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NEW TRENDS IN DYNAMICS AND STABILITY

Regular and singular kernel problems in magneto-viscoelasticity

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Abstract Magneto-viscoelastic materials find their interest in a variety of applications in which mechanical properties are coupled with magnetic ones. In particular, new materials such as magneto-rheological elastomers or, in general, magneto-sensitive polymeric composites are more and more widely employed in new materials. The deformation evolution is assumed to be viscoelastic, that is, the stress–strain relation depends on the deformation history of the material further to on the deformation at the *present* time. This is a characteristic feature of all *materials with memory*, namely those materials whose mechanical and/or thermodynamical response depends on time not only via the present time, but also through the whole past *history*. To describe this behaviour integro-differential model equations are adopted subject to the *fading memory* assumption which corresponds to require that, asymptotically, effects of past deformation events become negligible. Magneto-viscoelastic materials are modelled aiming to describe viscoelastic materials whose mechanical response is also influenced by the presence of a magnetic field. Thus, the model system is obtained on coupling the viscoelasticity linear integro-differential equation with a nonlinear partial differential equation which describes magnetic effects. The attention is focussed on the kernel of the integro-differential equation: both regular as well as a *singular* kernel, at $t = 0$, problems are analysed. Indeed, singular kernel models allow to describe a wider class of materials and are also connected to materials modelled via kernels described via fractional derivatives.

Keywords Materials with memory · Viscoelasticity · Magneto-viscoelasticity · Regular kernel integro-differential systems · Singular kernel integro-differential systems

Mathematics Subject Classification 74H20 · 35Q74 · 45K05 · 74D05

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1 Introduction

The study of materials with memory goes back to Boltzmann [5] and Volterra [41] in connection to the description of the strain–stress relation in the case when linear elasticity does not capture the physical behaviour of the material. An overview on non-classical memory kernels in linear viscoelasticity is comprised in [17] where non standard problems are considered aiming to describe effects of *age* of materials as well as singular kernel models.

The crucial feature of the model of material with memory is to take into account not only the instantaneous response, but also the so called *history* of the material. That is, consider that the mechanical behaviour of the material is determined not only by the actual action on it but also by the whole deformation history. The model of material with memory finds its application in a variety of different frameworks among which isothermal viscoelasticity as described in [28, 29] and rigid heat conduction with memory [2, 8–10, 26]; indeed, the analogy under the analytical viewpoint between rigid heat conduction with memory and isothermal viscoelasticity is well known, see for instance [7].

The interest in magneto-viscoelasticity problems is connected to consider model equations which are suitable to describe innovative materials. Specifically, the effects of a magnetic field which acts on a viscoelastic body needs to be modelled when, for instance, magneto-rheological elastomers or, in general, magneto-sensitive polymeric composites are investigated according to [34] and references therein.

The general framework of the models to describe the materials under investigation are given in [28, 38]. The two books by Borcherdt [6] and by Mainardi [36], represent an overview on new materials which include also viscoelastic ones and their mechanical behaviour. In particular [6] comprises applications of viscoelastic model to seismic problems while [36] shows the connection between fractional calculus and linear viscoelasticity. Such a connection, see [35], is later studied by Fabrizio [27], who analyses the relation between Volterra integro-differential problems and viscoelastic models when weaker constitutive assumptions the relaxation modulus must satisfy allow to consider also cases of *singular kernel* problems. These are of interest also under the perspective of bio-materials studied by Deseriet al. [23] who also consider fractional derivatives models.

Note that, when the relaxation modulus is assumed to satisfy regularity requirements which are weaker than the usual ones, then the existence and uniqueness results proved by Dafermos [20, 21] do not hold anymore. Hence, new analytical problems arise. Nevertheless, singular kernel models go back to Boltzmann [5] who was concerned about special viscoelastic behaviours. Later, investigations on the viscoelastic behaviour of polymers and/or bio-materials whose mechanical response can be modelled on introduction of a singular kernel are comprised in [1, 23–25, 38, 39]. Analytical developments in this direction are due to Berti [3] and Grasselli and Lorenzi [32] who study viscoelasticity problems characterised by a singular memory kernel; Giorgi and Morro [31], are concerned about the thermodynamical admissibility of a viscoelastic model with a singular viscoelastic relaxation modulus. Among the many further results, see [17] and references therein, [37] is concerned about stability results in this context.

The material is organised as follows. The opening Sect. 2 is devoted to a concise summary of the properties and, then, analytical assumptions needed to describe the model of viscoelastic body. The *viscoelastic body* is assumed to be homogeneous and isotropic: its reference configuration is represented by a bounded set $\Omega \subset \mathbb{R}^3$ whose boundary is a smooth surface. The crucial feature of the model is that the elastic behaviour of the body is assumed to depend on time not only via the instantaneous response at the time t considered, but also on the *past history* of the material. The subsequent Sect. 3 is concerned about magneto-viscoelastic bodies. In particular, the idea is to couple the viscoelastic behaviour of the material with a sensibility to magnetic effects. This requirement is suggested by new materials which are obtained by inserting magnetic defects into a solid body to have the opportunity to control and influence the mechanical behaviour of the body when a magnetic field is applied. Accordingly, in Sect. 3, the model of coupled response of a linear viscoelastic body subject to a magnetic field is briefly recalled. In Sect. 4, existence results of two different magneto-viscoelasticity problems, in turn, one and three dimensional, are given [13, 14]. The differences, both in the obtained results as well as in the technical tools needed to achieve them, are pointed out. Then, singular kernel problems are addressed to in Sect. 5. In Sect. 5.1, the viscoelastic singular kernel model is recalled together with an existence and uniqueness results [11], which refers to a 1-dimensional viscoelastic body. The closing Sect. 5.2 is concerned about a 1-dimensional singular magneto-viscoelasticity problem [16].

2 The viscoelastic material model

This section is devoted to a brief summary of the model of *viscoelastic body* under investigation. The model here considered relies on the thermodynamical assumptions, and in particular on the notion of free energy given by Gentili [29], applied to initial boundary value problems by Deseri, Fabrizio and Golden [22]. The model by Giorgi and Morro [31], was further developed by Gentili [29], and recently described in [17] in connection with new materials. The body is assumed to be homogeneous and isotropic and its crucial feature is that the stress response at time t linearly depends on the whole *past history* of the strain up to the present time t. In the three-dimensional case, all fields depend on the space-time pair $(\mathbf{x}, t) \in \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^3$ is the reference configuration. The displacement vector $\mathbf{u}(\mathbf{x}, t)$ is given by

$$\mathbf{u}(\mathbf{x}, t) = \boldsymbol{\mu}(\mathbf{x}, t) - \mathbf{x},$$

where $\boldsymbol{\mu}(\mathbf{x}, \cdot)$ is the motion of $\mathbf{x}$, and

$$\mathbb{E} = \frac{1}{2}\left[\nabla\mathbf{u} + \nabla\mathbf{u}^T\right],$$

is the infinitesimal strain tensor. Under the assumption that the material satisfies both the isotropy and homogeneity conditions, no space dependence is indicated, that is the $\mathbf{x}$-dependence is omitted. The quantities of interest can be listed:

$$\mathbb{E} = \mathbb{E}\,(t), \mathbb{T} = \mathbb{T}\,(t), \mathbb{G} = \mathbb{G}\,(t) \tag{2.1}$$

which represent, in turn, the *strain tensor*, the *stress tensor* and the *relaxation modulus*. Furthermore the *constitutive assumptions* are given [29, 41], via:

$$\mathbb{T}\,(t) = \int_0^\infty \mathbb{G}(\tau)\dot{\mathbb{E}}\,(t-\tau)d\tau, \tag{2.2}$$

or, equivalently, when $\mathbb{E}^t(\tau)$ denotes the *strain past history*, namely $\mathbb{E}^t(\tau) := \mathbb{E}(t-\tau)$, via

$$\mathbb{T}\,(t) = \mathbb{G}_0\mathbb{E}(t) + \int_0^\infty \dot{\mathbb{G}}(\tau)\,\mathbb{E}^t\,(\tau)d\tau, \text{where} \mathbb{G}_0 := \mathbb{G}(0) \tag{2.3}$$

and $\mathbb{G}_0$ denotes the *initial relaxation modulus*. According to Volterra [41], the following regularity requirements are assumed

$$\dot{\mathbb{G}} \in L^1(\mathbb{R}^+), \mathbb{G}(t) = \mathbb{G}_0 + \int_0^t \dot{\mathbb{G}}(s)\,ds, \mathbb{G}(\infty) = \lim_{t\to\infty}\mathbb{G}(t) \tag{2.4}$$

Hence, the relaxation modulus $\mathbb{G}$ enjoys the *fading memory property* which reads

$$\forall\varepsilon > 0\,\exists\,\tilde{a} = a\,(\varepsilon, \mathbb{E}^t) \in \mathbb{R}^+ s.t. \forall a > \tilde{a}, \quad \left|\int_0^\infty \dot{\mathbb{G}}(s+a)\mathbb{E}^t(s)ds\right| < \varepsilon. \tag{2.5}$$

3 The magneto-viscoelastic material model

This section aims to provide a sketch of the adopted model to describe the mechanical behaviour of a viscoelastic body when it is also influenced by the presence of a magnetic field. Accordingly, not only the magnetic effects, but also the interaction between the two different effects needs to be taken into account. Specifically, based on [17], summarizes some of the crucial features of the model of magnetic effects acting on a viscoelastic body, again, $\Omega \subset \mathbb{R}^3$ denotes a connected bounded set which represents the body configuration.

Such a body is termed *magneto-viscoelastic* body when its status is also characterized by the related magnetization, which, according to the Landau Lifshitz equation [4, 30] in Gilbert form, where $\mathbf{m}$ represents the magnetization vector

$$\gamma^{-1}\mathbf{m}_t - \mathbf{m} \times (a\,\Delta\mathbf{m} - \mathbf{m}_t) = 0, |\mathbf{m}| = 1, \gamma, a \in \mathbb{R}^+ \tag{3.1}$$

The quantities of interest, in the general 3-dimensional case, including also those ones already introduced to describe the viscoelastic behaviour, are here listed, where the convention that summation over repeated indices is tacitly understood.

$$\begin{aligned}
&\mathbf{u} := \mathbf{u}\,(\mathbf{x}, t)\\
&\mathbf{m} := \mathbf{m}\,(\mathbf{x}, t)\\
&\mathbb{G}(s) = \{G_{klmn}(s)\}, \quad s \in [0, T]\\
&\mathbb{L} = \{\lambda_{klmn}\}\\
&\mathbb{E} = \{\epsilon_{lm}\}\\
&G_{klmn}\epsilon_{kl}(\mathbf{u})\epsilon_{mn}(\mathbf{v}) = \mathbb{G}\,\nabla\mathbf{u}\cdot\nabla\mathbf{v},\\
&\{\lambda_{klmn}m_k m_l\} = \mathbb{L}\mathbf{m}\otimes\mathbf{m}\\
&\{\lambda_{klmn}m_k\epsilon_{lm}(\mathbf{u})\} = \mathbb{L}\mathbf{m}\otimes\nabla\mathbf{u}\\
&\lambda_{klmn}m_k m_l\epsilon_{mn}(\mathbf{u}) = \mathbb{L}\mathbf{m}\otimes\mathbf{m}\cdot\nabla\mathbf{u}\\
&\epsilon_{lm}(\mathbf{u}) = \frac{1}{2}\left(\mathbf{u}_{l,m} + \mathbf{u}_{m,l}\right)
\end{aligned}$$

where, $\mathbf{u}$ denotes the displacement vector, $\mathbf{m}$ the magnetization vector, $\mathbb{G}$ the visco-elasticity tensor, $\mathbb{L}$

the magneto-elasticity tensor, $\mathbb{E}$ the strain tensor; in addition, the coefficients λ_{klmn} of the magneto-elasticity tensor are subject to the condition

$$\lambda_{ijkl} = \lambda_1\delta_{ijkl} + \lambda_2\delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad (3.2)$$

Then, the following constitutive assumptions are adopted. Thus, the exchange magnetization energy is given by

$$E_{\text{ex}}(\mathbf{m}) = \frac{1}{2}\int_\Omega a_{ij}m_{k,i}m_{k,j}d\Omega \qquad (3.3)$$

where

- $a_{ij} = a_{ji}$ symmetric positive definite matrix
- $a_{ij} = a\,\delta_{ji}, a \in \mathbb{R}^+$ diagonal matrix (most materials).
  Then, the magneto-elastic energy is given by
  $$E_{\text{em}}(\mathbf{m},\mathbf{u}) = \frac{1}{2}\int_\Omega \lambda_{ijkl}m_im_j\epsilon_{kl}(\mathbf{u})d\Omega \qquad (3.4)$$
- $\lambda_{ijkl} = \lambda_1\delta_{ijkl} + \lambda_2\delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$;
- $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise;
- $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ constants.

The viscoelastic energy is given by

$$E_{\text{ve}}(\mathbf{u}) = \frac{1}{2}\int_\Omega G_{klmn}(0)\epsilon_{kl}\epsilon_{mn}d\Omega + \frac{1}{2}\int_0^t d\tau\left(\int_\Omega \dot{G}_{klmn}(t-\tau)\epsilon_{kl}(\tau)\epsilon_{mn}(\tau)d\Omega\right) \qquad (3.5)$$

where the tensor's entries of $\mathbb{G}$ satisfy

- $G_{klmn} = G_{mnkl} = G_{lkmn}$
- $G_{klmn}\, e_{kl}e_{mn} \geqslant \beta\, e_{kl}e_{kl}, \qquad \beta > 0, e_{kl} = e_{lk}$
- $\dot{G}_{klmn}e_{kl}e_{mn} \leqslant 0$
- $\ddot{G}_{klmn}e_{kl}e_{mn} \geqslant 0$

(3.6)

Then, the total energy of the system is represented by the sum of all the considered contributions, that is

$$E(\mathbf{m},\mathbf{u}) = E_{\text{ex}}(\mathbf{m}) + E_{\text{em}}(\mathbf{m},\mathbf{u}) + E_{\text{ve}}(\mathbf{u}). \qquad (3.7)$$

4 Regular magneto-viscoelasticity problems

This section is concerned about the regular problems, respectively, in the case of a 1 and 3-dimensional body. Indeed, the two cases which, under the physical viewpoint, can be modeled in the same way according to the previous section, can be treated analytically via different methods. Hence, the two different cases are considered in dedicated subsections.

4.1 One-dimensional problem

Consider the following model equation[1]

$$\begin{cases} u_{tt} - G(0)u_{xx} - \int_0^t \dot{G}(t-\tau)u_{xx}(\tau)d\tau - \frac{\lambda}{2}\Big(\Lambda(\mathbf{m})\cdot\mathbf{m}\Big)_x = f, \\ \mathbf{m}_t + \mathbf{m}\frac{|\mathbf{m}|^2-1}{\delta} + \lambda\Lambda(\mathbf{m})u_x - \mathbf{m}_{xx} = 0, \end{cases} \qquad (4.1)$$

where $0<\delta \ll 1$, and $\mathbf{m} = (m_1, m_2)$ is the magnetization vector, Λ is a linear operator defined by $\Lambda(\mathbf{m}) = (m_2, m_1)$, the scalar function u is the displacement in the direction of m_2 and λ is a positive parameter. Moreover f is an external force which also includes the past history.

In [13] a weak existence and uniqueness result is proved when the following initial and boundary conditions are prescribed

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad \mathbf{m}(\cdot,0) = \mathbf{m}_0, \quad |\mathbf{m}_0| = 1 \quad \text{in } \Omega, \qquad (4.2)$$

$$u = 0, \quad \frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Sigma = \partial\Omega \times (0,T), \qquad (4.3)$$

where $\boldsymbol{\nu}$ is the outer unit normal at the boundary $\partial\Omega$.

As stated in [13], under the further assumption

$$\begin{cases} u_0 \in H_0{}^1(\Omega),\ u_1 \in L^2(\Omega), m_0 \in H^1(\Omega), \\ f \in L^2(\Omega \times (0,T)), G(t) \in C^2(0,T), \end{cases} \qquad (4.4)$$

the following result holds.

Theorem 1 *Given $T > 0$ and ε small enough (i.e. $\varepsilon < \lambda^{-2}G(T)$), there exists a unique solution to the problem (4.1)–(4.3), s.t. $u \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ and $\mathbf{m} \in \mathbf{C}^0([0,T]; H^1(\Omega)) \cap \mathbf{L}^2(0,T; H^2(\Omega))$, $\mathbf{m}_t \in \mathbf{L}^2(0,T; L^2(\Omega))$.*

4.2 Three-dimensional problem

In this subsection, the 3-dimensional problem considered in [14] is revised. In particular, in the framework

[1] Details on the deduction of the integro-differential nonlinear system (4.1) are refereed to [18, 19, 33, 40].

of the magneto-viscoelasticity model in the previous section, the initial value problem

$$\begin{cases} \gamma^{-1}\dot{\mathbf{m}} - \mathbf{m} \times (a\Delta\mathbf{m} - \dot{\mathbf{m}} - \mathbb{L}\mathbf{m} \otimes \nabla\mathbf{u}) = 0 \\ \rho\ddot{\mathbf{u}} - \mathrm{div}\left(\mathbb{G}(0)\nabla\mathbf{u} + \int_0^t (\dot{\mathbb{G}}(t-\tau)\nabla\mathbf{u}(\tau)d\tau + \frac{1}{2}\mathbb{L}\mathbf{m} \otimes \mathbf{m}\right) = \mathbf{f} \end{cases} \tag{4.5}$$

$$\mathbf{m}(0) = \mathbf{m}^0, \ |\mathbf{m}^0| = 1, \quad \mathbf{u}(0) = \mathbf{u}^0, \quad \dot{\mathbf{u}}(0) = \mathbf{u}^1.$$

$$u = 0, \quad \frac{\partial\mathbf{m}}{\partial\boldsymbol{\nu}} = 0 \quad \text{on} \quad \Sigma = \partial\Omega \times (0, T), \tag{4.6}$$

is studied where the parameters $\rho, \gamma, a \in \mathbb{R}^+$ are given. Again, to write the viscoelastic integro-differential equation in Volterra form, $\mathbf{f}$, given, takes also into account also the material history.

Then, the following weak existence result [14] holds.

Theorem 2 *Given* $\mathbf{u}^0 \in H_0^1(\Omega; \mathbb{R}^3)$, $\mathbf{u}^1 \in L^2(\Omega; \mathbb{R}^3)$, $\mathbf{m}^0 \in H^1(\Omega; \mathbb{R}^3)$ *with* $|\mathbf{m}^0| = 1$ *a.e. in* Ω *and let* $\mathcal{Q} = \Omega \times [0, T]$. Assume $\mathbf{f} \in L^2(\mathcal{Q}; \mathbb{R}^3)$ and $\mathbb{G}(s) \in C^2[0, T]$ *verifying the assumptions* (3.6), *then there exists a weak solution* $(\mathbf{m}, \mathbf{u})$ *to the problem* (4.4), (4.5) *in the sense that*:

- $\mathbf{m} \in H^1(\mathcal{Q}; \mathbb{R}^3)$ with $|\mathbf{m}| = 1$ a.e. in $\mathcal{Q}$ $\mathbf{u} \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3))$ and $\dot{\mathbf{u}} \in L^2(\mathcal{Q}; \mathbb{R}^3)$
- for each couple $(\mathbf{p}, \mathbf{g})$ such that $\mathbf{g} \in H_0^1(\mathcal{Q}; \mathbb{R}^3)$ and $\mathbf{p} \in H^{1,\infty}(\mathcal{Q}; \mathbb{R}^3)$ vanishing at $t = 0$ and $t = T$, one has

$$\int_{\mathcal{Q}} [\gamma^{-1}\dot{\mathbf{m}} \cdot \mathbf{p} + a(\mathbf{m} \times \nabla\mathbf{m}) \cdot \nabla\mathbf{p} + \mathbf{m} \times (\dot{\mathbf{m}} + \mathbb{L}\mathbf{m} \otimes \mathbf{p} \cdot \nabla\mathbf{u})] \, d\Omega dt = 0 \tag{4.7}$$

$$\int_{\mathcal{Q}} [-\rho\,\dot{\mathbf{u}} \cdot \mathbf{g}_t + (\mathbb{G}(0)\nabla\mathbf{u} + \frac{1}{2}\,\mathbb{L}\mathbf{m} \otimes \mathbf{m}) \cdot \nabla\mathbf{g}] \, d\Omega dt + \int_{\mathcal{Q}} \left(\int_0^t \dot{\mathbb{G}}(t-\tau)\nabla\mathbf{u}(\tau) \cdot \nabla\mathbf{g}(\tau)d\tau\right) d\Omega dt - \int_{\mathcal{Q}} \mathbf{f} \cdot \mathbf{g} d\Omega dt = 0. \tag{4.8}$$

4.3 A comparison between 1 and 3 dimensional results

Notably, the existence result obtained in the 3-dimensional problem in [14] does not represent a trivial generalization of the previous 1-dimensional one in [13]. Specifically, first of all the two different model systems are different in the way the condition $|\mathbf{m}| = 1$ is imposed on the magnetization vector. Note that, in (4.1) a *penalization* term appears, while in the three dimensional system (4.4), in [14], the condition $\mathbf{m}$ is a unit vector needs to be explicitly imposed. Furthermore, in the 1-dimensional problem an existence and uniqueness result is proved [13], while, when the 3-dimensional extension is considered an existence result is established in [14], but no uniqueness is stated. A brief outline of the different techniques adopted to prove the obtained results follows. The details are provided, referring to the two different cases, in [13, 14].

Key tool turns out to be the the free viscoelastic energy which allows to establish an *apriori* estimate which holds for the coupled system. Thus, two Lemmas are proved, the latter of which provides a uniform a priori estimate. Finally, on application of a fixed point theorem, the existence and uniqueness proof of Theorem 1 is completed, [13].

When the three dimensional problem is studied, the constraint on the unit length of the magnetization vector is imposed on introduction of a suitable sequence of approximated penalty problems, which, on introduction of a small positive parameter, and of approximated i.b.v. problem in $\mathcal{Q}$, on application of the related Faedo-Galerkin approximation, after the relative proof of convergence, allow to establish the weak existence result in Theorem 2, according to the details in [14].

5 Singular memory kernel

This section is devoted to an overview on the generalised problem, studied in [16]: the generalisation concerns the viscoelastic behaviour of the body under investigation while the magnetic effects are modelled as in the previous section.

5.1 Singular viscoelastic model

In this subsection, the attention is focussed on the *relaxed* functional requirements in the 1-dimensional

Viscoelastic model. Specifically, in the regular case, in Sect. 2, the relaxation modulus G is assumed to satisfy the functional requirements (2.4) and, in addition, (3.6) (for further details, see, for instance [28, 29]):

$$G(t) > 0, \quad \dot{G}(t) \leqslant 0, \quad \ddot{G}(t) \geqslant 0, \quad t \in (0, \infty), \tag{5.1}$$

and

$$G \in L^1(0,T) \cap C^2(0,T), \forall T \in \mathbb{R}. \tag{5.2}$$

Now, the requirement $\dot{G} \in L^1(0,T)$ is removed so that, according to (5.1) and (5.2), the relaxation function $G(t)$ is not required to be finite at $t = 0$, but $\lim_{t \to 0^+} G(t) = +\infty$. As a consequence, Eq. $(4.1)_1$ loses its meaning. In [11], an *ad hoc* sequence of approximated regular problems is introduced, then, the corresponding solutions sequence is proved to converge to the solution of the original singular problem. Uniqueness is proved by contradiction. Here the idea of the approximation strategy, devised in [11], is briefly outlined.

- Introduce the following *approximated problems* problems $0 < \varepsilon < 1, G^\varepsilon(\cdot) := G(\varepsilon + \cdot)$

$$u^\varepsilon_{tt} = G^\varepsilon(0)\, u^\varepsilon_{xx} + \int_0^t \dot{G}^\varepsilon(t-\tau)\, u^\varepsilon_{xx}(\tau)\, d\tau + f\,, \tag{5.3}$$

  with associated initial and boundary conditions

$$\begin{aligned} & u^\varepsilon|_{t=0} = 0, \quad u^\varepsilon_t|_{t=0} = u_1(x), \\ & u^\varepsilon|_{\partial\Omega \times (0,T)} = 0, t < T. \end{aligned} \tag{5.4}$$

- construct the *integral* formulation of the problems

$$\begin{aligned} u^\varepsilon(t) = & \int_0^t K^\varepsilon(t-\tau) u^\varepsilon_{xx}(\tau) d\tau + u_1 t + u_0 \\ & + \int_0^t d\tau \int_0^\tau f(\xi) d\xi\,, \end{aligned} \tag{5.5}$$

  where $K(\xi) := \int_0^\xi G(\tau) d\tau$ is well defined since $G \in L^1(0,T), \forall T \in \mathbb{R}$.
- consider the integral problem P^ε

$$\begin{aligned} P^\varepsilon : u^\varepsilon(t) = & \int_0^t K^\varepsilon(t-\tau) u^\varepsilon_{xx}(\tau) d\tau + u_1 t + u_0 \\ & + \int_0^t d\tau \int_0^\tau f(\xi) d\xi\,, \end{aligned} \tag{5.6}$$

- then, introduce the test functions $\varphi \in H^1(\Omega \times (0,T))$ s.t. $\varphi = 0$ on $\partial\Omega$, and apply Lebesque's Theorem.

  Then, [11], it follows

Theorem 3 *Given u^ε solution to the integral problem P^ε*

$$\begin{aligned} P^\varepsilon : u^\varepsilon(t) = & \int_0^t K^\varepsilon(t-\tau) u^\varepsilon_{xx}(\tau) d\tau + u_1 t + u_0 \\ & + \int_0^t d\tau \int_0^\tau f(\xi) d\xi\,, \end{aligned} \tag{5.7}$$

$$\exists\, u(t) = \lim_{\varepsilon \to 0} u^\varepsilon(t) \text{ in } L^2(\mathcal{Q}), \; \mathcal{Q} = \Omega \times (0,T). \tag{5.8}$$

- Finally, the obtained weak solution is proved to be unique, by contradiction.

Notably, an existence and uniqueness result in the general three dimensional case, is obtained in [12] on the basis of the analogy between the two models of isothermal viscoelasticity and of rigid thermodynamics with memory, as pointed out for instance in [15].

5.2 Singular magneto-viscoelasticity problems

The coupling between the magnetic and the viscoelastic effects when the singularity at $t = 0$ is considered, since (4.1) loses validity, according to [16], can be modelled by the nonlinear integro-differential system

$$\begin{cases} u_t(t) - \int_0^t G(t-\tau) u_{xx}(\tau) d\tau - u_1 - \int_0^t \frac{\lambda}{2} (\Lambda(\mathbf{m}) \cdot \mathbf{m})_x d\tau = \int_0^t f(\tau) d\tau \\ \mathbf{m}_t + \mathbf{m} \frac{|\mathbf{m}|^2 - 1}{\delta} + \lambda \Lambda(\mathbf{m}) u_x - \mathbf{m}_{xx} = 0, \end{cases} \text{in } \mathcal{Q} \tag{5.9}$$

where $\Omega = (0,1)$, $\mathcal{Q} := \Omega \times (0,T)$; in addition, when the quantities of interest are written in $\mathbb{R}^3$, let $\mathcal{M} \equiv (0, \mathbf{m})$, so that $\mathbf{m} = (m_1, m_2)$, denotes the magnetization vector, which is orthogonal to the conductor and, hence, $\mathbf{u} \equiv (u, 0, 0)$; $\boldsymbol{\nu}$ indicates the outer unit normal at the boundary $\partial\Omega$, Λ is a linear operator defined by $\Lambda(\mathbf{m}) = (m_2, m_1)$, u represents the unique non trivial component of the displacement (parallel to the 1-dimensional conductor), here identified with the x-axis and λ is a positive parameter. In addition, as in

(4.1), the term f represents an external force which also includes the deformation history.

Under the further assumptions:

$$u_1 \in L^2(\Omega), \quad \mathbf{m}_0 \in \mathbf{H}^1(\Omega), \quad f \in L^2(\mathcal{Q}). \tag{5.10}$$

the problem (5.9) subject to the following initial and boundary conditions

$$u(\cdot,0) = u_0 = 0, \quad \mathbf{m}(\cdot,0) = \mathbf{m}_0, |\mathbf{m}_0| = 1 \quad \text{in } \Omega, \tag{5.11}$$

$$u = 0, \quad \frac{\partial \mathbf{m}}{\partial} \boldsymbol{v} = 0 \quad \text{on} \quad \Sigma = \partial\Omega \times (0,T)\,, \tag{5.12}$$

admits a weak solution. Specifically, as proved in [16], the following theorem holds.

Theorem 4 *For all $T > 0$, there exists a weak solution $(u, \mathbf{m})$ to the problem (5.9)–(5.11)–(5.12), that is a vector function $(u, \mathbf{m})$ s.t.*

- $u \in L^\infty(0,T;H^1_0(\Omega))$;
- $u_t \in L^\infty(0,T;L^2(\Omega))$;
- $\mathbf{m} \in L^\infty(0,T;H^1(\Omega))$;
- $\mathbf{m}_t \in L^2(\mathcal{Q})$.

which satisfies

$$\begin{aligned} &-\int_{\mathcal{Q}} \phi_t u^\varepsilon(t)dxdt + \int_{\mathcal{Q}}\int_0^t G^\varepsilon(t-\tau)u^\varepsilon_x(\tau)\phi_x d\tau dxdt \\ &+ \int_{\mathcal{Q}}\int_0^t \frac{\lambda}{2}\Lambda(\mathbf{m}^\varepsilon)\cdot\mathbf{m}^\varepsilon\phi_x d\tau\ dxdt \\ &- \int_{\mathcal{Q}}\left[u_1 + \int_0^t f(\tau)d\tau\ \right]\phi dxdt + \int_{\mathcal{Q}} \boldsymbol{\psi}_t \cdot \mathbf{m}^\varepsilon dxdt \\ &+ \int_{\mathcal{Q}} \mathbf{m}_0 \cdot \boldsymbol{\psi}(\cdot,0)dxdt + \int_{\mathcal{Q}}\left(\frac{|\mathbf{m}^\varepsilon|^2-1}{\delta}\right)\boldsymbol{\psi}\cdot\mathbf{m}^\varepsilon dxdt \\ &- \int_{\mathcal{Q}} \lambda u^\varepsilon_x \Lambda(\mathbf{m}^\varepsilon)\cdot\boldsymbol{\psi} dxdt - \int_{\mathcal{Q}} \mathbf{m}^\varepsilon_x \cdot \boldsymbol{\psi}_x dxdt = 0. \end{aligned} \tag{5.13}$$

$\forall \phi$ smooth s.t. $\phi(0,t) = \phi(1,t) = 0, \phi(\cdot,T) = 0$, and $\forall \boldsymbol{\psi} \equiv (\psi_1,\psi_2)$ s.t. $\boldsymbol{\psi}(x,T) = 0$.

6 Conclusions and perspectives

The problem of the analysis of the existence of solutions in the generic case of a magneto-viscoelastic three dimensional body which is modelled via a singular kernel viscoelastic behaviour at $t = 0$, remains open. Indeed, we expect that, under the technical viewpoint, the method to apply to obtain the result should, in analogy to what happened in the magneto-viscoelastic regular problems [13, 14], in one or three dimensions would be quite different from each other. On the other hand, the connection between singular viscoelastic problems and their modelling via the introduction of kernel which are expressed via fractional derivative deserves to be investigated to understand if it might be promising in the case of singular magneto-viscoelasticity problems.

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