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On the complementary energy in elasticity and its history: the Italian school of nineteenth century

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Abstract In the formulation of mechanical theories, physicists usually neglect complementary energy, while rational mechanicians and engineers widely use it, especially in continuum physics and structural mechanics. Indeed, in many cases the solutions of elastic problems are found in a simpler way by resorting to complementary, rather than potential, energy. Moseley and Cotterill in England, Menabrea and Castigliano in Italy were among the first to introduce complementary energy in their papers, though implicitly; a more explicit formulation is in Crotti's papers; and Engesser extended it to non-linear elasticity. In this work we run through the history of complementary energy and search for its possible mechanical meaning.

Keywords Complementary energy · Structural mechanics · Castigliano's theorem · History of mechanics

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1 Introduction

Elastic problems can be tackled by approaches in terms of the so-called generalised displacements or forces [2, 18, 25, 29, 31]. In both cases, two strategies to obtain the governing equations for the fields of interest may be adopted. The first one yields differential equations directly, calling for principles of conservation (of mass, quantity of motion, living force). By the second strategy, the relevant field equations are derived by suitable conditions of stationarity of certain functionals that may be interpreted as general energies and are supposed to depend on some measurable data. The stationarity of these energies expresses their invariance for suitable variations of (some of) the measurable physical data on which they depend in certain functional spaces. In a generalised displacements approach, the scalar quantity on which it is possible to operate, at least for conservative problems, is usually called *potential energy* and is widely used by physicists, rational mechanicians, engineers. In a generalised forces approach, the relevant scalar quantity is usually called *complementary energy* and in practice is used only by applied mechanicians and engineers, especially dealing with the resolution of redundant structures [16, 17, 26, 34, 38, 39]. Indeed, even though complementary energy was historically introduced almost at the same time as potential energy, under the name of force function, or work function, its use soon became restricted to the above quoted scientific community, and physicists prefer to use (generalised) potential functions.

Literature often dealt with this: [1, 4, 28, 30, 33, 41, 46–49] are monographs representing an apparently incomplete list of published works. A detailed historical account, together with the relevant comparisons with a contemporary point of view is missing, despite some relevant contributions [3]; the present work is thus intended to fill this gap. The paper is organized as follows. In Sect. 2 we introduce the potential and complementary energies from a contemporary point of view, and their reciprocity properties in terms of the Legendre transform, limited to the geometric linear case (a.k.a. "infinitesimal" or "small" displacements and strain). We present the use of both energies in elementary applications of linear elasticity; we stress how in structural mechanics the use of complementary energy immediately leads to technical solutions. In Sect. 3 we run through the historical birth and development of elastic complementary energy, with the discussions that arose in the scientific community of the time about the correct statement of its conditions of stationarity. Along with this critical presentation, we search for a possible mechanical interpretation of elastic complementary energy. Section 4 deals with the analysis of constitutively non-linear elastic systems. A final discussion is given in Sect. 5. Before listing the bibliography, two attachments complete the paper: "Appendix 1" contains the original sources from literature; "Appendix 2" presents a comprehensive list of the symbols adopted in Sect. 2.

2 A contemporary point of view

Let $\mathcal{B}$ be a continuous body; its boundary $\partial\mathcal{B}$ is supposed to be regular and composed of two complementary disjoint portions $\partial_1\mathcal{B}, \partial_2\mathcal{B} : \partial_1\mathcal{B} \cup \partial_2\mathcal{B} = \partial\mathcal{B}$, $\partial_1\mathcal{B} \cap \partial_2\mathcal{B} = \emptyset$, see Fig. 1. The body is supposed to be subjected to the following external data:

(1) an assigned displacement $\boldsymbol{u}_o$ over $\partial_1\mathcal{B}$;
(2) a load system $\boldsymbol{f} = (\boldsymbol{s}_o, \boldsymbol{b}_o)$; $\boldsymbol{s}_o$ is the surface density over $\partial_2\mathcal{B}$, and $\boldsymbol{b}_o$ the volume density over $\mathcal{B}$.

The elastic problem looks for the triplet $\mathcal{S} = \{\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{S}\}$ of displacement, strain, and stress characterizing the *state* of the body; the traction $\boldsymbol{s}$ is then determined by Cauchy's theorem $\boldsymbol{s} = \boldsymbol{S}\boldsymbol{n}$, with $\boldsymbol{n}$ the outward unit normal to $\partial\mathcal{B}$. Let us define the real spaces:

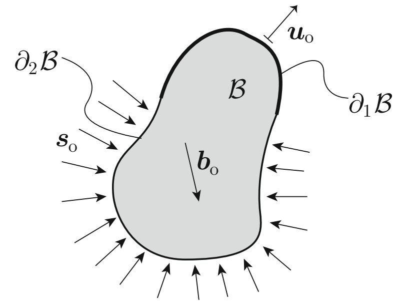

Fig. 1 A body $\mathcal{B}$, with applied loads and imposed displacement

1. $\mathscr{K}$ of *kinematically admissible displacement and strain*:

$$\mathscr{K} := \{(\boldsymbol{u},\boldsymbol{E}) \in \left(C^1(\mathcal{B}) \cap C^0(\bar{\mathcal{B}})\right) \times C^0(\bar{\mathcal{B}}) | \mathfrak{D}[\boldsymbol{u}] = \boldsymbol{E} \text{ in } \mathcal{B}, \mathfrak{B}_o[\boldsymbol{u}] = \boldsymbol{u}_o \text{ in } \partial_1\mathcal{B}\},$$

2. $\mathscr{S}$ of *statically admissible stresses*:

$$\mathscr{S} := \{\boldsymbol{S} \in C^1(\mathcal{B}) \cap C^0(\bar{\mathcal{B}}) \mid \mathfrak{S}[\boldsymbol{S}] = -\boldsymbol{b}_o \text{ in } \mathcal{B}, \mathfrak{B}_1[\boldsymbol{S}] = \boldsymbol{s}_o \text{ in } \partial_2\mathcal{B}\}.$$

The operators introduced so far act as follows: $\mathfrak{D}$ ensures kinematical compatibility, $\mathfrak{S}$ equilibrium, $\mathfrak{B}_o, \mathfrak{B}_1$ the fulfillment of boundary conditions. In linear elasticity, we have

$$\begin{aligned} &\mathrm{sym}(\nabla \boldsymbol{u}) = \boldsymbol{E} \text{ in } \mathcal{B}, \ \boldsymbol{u} = \boldsymbol{u}_o \text{ in } \partial_1\mathcal{B}, \\ &\boldsymbol{S} \in \mathrm{Sym} \text{ in } \bar{\mathcal{B}}, \ \mathrm{Div}\,\boldsymbol{S} = -\boldsymbol{b}_o \text{ in } \mathcal{B}, \ \boldsymbol{S}\boldsymbol{n} = \boldsymbol{s}_o \text{ in } \partial_2\mathcal{B}. \end{aligned} \tag{1}$$

The bilinear form of the *external work* spent by the loads $\boldsymbol{f}$ on the displacement $\boldsymbol{u}$ makes $\boldsymbol{f}$ and $\boldsymbol{u}$ *dual*:

$$\mathcal{W}^e := \int_{\mathcal{B}} \boldsymbol{b}_o \cdot \boldsymbol{u} + \int_{\partial_2\mathcal{B}} \boldsymbol{s}_o \cdot \boldsymbol{u}; \tag{2}$$

the bilinear form of the *internal work* spent by the stresses $\boldsymbol{S}$ on strains $\boldsymbol{E}$ makes $\boldsymbol{S}$ and $\boldsymbol{E}$ *dual*:

$$\mathcal{W}^i := \int_{\mathcal{B}} \boldsymbol{S} \cdot \boldsymbol{E}. \tag{3}$$

The *principle of virtual work* states that $\mathcal{W}^e = \mathcal{W}^i$ and expresses the physical requirement that the work spent by the external actions in the considered process cannot be lost; it puts in duality $\mathfrak{S}$ and $\mathfrak{D}$, in the sense that $\mathfrak{S} = \mathfrak{D}^*$, the adjoint of $\mathfrak{D}$. Synthetically, we may infer the following relations:

$$\boldsymbol{f} = \boldsymbol{u}^*, \quad \boldsymbol{S} = \boldsymbol{E}^*, \quad \mathfrak{S} = \mathfrak{D}^*, \quad \mathscr{S} = \mathscr{K}^*, \tag{4}$$

where $(\cdot)^*$ denotes the dual of $(\cdot)$ in the sense stated by (2) and (3).

The *constitutive operator* relates strain and stress at any point of the body according to:

$$\mathfrak{C} : \boldsymbol{E} \mapsto \boldsymbol{S} = \mathfrak{C}\boldsymbol{E}\,; \tag{5}$$

we remark that $\mathfrak{C}$ is not necessarily linear. In the so-called hyper-elastic materials, the constitutive relation expressed by (5) derives from a volume *density of potential energy*, here assumed to be convex, so that:

$$\boldsymbol{S} = \partial_{\boldsymbol{E}}(\mathfrak{C}(\boldsymbol{E}) \cdot \boldsymbol{E}).$$

Thus, the *potential energy* turns out to be

$$U(\boldsymbol{E}) := \int_{\mathcal{B}} \mathfrak{C}(\boldsymbol{E}) \cdot \boldsymbol{E}. \tag{6}$$

Formally, the *complementary energy*[1] $U^*(\boldsymbol{E}^*)$ is defined as the *Legendre transform*[2] of the potential energy (6):

$$U^*(\boldsymbol{E}^*) := \sup_{\boldsymbol{E}}\{\boldsymbol{E} \cdot \boldsymbol{E}^* - U(\boldsymbol{E})\}. \tag{7}$$

A simple geometric interpretation of the definitions in (6), (7) can be provided in a one-dimensional setting. Let $\sigma, \varepsilon, u, f$ denote stress, strain, displacement, and loads, respectively. Then, according to (6), $U(\varepsilon)$ provides the area under the graph of $\mathfrak{C}\varepsilon$ (Fig. 2).

The curve $\varepsilon^* = \mathfrak{C}\varepsilon$ can be also seen as $\varepsilon = \mathfrak{C}^{-1}\varepsilon^*$, according to which independent variable is considered. The area $U^*(\varepsilon^*)$ provides complementary energy. At any point $(\varepsilon, \varepsilon^*)$ of the curve, the sum of the two areas U, U^* provides the area of the rectangle with sides $\varepsilon, \varepsilon^*$:

$$U(\varepsilon) + U^*(\varepsilon^*) = \varepsilon\varepsilon^*. \tag{8}$$

Figure 2 shows that, at a point that does not lie on the curve $\mathfrak{C}\varepsilon$,

$$U(\varepsilon) + U^*(\varepsilon^*) \geq \varepsilon\varepsilon^*. \tag{9}$$

For any ε^*, by (8), (9) the difference $\varepsilon\varepsilon^* - U(\varepsilon)$ is largest on the curve, and equals $U^*(\varepsilon^*)$. Thus, the explicit relation between $U(\varepsilon)$ and $U^*(\varepsilon^*)$ is

$$U^*(\varepsilon^*) := \sup_{\varepsilon}\{\varepsilon\varepsilon^* - U(\varepsilon)\},$$

which is nothing but the one-dimensional version of (7).

By convex analysis [42], it is known that the following statements are equivalent:

$$\boldsymbol{E}^* \in \partial U(\boldsymbol{E}),\ \boldsymbol{E} = \partial U^*(\boldsymbol{E}^*),\ \mathcal{W}^i = U(\boldsymbol{E}) + U^*(\boldsymbol{E}^*), \tag{10}$$

where $\partial U(\boldsymbol{E})$ is the subdifferential of $U(\boldsymbol{E})$:

$$\partial U(\boldsymbol{E}) = \{\boldsymbol{E}^* \in \mathscr{K}^* \mid \langle \boldsymbol{E} - \boldsymbol{E}_0, \boldsymbol{E}^* \rangle \geq U(\boldsymbol{E}) - U(\boldsymbol{E}_0), \forall \boldsymbol{E}_0 \in \mathscr{K}\}.$$

If $U(\boldsymbol{E})$ is smooth, by (4) the conditions (10) become

$$\boldsymbol{S} = \partial_{\boldsymbol{E}} U(\boldsymbol{E}),\ \boldsymbol{E} = \partial_{\boldsymbol{S}} V(\boldsymbol{S}),\ \int_{\mathcal{B}} \boldsymbol{S} \cdot \boldsymbol{E} = U(\boldsymbol{E}) + V(\boldsymbol{S}),$$

where we set $V = U^*$ for the complementary energy.

Let us introduce:

1. the *load potential* functional

$$F(\boldsymbol{u}) := -\left(\int_{\mathcal{B}} \boldsymbol{b}_o \cdot \boldsymbol{u} + \int_{\partial_2\mathcal{B}} \boldsymbol{s}_o \cdot \boldsymbol{u}\right) = -\mathcal{W}^e, \tag{11}$$

2. the *displacement potential* functional

$$G(\boldsymbol{s}) := -\int_{\partial_1\mathcal{B}} \boldsymbol{s} \cdot \boldsymbol{u}_o, \quad \boldsymbol{s} = \boldsymbol{S}\boldsymbol{n}, \tag{12}$$

i.e., the opposite of the work spent by the traction $\boldsymbol{s}$ on the displacement of $\partial_1\mathcal{B}$. To a structural mechanician, used to constrained structures, G is the work spent by the reactive forces maintaining the constraints on the displacement they impose.

[1] This term (better, complementary work, *Ergänzungsarbeit*), but not its definition and possible physical interpretations, is due to Friedrich Engesser (1848–1931), see [26]. Details are in Sect. 4.

[2] Adrien-Marie Legendre (1752–1833) introduced his transform in a memoir on differential equations [32], on pp. 346–347; the conjugated quantity is defined as $\boldsymbol{E}^* = \partial_{\boldsymbol{E}} U(\boldsymbol{E})$. The first to use a Legendre transform in dynamics to turn 'Lagrange's characteristic function' into what we now call action was William Rowan Hamilton (1805–1865) in his *On a general method in dynamics*, Phil. Trans. of the R. Society, part II for 1834, 247–308.

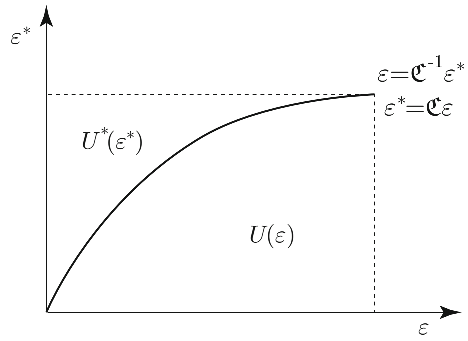

Fig. 2 1-D elasticity: potential and complementary energy

The *total external potential* is the opposite of the work spent by the body and surface actions [28, 41]:

$$\begin{aligned}\mathcal{L}(\boldsymbol{u},\boldsymbol{s}) &:= -\left(\int_{\mathcal{B}} \boldsymbol{b}_o \cdot \boldsymbol{u} + \int_{\partial\mathcal{B}} \boldsymbol{S}\boldsymbol{n} \cdot \boldsymbol{u}\right) \\ &= -\left(\int_{\mathcal{B}} \boldsymbol{b}_o \cdot \boldsymbol{u} + \int_{\partial_2\mathcal{B}} \boldsymbol{s}_o \cdot \boldsymbol{u} + \int_{\partial_1\mathcal{B}} \boldsymbol{s} \cdot \boldsymbol{u}_o\right).\end{aligned} \tag{13}$$

The definitions (11)–(13) imply at once that G and F are related through Legendre transform:

$$G(\boldsymbol{s}) = \sup_{\boldsymbol{u}}\{\mathcal{L}(\boldsymbol{u},\boldsymbol{s}) - F(\boldsymbol{u})\}.$$

Then, the *total potential energy* Φ and the *total complementary energy* Ψ

$$\Phi(\boldsymbol{u},\boldsymbol{E}) = U(\boldsymbol{E}) + F(\boldsymbol{u}), \quad \Psi(\boldsymbol{S},\boldsymbol{s}) = V(\boldsymbol{S}) + G(\boldsymbol{s}),$$

are Legendre conjugated as well.

It is also easy to prove the two *minimum theorems* (see [28, 41] for a modern proof):

(i) Let $\mathscr{K}$ be the set of all kinematically admissible states and $\Phi(\boldsymbol{E},\boldsymbol{u})$ the functional on $\mathscr{K}$ defined by

$$\Phi\{\boldsymbol{E},\boldsymbol{u}\} = U(\boldsymbol{E}) + F(\boldsymbol{u}), \quad \forall\{\boldsymbol{u},\boldsymbol{E}\} \in \mathscr{K}.$$

Further, if $\boldsymbol{u}$ is a solution of (1), (5), then

$$\Phi\{\boldsymbol{E},\boldsymbol{u}\} \le \Phi\{\bar{\boldsymbol{E}},\bar{\boldsymbol{u}}\}, \quad \forall\{\bar{\boldsymbol{u}},\bar{\boldsymbol{E}}\} \in \mathscr{K};$$

the equality holds only if $\{\bar{\boldsymbol{u}},\bar{\boldsymbol{E}}\} \equiv \{\boldsymbol{u},\boldsymbol{E}\}$, modulo a rigid displacement. That is, the total potential energy attains a global minimum at the solution of the elastic problem.

(ii) Let $\mathscr{S}$ be the set of all statically admissible stress fields and $\Psi\{\boldsymbol{S},\boldsymbol{s}\}$ the functional on $\mathscr{S}$ defined by

$$\Psi\{\boldsymbol{S},\boldsymbol{s}\} = V(\boldsymbol{S}) + G(\boldsymbol{s}), \quad \forall\{\boldsymbol{S},\boldsymbol{s}\} \in \mathscr{S}.$$

Further, if $\{\boldsymbol{S},\boldsymbol{s}\}$ is the solution of (1),(5), then

$$\Psi\{\boldsymbol{S},\boldsymbol{s}\} \le \Psi\{\bar{\boldsymbol{S}},\bar{\boldsymbol{s}}\}, \quad \forall\{\bar{\boldsymbol{S}},\bar{\boldsymbol{s}}\} \in \mathscr{K};$$

the equality holds only if $\{\bar{\boldsymbol{S}},\bar{\boldsymbol{s}}\} \equiv \{\boldsymbol{S},\boldsymbol{s}\}$. That is, the total complementary energy attains a global minimum at the solution of the elastic problem.

2.1 Linearly elastic systems

For the linear elastic spring in Fig. 3a, the strain is $\varepsilon = u_B - u_A$, with u the axial displacement; σ is the stress and P_o the load applied at B. If κ is the stiffness of the spring, it is $\sigma = \kappa\varepsilon$. The stored energy, load potential, and total potential energy are

$$U(\varepsilon) = \frac{1}{2}\kappa\varepsilon^2, \; F(u_B) = -P_o u_B, \; \Phi(\varepsilon, u_B) = \frac{1}{2}\kappa\varepsilon^2 - P_o u_B.$$

The minimum for Φ has to be sought in the space of compatible strains, i.e., $\{\varepsilon : \varepsilon = u_B - u_A, \, u_A = 0\}$. Thus,

$$\Phi(\varepsilon, u_B) = \widehat{\Phi}(u_B) = \frac{1}{2}\kappa u_B^2 - P_o\, u_B,$$

whence the minimum of $\widehat{\Phi}$ provides the unknown u_B:

$$\partial_{u_B}\widehat{\Phi}(u_B) = 0 \quad \Rightarrow \quad u_B = \frac{P_o}{\kappa}.$$

Here, the control variable (the *datum*) is the force P_o applied at the boundary $\partial_2\mathcal{B} \equiv \{B\}$, with the same role as $\boldsymbol{s}_o$ in (11). In the dual problem, the control variable (the *datum*) is the imposed displacement $\boldsymbol{u}_o$ at $\partial_1\mathcal{B} \equiv \{A\} \cup \{B\}$ (Fig. 3b), and P has the same role as $\boldsymbol{s}$ in

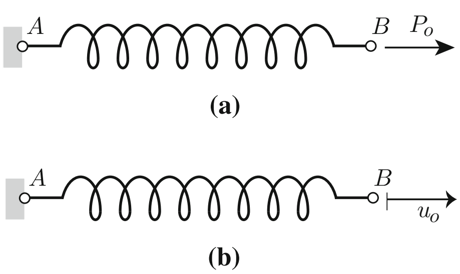

Fig. 3 A linearly elastic spring

(12). An interesting and thorough discussion on the role of Legendre transform in physics with reference to the variables controlling the observed phenomena is in [50].

The complementary stored energy, load potential, and total complementary energy are

$$V(\sigma)=\frac{1}{2}\frac{\sigma^2}{\kappa},\quad G(P)=-Pu_o,\quad \Psi(\sigma,P)=\frac{1}{2}\frac{\sigma^2}{\kappa}-Pu_o.$$

The minimum for Ψ has to be sought in the space of balanced stresses, i.e., $\{\sigma:\sigma=P\}$. Thus,

$$\Psi=\widehat{\Psi}(P)=\frac{1}{2}\frac{P^2}{\kappa}-Pu_o,$$

whence the requirement that $\widehat{\Psi}$ be a minimum provides the unknown P:

$$\partial_P\widehat{\Psi}(P)=0\quad\Rightarrow\quad P=\kappa u_o.$$

In structural mechanics, this approach is expedient to find the elastic state of a structure, be it statically determined or not. A simple example is the redundant truss in Fig. 4, subjected to the load P_o and the displacement u_o. Let the axial stiffnesses of the bars be $\kappa_1=E_1A_1$ for BD and $\kappa_2=E_2A_2$ for AD, CD, with E_i $(i=1,2)$ Young's modulus and A_i the cross-section area of the i-th bar. The stored energy is then

$$U(\varepsilon_1,\varepsilon_2)=\frac{1}{2}\kappa_1\varepsilon_1^2L_1+2\times\frac{1}{2}\kappa_2\varepsilon_2^2L_2,$$

with ε_i the axial strain and L_i the length of the bars. The load potential and total potential energy are then

$$F(u_D)=-P_ou_D,\ \Phi(\varepsilon_1,\varepsilon_2,u_D)=U(\varepsilon_1,\varepsilon_2)+F(u_D).$$

The minimum of Φ has to be sought in the space of compatible strains, i.e.

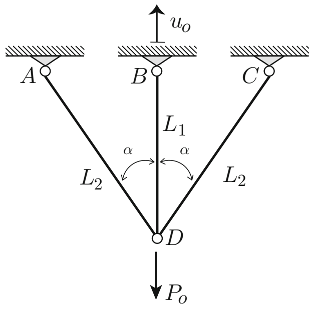

Fig. 4 A linearly elastic reticular system

$$\varepsilon_1=\frac{u_D+u_o}{L_1},\quad \varepsilon_2=\frac{u_D}{L_2}\cos\alpha,$$

whence

$$\Phi=\widehat{\Phi}(u_D)=\frac{1}{2}\frac{\kappa_1}{L_1}(u_D+u_o)^2+\frac{\kappa_2}{L_2}\cos^2\alpha\,u_D^2-P_ou_D.$$

On requiring that $\widehat{\Phi}$ be a minimum, we get:

$$\partial_{u_D}\widehat{\Phi}(u_D)=0\ \Rightarrow\ u_D=\left(P_o-\frac{\kappa_1}{L_1}u_o\right)\left(\frac{\kappa_1}{L_1}+2\frac{\kappa_2}{L_2}\cos^2\alpha\right)^{-1}.$$

If N_i is the normal force in the ith bar, the complementary stored energy, load potential, and total complementary energy are:

$$V(N_1,N_2)=\frac{1}{2}\frac{L_1}{\kappa_1}N_1^2+2\times\frac{1}{2}\frac{L_2}{\kappa_2}N_2^2,$$
$$G(P)=-Pu_o,\quad \Psi(N_1,N_2,P)=V(N_1,N_2)+G(P),$$

where P is the value of the normal force at the boundary point B, which turns out to be $P=N_1$, since the normal force is constant on the bar; then

$$\Psi(N_1,N_2,P)=\widetilde{\Psi}(N_1,N_2).$$

The minimum for Ψ has to be sought in the space of balanced stresses, i.e., of all N_i assuring balance at D:

$$N_1+2N_2\cos\alpha=P_o.$$

Thus, we get:

$$\widetilde{\Psi}(N_1,N_2)=\widehat{\Psi}(N_1)=\frac{1}{2}\frac{L_1}{\kappa_1}N_1^2+\frac{L_2}{\kappa_2}\left(\frac{P_o-N_1}{2\cos\alpha}\right)^2-N_1u_o,$$

whence the requirement that $\widehat{\Psi}$ be minimum yields:

$$N_1=\frac{\kappa_1}{L_1}\left(P_o+2\frac{\kappa_2}{L_2}\cos^2\alpha\,u_o\right)\left(\frac{\kappa_1}{L_1}+2\frac{\kappa_2}{L_2}\cos^2\alpha\right)^{-1}.$$

3 The Italian school of the nineteenth century

Potential energy and the relevant stationarity theorem date back at least to the end of the eighteenth century. We find their rather defined form in Lagrange's *Mécanique analytique* (1788), and even before.

Complementary energy and the relevant stationarity theorem date back to the twentieth century, at least for a complete formulation. However, in the historical development of structural mechanics this latter theorem was the first used, with no clear idea of the procedures being applied.

Indeed, the pioneers in applying energetic approaches for solving structural problems soon realised that the elastic energy U is independent of the external load $\boldsymbol{f}$, which is a fundamental datum of the problem; thus, U seems useless and, moreover, the expression of $\Phi(\boldsymbol{u}, \boldsymbol{E})$ may be quite complex. On the contrary, the complementary energy $\Psi(\boldsymbol{S}, \boldsymbol{s})$ easily incorporates such information, since tractions and stresses shall balance the loads $\boldsymbol{f}$. Thus, minimizing $\Psi(\boldsymbol{S}, \boldsymbol{s})$ quite likely seemed most natural for the structural engineers of the nineteenth century, who did exactly the opposite of what is done nowadays. Indeed, the total elastic potential energy seemed to be devoid of any mechanical meaning, contrary to the total elastic complementary energy, which lets a theorem of minimum be naturally formulated. On the contrary, nowadays U, and consequently Φ, is attributed a precise mechanical meaning, while the same is not for V and Ψ, at least in those theoretical formulations for which the mechanical meaning of the terms is important.

The best known contributions in structural mechanics about complementary energy are by the Italian engineers and scientists Luigi Federico Menabrea and Carlo Alberto Castigliano, who stated theorems still called by their names; later on, Valentino Cerruti (1850–1909), Francesco Crotti (1839–1896), Silvio Canevazzi (1852–1918), Luigi Donati (1846–1932), Gustavo Colonnetti (1886–1968) joined the precursors. These contributions were not isolated in Europe, though: Augustin Cournot (1801–1877), Henry Moseley (1801–1872), James Henry Cotterill (1836–1922) in the first half of the eighteenth century studied structural mechanics by a work function depending on inner or outer actions; the German school made fundamental studies in the second half of the century. The contribution of the Italian school seems in any case the most relevant and lasting, and for this reason we will focus on it, with its main papers on prototype linear elastic truss structures [7, § 4.3.1].

In a series of papers [34–37], Menabrea considered a system of hinged elastic bars undergoing very small displacements as representative of a linear elastic body, for which he stated the following 'energy based' theorem (hereinafter: *Menabrea's theorem*):

> When an elastic system is equilibrated under the action of external forces, the work spent by the tensions, or compressions, of the links joining the various points of the system is a minimum.[3]

With Menabrea's symbols, the (internal) work spent by the axial forces T (tensions) of all the bars is:

$$\frac{1}{2}\sum\frac{1}{\epsilon}T^2$$

where ϵ is the coefficient of elasticity (with the contemporary standard symbols of Sect. 2.1, $\epsilon = EA/L$).

Menabrea solved the problem of minimum as a conditioned problem, supplementing it with equilibrium equations and using Lagrange multipliers. Subsequently, the standard approach was to solve an unconditioned minimum problem, where the forces in the bars are expressed in terms of the unknown reactions of the redundant bars (see [7], § 4.4, [26]).

Menabrea's theorem, which he called 'principle of elasticity', is that of minimum elastic complementary energy stated in Sect. 2 for the particular case of fixed constraints, whence $G(\boldsymbol{s}) = 0$ in the expression of $\Phi(\boldsymbol{S}, \boldsymbol{s})$. Menabrea's theorem is, thus, correct in itself; its proof, however, was based on the principle of virtual work and was all but impeccable, leading to vivacious discussions (see [44] and [7], pp. 195–197). Indeed, it is not clear whether one shall operate in the spaces of kinematically admissible displacements or of balanced forces.

In spite of its unclear proof, there was at least an immediate application of Menabrea's theorem to a real steel structure, the truss roof of the railway station in Arezzo, Tuscany [20, 40]. This truss was poorly designed, even according to the standards of the time: the number of bars was insufficient to make the structure statically determined. The truss exhibited settlements, but remained standing; to explain the fact Giovanni Sacheri, professor of drawing at the School of engineering application of Turin, in 1872 admitted that the bars provide shearing forces N (normal to the axis) as well as axial forces T (tangent to the axis: remark the opposite terminology with respect to the present one). In this way, the truss became redundant,

3 Menabrea [34], p. 1056; source in "Appendix 1".

and Sacheri decided to tackle the structural calculations by Menabrea's theorem, generalising it to shear forces and bending moments. To this purpose, the internal work, which he called molecular action, was written as

$$\frac{1}{2}\sum\frac{1}{\tau}T^2+\frac{1}{2}\sum\frac{1}{v}N^2$$

where $\tau = E\omega/l$ is the coefficient of axial elasticity, ω being the section of the bar, and $v = 3EI/l^3$ is the coefficient of transverse elasticity; the other symbols are usual [45, pp. 99, 103]. Remark that Sacheri introduced the internal work in bending before Castigliano; however, Moseley and Cotterill had already proposed the same approach [7, pp. 54–55].

In 1876 Giovanni Battista Rombaux, a professional engineer, reconnected to the roof of the railway station in Arezzo to propose a thorough discussion on the 'principle of elasticity', and account for the internal work in bending in a more systematic way than Sacheri [6, 7, 43]. A more precise and correct proof of Menabrea's statement is in Castigliano's works [6–13]. In his graduation thesis [8], Castigliano adopted the same truss model as Menabrea, and stated that

> If I determine the tensions T_{pq} so that they make the expression T_{pq}^2/ϵ_{pq} a minimum, by supposing that those tensions satisfy [the balance equations], in which, however, all the external forces X_p, Y_p, Z_p and all the angles $\alpha_{pq}, \beta_{pq}, \gamma_{pq}$ are considered constant, the values of the tensions so obtained coincide with those obtained by the method of displacements.[4]

Here ϵ_{pq}, T_{pq} are the stiffnesses and the axial forces of the bars. Once Castigliano verified that the minimum of what he called *molecular work* T_{pq}^2/ϵ_{pq} provides the same balance equations as the usual, well-established, non-problematic method of displacements (or deformation), the proof is found that such a minimum yields the (unique) solution of the linear elastic problem. He went further, formulating what are now called Castigliano's second and first theorem, respectively:

> *First Part* [second theorem] - If we express the strain work of an articulated system as a function of the relative displacements of the external forces applied to its vertexes, we obtain a formula, the derivatives of which with respect to such displacements provide the values of the corresponding forces.
>
> *Second Part* [first theorem] - If, on the other hand, we express the strain work of an articulated system as a function of the external forces, we obtain a formula, the derivatives of which with respect to such forces, provide the relative displacements of their points of application.[5]

By Castigliano's symbols, these statements lead to

$$\frac{dL}{dr_p}=R_p, \quad \frac{dL}{dR_p}=r_p$$

where L is the molecular work, and r_p, R_p are the components of displacements and forces, respectively. Castigliano extended this result to systems with beams in bending, shearing, and torsion. Basing on his theorems on the derivatives of the molecular work, especially the first one with respect to forces, Castigliano proved Menabrea's statement in a rigorous way, by an approach different from that of 1873.

3.1 Complementary energy in historic literature

Section 2 follows a contemporary axiomatic-deductive formal approach: the physical interpretation of the terms entering the theory is reduced to a minimum, thus aiming to ignore their ontological status, leaving the problem to the philosophers of science. From this point of view, the total potential energy Φ is merely a function of some parameters, the displacements. As a consequence of the axioms of mechanics, and under a series of assumptions, this function satisfies some theorems. One of these states that a mechanical state called equilibrium corresponds to stationary Φ when the parameters vary in a suitably defined space of admissible displacements. By this 'aseptic' approach, the total complementary energy Ψ has an ontological state similar to that of total potential energy. By following the same axioms of mechanics other theorems hold, and one of these states that a mechanical state called of compatible displacements corresponds to a stationary Ψ when its parameters vary in a suitably defined space of balanced forces.

[4] Castigliano [8], p. 14; source in "Appendix 1".

[5] Castigliano [12], p. 26; source in "Appendix 1".

In less formal approaches to mechanics, like that of nineteenth century engineers and mathematicians, we find a different situation: the energies were provided with a physical meaning, but the procedure was not fully consistent, in that there was no difference between potential and complementary energy. Indeed, a modern scholar of structural mechanics remarks the absence of a clear distinction between compatibility (of displacements and strain) and balance (of external actions and internal stresses) in Castigliano and Menabrea's works. This leads, at least to our eyes, to true errors (Menabrea's first proof of his principle), or to lexical ambiguities (Castigliano). A modern reader, indeed, would find difficulties in Castigliano's proof of the theorem of least work of 1873, due to an ambiguous use of strain work.

In linear elasticity, engineers of the second half of the nineteenth century dealt with $V(\mathbf{S}), U(\mathbf{E})$ indifferently, in that the two functions are interchangeable via the simple constitutive relation. So, they seemed the same quantity, called molecular work, internal work, work function according to the jargon of the age. Their variations as well, the one with respect to balanced forces, the other with respect to compatible displacements, were seen as variations of energy of a unique structural system.

For instance, Cerruti [14] reduced Menabrea's principle to Green's theorem of elastic forces [14, p. 571]. Adopting the usual paradigmatic truss, he assumed as a 'potential' V (his term) the quantity

$$V = -\frac{1}{2}\sum \int \frac{\tau^2}{e}\omega d\sigma$$

where σ is the length, e the elastic modulus, ω the area, and τ the stress in each bar. Leaving details aside, we read that "In order to have equilibrium, it must be

$$\delta V = 0 \tag{14}$$

in accord with the equations imposed at the boundary" ([14], p. 572: those balance outer forces and inner elastic stresses at the nodes). Then, it was not difficult for Cerruti to interpret Eq. (14) as Menabrea's theorem.

Cerruti's considering V a potential energy of a system is a conceptual error, however: if V depends on balanced forces and stresses, in general displacements are not compatible with strains. Thus, since potential energy is a function of admissible (compatible) states of the system, one cannot say that V is a form of its potential energy. The total potential energy is actually stationary with respect to variations of compatible displacements and strains, but no proof exists that stationarity remains with respect to variations of inner stresses balanced with outer forces: there are hypotheses assuring stationarity, but Cerruti's proof is inconsistent.

A similar error was done in 1889 by Silvio Canevazzi, professor of structural mechanics, bridges and hydraulic structures in Bologna, in a paper on strength of materials [5]. He introduced the strain work (potential energy of inner forces) L and the potential energy J of the active forces R for a linearly elastic structure, and admitted that the stationarity condition

$$\delta(L + J) = 0 \tag{15}$$

assures balance. In (15) the variation is with respect to the admissible displacements of the nodes subjected to the active forces, which induce strains in bars and provide the relevant stresses [5, p. 108]. Denoting Λ the expression of the strain work in terms of R, Canevazzi wrote the balance condition as [5, p. 108]

$$\delta(\Lambda + J) = 0 \tag{16}$$

However, this is incorrect: indeed, Λ is a function of nodal displacements, and can be easily written in terms of *all* external forces, which determine the strain state and the displacements of the nodes accounted for in (15). On the other hand, J depends *only* on active forces (supposed 'dead'), thus the variation in (15), (16) cannot be operated with respect to the same quantities.

For a constrained system, Canevazzi denoted the relevant reactions by R': they are unknown outer forces that spend no work in the usual cases of fixed constraints, thus do not enter the expression for J. On the other hand, Λ is a function of R' as well, since both constraint reactions and active forces determine the stresses, hence the strains, in the bars. By admitting that (16) still holds, Canevazzi wrote balance as

$$\frac{(d\Lambda + J)}{dR'} = 0 \tag{17}$$

Such equation makes sense only if the considered structure is redundant, since then the R' may vary freely in the given admissible configuration, still assuring balance. If the R' are considered as independent variables, then (15) does not imply (16), since for Λ to represent the actual potential energy of a system, its configurations shall be admissible: but a generic free variation of the R' cannot assure compatibility, hence the considered object is not a structural system in the physical sense. For fixed constraints, J does not depend on R', the active forces are 'dead', thus (17) implies

$$\frac{(d\Lambda)}{dR'} = 0$$

i.e., Menabrea's theorem. Canevazzi remarked

> We could not get to this result if the constraints imposed to the system were movable, or the relevant reactions expended work, exactly as it could happen if some points were forced to remain over surfaces exhibiting a frictional resistance to their movement.[6]

In the cases of movable constraints he added

> If the unknown reactions are such that an external work is spent [...] then Menabrea's theorem holds no more, and in order to determine the unknown [inner] forces and constraint reactions one must resort to the methods of the derivatives of work, or of deformations, which we presented in the previous chapter.[7]

thus admitting the impossibility to derive a minimum property for complementary energy in a general case.

Castigliano also had doubts on the derivation of Menabrea's theorem from energetic considerations. In his graduation thesis of 1873 [8] he proved that, by searching the minimum of the work function L proposed by Menabrea, one obtains equations that must be added to those of balance of force and moment in order to solve the linear elastic problem; these equations are the same that would be obtained by the method of displacements, and *vice versa*, thus Menabrea's theorem was proved. In his famous monograph of 1879 [12] Castigliano came to Menabrea's theorem starting from one of his theorems on the derivatives of elastic work (the first theorem bringing his name):

$$\frac{dl}{dR'_p} = r_p \tag{18}$$

where R'_p is the unknown redundant reaction and r_p the displacement of the pth simple constraint. In the case of fixed constraints, $r_p = 0$, thus by (18) L is stationary and Menabrea's theorem holds.

Luigi Donati, professor of mathematical physics in Bologna, took a different approach in a series of papers from 1888 to 1894 [22–24]. In his paper dated 1888, *Sul lavoro di deformazione dei sistemi elastici*, Donati quoted Menabrea, Castigliano, Cerruti, and Canevazzi [22, p. 345]; however, he did not quote Cotterill, to which his investigation seems to be somehow inspired. Even though understanding this paper is not easy for a contemporary, it is apparent that Donati makes it clear, also from the point of view of the precision of terms, that one should talk of strain work, or energy, only when the strains of the single particles of a body obey certain differential relations, in order to be compatible as a deformation of the whole body. When strains are not so, like in the case examined by Menabrea, he stated that:

> Then, however, one cannot anymore talk of displacements of the body as a whole, nor the strain work can be seen as a function of displacements and local stress fluxes, but simply as the sum, or integral $\int \Phi(a, b, \ldots) d\tau$, of the work relative to the body elements taken separately.[8]

Donati developed this thesis in his following papers of 1889 and 1894; in particular, in the last one he thoroughly presented the various statements of stationarity of various energetic expressions, as remarked also in the known historical monograph by Benvenuto [3, vol. 2, p. 508]. In addition, Donati proved Menabrea's theorem rigorously and in an elegant way, with no necessity to resort to energetic considerations, and put into clear evidence the ambiguities present in its original statement. He pointed out that Menabrea always referred, either explicitly or implicitly, to structural systems composed of simple assemblies of bars and beams, for each of which the inner compatibility of displacement and strain is granted. On the other hand, nothing is said about more general

6 Canevazzi [5], p. 109; source in "Appendix 1".

7 Canevazzi [5], p. 378; source in "Appendix 1".

8 Donati [22], p. 363; source in "Appendix 1".

systems, for which this inner compatibility is not assured in general:

> The fact that the expression of [strain] work Π, due to the constitution of articulated systems, spontaneously appears as the sum of terms relevant to the single composing parts taken separately, explains why, even though applications were possible, the true sense of the statement [of Menabrea's theorem], which has at its basis the consideration that the strain of the parts may be regarded as independent, was generally not felt, or misunderstood.
> Actually, it is useful to stress, without this the statement [of Menabrea's theorem] loses its meaning. Indeed, if we consider the system in its constitution as a whole continuum, its state for given stress fluxes (or its equilibrium state under the action of given external forces) is fully determined and unique, and thus no variation of its elements is compatible anymore, and one cannot talk of minimum work with respect to the given values of stress fluxes (or external forces) anymore.[9]

The ambiguity in dealing with complementary energy was fully eliminated by Gustavo Colonnetti, an important figure of the Italian school of elasticity applied to constructions of the first half of twentieth century. He graduated in engineering and mathematics in Turin, where he taught Mechanics applied to constructions, Rational and superior mechanics, Strength of materials. He also was director of the Polytechnic of Turin, of the local Laboratory of strength of materials and of the Institute of Italian dynamic measurements (*Istituto dinamometrico italiano*). He is known for the formulation of a reciprocity theorem in elasticity (similar to Betti's) and for investigations on elastic co-actions and elastic–plastic equilibrium. In his monograph of 1912 [15], he quoted people involved in the solution of elastic system by means of elastic energies, Donati included. After having presented enlightening considerations on the difference between potential and complementary energy, Colonnetti concluded stating that the latter can be given only a very weak mechanical meaning:

> In this proposition, stated first by Menabrea at the R. Academy of Sciences of Turin in 1857 under the name of *principle of elasticity*, or *of minimum work*, we kept the name of strain work for the quantity Φ, even though, when we are not dealing with a real deformation of the body, the same Φ has no physical meaning, nor can be considered as a real increment of energy due to strain.
> In that case we shall then attribute to the expression *strain work* only an abstract meaning of ideal sum of the elastic energies of the body single elements, considered as independent.
> At most, in the applications one can attribute a more concrete physical meaning to the function Φ, by imagining the system suitably divided into a well determined number of parts, and considering variations corresponding to *possible* deformations of each part *separately*; by this, it will represent the sum of the strain energies of the single parts seen as independent.[10]

In a footnote to this quotation, Colonnetti added that

> It is sometimes possible to replace the ideal cuts discussed here with suitable variations of the constraint conditions. All in all, we do not exclude by this other possible physical interpretations of the function Φ. So for instance in the case of trusses with redundant bars it has also been interpreted as the strain work which could actually be produced in the system once the effects of variations of temperature different from bar to bar were superposed to the action of given external forces.[11]

and recalled the well known monograph by Mohr [38] where such an approach, following Engesser's suggestion, was adopted. Colonnetti also remarked that only in a balanced and compatible configuration the work function Φ, or complementary work, or complementary energy, is actually a strain work in the physical sense it is usually provided with. On this purpose, he quoted the already discussed comprehensive monograph by Canevazzi [5], which appeared at the same time as Engesser's [26] and came, more or less, to the same conclusions.

[9] Donati [24], p. 465; source in "Appendix 1".

[10] Colonnetti [15], p. 14; source in "Appendix 1".

[11] Colonnetti [15], p. 14; source in "Appendix 1".

4 Non-linear elastic systems

In the case of non-linear elastic system, which began to be investigated shortly after Castigliano's works, the attempts to obtain the theorem of minimum complementary energy from energetic considerations turned out to be ineffective. This still holds true even when non linearities are related only to constitutive assumptions, as in those considered at the end of nineteenth century, and which we will consider in the present section. This is due not so much to conceptual difficulties in correctly dealing with balance and compatibility conditions, but rather to the fact that the stationarity condition of the considered energy obtained by varying balanced forces does not lead to correct results. Menabrea's theorem, thus, cannot be at once extended to non-linear elasticity, and the first Castigliano's theorem is not valid.

The first who extended Castigliano's theorems to non-linear elasticity was Castigliano's friend Francesco Crotti. He stated that, according to him, the mathematical aspects of Castigliano's work, as well as its extension to non-linear elasticity, were already present in Legendre's treatise on differential equations, where he introduced his transform [32]:

> Let us stop for a while to consider what is, from the scientific point of view, the novelty, the scope, and the usefulness of this theorem of the derivatives of work, and of the other, we may say its twin, of minimum work. Well then, these theorems, if considered from the point of view of the general theory, do not constitute substantial new statements. Legendre had already proved that, given a function ϕ of n variables x, one can form by its partial derivatives a function ψ the partial derivatives of which are equal to the variables x, respectively. It had also been recognized that, if ϕ is quadratic, it turns out that $\phi = \psi$. Later on, the famous English mathematician George Green was lead, by considerations on the impossibility of perpetual motion, to establish that the work of an elastic system was represented by a potential of the displacements, and this in the two illustrious memoirs on light of 1839. The analytical background expressing the properties of the two theorems of which I talk was, then, completely known; I do not believe, however, that they have been formally stated, maybe perhaps they did not concur to the progress of the general theory, which, by the considerations on the displacements, comes to use the same formulas to which those two theorems lead.[12]

Crotti used the term 'work function' to introduce the elastic potential energy of both linear and non-linear conservative systems, expressed in terms of either displacements $u_1, u_2, \ldots, u_n$ [19, p. 60]

$$L = \phi(u_1, u_2, \ldots, u_n)$$

or forces $f_1, f_2, \ldots, f_n$ that produce these displacements:

$$L = \psi(f_1, f_2, \ldots, f_n)$$

Crotti actually dealt with a system of hinged bars (like Menabrea and Castigliano) that in principle could even be non-elastic; in such a system, compatibility is understood, since the nodes common to various bars cannot detach in any admissible configuration. Thus, it seems that also Crotti did not realize the necessity to distinguish between compatibility and balance. He introduced elastic complementary energy without giving it a name, but only denoting it by the symbol λ [13]

$$\lambda = f_1 u_1 + f_2 u_2 + \cdots + f_n u_n - L \tag{19}$$

and proved the extension of Castigliano's first theorem to non-linear cases

$$\partial\lambda/\partial f_1 = u_1, \partial\lambda/\partial f_2 = u_2, \ldots, \partial\lambda/\partial f_n = u_n$$

Crotti remarked that in linear elasticity $\lambda = L$ and one re-obtains Castigliano's results [8–10, 19]. However, in no place did he formulate a theorem of stationarity of his new functional λ; this was the task of the German school of structural mechanics.

Interesting researches on the applications of 'work functions' and the relevant theorems were indeed in German-speaking countries, where between the end of the nineteenth and the beginning of the twentieth century we see them flourish in the works of Fränkel, Engesser, Mohr, Müller-Breslau. Since the first to

[12] Crotti [19], pp. 5–6; source in "Appendix 1".

[13] Because of his quotation to [32], it is apparent that Crotti should know that (19) actually represented the Legendre's transform of L, but in fact he did not explicitly stated this.

introduce the adjective *complementary* for the ‘work function’ was Engesser, we will focus on him. Engesser was actually interested in solving statically redundant systems under non-linear constitutive assumptions:

> The following considerations apply to the behaviour of statically undetermined beams under any constitutive law; in addition, statically undetermined trusses will be more easily dealt with.
> For the solution of the given task the theorem of virtual displacements offers the most easy and sure way, while the theorem of minimum strain work appears inadequate, since its validity is bound to given constitutive laws. In its place enters the theorem of minimum “complementary work”.[14]

In a truss with m redundant bars, by removing these it is possible to obtain a statically determined auxiliary truss. A simple application of the theorem of virtual displacements, or velocities, in the case of smooth constraints let Engesser write the m compatibility conditions [26, col. 734]

$$0 = \sum_j \varsigma_j^i e_j = \sum_j \frac{\partial S_j}{\partial X^i} e_j \tag{20}$$

to solve the system. In Eq. (20) X^i, $i = 1, \ldots, m$ are the values of the forces in the m redundant bars; ς^i are the values of the forces in the bars of the auxiliary truss is subjected to a unit action of the i-th redundant bar; S_j is the actual force in the j-th bar of the auxiliary system; and e_j is its actual elongation, provided by a constitutive law whatsoever, which Engesser provided according to the sum of a non-linear elastic and a linear thermal law. Engesser then introduced complementary work without caring of describing its possible physical meaning:

> The difference between the virtual work A_v and the real work A is called complementary work.[15]

For Engesser, the virtual work is that spent by the actual force in a bar if it acted with its final value on the corresponding elongation, while the actual work is the area under the graph of the generally non-linear constitutive law; the situation is the same as depicted in Fig. 2, and Engesser obtained [26, eq. (10), col. 739]

$$\mathrm{O} = \frac{\partial B}{\partial X_i} = \sum \frac{\partial}{\partial X_i} \int_0^S dS \cdot e = \sum \frac{\partial S}{\partial X_i} \cdot e \tag{21}$$

where the left hand side vanishes since the right hand side coincides with the compatibility conditions in eq. (20). Thus, after some remarks on the generality on the constitutive law expressing the actual value of the elongations in Eq. (21), Engesser stated that

> The redundant quantities X of a statically undetermined truss assume those values that let the complementary work of the whole construction [truss and ‘ground’] attain a minimum value.[16]

This is a general theorem, and by this Engesser managed to obtain Castigliano’s and Menabrea’s results, as well as those reciprocity conditions usually attributed to Maxwell and Betti [7]. It is apparent, however, that Engesser was not interested in the interpretation of the quantity he introduced: it was simply pivotal for his aim. The same attitude was shared by other German scientist of the theory of elasticity later on, see for instance Mohr [38] and Domke [21]. Thus, the question on whether complementary work-energy has a physical meaning was not tackled by those who first explicitly introduced it, and remained open.

5 Final remarks

A modern scholar of mechanics of continua and of structures has no difficulties in employing both potential (Φ) and complementary energy (Ψ), since each provides a means of posing and solving the elastic problem for objects of interest in applications. Indeed, many modern computer codes find numerical solutions of the elastic problem by suitable variations of the one energy or the other, depending simply on computing costs and convergence criteria, and not caring too much on the effective mechanical meaning of the manipulated quantity. Thus, one may say that from the point of view of the application, there is no

[14] Engesser [26], col. 733; source in “Appendix 1”.

[15] Engesser [26], col. 738; source in “Appendix 1”.

[16] Engesser [26], col. 740; source in “Appendix 1”.

debate on the physical interpretation of complementary energy.

Things are different if we want a mechanical interpretation for Φ and Ψ, as in the nineteenth century (when these quantities and their minimum theorems were introduced), or if we take energy, not force, as the fundamental magnitude of mechanics. Actually, Φ and Ψ, together with their relevant stationarity theorems, are given by well-defined mathematical objects and transformations: thus, from a purely logical point of view they are equivalent. However, they are not liable to the same mechanical interpretation: there is difference when considering the linear or non-linear elastic cases, and the values of the two energies in a balanced and kinematically admissible state or in another situation.

The pioneers in applying energetic approaches for solving statically undetermined structural problems soon realized that the elastic energy U could not be used for a minimum criterion, because the minimum in the space of displacements compatible with the strains constitutively linked with inner actions is always zero. Indeed, U is independent of the external load $\boldsymbol{f}$, which is a fundamental datum of the problem; thus, U seemed useless and, moreover, the expression of $U(\boldsymbol{u}, \boldsymbol{E})$ may be quite complex. On the contrary, the complementary energy $V(\boldsymbol{S}, \boldsymbol{s})$ easily incorporates such information, since stresses shall balance the loads $\boldsymbol{f}$. Thus, minimizing $V(\boldsymbol{S}, \boldsymbol{s})$ in the space of stresses balanced with given external forces quite likely seemed most natural for the structural engineers of the nineteenth century, who did exactly the opposite of what is done nowadays. The minimum was easily carried out by assuming as unknowns the redundant forces, usually ranging in a small set.

In the linear elastic case, treated first in the mechanics of structures, for example by Menabrea and Castigliano, potential and complementary elastic energy are the same thing, at least in the balanced and kinematically admissible configuration. This coincidence was the origin of a series of misunderstandings that led to the correct formulation of the theorem of stationarity of total complementary energy by inconsistent reasonings. The misunderstandings arose because of a not clear perception of the difference when considering virtual variations of Φ and Ψ. Indeed, it was natural to assume for Φ the space of admissible displacements and for Ψ that of balanced stresses. In general the two spaces are however distinct, therefore also Φ and Ψ, albeit formally identical, are in fact different. One cannot then move from the theorem of stationarity of Φ to that of Ψ, so naïvely as done by Cerruti and Canevazzi for example.

The variations of Φ with the admissible displacements are still provided with the mechanical meaning of a possible potential energy, and we can even imagine of performing a real or ideal experiment in which this potential energy is used to provide work. On the other hand, it is not possible to give any mechanical meaning to the variations of Ψ with balanced stresses: indeed, kinematical compatibility can be violated in statically indeterminate systems, therefore it is impossible to imagine any possible geometry of the system, and consequently any experiment, even ideal.

In the non-linear case there is a clear difference between Φ and Ψ. They are complementary in the sense defined by Engesser: the introduction of a quantity and a specific name appeared necessary since there were difficulties to distinguish between balanced and kinematically admissible configurations. We can only use Ψ to solve structural problems using stresses as unknowns.

The physical meaning of Ψ is not clear, not even in equilibrated and kinematically admissible configurations, and in general it is not possible to give it one, at least remaining in a purely mechanical context. To our knowledge, the only attempts to provide complementary energy and its theorem of stationarity a mechanical meaning are found in the paper of Donati and Colonnetti, who concluded for no mechanical meaning, and in the textbook of Vincenzo Franciosi (1925–1989) [27][17] who, however, referred to a very particular load situation-thus, his conclusions have no general value.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

[17] Vol. 2, pp. 281–284.

Appendix 1: Sources

1. Lorsqu'un système élastique se met en équilibre sous l'action de forces extérieures, le travail développé par l'effet des tensions ou des compressions des liens qui unissent les divers points du système est un minimum.
2. Se determino le tensioni T_{pq} in modo che rendano minima l'espressione $\sum T_{pq}^2/\epsilon_{pq}$, supponendo che tra quelle tensioni debbano aver luogo le equazioni [di bilancio meccanico], nelle quali però si considerano costanti tutte le forze esterne X_p, Y_p, Z_q, e tutti gli angoli $\alpha_{pq}, \beta_{pq}, \gamma_{pq}$, i valori delle tensioni che così si ottengono, coincidono con quelli ottenuti con il metodo degli spostamenti.
3. Première partie - Si l'on exprime le travail de déformation d'un système articulé, en fonction des *déplacements relatifs* des forces extérieures appliquées à ses sommets, on obtient une formule, dont les dérivées, par rapport à ces déplacements, donnent la valeur des forces correspondantes.
   Seconde partie - Si l'on exprime, au contraire, le travail de déformation d'un système articulé en fonction des forces extérieures on obtient une formule, dont les dérivées, par rapport à ces forces, donnent les *déplacements relatifs* de leurs points d'application.
4. A questo risultato non si sarebbe potuti arrivare qualora i vincoli imposti al sistema fossero variabili o le reazioni corrispondenti producessero lavoro, come appunto potrebbe accadere se alcuni punti fossero obbligati a mantenersi sopra superfici presentanti una resistenza d'attrito al loro movimento.
5. Se le reazioni incognite sono tali da dar luogo a un lavoro esterno [...] allora il teorema di Menabrea non ha più luogo e per determinare le forze e le reazioni incognite bisogna ricorrere al metodo delle derivate del lavoro o a quello delle deformazioni svolti nel capitolo precedente.
6. Allora però non si può parlar più di spostamenti dei punti del corpo nel suo insieme, né il lavoro di deformazione può più riguardarsi come funzione degli spostamenti o degli elateri, ma semplicemente come la somma o integrale $\int \Phi(a, b, \ldots) d\tau$ dei lavori relativi agli elementi del corpo considerati partitamente.
7. La circostanza che l'espressione del lavoro Π, in grazia della costituzione dei sistemi considerati, si presentava spontaneamente quale somma di termini spettanti alle singole parti costituenti distintamente prese, spiega perché, pur facendone l'applicazione, il senso vero della proposizione, che ha per base la considerazione delle deformazioni delle parti riguardate come indipendenti, passasse generalmente inavvertito o fosse frainteso.
   Perché, giova insistervi, senza di ciò la proposizione perde ogni significato. Infatti considerato il sistema colla sua compagine come un tutto continuo, il suo stato per dati elateri (o lo stato di equilibrio sotto l'azione di date forze esterne) è pienamente determinato ed unico, e quindi nessuna variazione degli elementi ad esso relativi è più compatibile, e non si può perciò parlare di minimo dei lavoro compatibilmente coi dati valori degli elateri (o delle forze esterne).
8. In questa proposizione, annunciata per la prima volta dal Menabrea alla R. Accademia delle Scienze di Torino nel 1857, sotto il nome di *principio di elasticità* o *del minimo lavoro*, noi abbiamo conservato il nome di lavoro di deformazione alla quantità Φ sebbene, quando non si tratta di una vera e propria deformazione del corpo, essa Φ non abbia più alcun significato fisico, né possa riguardarsi come un reale incremento di energia dovuto alla deformazione.
   All'espressione *lavoro di deformazione* dovrà in tal caso attribuirsi soltanto un significato astratto di somma ideale delle energie elastiche dei singoli elementi del corpo considerati come indipendenti.
   Tutto al più si potrà attribuire, nelle applicazioni, un significato fisico più concreto alla funzione Φ immaginando il sistema convenientemente diviso in un certo numero ben determinato di parti, e considerando delle variazioni corrispondenti a deformazioni possibili di ciascuna parte presa separatamente; con ciò verrà a rappresentare la somma delle

energie di deformazione possedute dalle singole parti riguardate come indipendenti.

9. Ai tagli ideali, a cui qui si allude, possono, a volte, sostituirsi opportune variazioni nelle condizioni di vincolo. Non si escludono del resto con ciò altre eventuali interpretazioni fisiche della funzione Φ. Così per es.: nel caso di travature reticolari ad aste sovrabbondanti essa è stata anche interpretata come il lavoro di deformazione che nel sistema potrebbe effettivamente prodursi qualora all'azione delle forze esterne date si sovrapponessero gli effetti di variazioni di temperatura diverse da asta ad asta.
10. Arrestiamoci alquanto a considerare quale sia dal punto di vista scientifico, la novità, la portata e la utilità di questo teorema delle derivate del lavoro e dell'altro, che si può dire gemello, del minimo lavoro. Or bene questi teoremi se bene si considerano dal punto di vista della teoria generale non costituiscono enunciati essenzialmente nuovi. Già Legendre aveva dimostrato che data una funzione ϕ di n variabili x, si può formare colle sue derivate parziali una funzione ψ le di cui derivate parziali sono rispettivamente eguali alle variabili x. Era anche stato riconosciuto che se la ϕ è funzione quadratica, risulta $\phi = \psi$. Più tardi l'illustre matematico inglese Giorgio Green da considerazioni sull'impossibilità del moto perpetuo fu condotto a stabilire che il lavoro di un sistema elastico era rappresentato da un potenziale degli spostamenti, e ciò nelle due celebri memorie sulla luce del 1839. Era quindi completamente noto il substrato analitico che esprime la proprietà dei due teoremi di cui discorro; non credo però che siano mai stati formalmente enunciati forse perché in fondo non occorrevano al progresso della teoria generale, la quale colle considerazioni degli spostamenti viene a far uso delle stesse formole a cui quei due teoremi conducono.
11. Die folgenden Betrachtungen beziehen sich auf das Verhalten, statisch unbestimmter Träger bei beliebigem Formänderungs-Gesetze; insbesondere werden statisch unbestimmte Fachwerkträger einer eingehenderen Behandlung unterzogen.
Zur Lösung der gestellten Aufgabe bietet der Satz der virtuellen Verschiebungen den bequemsten und sichersten Weg, während der Satz von der kleinsten Formänderungsarbeit sich als unzulänglich erweist, da seine Gültigkeit an bestimmte, Formänderungs-Gesetze gebunden ist. An seine Stelle tritt der allgemeinere Satz von der kleinsten "Ergänzungsarbeit".
12. Der Unterschied zwischen virtuller Arbeit A_v und wirklicher Arbeit A werde Ergänzungsarbeit genannt.
13. Die überzahligen Größen X eines statisch unbestimmten Fachwerkes nehmen diejenigen Werthe an, welche die Ergänzungsarbeit der gesammten Konstruktion zu einem Kleinstwerthe machen.

Appendix 2: Symbols used in Sect. 2

Symbol	Name
$\boldsymbol{b}_o$	Assigned volume load
$\boldsymbol{f}$	Load system
$\boldsymbol{n}$	Outward unit normal to $\partial\mathcal{B}$
$\boldsymbol{s}$	Traction vector
$\boldsymbol{s}_o$	Assigned surface traction
$\boldsymbol{u}$	Displacement field
$\boldsymbol{u}_o$	Assigned displacement
$\boldsymbol{E}$	Strain tensor
$\boldsymbol{S}$	Stress tensor
f	One-dimensional external load
F	Load potential
G	Displacement potential
U	Elastic potential energy
V	Elastic complementary energy
$\mathcal{B}$	Continuous body
$\partial\mathcal{B}$	Boundary of $\mathcal{B}$
$\mathcal{L}$	Total external potential
$\mathcal{W}^e$	External work
$\mathcal{W}^i$	Internal work
$\mathscr{K}$	Space of kinematically admissible displacement and strains
$\mathscr{S}$	Space of statically admissible stresses
$\mathfrak{B}_o, \mathfrak{B}_1$	Boundary operators
$\mathfrak{C}$	Constitutive operator

Symbol	Name
$\mathfrak{D}$	Compatibility operator
$\mathfrak{S}$	Equilibrium operator
ε	One-dimensional strain measure
σ	One-dimensional stress measure
Φ	Total potential energy
Ψ	Total complementary energy

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