Meccanica (2020) 55:103–112
https://doi.org/10.1007/s11012-019-01102-3

Some properties of solutions in linear theory for semi-strongly elliptic porous elastic materials

Ada Amendola · Francesca Passarella · Vincenzo Tibullo

Received: 16 August 2019 / Accepted: 28 November 2019 / Published online: 11 December 2019

Abstract This paper is concerned with the linear theory of elastodynamics for homogeneous, isotropic, porous elastic materials with memory effects for the intrinsic equilibrated body forces. We are able to relax the conditions on constitutive coefficients and to determine the wider class of materials for which the internal energy is positive semi-definite, when boundary conditions are homogeneous. We found the class of semi-strongly elliptic porous elastic materials. For this class of materials, the above conditions may be relaxed without loss of some well-posedness properties of the solutions. In particular, we obtain uniqueness of the solutions and we study the spatial behavior problem.

Keywords Porous materials · Memory effect · Semi-strong ellipticity · Uniqueness result · Spatial behavior

A. Amendola
Dipartimento di Ingegneria Civile, Università di Salerno, Fisciano, Italy
e-mail: aamendola1@unisa.it

F. Passarella · V. Tibullo (✉)
Dipartimento di Matematica, Università di Salerno, Fisciano, Italy
e-mail: vtibullo@unisa.it

F. Passarella
e-mail: fpassarella@unisa.it

1 Introduction

In the study of the uniqueness of solutions or in wave propagation, Gurtin [1] shows how the condition of strong ellipticity of the elasticity tensor is applicable. Ericksen and Toupin [2] prove, in the equilibrium case, that there is uniqueness of solutions if and only if we assume constitutive coefficients of the strongly elliptic type. The hypothesis of strong ellipticity, in the dynamical case, may be relaxed without loss of uniqueness, or without loss of some other well-posedness properties of the solutions. Examples of strongly elliptic materials are auxetic materials (see e.g. Lakes [3] and Caddock and Evans [4]); these materials have particular structures that expand laterally when stretched, in contrast to the behavior of ordinary materials, see [5]. Some anisotropic polymer foams have been prepared which exhibit a Poisson's ratio exceeding 1 (see [6]). Materials of the above sorts are expected to have interesting mechanical properties, such as high energy absorption and fracture resistance, which may be useful in applications. Possible applications of such materials in prevention of pressure sores or ulcers are outlined by Wang and Lakes [7]. The ellipticity analysis is relevant in studying wave propagation [1, 8, 9] and has important applications in several contexts [10–13].

Further, Gurtin and Sternberg [14] and Gurtin and Toupin [15] have obtained the uniqueness of solutions for semi-strongly elliptic elastic bodies for the first boundary value problem of elastodynamics, surface displacements prescribed, and for bounded domains. The concept of semi-strongly elliptic materials is not new, but has already been object of investigation by many and has theoretical interest. Now, we have to outline that the method used in the uniqueness theorem (see also [1, 16, 17]) can be used also for appropriate classes of weak solutions.

Porous media may be studied by mathematical models where the bulk density is written as the product of two fields: the matrix material density field and the volume fraction field (see Cowin and Nunziato [18, 19]). The Cowin and Nunziato theory has enjoyed much success and its applications are to geological materials, such as rock, soils and to manufactured porous materials. The literature dealing with these phenomena is very rich and covers different research fields, see, for example, [20–26].

In [27] Chiriță and Ciarletta obtain a spatial decay estimate of exponential type with a factor independent of time using the time-weighted surface power function method under the positive definiteness condition on the elasticity tensor. In [28] under the positive definiteness of internal energy density, Scalia studies the spatial and temporal behavior of the solution to the initial-boundary value problems associated with the linear theory of porous elastic materials with memory effects for the intrinsic equilibrated body forces.

Important results have been achieved to describe the spatial behavior of the solutions for some classes of materials for which the internal energy density is not necessarily positive definite. There are been several attempts to relax the conditions on constitutive coefficients, while still maintaining well-posedness results. For example, following [27, 29], in [30] the authors obtain a result of uniqueness and spatial behavior for two classes of isotropic porous materials whose internal energy density is not always a positive definite quadratic form. Important results have been achieved also for other types of materials (see [31–36] and references therein.)

Chiriţă and Ghiba [37] have established the necessary and sufficient conditions characterizing the strong ellipticity of materials with voids and have investigated a model for propagation of progressive waves associated with this class of materials.

In [38, 39], the authors study the spatial behavior for materials whose elasticity tensor is semi-strongly elliptic, in linear elastodynamics and linear thermoelastodynamics, respectively.

The work plan of this article is as follows: in Sect. 2, we present the linear theory of porous elastic materials taking into account memory effects for the intrinsic equilibrated body forces [19]; in Sect. 3, we study how to relax the conditions on constitutive coefficients and determine a wider class of materials for which the internal energy density is not always positive definite but the internal energy is non negative when the boundary data are null; in Sect. 4, we establish a uniqueness result for the initial-boundary value problem for the class of isotropic homogeneous and semi-strongly elliptic porous materials; in Sect. 5, a family of appropriate surface integral measures and a set of properties are established; moreover, we establish a result describing the domain of influence and a spatial decay estimate of exponential type for a class of materials that is wider with respect to [30], by using a single family of measures, so simplifying the whole process.

2 Formulation of the problem

Throughout this article, we refer the motions of a continuum body to a fixed orthonormal frame in the physical 3-dimensional space $\mathbb{R}^3$. We deal with functions of position and time. We denote the tensor components of order $p \geq 1$ by Latin subscripts ranging over the integers $\{1, 2, 3\}$. Summation over repeated subscripts is implied. Superposed dots or subscripts preceded by a comma will mean partial derivative with respect to the time or the corresponding coordinate. Further, we suppress the dependence upon the spatial variable when no confusion may occur and, occasionally, we shall use bold-face characters and typical notations for vectors and operations upon them. All involved functions are supposed sufficiently regular to ensure analysis to be valid.

Let B be a bounded regular region of the physical space $\mathbb{R}^3$ with piecewise smooth boundary surface ∂B. The set $\overline{B}$ represents the closure of B. We designate by $\mathbf{n}$ the outward unit normal vector to the boundary. The region B is filled with an isotropic and homogeneous elastic porous material with memory effects for the intrinsic equilibrated body forces. In the context of the linear theory [19, 22, 28], the behavior is governed by the equations of motion

$$\begin{aligned} &t_{ji,j} + \rho b_i = \rho \ddot{u}_i, \\ &h_{j,j} + g + \rho l = \rho\chi\ddot{\phi}, \end{aligned} \quad \text{in } B \times (0,\infty), \tag{1}$$

the constitutive equations

$$\begin{aligned} &t_{ji} = \lambda e \delta_{ij} + 2\mu e_{ij} + \beta\phi\delta_{ij}, \\ &h_j = \alpha\phi_{,j}, \qquad\qquad \text{in } \bar{B} \times [0,\infty), \\ &g = \tilde{g} - \tau\dot{\phi}, \qquad \tilde{g} = -\beta e - \xi\phi, \end{aligned} \tag{2}$$

where e_{ij} are the components of the strain tensor given by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{3}$$

and $e = e_{kk}$ and $\tau \geq 0$. Here u_i, t_{ij} and b_i are the components of the displacement field, the stress tensor and the body force, respectively; ϕ, h_i, g and l are the change in volume fraction from the reference volume fraction, the equilibrated stress, the intrinsic equilibrated force and the extrinsic equilibrated body force, respectively. Further, ρ and χ are the positive bulk mass density and the positive equilibrated inertia in the reference state and λ, μ, α, β and ξ are constant constitutive coefficients.

We denote by $\mathscr{P}$ the initial-boundary value problem defined by the following system in the variables u_i, ϕ equivalent to Eqs. (1)–(3)

$$\begin{aligned} &\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \beta\phi_{,i} + \rho b_i = \rho\ddot{u}_i, \\ &\alpha\phi_{,jj} - \beta u_{j,j} - \xi\phi + \rho l = \rho\chi\ddot{\phi} + \tau\dot{\phi}, \end{aligned}$$

and by the following initial conditions

$$u_i = u_i^0, \quad \dot{u}_i = \dot{u}_i^0, \quad \phi = \phi^0, \quad \dot{\phi} = \dot{\phi}^0, \quad \text{on } \overline{B} \times \{0\},$$

and boundary conditions

$$u_i = \tilde{u}_i, \quad \phi = \tilde{\phi}, \quad \text{on } \partial B \times [0,\infty),$$

where $u_i^0, \dot{u}_i^0, \phi^0, \dot{\phi}^0, \tilde{u}_i$ and $\tilde{\phi}$ are prescribed functions. A regular solution of $\mathscr{P}$ is denoted by $\mathscr{U} = \{u_i, \phi\}$.

The positive definite kinetic energy density and the internal energy density $\mathscr{W}$ associated with $\mathscr{U}$ are defined by (see Cowin and Nunziato [19])

$$\mathscr{K} = \frac{1}{2}\rho\left(\dot{u}_i\dot{u}_i + \chi\dot{\phi}^2\right),$$

and

$$\mathscr{W} = \frac{1}{2}\left(\lambda e^2 + 2\mu e_{ij}e_{ij} + 2\beta e\phi + \xi\phi^2 + \alpha\phi_{,j}\phi_{,j}\right).$$

It is well known that

$$\mathscr{W} = \frac{1}{2}\left(t_{ji}u_{i,j} + h_j\phi_{,j} - \tilde{g}\phi\right), \quad \frac{\partial\mathscr{W}}{\partial t} = t_{ji}\dot{u}_{i,j} + h_j\dot{\phi}_{,j} - \tilde{g}\dot{\phi}.$$

The internal energy $\mathscr{W}$ is positive definite if and only if the constitutive coefficients satisfy the inequalities (see [19])

$$\alpha > 0 \quad \mu > 0, \quad \xi > 0, \quad \left(\lambda + \frac{2}{3}\mu\right)\xi > \beta^2.$$

In [37], the authors prove that an isotropic, homogeneous porous material is strongly elliptic if and only if

$$\alpha > 0 \quad \mu > 0, \quad \xi > 0, \quad (\lambda + 2\mu)\xi > \beta^2.$$

3 Semi-strongly elliptic porous materials

In this section, we study how to relax the conditions on constitutive coefficients to have that the internal energy associated with B be non negative, even if the internal energy density is not positive. Our analysis is motivated by the existence of novel foam structures for which the internal energy density is not always a positive definite quadratic form and therefore the constitutive coefficients satisfy relaxed conditions, as for example auxetic or anti-rubber materials.

The porous material satisfying

$$\mu \geq 0, \quad (\lambda + 2\mu)\xi \geq \beta^2, \tag{4}$$

$$\alpha \geq 0, \quad \xi \geq 0, \tag{5}$$

will be called semi-strongly elliptic material.

We denote by $\mathscr{D}^*$ and by $\mathscr{D}^+$ the classes of semi-strongly elliptic materials and the class of materials for which the internal energy $\mathscr{W}$ is positive semi-definite, respectively. It is obvious that $\mathscr{D}^+ \subseteq \mathscr{D}^*$ (see Fig. 1).

Now, we write the equations of motion (1) in the following equivalent form

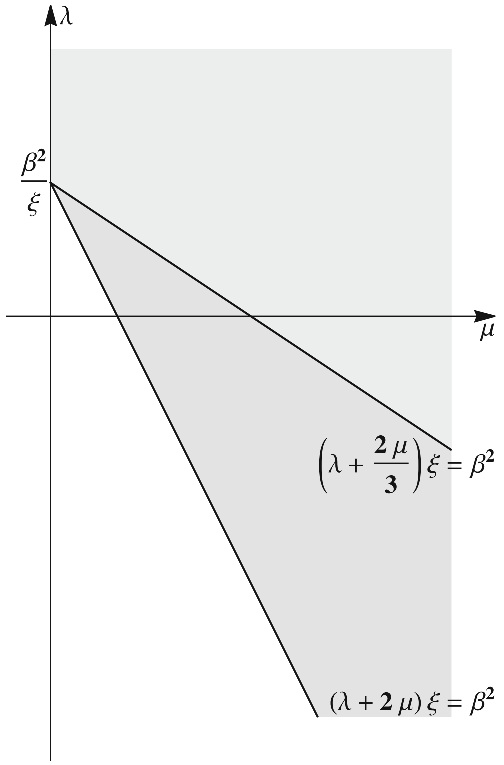

Fig. 1 Areas defining $\mathscr{D}^*$, $\mathscr{D}^+$ in the $\mu\lambda$ plane, for β, ξ constant and $\xi > 0$

$$s_{ji,j} + \rho b_i = \rho \ddot{u}_i, \qquad h_{j,j} + \tilde{g} + \rho l = \rho\chi\ddot{\phi} + \tau\dot{\phi}, \qquad \text{in } B \times (0,\infty), \tag{6}$$

with

$$\begin{aligned} s_{ji} &= \alpha_1 u_{r,r}\delta_{ij} + \alpha_2 u_{j,i} + \alpha_3 u_{i,j} + \beta\phi\delta_{ij}, \\ h_j &= \alpha\phi_{,j}, \\ \tilde{g} &= -\beta e - \xi\phi, \end{aligned} \tag{7}$$

where the coefficients α_1, α_2, $\alpha_3 \in \mathbb{R}$ are such that

$$t_{ji,j} = s_{ji,j}. \tag{8}$$

From Eqs. $(2)_1$, (3) and $(7)_1$ we prove that

$$s_{ji} - t_{ji} = (\alpha_1 - \lambda)u_{r,r}\delta_{ij} - (\mu - \alpha_2,)u_{j,i} + (\alpha_3 - \mu)u_{i,j},$$

and, consequently,

$$s_{ji,j} - t_{ji,j} = (\alpha_1 - \lambda)u_{j,ji} - (\mu - \alpha_2,)u_{j,ij} + (\alpha_3 - \mu)u_{i,jj}.$$

Then, it is evident that Eq. (8) is satisfied when

$$\alpha_1 - \lambda = \mu - \alpha_2, \quad \mu - \alpha_3 = 0.$$

If we introduce the parameter ε such that

$$\varepsilon = \alpha_1 - \lambda = \mu - \alpha_2, \tag{9}$$

the coefficients α_1, α_2, α_3 are

$$\alpha_1 = \lambda + \varepsilon, \qquad \alpha_2 = \mu - \varepsilon, \qquad \alpha_3 = \mu,$$

and, replacing the last relations in Eq. $(7)_1$, we get the following family

$$s_{ji}^{(\varepsilon)} = (\lambda + \varepsilon)u_{r,r}\delta_{ij} + (\mu - \varepsilon)u_{j,i} + \mu u_{i,j} + \beta\phi\delta_{ij}. \tag{10}$$

Now, we put

$$\mathscr{W}^{(\varepsilon)} = \frac{1}{2}\left(s_{ji}^{(\varepsilon)}u_{i,j} + h_j\phi_{,j} - \tilde{g}\phi\right). \tag{11}$$

We remark that when $\varepsilon = 0$ we have

$$s_{ji}^{(0)} = t_{ji} \qquad \text{and} \qquad \mathscr{W}^{(0)} = \mathscr{W}. \tag{12}$$

Using Eqs. (10) and $(7)_{2,3}$, we get the following quadratic form

$$\begin{aligned} \mathscr{W}^{(\varepsilon)} = \frac{1}{2}\big[&(\lambda + \varepsilon)u_{r,r}u_{j,j} + (\mu - \varepsilon)u_{j,i}u_{i,j} + \mu u_{i,j}u_{i,j} \\ &+ 2\beta\phi u_{r,r} + \xi\phi^2 + \alpha\phi_{,j}\phi_{,j}\big], \end{aligned} \tag{13}$$

in the variables

$$\begin{aligned} \boldsymbol{\psi} = \big\{&u_{1,1}, u_{2,2}, u_{3,3}, \phi, u_{2,1}, u_{1,2}, u_{3,1}, u_{1,3}, u_{3,2}, u_{2,3}, \\ &\sqrt{\chi}\phi_{,1},\ \sqrt{\chi}\phi_{,2},\ \sqrt{\chi}\phi_{,3}\big\}. \end{aligned}$$

We define $\mathbf{A}^{(\varepsilon)}$ the matrix associated to the quadratic form $\mathscr{W}^{(\varepsilon)}$, such that $\mathscr{W}^{(\varepsilon)} = \boldsymbol{\psi}\cdot\mathbf{A}^{(\varepsilon)}\boldsymbol{\psi}$, so that it is

$$\mathbf{A}^{(\varepsilon)} = \begin{pmatrix} \mathbf{B}^{(\varepsilon)} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{C}^{(\varepsilon)} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{C}^{(\varepsilon)} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{C}^{(\varepsilon)} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{\chi}\mathbf{I} \end{pmatrix}, \tag{14}$$

with

$$\mathbf{B}^{(\varepsilon)} = \begin{pmatrix} \lambda+2\mu & \lambda+\varepsilon & \lambda+\varepsilon & \beta \\ \lambda+\varepsilon & \lambda+2\mu & \lambda+\varepsilon & \beta \\ \lambda+\varepsilon & \lambda+\varepsilon & \lambda+2\mu & \beta \\ \beta & \beta & \beta & \xi \end{pmatrix},$$

$$\mathbf{C}^{(\varepsilon)} = \begin{pmatrix} \mu & \mu-\varepsilon \\ \mu-\varepsilon & \mu \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

We can prove that $\mathscr{W}^{(\varepsilon)}$ is positive semi-definite if and only if the elastic coefficients satisfy the inequalities (5) and the following inequalities in substitution of (4)

$$0 \le \varepsilon \le 2\mu, \quad \left(\lambda + \frac{2}{3}(\mu+\varepsilon)\right)\xi \ge \beta^2. \tag{16}$$

It is obvious that $\xi = 0$ implies $\beta = 0$.

If we consider these as inequalities with respect to ε with λ, μ, β, ξ fixed ($\xi > 0$), the second of them can be rewritten as

$$\varepsilon \ge \frac{3}{2}\left(\frac{\beta^2}{\xi} - \lambda\right) - \mu,$$

and the two inequalities are compatible if and only if

$$\frac{3}{2}\left(\frac{\beta^2}{\xi} - \lambda\right) - \mu \le 2\mu \quad \Leftrightarrow \quad (\lambda+2\mu)\xi \ge \beta^2.$$

We can conclude that there is no possibility to further relax the semi-strong ellipticity conditions.

The set of material, such that Eqs. (5), (16) holds, is denoted by $\mathscr{D}^{(\varepsilon)}$ and we have (see Fig. 2).

$$\mathscr{D}^{(\varepsilon)} \subseteq \mathscr{D}^*,$$

for each ε satisfying inequalities (16) and in particular

$$\mathscr{D}^{(2\mu)} = \mathscr{D}^*,$$

On the other hand, we introduce the antisymmetric part of $\nabla\mathbf{u}$ and the corresponding axial vector

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}), \quad \omega_i = -\frac{1}{2}\varepsilon_{ijk}\omega_{jk}, \tag{17}$$

and we observe that

$$\omega_{ij} = -\varepsilon_{ijk}\omega_k, \quad \omega_{ij}\omega_{ij} = 2\omega_i\omega_i. \tag{18}$$

Further, it is well known that

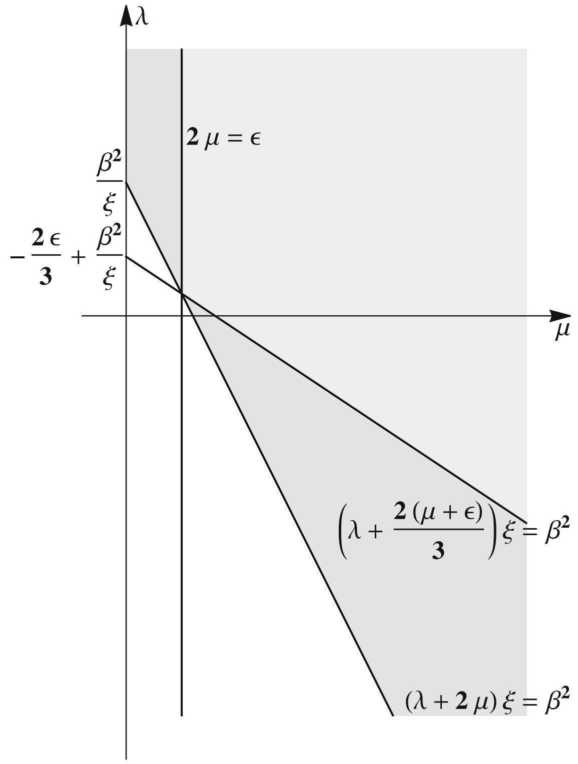

Fig. 2 Areas defining $\mathscr{D}^{(\varepsilon)}$, $\mathscr{D}^*$ in the $\mu\lambda$ plane, for ε, β, ξ constant and $\xi > 0$

$$u_{i,j} = e_{ij} + \omega_{ij}, \quad u_{j,i} = e_{ij} - \omega_{ij}, \tag{19}$$

so that, replacing these relations in Eq. (13), we obtain

$$\mathscr{W}^{(\varepsilon)} = \frac{1}{2}\big[(\lambda+\varepsilon)e^2 + (2\mu-\varepsilon)e_{ij}e_{ij} + 2\varepsilon\,\omega_i\omega_i + 2\beta\phi e + \xi\phi^2 + \alpha\phi_{,j}\phi_{,j}\big].$$

With the help of the following relation

$$e_{ij}e_{ij} = 2\omega_i\omega_i + e^2 + [u_{j,i}u_i - u_j u_{i,i}]_{,j}, \tag{20}$$

we arrive to

$$\mathscr{W}^{(\varepsilon)} = \mathscr{W}^* + \frac{1}{2}(2\mu-\varepsilon)[u_{j,i}u_i - u_j u_{i,i}]_{,j}, \tag{21}$$

where

$$\mathscr{W}^* = \frac{1}{2}\big[4\mu\omega_i\omega_i + (\lambda+2\mu)e^2 + 2\beta\phi e + \xi\phi^2 + \alpha\phi_{,j}\phi_{,j}\big].$$

We see that $\mathscr{W}^*$ is a quadratic form in the variables

$$\boldsymbol{\Psi}^* = \left\{\sqrt{2}\omega_1, \sqrt{2}\omega_2, \sqrt{2}\omega_3, \frac{e}{\sqrt{3}}, \phi, \sqrt{\chi}\phi_{,1}, \sqrt{\chi}\phi_{,2}, \sqrt{\chi}\phi_{,3}\right\}, \tag{22}$$

while $\mathscr{W}^{(\varepsilon)}$ also depends on the variables e_{ij} except for $\varepsilon = 2\mu$, in fact

$$\mathscr{W}^* = \mathscr{W}^{(2\mu)}.$$

Denoted by $\mathbf{A}^*$ the matrix associated to the quadratic form $\mathscr{W}^*$, i.e.

$$\mathbf{A}^* = \begin{pmatrix} 2\mu\mathbf{I} & 0 & 0 \\ 0 & \mathbf{B}^* & 0 \\ 0 & 0 & \frac{\alpha}{\chi}\mathbf{I} \end{pmatrix}, \quad \mathbf{B}^* = \begin{pmatrix} 3(\lambda+2\mu) & \sqrt{3}\beta \\ \sqrt{3}\beta & \xi \end{pmatrix}, \tag{23}$$

it is obvious that $\mathscr{W}^*$ is positive semi-definite if and only if the inequalities (4), (5) are satisfied.

Then, the set of materials for which $\mathscr{W}^*$ is positive semi-definite coincides with the set of semi-strongly elliptic porous materials $\mathscr{D}^*$ and, so, include all sets $\mathscr{D}^{(\varepsilon)}$ with $\varepsilon \in [0, 2\mu]$.

On the other hand, taking into account (12) and (21) it is easy to verify that

$$\mathscr{W} = \mathscr{W}^{(0)} = \mathscr{W}^* + \mu[u_{j,i}u_i - u_j u_{i,i}]_{,j},$$

and the following theorem holds

Theorem 1 *If the boundary data are null, the internal energy W associated with B is non negative when we consider a semi-strongly elliptic material. In particular, we have*

$$W = \int_B \mathscr{W} dv = \int_B \mathscr{W}^* dv \geq 0. \tag{24}$$

We emphasize the fact that ε enter into the expression of $\mathscr{W}^{(\varepsilon)}$ only through the term $[u_{j,i}u_i - u_j u_{i,i}]_{,j}$ whose integral vanishes in our hypotheses, so that Eqs. (21) and (24) imply

$$W = \int_B \mathscr{W}^{(\varepsilon)} dv \geq 0, \tag{25}$$

for any value of ε, also if the internal energy density is not positive.

In order to prove the results of the following sections, we multiply Eq. $(6)_1$ by $\dot{u}_i$ and Eq. $(6)_2$ by $\dot{\phi}$, so we arrive to

$$[s_{ji}^{(\varepsilon)}\dot{u}_i + h_j\dot{\phi}]_{,j} - [s_{ji}^{(\varepsilon)}\dot{u}_{i,j} + h_j\dot{\phi}_{,j} - \tilde{g}\dot{\phi}] + \rho b_i\dot{u}_i + \rho l\dot{\phi} = \frac{\partial \mathscr{K}}{\partial t} + \tau\dot{\phi}^2. \tag{26}$$

On the other hand, using Eqs. (10), (13) and (21), we observe that

$$\begin{aligned} s_{ji}^{(\varepsilon)}\dot{u}_{i,j} + h_j\dot{\phi}_{,j} - \tilde{g}\dot{\phi} &= (\lambda+\varepsilon)u_{r,r}\dot{u}_{j,j} + (\mu-\varepsilon)u_{j,i}\dot{u}_{i,j} + \mu u_{i,j}\dot{u}_{i,j} \\ &\quad + \beta(\phi\dot{u}_{r,r} + \beta u_{r,r}\dot{\phi}) + \alpha\phi_{,j}\dot{\phi}_{,j} + \xi\phi\dot{\phi} = \frac{\partial \mathscr{W}^{(\varepsilon)}}{\partial t}, \end{aligned}$$

and

$$\frac{\partial \mathscr{W}^{(\varepsilon)}}{\partial t} = \frac{\partial \mathscr{W}^*}{\partial t} + \frac{1}{2}(2\mu-\varepsilon)\frac{\partial}{\partial t}[u_{j,i}u_i - u_j u_{i,i}]_{,j}. \tag{27}$$

Combining Eqs. (26), (27), we obtain

$$\frac{\partial \mathscr{E}^*}{\partial t} + \tau\dot{\phi}^2 = [s_{ji}^{(\varepsilon)}\dot{u}_i + h_j\dot{\phi}]_{,j} - \frac{1}{2}(2\mu-\varepsilon)\frac{\partial}{\partial t}[u_{j,i}u_i - u_j u_{i,i}]_{,j} + \rho b_i\dot{u}_i + \rho l\dot{\phi}, \tag{28}$$

where

$$\mathscr{E}^* = \mathscr{K} + \mathscr{W}^*.$$

Taking into account Eq. (28), it is easy to prove the following lemma, that will be useful in the following

Lemma 1 *Let D be any regular subregion of B with regular boundary ∂D. For any $\sigma \geq 0$ it is*

$$\begin{aligned} \int_D e^{-\sigma t}\left(\frac{\partial \mathscr{E}^*}{\partial t} + \tau\dot{\phi}^2\right)dv &= \int_D e^{-\sigma t}\rho(b_i\dot{u}_i + l\dot{\phi})dv \\ &\quad - \frac{1}{2}(2\mu-\varepsilon)\int_{\partial D} e^{-\sigma t}\frac{\partial}{\partial t}[u_{j,i}u_i - u_j u_{i,i}]n_j da \\ &\quad + \int_{\partial D} e^{-\sigma t}[s_{ji}^{(\varepsilon)}\dot{u}_i + h_j\dot{\phi}]n_j da. \end{aligned}$$

4 Uniqueness result

In this section, for the class of isotropic, homogeneous and semi-strongly elliptic materials with voids we prove a uniqueness result for the initial-boundary value problem $\mathscr{P}$.

Theorem 2 (Uniqueness) *If we consider a semi-strongly elliptic material, there exists at most one solution to the initial-boundary value problem* $\mathscr{P}$.

Proof Given two solutions to $\mathscr{P}$ for the same initial data, same boundary data and for the same external supplies, due to the linearity of the problem the difference of them satisfies the associated problem $\mathscr{P}^*$ with null initial and boundary data and null external supplies. So we have to show that the only solution $\mathscr{U} = \{\mathbf{u}, \phi\}$ to $\mathscr{P}^*$ is the null solution.

If we take the relation of Lemma 1 with $D = B$ and $\sigma = 0$ and integrate on the time interval $[0, t]$, thanks to the null given data, we have

$$\int_B \mathscr{E}^*(t)dv = -\int_0^t \int_B \tau\dot{\phi}^2(s)dvds \leq 0, \qquad \forall t \geq 0.$$

Since $\mathscr{K}$ and $\mathscr{W}^*$ are non negative functions and using the null initial conditions again, we arrive to

$$\mathscr{K}(t) = \mathscr{W}^*(t) = 0 \quad \Rightarrow \quad u_i(t) = 0, \quad \phi(t) = 0,$$

for any $t \geq 0$. □

We point out that this results can be obtain in more general hypotheses with other techniques, for example with the Lagrange-Brun identity method or the logarithmic convexity. Nevertheless our setup will also be useful in other type of problems where these techniques cannot be applied.

5 Spatial behavior of solutions

In this section, we describe the spatial behavior of solutions for semi-strongly elliptic materials with voids by following the approach used in [27, 28, 30].

For a fixed $T > 0$, we suppose there exist a nonempty regular region $D_0 \subseteq B$ such that the given data (initial, boundary and external data) is localized in D_0 in the time interval $[0, T]$, i.e. the given data vanish outside of D_0 for every $t \in [0, T]$. More formally

- initial data vanish in $\overline{B - \overline{D}_0}$,
- boundary data vanish in $\partial B - \partial D_0$ for every $t \in [0, T]$,
- external data vanish in $\overline{B - \overline{D}_0}$ for every $t \in [0, T]$.

We will call D_0 the support of the external data on the time interval $[0, T]$. Let us further define, for $r \geq 0$,
$B_r = \{\mathbf{x} \in B : d(\mathbf{x}, D_0) \geq r\}^\circ$,

where d denote the distance, S° denote the interior of a set S. Let us define
$L = \sup\{r : B_r \neq \varnothing\}$.

We suppose that B_r is a nonempty regular regions with piecewise regular boundary for every $0 \leq r < L$, while for $r = L$ we have $B_L = \varnothing$.

In order to study the spatial behavior, we introduce the following time-weighted surface power function associated with $\mathscr{U}$, for $0 \leq r \leq L$ and $0 \leq t \leq T$

$$J(r, t) = \int_0^t \int_{\partial B_r} e^{-\sigma s}\left[s_{ji}^* \dot{u}_i + h_j \dot{\phi}\right] n_j da ds, \tag{29}$$

where σ is a fixed positive parameter and s_{ij}^* is given by

$$s_{ij}^* = s_{ij}^{(2\mu)} = [(\lambda + 2\mu)e + \beta\phi]\delta_{ij} - 2\mu\varepsilon_{ijk}\omega_k.$$

We now prove some useful proprieties of the time-weighted surface power function $J(r, t)$

Theorem 3 *If we consider a semi-strongly elliptic material, then* $J(r, t)$ *has the following properties:*

(i) $J(r, t)$ *is a continuous differentiable function and*

$$\frac{\partial J}{\partial r} = -\int_{\partial B_r} e^{-\sigma t}\mathscr{E}^* da - \int_0^t \int_{\partial B_r} e^{-\sigma s}\left[\sigma\mathscr{E}^*(s) + \tau\dot{\phi}^2(s)\right] da ds, \tag{30}$$

$$\frac{\partial J}{\partial t} = +\int_{\partial B_r} e^{-\sigma s}\left[s_{ji}^* \dot{u}_i + h_j \dot{\phi}\right] n_j da; \tag{31}$$

(ii) $J(r, t)$ *is a non-increasing function with respect to* r, *i.e.*

$$J(r_1, t) \leq J(r_2, t), \qquad \text{with } r_1 \geq r_2; \tag{32}$$

(iii) $J(r, t)$ *is a measure associated to the solution* $\mathscr{U}$ *of* $\mathscr{P}$ *and can be expressed as*

$$J(r,t) = + \int_{B_r} e^{-\sigma t} \mathscr{E}^* dv + \int_0^t \int_{B_r} e^{-\sigma s} \left[\sigma \mathscr{E}^*(s) + \tau \dot{\phi}^2(s) \right] dv ds \geq 0.$$

Proof We proceed as follows

(i) From Eq. (29) we arrive to

$$\frac{\partial J}{\partial r} = - \lim_{h \to 0^+} \frac{1}{h} \int_0^t \int_{B_r - B_{r+h}} e^{-\sigma s} \left[s^*_{ji} \dot{u}_i + h_j \dot{\phi} \right]_{,j} dv ds = - \int_0^t \int_{\partial B_r} e^{-\sigma s} \left[s^*_{ji} \dot{u}_i + h_j \dot{\phi} \right]_{,j} da ds.$$

By the definitions given the body force, the extrinsic equilibrated body force and the initial data vanish on B_r and ∂B_r. Using the divergence theorem and Lemma X it follows Eq. (30).
Further, Eq. (31) immediately follows from the definition (29).

(ii) For semi-strongly elliptic material, $\mathscr{K}$, $\mathscr{W}^*$ are non negative functions, consequently we have $\mathscr{E}^* \geq 0$ and from Eq. (30) we obtain

$$\frac{\partial J}{\partial r}(r,t) \leq 0,$$

and consequently Eq. (32)

(iii) From Eqs. (28) and (29) we have

$$J(r,t) - J(L,t) = \int_0^t \int_{B_r - B_L} e^{-\sigma s} \left[\frac{\partial \mathscr{E}^*}{\partial s} + \tau \dot{\phi}^2(s) \right] dv ds \geq 0,$$

and taking into account that $B_L = \emptyset$, we have $J(L,t) = 0$ and this proves the thesis.

□

Next, we can establish the following

Theorem 4 *Assumed that the conditions* (4) *and* (5) *hold, and that the maximum eigenvalues* κ *of* $\mathbf{A}^*$ *is strictly positive, for* $\sigma > 0$ *there exists* $\xi > 0$ *such that*

$$J(r,t) \leq \xi \int_0^t \int_{\partial B_r} e^{-\sigma s} \mathscr{E}^*(s) da ds,$$

and

$$\left| \frac{\partial J}{\partial t}(r,t) \right| \leq \xi \int_{\partial B_r} e^{-\sigma t} \mathscr{E}^* da. \tag{33}$$

Proof Remembering that $\boldsymbol{\Psi}^*$, $\mathbf{A}^*$ are defined in Eqs. (22) and (23), we have

$$\boldsymbol{\Psi}^* \cdot \mathbf{A}^* \boldsymbol{\Psi}^* = 2\mathscr{W}^*, \qquad \mathbf{S}^* = \mathbf{A}^* \boldsymbol{\Psi}^*, \tag{34}$$

where

$$\mathbf{S}^* = \left\{ \sqrt{2} s^*_1, \sqrt{2} s^*_2, \sqrt{2} s^*_3, \frac{s^*}{\sqrt{3}}, -\tilde{g}, \frac{1}{\sqrt{\chi}} h_1, \frac{1}{\sqrt{\chi}} h_2, \frac{1}{\sqrt{\chi}} h_3 \right\},$$

with

$$s^*_i = -\frac{1}{2} \varepsilon_{ijk} s^*_{jk}, \quad s^* = s^*_{kk}.$$

We can prove that

$$s^*_{ji} s^*_{ji} + \tilde{g}^2 + \frac{1}{\chi} h_i h_i = 2 s^*_i s^*_i + \frac{s^{*2}}{3} + \tilde{g}^2 + \frac{1}{\chi} h_i h_i = \mathbf{S}^{*2}. \tag{35}$$

Use of Cauchy–Schwarz' inequality, with respect to the positive semi-definite symmetric bilinear form associated to $\mathbf{A}^*$, and Eqs. (34) leads to

$$\mathbf{S}^{*2} = \boldsymbol{\Psi}^* \cdot \mathbf{A}^* \mathbf{A}^* \boldsymbol{\Psi}^* \leq \kappa \boldsymbol{\Psi}^* \cdot \mathbf{A}^* \boldsymbol{\Psi}^* = 2\kappa \mathscr{W}^*. \tag{36}$$

Taking into account the Cauchy–Schwarz's and arithmetic–geometric mean inequalities, Eqs. (35) and (36) lead to

$$\left| s^*_{ji} \dot{u}_i n_j + h_j \dot{\phi} n_j \right| \leq \frac{1}{2} \xi \left(\rho \dot{u}_i \dot{u}_i + \rho \chi \dot{\phi}^2 \right) + \frac{1}{2\rho\xi} \left(s^*_{ji} s^*_{ji} + \tilde{g}^2 + \frac{1}{\chi} h_j h_j \right) \leq \xi \mathscr{K} + \frac{\kappa}{\rho \xi} \mathscr{W}^*, \tag{37}$$

where ξ is an arbitrary positive constant. Choosing $\xi = \sqrt{\kappa/\rho}$ in Eq. (37), then Eqs. (30), (31) yield Eq. (33). □

Following the procedure developed in [27], one can prove the following

Theorem 5 *Under the hypotheses of Theorem* 4, *it follows that*

$$\sigma J(r,t) + \xi \frac{\partial J}{\partial r}(r,t) \leq 0,$$

and

$$\pm \frac{\partial J}{\partial t}(r,t) + \xi \frac{\partial J}{\partial r}(r,t) \leq 0.$$

As an immediate consequence of Theorem 5, we have

$$\frac{\partial}{\partial r}\left[e^{\sigma r/\xi} J(r,t)\right] \leq 0,$$

and

$$\frac{d}{dr}\left[J\left(r, t_0 \pm \frac{r - r_0}{\xi}\right)\right] \leq 0, \qquad r_0 \geq \xi t_0.$$

Finally it is easy to prove the following theorem

Theorem 6 *Under the hypotheses of Theorem 4, the spatial behavior of solution $\mathscr{U}$ of $\mathscr{P}$ outside of the support D_0 is described by:*

(i) *for $0 \leq r \leq \xi t$ we have*

$$J(r,t) \leq e^{-\sigma r/\xi} J(0,t);$$

(ii) *for $r \geq \xi t$ we have*

$$u_i = 0, \quad \phi = 0.$$

6 Conclusions

The linear theory of elastodynamics for homogeneous and isotropic, porous elastic materials with memory effects for the intrinsic equilibrated body forces has been studied. A family of energy densities $\mathscr{W}^{(\varepsilon)}$ has been considered, to which the classical internal strain energy density belongs, and the conditions of positive semi-definiteness for these energy densities has been investigated. As relaxed conditions on the constitutive coefficients, the conditions of semi-strong ellipticity have been obtained, for which the classical internal strain energy density is not necessarily positive.

For the class of linear porous elastic materials with memory effects considered, we obtained a uniqueness result for the solutions, with relaxed conditions on the constitutive coefficients, in order to consider also materials for which the internal energy density is not always positive definite.

We deduced a result for describing the domain of influence by using an appropriate and single time-weighted surface measure family, differently from other attempts where they need two families to cover the whole class of strongly elliptic materials, still not covering the semi-strongly elliptic case. Moreover, we obtained a spatial decay of exponential type into the domain of influence.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Gurtin ME (1972) The linear theory of elasticity. In: Truesdell C (ed) Handbuch der physik, vol VIa/2. Springer, Berlin, pp 1–292
2. Ericksen JL, Toupin RA (1956) Implications of Hadamard's conditions for elastic stability with respect to uniqueness theorems. Can J Math 8:432–436. https://doi.org/10.4153/CJM-1956-051-2
3. Lakes RS (1987) Foam structures with a negative Poisson's ratio. Science 235:1038–1040. https://doi.org/10.1126/science.235.4792.1038
4. Caddock BD, Evans KE (1989) Microporous materials with negative Poisson's ratios. I. Microstructure and mechanical properties. J Phys D Appl Phys 22(12):1877–1882. https://doi.org/10.1088/0022-3727/22/12/012
5. Park J, Lakes RS (2007) Biomaterials: an introduction. Springer, New York. https://doi.org/10.1007/978-0-387-37880-0. ISBN 978-0-387-37879-4
6. Lee T, Lakes RS (1997) Anisotropic polyurethane foam with Poisson's ratio greater than 1. J Mater Sci 32(9):2397–2401. https://doi.org/10.1023/A:1018557107786
7. Wang Y, Lakes RS (2002) Analytical parametric analysis of the contact problem of human buttocks and negative Poisson's ratio foam cushions. Int J Solids Struct 39(18):4825–4838. https://doi.org/10.1016/S0020-7683(02)00379-7
8. Ciarletta M, Passarella F, Tibullo V (2015) Plane harmonic waves in strongly elliptic thermoelastic materials with microtemperatures. J Math Anal Appl 424(2):1186–1197. https://doi.org/10.1016/j.jmaa.2014.11.065
9. Passarella F, Tibullo V (2016) On the propagation of Rayleigh waves in a strongly elliptic thermoelastic material with microtemperatures. J Therm Stress 39(9):1111–1118. https://doi.org/10.1080/01495739.2016.1192861
10. Merodio J, Ogden RW (2003) A note on strong ellipticity for transversely isotropic linearly elastic solids. Q J Mech Appl Math 56(4):589–591. https://doi.org/10.1093/qjmam/56.4.589

11. Merodio J, Ogden RW (2005) On tensile instabilities and ellipticity loss in fiber-reinforced incompressible non-linearly elastic solids. Mech Res Commun 32(3):290–299. https://doi.org/10.1016/j.mechrescom.2004.06.008
12. Tibullo V, Vaccaro M (2008) Spatial behaviour for constrained motion of a cylinder made of a strongly elliptic anisotropic material. J Mech Mater Struct 3(5):983–993. https://doi.org/10.2140/jomms.2008.3.983
13. Passarella F, Tibullo V, Zampoli V (2011a) On the strong ellipticity for orthotropic micropolar elastic bodies in a plane strain state. Mech Res Commun 38(7):512–517. https://doi.org/10.1016/j.mechrescom.2011.06.006
14. Gurtin ME, Sternberg E (1961) A note on uniqueness in classical elastodynamics. Q Appl Math 19(2):169–171. https://doi.org/10.1090/qam/129226
15. Gurtin ME, Toupin RA (1965) A uniqueness theorem for the displacement boundary-value problem of linear elastodynamics. Q Appl Math 23(1):79–81. https://doi.org/10.1090/qam/177557
16. Borchardt CW (1873) Ueber die transformation der elasticitätsgleichungen in allgemeine orthogonale coordinaten. J Reine Angew Math 1873(76):45–58. https://doi.org/10.1515/crll.1873.76.45
17. Thomson W (1888) XLVI. On the reflexion and refraction of light. Philos Mag 26(162):414–425. https://doi.org/10.1080/14786448808628288
18. Nunziato JW, Cowin SC (1979) A nonlinear theory of elastic materials with voids. Arch Ration Mech Anal 72(2):175–201. https://doi.org/10.1007/BF00249363
19. Cowin SC, Nunziato JW (1983) Linear elastic materials with voids. J Elast 13(2):125–147. https://doi.org/10.1007/BF00041230
20. Ieşan D (1985) Some theorems in the theory of elastic materials with voids. J Elast 15(2):215–224. https://doi.org/10.1007/BF00041994
21. Ieşan D (1986) A theory of thermoelastic materials with voids. Acta Mech 60(1–2):67–89. https://doi.org/10.1007/BF01302942
22. Ciarletta M, Ieşan D (1993) Non-classical elastic solids, volume 293 of Research Notes in Mathematics. CRC Press
23. Ieşan D (2004) Thermoelastic models of continua, volume 118 of solid mechanics and its applications. Springer, Dordrecht. ISBN 978-90-481-6634-3. https://doi.org/10.1007/978-1-4020-2310-1
24. Passarella F, Tibullo V, Zampoli V (2011b) On the heat-flux dependent thermoelasticity for micropolar porous media. J Therm Stress 34(8):778–794. https://doi.org/10.1080/01495739.2011.564041
25. Hassan M, Marin M, Ellahi R, Alamri SZ (2018) Exploration of convective heat transfer and flow characteristics synthesis by Cu–Ag/Water hybrid-nanofluids. Heat Transf Res 49(18):1837–1848. https://doi.org/10.1615/HeatTransRes.2018025569
26. Othman MIA, Marin M (2017) Effect of thermal loading due to laser pulse on thermoelastic porous medium under G-N theory. Results Phys 7:3863–3872. https://doi.org/10.1016/j.rinp.2017.10.012
27. Chiriţă S, Ciarletta M (1999) Time-weighted surface power function method for the study of spatial behaviour in dynamics of continua. Eur J Mech A Solids 18(5):915–933. https://doi.org/10.1016/S0997-7538(99)00121-7
28. Scalia A (2001) Spatial and temporal behavior in elastic materials with voids. Acta Mech 151(1–2):47–60. https://doi.org/10.1007/BF01272524
29. Chiriţă S (2004) Further results on the spatial behavior in linear elastodynamics. An Stiint Univ Al I Cuza Iasi Mat 50:289–304
30. Ciarletta M, Chiriţă S, Passarella F (2005) Some results on the spatial behaviour in linear porous elasticity. Arch Mech 57(1):43–65
31. D'Apice C, Tibullo V, Chiriţă S (2005) On the spatial behavior in the dynamic theory of mixtures of thermoelastic solids. J Therm Stress 28(1):63–82. https://doi.org/10.1080/014957390523732
32. Passarella F, Zampoli V (2006) Some results on the spatial behaviour for elastic mixtures. Eur J Mech A Solids 25(6):1031–1040. https://doi.org/10.1016/j.euromechsol.2006.01.004 ISSN 0997-7538
33. Passarella F, Zampoli V (2009a) Spatial estimates for transient and steady-state solutions in transversely isotropic plates of Mindlin-type. Eur J Mech A Solids 28(4):868–876. https://doi.org/10.1016/j.euromechsol.2009.01.004 ISSN 0997-7538
34. Passarella F, Zampoli V (2009b) Some results concerning the state of bending for transversely isotropic plates. Math Methods Appl Sci 32(14):1828–1843. https://doi.org/10.1002/mma.1113
35. Passarella F, Tibullo V, Zampoli V (2010) Decay properties of solutions of a Mindlin-type plate model for rhombic systems. J Mech Mater Struct 5(2):323–339. https://doi.org/10.2140/jomms.2010.5.323
36. Ghiba ID (2013) On the spatial behaviour in the bending theory of porous thermoelastic plates. J Math Anal Appl 403(1):129–142. https://doi.org/10.1016/j.jmaa.2013.02.023
37. Chiriţă S, Ghiba ID (2010) Strong ellipticity and progressive waves in elastic materials with voids. Proc R Soc Lond Ser A 466(2114):439–458. https://doi.org/10.1098/rspa.2009.0360
38. Passarella F (2018) New results for semi-strongly elliptic materials in linear elastodynamics. Mech Res Commun 94:53–57. https://doi.org/10.1016/j.mechrescom.2018.09.002
39. Passarella F, Tibullo V (2019) New decay results in linear thermoelastodynamics. Math Methods Appl Sci 42(4):1114–1121. https://doi.org/10.1002/mma.5417