\varepsilon}_x^{(1)}(x)$ leads to

$$\begin{aligned}
&\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad= \left[\left(1-(c_1+c_2)\frac{d}{dx}+c_1c_2\frac{d^2}{dx^2}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=0} = -\frac{c_2}{c_1}\varepsilon_x(0),\\
&\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad= \left[\left(1-c_2\frac{d}{dx}-c_1^2\frac{d^2}{dx^2}+c_1^2c_2\frac{d^3}{dx^3}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=0} = \varepsilon_x(0)-c_2\varepsilon_x'(0),\\
&\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=L}\\
&\quad= \left[\left(1+(c_1+c_2)\frac{d}{dx}+c_1c_2\frac{d^2}{dx^2}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=L} = -\frac{c_2}{c_1}\varepsilon_x(L),\\
&\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=L}\\
&\quad= \left[\left(1+c_2\frac{d}{dx}-c_1^2\frac{d^2}{dx^2}-c_1^2c_2\frac{d^3}{dx^3}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=L} = \varepsilon_x(L)+c_2\varepsilon_x'(L).
\end{aligned} \tag{127}$$

By extending this result, the nonstandard boundary conditions for $n = 3$ as per (44) and (45) applied to $\bar{\varepsilon}_x^{(1)}(x)$ leads to

$$
\begin{aligned}
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad=\left[\left(1-(c_1+c_2+c_3)\frac{d}{dx}+(c_1c_2+c_1c_3+c_2c_3)\frac{d^2}{dx^2}-c_1c_2c_3\frac{d^3}{dx^3}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=0}\\
&\quad=-\frac{(c_2c_3+c_1c_2+c_1c_3)}{c_1^2}\varepsilon_x(0)+\frac{c_1c_2c_3}{c_1^2}\varepsilon_x^{'}(0),\\
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad=\left[\left(1-(c_2+c_3)\frac{d}{dx}+(c_2c_3-c_1^2)\frac{d^2}{dx^2}+(c_1^2c_2+c_1^2c_3)\frac{d^3}{dx^3}-c_1^2c_2c_3\frac{d^4}{dx^4}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=0}\\
&\quad=\varepsilon_x(0)-(c_2+c_3)\varepsilon_x^{'}(0)+c_2c_3\varepsilon_x^{''}(0),\\
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}=\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad=\left[\left(1-c_3\frac{d}{dx}-(c_1^2+c_2^2)\frac{d^2}{dx^2}+(c_1^2c_3+c_2^2c_3)\frac{d^3}{dx^3}+c_1^2c_2^2\frac{d^4}{dx^4}-c_1^2c_2^2c_3\frac{d^5}{dx^5}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=0}\\
&\quad=\varepsilon_x(0)-c_3\varepsilon_x^{'}(0)-c_2^2\varepsilon_x^{''}(0)+c_2^2c_3\varepsilon_x^{'''}(0),\\
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=L}\\
&\quad=\left[\left(1+(c_1+c_2+c_3)\frac{d}{dx}+(c_1c_2+c_1c_3+c_2c_3)\frac{d^2}{dx^2}+c_1c_2c_3\frac{d^3}{dx^3}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=L}\\
&\quad=-\frac{(c_2c_3+c_1c_2+c_1c_3)}{c_1^2}\varepsilon_x(L)+\frac{c_1c_2c_3}{c_1^2}\varepsilon_x^{'}(L),\\
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=L}\\
&\quad=\left[\left(1+(c_2+c_3)\frac{d}{dx}+(c_2c_3-c_1^2)\frac{d^2}{dx^2}-(c_1^2c_2+c_1^2c_3)\frac{d^3}{dx^3}-c_1^2c_2c_3\frac{d^4}{dx^4}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=L}\\
&\quad=\varepsilon_x(L)+(c_2+c_3)\varepsilon_x^{'}(L)+c_2c_3\varepsilon_x^{''}(L),\\
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=L}=\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=L}\\
&\quad=\left[\left(1+c_3\frac{d}{dx}-(c_1^2+c_2^2)\frac{d^2}{dx^2}-(c_1^2c_3+c_2^2c_3)\frac{d^3}{dx^3}+c_1^2c_2^2\frac{d^4}{dx^4}+c_1^2c_2^2c_3\frac{d^5}{dx^5}\right)\bar{\varepsilon}_x^{(1)}(x)\right]\Big|_{x=L}\\
&\quad=\varepsilon_x(L)+c_3\varepsilon_x^{'}(L)-c_2^2\varepsilon_x^{''}(L)-c_2^2c_3\varepsilon_x^{'''}(L).
\end{aligned}
\tag{128}
$$

And, finally, the nonstandard boundary conditions for $n = 4$ as per (46) applied to $\bar{\varepsilon}_x^{(1)}(x)$ leads to

$$
\begin{aligned}
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad= -\frac{\left(c_2c_3c_4 + c_1c_3c_4 + c_1c_2c_3 + c_1c_2c_4 + c_1^2(c_2+c_3+c_4)\right)}{c_1^3}\varepsilon_x(0)\\
&\quad -\frac{\left(c_1^2c_2c_3 - c_1^2c_2c_4 - c_1^2c_3c_4 - c_1c_2c_3c_4\right)}{c_1^3}\varepsilon_x^{'}(0) - \frac{(c_2c_3c_4)}{c_1}\varepsilon_x^{''}(0),\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad= \varepsilon_x(0) - (c_2+c_3+c_4)\varepsilon_x^{'}(0) + (c_2c_3+c_2c_4+c_3c_4)\varepsilon_x^{''}(0) - (c_2c_3c_4)\varepsilon_x^{'''}(0),\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad= \varepsilon_x(0) - (c_3+c_4)\varepsilon_x^{'}(0) + \left(c_3c_4 - c_2^2\right)\varepsilon_x^{''}(0) + \left(c_2^2c_3 + c_2^2c_4\right)\varepsilon_x^{'''}(0) - \left(c_2^2c_3c_4\right)\varepsilon_x^{''''}(0),\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(1)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0} = \mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_1,c_2,c_3}^{(3)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0}\\
&\quad= \varepsilon_x(0) - (c_4)\varepsilon_x^{'}(0) - \left(c_2^2 - c_3^2\right)\varepsilon_x^{''}(0) + \left(c_2^2c_4 + c_3^2c_4\right)\varepsilon_x^{'''}(0) + \left(c_2^2c_3^2\right)\varepsilon_x^{''''}(0) - \left(c_2^2c_3^2c_4\right)\varepsilon_x^{'''''}(0),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=L}\\
&\quad= -\frac{\left(c_2c_3c_4 + c_1c_3c_4 + c_1c_2c_3 + c_1c_2c_4 + c_1^2(c_2+c_3+c_4)\right)}{c_1^3}\varepsilon_x(L)\\
&\quad +\frac{\left(c_1^2c_2c_3 - c_1^2c_2c_4 - c_1^2c_3c_4 - c_1c_2c_3c_4\right)}{c_1^3}\varepsilon_x^{'}(0) - \frac{(c_2c_3c_4)}{c_1}\varepsilon_x^{''}(L),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(4)}(x)\big|_{x=L}\\
&\quad= \varepsilon_x(L) + (c_2+c_3+c_4)\varepsilon_x^{'}(L) + (c_2c_3+c_2c_4+c_3c_4)\varepsilon_x^{''}(L) + (c_2c_3c_4)\varepsilon_x^{'''}(L),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(4)}(x)\big|_{x=L} = \mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(4)}(x)\big|_{x=L}\\
&\quad= \varepsilon_x(L) + (c_3+c_4)\varepsilon_x^{'}(L) + \left(c_3c_4 - c_2^2\right)\varepsilon_x^{''}(L) - \left(c_2^2c_3 + c_2^2c_4\right)\varepsilon_x^{'''}(L) - \left(c_2^2c_3c_4\right)\varepsilon_x^{''''}(L),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(1)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(4)}(x)\big|_{x=L} = \mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_1,c_2,c_3}^{(3)}\bar{\varepsilon}_x^{(4)}(x)\big|_{x=L}\\
&\quad= \varepsilon_x(0) + (c_4)\varepsilon_x^{'}(L) - \left(c_2^2 + c_3^2\right)\varepsilon_x^{''}(L) - \left(c_2^2c_4 + c_3^2c_4\right)\varepsilon_x^{'''}(L) + \left(c_2^2c_3^2\right)\varepsilon_x^{''''}(L) + \left(c_2^2c_3^2c_4\right)\varepsilon_x^{'''''}(L).
\end{aligned}
\tag{129}
$$

Now, let us consider the expression of $\bar{\varepsilon}_x^{(2)}(x)$ given in (37). The nonstandard boundary conditions for $n = 3$ as per (44) and (45) applied to $\bar{\varepsilon}_x^{(2)}(x)$ leads to

$$
\begin{aligned}
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0} = \cdots = 0,\\
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0} = \cdots = -\frac{c_3}{c_2}\varepsilon_x(0),\\
&\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0} = \mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=0} = \cdots = \varepsilon_x(0) - c_3\varepsilon_x^{'}(0),\\
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L} = \cdots = 0,\\
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L} = \cdots = -\frac{c_3}{c_2}\varepsilon_x(L),\\
&\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L} = \mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(1)}(x)\big|_{x=L} = \cdots = \varepsilon_x(L) + c_3\varepsilon_x^{'}(L).
\end{aligned}
\tag{130}
$$

And, finally, the nonstandard boundary conditions for $n = 4$ as per (46) applied to $\bar{\varepsilon}_x^{(2)}(x)$ leads to

$$
\begin{aligned}
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0}=\cdots=\frac{c_3c_4}{c_1c_2}\varepsilon_x(0),\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0}=\cdots=-\frac{(c_3c_4+c_2c_3+c_2c_4)}{c_2^2}\varepsilon_x(0)+\frac{(c_2c_3c_4)}{c_2^2}\varepsilon_x'(0),\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0}=\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0}\\
&\quad=\varepsilon_x(0)-(c_3+c_4)\varepsilon_x^{'}(0)+(c_3c_4)\varepsilon_x^{''}(0),\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(1)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0}=\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_1,c_2,c_3}^{(3)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=0}\\
&\quad=\varepsilon_x(0)-c_4\varepsilon_x^{'}(0)-c_3^2\varepsilon_x^{''}(0)+\left(c_3^2c_4\right)\varepsilon_x^{'''}(0),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L}=\cdots=\frac{c_3c_4}{c_1c_2}\varepsilon_x(0),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L}=\cdots=-\frac{(c_3c_4+c_2c_3+c_2c_4)}{c_2^2}\varepsilon_x(L)-\frac{(c_2c_3c_4)}{c_2^2}\varepsilon_x'(L),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L}=\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L}\\
&\quad=\varepsilon_x(L)+(c_3+c_4)\varepsilon_x^{'}(L)+(c_3c_4)\varepsilon_x^{''}(L),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(1)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L}=\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_1,c_2,c_3}^{(3)}\bar{\varepsilon}_x^{(2)}(x)\big|_{x=L}\\
&\quad=\varepsilon_x(L)+c_4\varepsilon_x^{'}(L)-c_3^2\varepsilon_x^{''}(L)-\left(c_3^2c_4\right)\varepsilon_x^{'''}(L).
\end{aligned}
\tag{131}
$$

Now, let us consider the expression of $\bar{\varepsilon}_x^{(3)}(x)$ given in (43). The nonstandard boundary conditions for $n = 4$ as per (46) applied to $\bar{\varepsilon}_x^{(3)}(x)$ leads to

$$
\begin{aligned}
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(-)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=0}=0,\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(-)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=0}=0,\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=0}=\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(-)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=0}=-\frac{c_4}{c_3}\varepsilon_x(0),\\
&\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_3}^{(1)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=0}=\mathscr{L}_{c_4}^{(-)}\mathscr{L}_{c_1,c_2,c_3}^{(3)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=0}=\varepsilon_x(0)-c_4\varepsilon_x'(0),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(+)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=L}=0,\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(+)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=L}=0,\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=L}=\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(+)}\mathscr{L}_{c_1,c_2}^{(2)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=L}=-\frac{c_4}{c_3}\varepsilon_x(L),\\
&\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_3}^{(1)}\mathscr{L}_{c_2}^{(1)}\mathscr{L}_{c_1}^{(1)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=L}=\mathscr{L}_{c_4}^{(+)}\mathscr{L}_{c_1,c_2,c_3}^{(3)}\bar{\varepsilon}_x^{(3)}(x)\big|_{x=L}=\varepsilon_x(L)+c_4\varepsilon_x'(L).
\end{aligned}
\tag{132}
$$

By extension, for a generic order $n$ one should calculate the nonstandard boundary conditions applied to $\bar{\varepsilon}_x^{(k)}(x)\,(k=1,\ldots,n-1)$ by repeatedly applying the following relations

$$
\begin{cases}
\mathscr{L}_{c_n,c_{n-1},\ldots,c_j}^{(p-)}\mathscr{L}_{c_1,c_2,\ldots,c_{j-1}}^{(j-1)}\bar{\varepsilon}_x^{(k)}(x)\Big|_{x=0}\\
\mathscr{L}_{c_n,c_{n-1},\ldots,c_j}^{(p+)}\mathscr{L}_{c_1,c_2,\ldots,c_{j-1}}^{(j-1)}\bar{\varepsilon}_x^{(k)}(x)\Big|_{x=L}
\end{cases}
\left(\begin{array}{r}\text{for } j=1,2,\ldots,n-1;\\ p=n-j+1;\quad n\geq j;\\ k=1,\ldots,n-1\end{array}\right)
\tag{133}
$$

where the differential operators $\mathscr{L}_{c_n,c_{n-1},\ldots,c_j}^{(p-)}$ and $\mathscr{L}_{c_n,c_{n-1},\ldots,c_j}^{(p+)}$ are defined in (48). These higher order boundary conditions are useful for the determination of the eigenfunctions $\phi(x)$ discussed in subsection 4.2. In this Appendix, we have explicitly reported the nonstandard boundary conditions up to order $n = 4$. It is worth noting that all the presented expressions are simplified in the case $c_1 = c_2 = \cdots = c_n = \ell$.

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