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On temperature and stresses in a thermoelastic half-space with temperature dependent properties

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Abstract The paper deals with the axisymmetric problem of the thermoelastic half-space with temperature dependent properties. The thermal coefficients: heat conductivity and coefficient of linear expansion are assumed to be functions of temperature. The mechanical properties: Young modulus and Poisson ratio are taken into account as constants. Two cases of boundary conditions are considered: a normal heat flux acting on a circle with given radius and two variants of the boundary conditions on the outside of the heated region: (1) a thermal insulation, or (2) a constant temperature, taken as reference. The boundary is assumed to be free of mechanical loadings. The linear dependences of thermal properties on temperature is considered as a special case. The obtained exact results are presented in the forms of multiple integrals and the detailed analysis are derived for linear dependences of the thermal properties on temperature.

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1 Introduction

Nonhomogeneous materials, whose material properties vary continuously, have received considerable technical interest in the engineering applications. The design of elements of structures, machines subjected to extremely high thermal loadings should consider changes of material properties under temperatures. The solids, which in the isothermal state are characterized by constant thermal and mechanical parameters, can be treated as homogeneous bodies, but if they are subjected to high thermal loadings then their properties are dependent on temperature and indirectly vary continuously with respect to spatial variables and time. The thermoelasticity of bodies with temperature dependent properties was developed by Nowiński [1–4]. The monograph [4] includes some wide scientific descriptions of the author's results as well as other investigators. The papers [5, 6] deal with the problems of stress distributions in the thermoelastic plate with temperature dependent properties weakened by a Griffith crack. The problem of stress distributions in an elastic layer with temperature dependent properties caused by concentrated loads is considered in [7]. The review on thermal stresses in materials with temperature dependent properties for papers published after

1980 is presented in [8]. The problems of an annular cylinder based on the finite element method is solved in [9]. The paper [10] deals with the problem of SH harmonic wave propagation in an elastic layer whose shear modulus and mass density are linearly dependent on temperature. In the paper [11] the wave fronts propagated in thermoelastic bodies with temperature dependent properties are analysed. Some problems of thermoelasticity for thermosensitive bodies are investigated in papers [12–15]. The authors assumed that the considered problems are axisymmetric or point-symmetrical, so it is useful to introduce the cylindrical or spherical coordinates and to reduce the dimensions of the boundary value problems. Boundary value problems of thermoelasticity with both thermal and mechanical properties dependent on temperature are rather too complicated for analytical approaches in the two-dimensional or three-dimensional cases. So, in the paper [12] the stresses caused by thermal loadings in a layer with only mechanical properties dependent on temperature are investigated.

In this paper the axisymmetrical problem of thermal loadings of an elastic half-space with temperature dependent thermal properties is considered. The mechanical properties are assumed to be independent of temperature (Young modulus and Poisson ratio are taken into account as constants). The elastic half-space is heated by a given normal heat flux on a circle and two cases of boundary conditions on the outside of the heated region: (1°) a thermal insulation, or (2°) a zero temperature, are investigated. The boundary is assumed to be free of mechanical loadings. The considered problem is stationary and axisymmetric. The problem is solved for arbitrary given a priori functions dependent on temperature being the thermal conductivity and coefficient of linear expansion. The linear dependences of thermal properties on temperature is analysed as a special case. The obtained numerical results are presented in the form of figures for both boundary cases. The influence of parameters that determine the thermal properties of the half-space on the stress distributions on the boundary is investigated.

2 Formulations of the problems

Consider a thermoelastic half-space with temperature dependent thermal coefficients and mechanical coefficients being constants. Let (r, φ, z) denote the cylindrical coordinate system, such that the plane $z = 0$ is the boundary surface of the half-space $z > 0$. Let T denote the temperature and $\mathbf{q} = (q_r, q_\varphi, q_z)$ denote the heat flux vector. Let K and α be the thermal conductivity and the linear expansion coefficients, respectively. The mechanical properties will be denoted as follows: E be Young modulus, ν be Poisson ratio. In the paper the thermal and mechanical properties will be taken into account in the form:

$$K(T) = K_0 f(T), \quad \alpha(T) = \alpha_0 g(T), \quad E = const., \quad \nu = const., \tag{2.1}$$

where K_0, α_0 are constants being the thermal properties of the body in the reference temperature. The functions $f(T), g(T)$ are a priori given functions describing changes of thermal properties under influence of temperature. The functions are determined experimentally and are dependent on the kind of materials [16, 17].

The half-space is heated by a normal heat flux on the circle with given radius a dependent only on variable r and two cases of the boundary conditions on the outside of heated region are considered:

(1°) a thermal insulation, or
(2°) zero temperature.

Moreover, the half-space is assumed to be free of mechanical loadings. The considered problems are stationary and axisymmetric, independent on φ and from the boundary conditions and symmetry of equation it follows that $q_\varphi = 0$. The two following cases of the thermal boundary conditions will be taken into account:

Problem 1

$$q_z(r, 0) = q_0\, q^*(r), \text{ for } r < a \text{ and } q_z(r, 0) = 0 \text{ for } r \geq a, \tag{2.2}$$

where $q^*(\cdot)$ is a given function, q_0 a given constant. Moreover, the condition $q_r(r, 0) = 0$, $q_\varphi(r, 0) = 0$ that correspond to normal flux vector are considered.

Problem 2

$$q_z(r, 0) = q_0\, q^*(r), \text{ for } r < a, \quad \text{and } T(r, 0) = 0, \text{ for } r \geq a. \tag{2.3}$$

The solutions of both problems should satisfy the condition at infinity

$$T(r,z) \to 0 \quad \text{for } r^2 + z^2 \to \infty. \tag{2.4}$$

Denote by $\mathbf{u}(r,z) = (u_r, 0, u_z)$ the displacement vector and by $\boldsymbol{\sigma}(r,z)$ the stress tensor with nonzero components $\sigma_{rr}, \sigma_{\varphi\varphi}, \sigma_{zz}, \sigma_{rz}$. The boundary plane is assumed to be free of loadings, so the mechanical boundary conditions can be written:

$$\sigma_{rz}(r,0) = 0, \; \sigma_{zz}(r,0) = 0, \quad r \geq 0. \tag{2.5}$$

The regularity conditions at infinity take the form:

$$\boldsymbol{\sigma}(r,z) \to 0 \text{ for } r^2 + z^2 \to \infty. \tag{2.6}$$

The temperature T and displacements u_r, u_z besides the thermal and mechanical boundary conditions and the conditions at infinity should satisfy the following equations of thermoelasticity [4]:

(a) the stationary equation of heat conduction

$$\frac{1}{r}\frac{\partial}{\partial r}\left(K(T) r \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(K(T)\frac{\partial T}{\partial z}\right) = 0,$$
$$r \geq 0, \quad z > 0, \tag{2.7}$$

and

(b) the equilibrium equations

$$2(1-\nu)D_1^2 u_r + (1-2\nu)\frac{\partial^2 u_r}{\partial z^2} + \frac{\partial^2 u_z}{\partial r \partial z}$$
$$= 2(1+\nu)\frac{\partial}{\partial r}\int_0^T \alpha(\vartheta)d\vartheta, \quad r \geq 0, \quad z > 0,$$
$$(1-2\nu)D_0^2 u_z + 2(1-\nu)\frac{\partial^2 u_z}{\partial z^2} + \frac{\partial}{\partial z}D u_r$$
$$= 2(1+\nu)\frac{\partial}{\partial z}\int_0^T \alpha(\vartheta)d\vartheta, \quad r \geq 0, \quad z > 0, \tag{2.8}$$

where ν is Poisson's ratio and

$$D_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}, \quad D_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r},$$
$$D = \frac{\partial}{\partial r} + \frac{1}{r}. \tag{2.9}$$

3 Solutions and analysis of results

First, the temperature T satisfying Eq. (2.7) with the boundary conditions (2.2) and (2.4) (for Problem 1) or (2.3) with (2.4) (for Problem 2) should be determined. For this aim to a linearization of the considered problems the integral Kirchhoff's transform will be applied (see [22])

$$\Psi = \int_0^T \frac{K(\vartheta)}{K_0} d\vartheta. \tag{3.1}$$

Substituting (3.1) into (2.7) the thermal potential Ψ should satisfy the linear partial differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) + \frac{\partial^2 \Psi}{\partial z^2} = 0. \tag{3.2}$$

Because the components of heat flux q_r, q_z are expressed by the potential Ψ as follows

$$q_r = -K\frac{\partial T}{\partial r} = -K_0\frac{\partial \Psi}{\partial r},$$
$$q_z = -K\frac{\partial T}{\partial z} = -K_0\frac{\partial \Psi}{\partial z}, \tag{3.3}$$

the boundary conditions (2.2)–(2.4) can be rewritten in the form:

Problem 1

$$-K_0\frac{\partial \Psi(r,0)}{\partial z} = q_0 q^*(r) H(a-r), \tag{3.4}$$

and

Problem 2

$$-K_0\frac{\partial \Psi(r,0)}{\partial z} = q_0 q^*(r),$$
$$\text{for } 0 \leq r < a, \; \Psi(r,0) = 0, \quad \text{for } r > a, \tag{3.5}$$

with the condition at infinity

$$\Psi(r,z) \to 0, \quad \text{for } r^2 + z^2 \to \infty. \tag{3.6}$$

The boundary value problems for potential Ψ take the same form as for the well-known problem of temperature in the case of linear theory of heat conduction [19]. The solution of Problem 1 takes the form

$$\Psi(r,z) = \frac{q_0}{K_0}\int\limits_0^\infty \bar{q}^*(s)e^{-sz}J_0(sr)ds, \tag{3.7}$$

where

$$\bar{q}^*(s) = \int\limits_0^a rq^*(r)J_0(sr)dr. \tag{3.8}$$

Problem 2 is the well-known mixed boundary value problem which can be reduced to dual integral equations and next, to the Abel integral equation [20]. The final solution for potential Ψ is given by

$$\Psi(r,z) = \int\limits_0^\infty A(s)e^{-sz}J_0(sr)ds, \tag{3.9}$$

where

$$A(s) = \int\limits_0^a g(t)\sin(st)dt, \tag{3.10}$$

and

$$g(t) = \frac{2}{\pi}\frac{q_0}{K_0}\int\limits_0^t \frac{xq^*(x)dx}{\sqrt{t^2-x^2}}. \tag{3.11}$$

The displacements u_r, u_z should satisfy Eqs. (2.8) together with conditions (2.5) and (2.6). The problem for displacements is linear, so the solution can be written in the form

$$\begin{aligned} u_r(r,z) &= u_r^e(r,z) + u_r^{th}(r,z),\\ u_z(r,z) &= u_z^e(r,z) + u_z^{th}(r,z). \end{aligned} \tag{3.12}$$

where u_r^e, u_z^e are the components of displacement vector for the problem of elasticity (under assumption that the temperature is zero—general solution) and u_r^{th}, u_z^{th} are the displacements being a special solution of Eq. (2.8).

The general solution of the homogeneous equations [Eq. (2.8) with the right hand side equals zero] takes the form [18, p. 40]:

$$\begin{aligned}
2u_r^e(r,z) &= -\int\limits_0^\infty \{(2+d_1-d_1sz)a_1(s)+2a_2(s)s\}J_1(sr)\\
&\quad\times\exp(-sz)ds,\\
2u_z^e(r,z) &= \int\limits_0^\infty \{d_1za_1(s)-2a_2(s)\}sJ_0(sr)\exp(-sz)ds,\\
\frac{\sigma_{rr}^e(r,z)}{\mu} &= -\int\limits_0^\infty \{(2d_1+1-d_1sz)a_1(s)+2a_2(s)s\}sJ_0(sr)\\
&\quad\times\exp(-sz)ds+\frac{1}{r}\int\limits_0^\infty \{(2+d_1-d_1sz)a_1(s)\\
&\quad+2a_2(s)s\}J_1(sr)\exp(-sz)ds,\\
\frac{\sigma_{\varphi\varphi}^e(r,z)}{\mu} &= \int\limits_0^\infty \{(1-d_1)a_1(s)\}sJ_0(sr)\exp(-sz)ds+\\
&\quad-\frac{1}{r}\int\limits_0^\infty \{(2+d_1-d_1sz)a_1(s)+2a_2(s)s\}\\
&\quad\times J_1(sr)\exp(-sz)ds,\\
\frac{\sigma_{zz}^e(r,z)}{\mu} &= \int\limits_0^\infty \{(1-d_1sz)a_1(s)+2a_2(s)s\}sJ_0(sr)\\
&\quad\times\exp(-sz)ds,\\
\frac{\sigma_{rz}^e(r,z)}{\mu} &= \int\limits_0^\infty \{(1+d_1-d_1sz)a_1(s)+2a_2(s)s\}sJ_1(sr)\\
&\quad\times\exp(-sz)ds,
\end{aligned} \tag{3.13}$$

where $d_1 = \frac{1}{1-2\nu}$, μ—shear modulus, and $J_0(\cdot)$, $J_1(\cdot)$ are the Bessel functions of first kind, $a_1(s)$, $a_2(s)$ are unknowns which will be determined from mechanical boundary conditions (2.5).

To obtain a special solution of Eqs. (2.6) the following thermoelastic potential Φ is introduced [19]:

$$u_r^{th} = \frac{\partial\Phi}{\partial r},\quad u_z^{th} = \frac{\partial\Phi}{\partial z}. \tag{3.14}$$

The following relations for the stress tensor components and potential Φ can be written

$$\sigma_{zz}^{th}(r,z) = -2\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi(r,z)}{\partial r}\right),$$
$$\sigma_{rz}^{th}(r,z) = 2\mu\left(\frac{\partial^2 \Phi(r,z)}{\partial r \partial z}\right). \tag{3.15}$$

Substituting (3.14) into Eqs. (2.8) we obtain

$$\Delta\Phi(r,z) = \frac{1+\nu}{1-\nu}\int_0^T \alpha(\vartheta)d\vartheta. \tag{3.16}$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

Special solution of Eq. (3.16) takes the form

$$\Phi(r,z) = \frac{1+\nu}{1-\nu}\int_0^\infty J_0(sr)ds\int_z^\infty \bar{T}^*(s,\xi)\sinh[s(\xi - z)]d\xi, \tag{3.17}$$

where $\bar{T}^*(s,\xi)$ is the Hankel transform of the zero order of function $\int_0^T \alpha(\vartheta)d\vartheta$, so

$$\bar{T}^*(s,\xi) = \int_0^\infty xJ_0(sx)dx \int_0^{T(x,\xi)} \alpha(\vartheta)d\vartheta. \tag{3.18}$$

Knowing potential Φ displacements u_r^{th}, u_z^{th} being the special solution of Eq. (2.6) can be determined by using (3.14) and (3.15). Substituting obtained radial and normal displacements u_r^{th}, u_z^{th} into (3.12) and using (3.13)–(3.15) and (3.17) from the boundary conditions (2.5) we obtain the unknown functions $a_1(s)$, $a_2(s)$ which are given in the general solution (3.13):

$$a_1(s) = -2(1-2\nu)s\frac{1+\nu}{1-\nu}\int_0^\infty \bar{T}^*(s,\xi)\exp(-s\xi)d\xi,$$
$$a_2(s) = \frac{1+\nu}{1-\nu}\int_0^\infty \bar{T}^*(s,\xi)\{(1-2\nu)\exp(-s\xi) - \sinh(s\xi)\}d\xi. \tag{3.19}$$

The Hankel transforms of the first order in the case of radial displacement u_r and zero order for normal displacement u_z representing the final solution (after summing u_r^e and u_r^{th} as well as u_z^e and u_z^{th}) take the following form

$$\bar{u}_r(s,z) = \frac{1+\nu}{1-\nu}\left\{\exp(-sz)\int_0^z \bar{T}^*(s,\xi)\sinh(s\xi)d\xi + \sinh(sz)\int_z^\infty \bar{T}^*(s,\xi)\exp(-s\xi)d\xi + +(2(1-\nu) - sz\exp(-sz))\int_0^\infty \bar{T}^*(s,\xi)\exp(-s\xi)d\xi\right\},$$
$$\bar{u}_z(s,z) = \frac{1+\nu}{1-\nu}\left\{\exp(-sz)\int_0^z \bar{T}^*(s,\xi)\sinh(s\xi)d\xi - \cosh(sz)\int_z^\infty \bar{T}^*(s,\xi)\exp(-s\xi)d\xi - (1-2\nu+sz)\exp(-sz)\int_0^\infty \bar{T}^*(s,\xi)\exp(-s\xi)d\xi\right\}. \tag{3.20}$$

The displacements u_r, u_z can be obtained from (3.20) by using inverse Hankel transforms of first and zero order, respectively.

In the future analysis we focus considerations on the stresses and displacements on the boundary plane $z = 0$. For this reason from Eq. (3.20) and inverse transforms it follows that

$$u_r(r,0) = 2(1+\nu)\int_0^\infty sJ_1(sr)ds\int_0^\infty \exp(-s\xi)d\xi \int_0^\infty xJ_0(sx)dx\int_0^{T(x,\xi)} \alpha(\vartheta)d\vartheta,$$
$$u_z(r,0) = -2(1+\nu)\int_0^\infty sJ_0(sr)ds\int_0^\infty \exp(-s\xi)d\xi \int_0^\infty xJ_0(sx)dx\int_0^{T(x,\xi)} \alpha(\vartheta)d\vartheta. \tag{3.21}$$

Because $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = 0$ we confine on the calculation of $\sigma_{\varphi\varphi}(r,0)$ and $\sigma_{rr}(r,0)$. Assuming that $\sigma_{zz}(r,0) = 0$ from the constitutive relations [4] for $z = 0$ we have

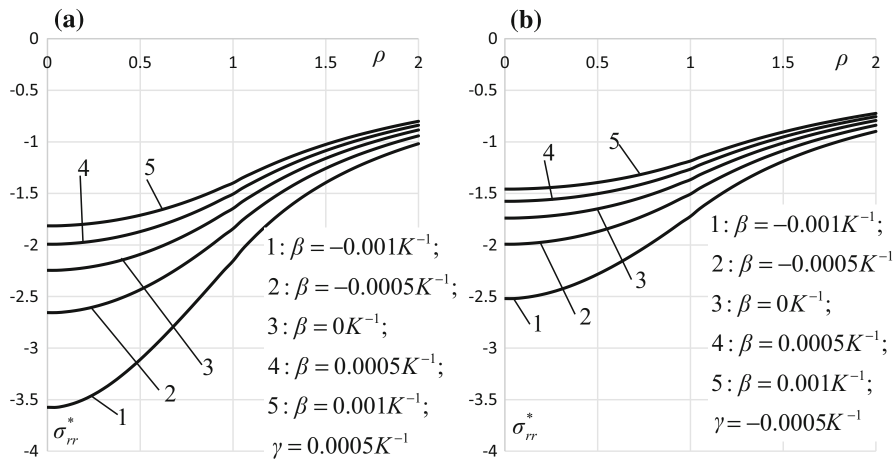


Fig. 1 The dimensionless stress tensor component σ^*_{rr} on boundary surface $z = 0$ as a function of parameter β

$$\frac{1}{2\mu}\sigma_{rr}(r,0) = \frac{1}{1-\nu}\frac{\partial u_r}{\partial r} + \frac{\nu}{1-\nu}\frac{u_r}{r} - \frac{1+\nu}{1-\nu}\int_0^T \alpha(\vartheta)d\vartheta,$$

$$\frac{1}{2\mu}\sigma_{\varphi\varphi}(r,0) = \frac{\nu}{1-\nu}\frac{\partial u_r}{\partial r} + \frac{1}{1-\nu}\frac{u_r}{r} - \frac{1+\nu}{1-\nu}\int_0^T \alpha(\vartheta)d\vartheta. \tag{3.22}$$

From Eq. (3.22) it follows that the stress components $\sigma_{rr}(r,0)$ and $\sigma_{\varphi\varphi}(r,0)$ are based on the displacement u_r. Taking into account Eq. (3.21) and introducing the following notation

$$K(r,x,\xi) = \int_0^\infty sJ_1(sr)J_0(sx)\exp(-s\xi)ds, \tag{3.23}$$

the radial displacement $u_r(r,0)$ can be written in the form

$$u_r(r,0) = 2(1+\nu)\int_0^\infty d\xi \int_0^\infty x\left(\int_0^{T(x,\xi)} \alpha(\vartheta)d\vartheta\right)K(r,x,\xi)dx. \tag{3.24}$$

The integral in Eq. (3.23) is calculated from the relation

$$K(r,x,\xi) = -\frac{\partial}{\partial r}\int_0^\infty J_0(sr)J_0(sx)\exp(-s\xi)ds. \tag{3.25}$$

The integral in (3.25) has the form [21]

$$\int_0^\infty J_0(sr)J_0(sx)\exp(-s\xi)ds = \frac{1}{\sqrt{\xi^2+r^2+x^2}}F\left(\frac{3}{4},\frac{1}{4};1;\frac{4x^2r^2}{\left(\xi^2+x^2+r^2\right)^2}\right), \tag{3.26}$$

where $F(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function.

Substituting (3.26) into (3.25) we obtain

$$K(r,x,\xi) = \frac{r}{\sqrt{\left(\xi^2+r^2+x^2\right)^3}}F\left(\frac{3}{4},\frac{1}{4};1;\frac{4r^2x^2}{\left(\xi^2+r^2+x^2\right)^2}\right) - \frac{3}{2}\frac{rx^2\left(\xi^2+x^2-r^2\right)}{\sqrt{\left(\xi^2+x^2+r^2\right)^7}}F\left(\frac{7}{4},\frac{5}{4};2;\frac{4r^2x^2}{\left(\xi^2+r^2+x^2\right)^2}\right). \tag{3.27}$$

The derivative $\frac{\partial u_r}{\partial r}$ will be calculated numerically. The above presented solutions are derived for arbitrary forms of $\alpha(T)$ and $K(T)$.

4 Special case

In the further analysis and numerical calculations the following coefficients of heat conduction and linear expansion are taken into account:

$$\alpha = \alpha_0(1+\gamma T), \quad K = K_0(1+\beta T), \tag{4.1}$$

where α_0, γ, K_0, β are given constants.

From Eq. (4.1) and (3.1) it follows that

$$\Psi = \int_0^T \frac{K(\vartheta)}{K_0} d\vartheta = T + \frac{\beta T^2}{2}. \tag{4.2}$$

Knowing potential Ψ from Eq. (4.2) we obtain

$$\beta T^2 + 2T - 2\Psi = 0, \tag{4.3}$$

and

$$T = \frac{-1 + \sqrt{1 + 2\beta\Psi}}{\beta}. \tag{4.4}$$

Having temperature and using (4.1) the following integral can be determined

$$\int_0^T \alpha(\vartheta) d\vartheta = \alpha_0 \left(T + \gamma \frac{T^2}{2} \right). \tag{4.5}$$

Remark It can be observed that in the case when

$$\gamma = \beta, \tag{4.6}$$

then

$$\int_0^T \alpha(\vartheta) d\vartheta = \alpha_0 \Psi, \tag{4.7}$$

what it means that the considered case presents the analogical problem to the temperature and stresses distributions for a homogeneous half-space investigated within the framework of the linear theory of thermal stresses with boundary conditions given in (2.2)–(2.6).

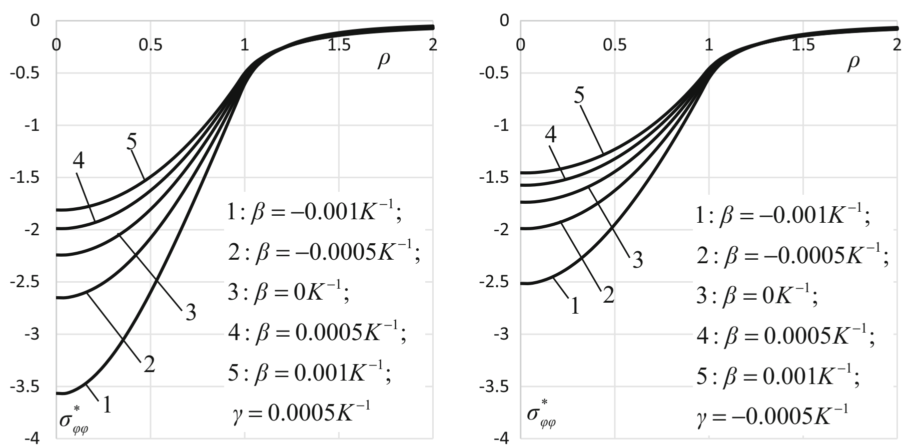


Fig. 2 The dimensionless stress tensor component $\sigma^*_{\varphi\varphi}$ on boundary surface $z = 0$ as a function of parameter β

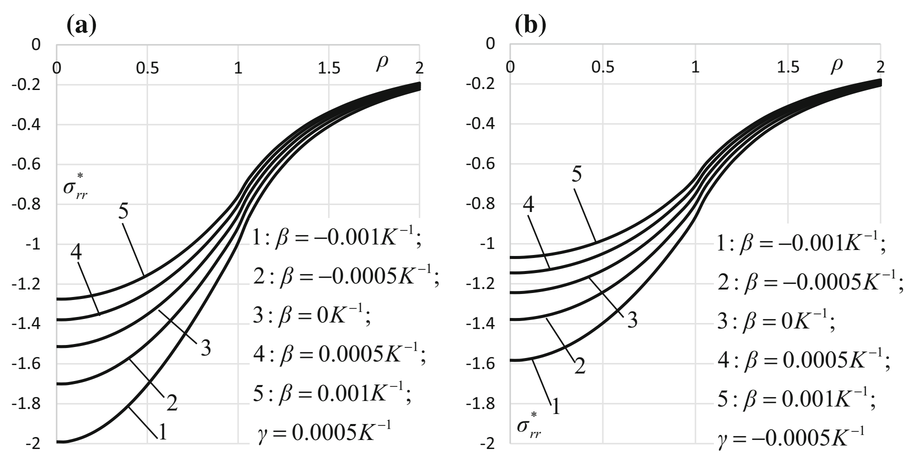


Fig. 3 The dimensionless stress tensor component σ^*_{rr} on boundary surface $z = 0$ as a function of parameter β

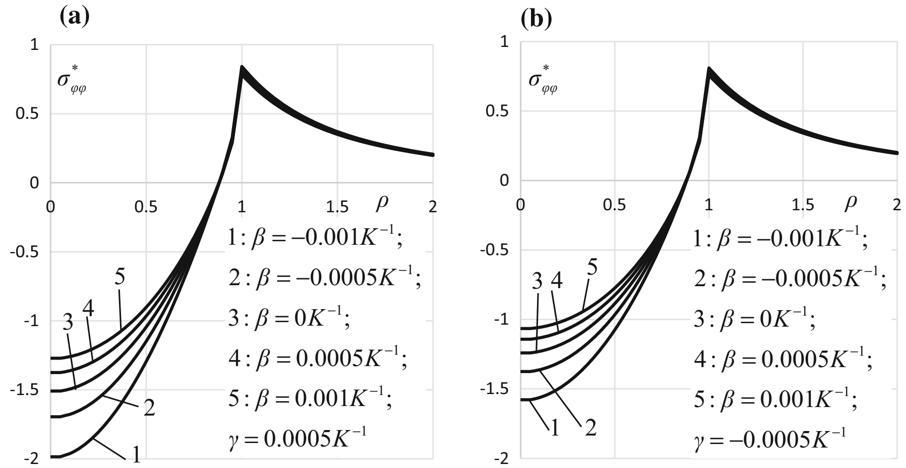


Fig. 4 The dimensionless stress tensor component $\sigma^*_{\varphi\varphi}$ on boundary surface $z = 0$ as a function of parameter β

For further calculations the following heat flux $q^*(r)$ is taken for both problems [boundary conditions (3.4)—Problem 1, and (3.5)—Problem 2]:

$$q^*(r) = \sqrt{1 - \frac{r^2}{a^2}}. \tag{4.8}$$

From Eqs. (4.8) and (3.7) it follows that [21]:

Problem 1

$$\Psi(r,0) = \frac{q_0 a}{K_0}\begin{cases} \frac{\pi}{4}\left(1 - \frac{1}{2}\frac{r^2}{a^2}\right), & r < a \\ \frac{a}{3r}F\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{a^2}{r^2}\right), & r > a, \end{cases} \tag{4.9}$$

and

Problem 2 From Eqs. (3.9)–(3.11) and (4.8), using the following integral [21]:

$$\int_0^t \frac{x\sqrt{a^2 - x^2}}{\sqrt{t^2 - x^2}}dx = \frac{1}{2}\left(at + \frac{a^2 - t^2}{2}\ln\frac{a+t}{a-t}\right), \tag{4.10}$$

we obtain

$$\Psi(r,0) = \frac{q_0}{K_0}\frac{1}{\pi}\int_r^a \left(t + \frac{a^2 - t^2}{2a}\ln\frac{a+t}{a-t}\right)\frac{dt}{\sqrt{t^2 - r^2}}. \tag{4.11}$$

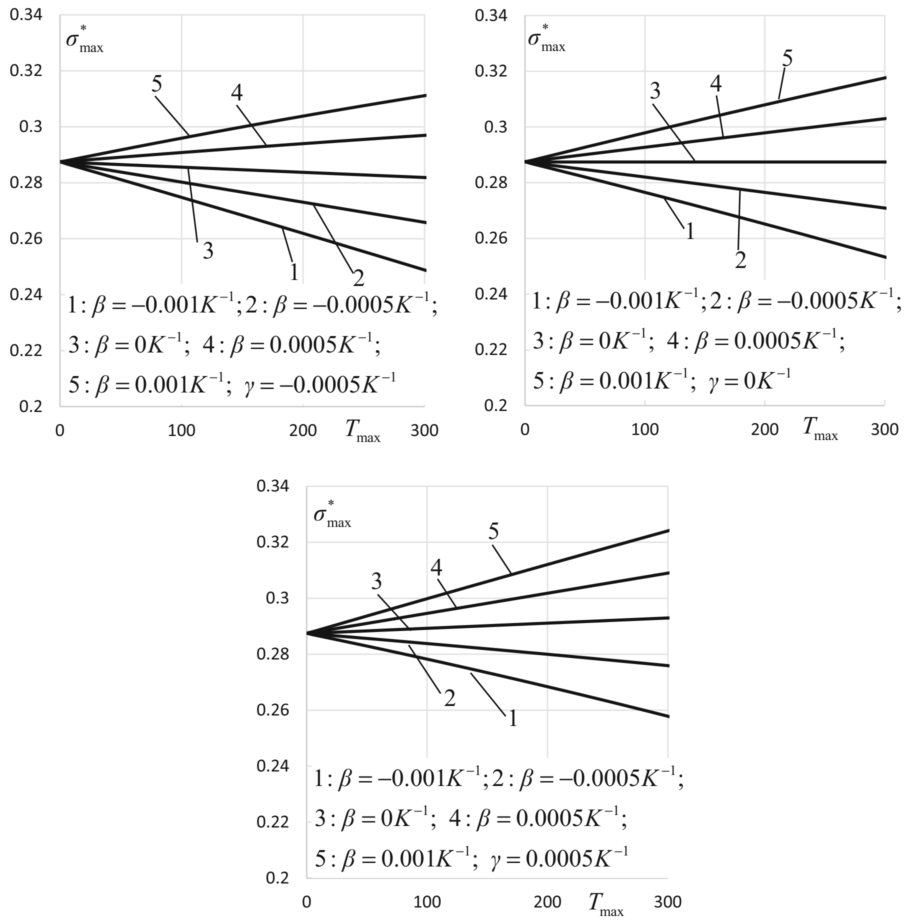
The integral in (4.11) will be calculated numerically by using dimensionless variable $\rho = \frac{r}{a}$, so potential given in (4.11) can rewritten in the form

$$\Psi(\rho,0) = \frac{q_0 a}{K_0}\frac{1}{\pi}\int_\rho^1 \left(t + \frac{1 - t^2}{2}\ln\frac{1+t}{1-t}\right)\frac{dt}{\sqrt{t^2 - \rho^2}},$$
$$\rho = \frac{r}{a}, \tag{4.12}$$

and the following algorithm is applied to separate a singular (logarithmic) part of integral (4.12)

$$\begin{aligned}\frac{1}{\pi}\int_\rho^1 \frac{f(t)dt}{\sqrt{t^2 - \rho^2}} &= \frac{1}{\pi}f(\rho)\int_\rho^1 \frac{dt}{\sqrt{t^2 - \rho^2}} \\ &\quad + \frac{1}{\pi}\int_\rho^1 (f(t) - f(\rho))\frac{dt}{\sqrt{t^2 - \rho^2}} \\ &= \frac{1}{\pi}f(\rho)\left[\ln\left(1 + \sqrt{1 - \rho^2}\right) - \ln\rho\right] \\ &\quad + \frac{1}{\pi}\int_\rho^1 (f(t) - f(\rho))\frac{dt}{\sqrt{t^2 - \rho^2}}.\end{aligned} \tag{4.13}$$

Fig. 5 The maximal dimensionless tensile stress $\sigma^*_{\max} = 100\sigma^*_{\varphi\varphi}(1,0)/T_{\max}$ as a function of maximal temperature $T_{\max} = T(0,0)$



Knowing potential Ψ from Eq. (4.4) we have temperature T, what leads to determination of $\int_0^T \alpha(\vartheta)d\vartheta$ from (4.5). Next, using (3.24), (3.27) and (3.12) after numerical calculations the results obtained for dimensionless stress components $\sigma^*_{rr}(\rho,0)$, $\sigma^*_{\varphi\varphi}(\rho,0)$, where

$$\left(\sigma^*_{rr}, \sigma^*_{\varphi\varphi}\right) = \frac{\left(\sigma_{rr}, \sigma_{\varphi\varphi}\right)}{2\mu(1+v)\alpha_0 100K}, \tag{4.14}$$

are presented in the form of figures.

Further analysis of stresses will be derived numerically. For this aim it can be concluded that the dimensionless stress components are dependent on four parameters $q^*_0 = q_0a/K_0, \beta, \gamma$ and v for calculations it will be taken $v = 0.3$ and $q^*_0 = 500$ (for Figs. 1, 2, 3, 4).

Problem 1 Figure 1a presents the dimensionless stress component σ^*_{rr} on the boundary plane $z = 0$ for $\beta = -0.001;\ -0.0005;\ 0;\ 0.0005;\ 0.001\ \mathrm{K}^{-1}$ and $\gamma = 0.0005\ \mathrm{K}^{-1}$. It can be observed that the values of σ^*_{rr} decrease together with decrease of parameter β. The biggest differences between the values of σ^*_{rr} are in the centre of heating, for $\rho \to \infty$ the values of σ^*_{rr} tend to zero. Figure 1b shows $\sigma^*_{rr}(\rho,0)$ for $\beta = -0.001;\ -0.0005;\ 0;\ 0.0005;\ 0.001\ \mathrm{K}^{-1}$ and $\gamma = -0.0005\ \mathrm{K}^{-1}$. It is seen that for $\beta = -0.001\ \mathrm{K}^{-1}$ we have the smallest values of σ^*_{rr}. Comparing Fig. 1a with Fig. 1b we observe some increase of σ^*_{rr} for the same β and small values of γ.

The dimensionless stress component $\sigma^*_{\varphi\varphi}$ is shown in Fig. 2. Figure 2a presents $\sigma^*_{\varphi\varphi}$ for $\beta = -0.001;\ -0.0005;\ 0;\ 0.0005;\ 0.001\ \mathrm{K}^{-1}$ and $\gamma = 0.0005\ \mathrm{K}^{-1}$, Fig. 2b for $\gamma = -0.0005\ \mathrm{K}^{-1}$. We observe analogical behaviour of $\sigma^*_{\varphi\varphi}$ as σ^*_{rr} in the heating centre. For $\rho > 1$ the differences between the curves for different values of β are very small and $\sigma^*_{\varphi\varphi} \to 0$ for $\rho \to \infty$.

Problem 2 The results for the mixed boundary value problem are presented in Figs. 3 and 4. Figures 3a, b presents dimensionless stress component σ^*_{rr} for $\beta = -0.001;\ -0.0005;\ 0;\ 0.0005;\ 0.001\,\mathrm{K}^{-1}$ and $\gamma = 0.0005\,\mathrm{K}^{-1}$ as well as $\gamma = 0.0005\,\mathrm{K}^{-1}$, respectively. The greater differences between the curves for adequate different values of β are observed in the heating region and $\sigma^*_{rr} \to 0$ for $\rho \to \infty$.

Figures 4a, b shows the dimensionless stress component $\sigma^*_{\varphi\varphi}$ on the boundary plane for $\beta = -0.001;\ -0.0005;\ 0;\ 0.0005;\ 0.001\,\mathrm{K}^{-1}$ and $\gamma = 0.0005\,\mathrm{K}^{-1}$ (Fig. 4a) or $\gamma = 0.0005\,\mathrm{K}^{-1}$ (Fig. 4b). In these cases $\sigma^*_{\varphi\varphi}$ changes sign for $\rho \approx 0.9$ and achieves maximal value for $\rho = 1$ (on the boundary of heated region). Moreover $\sigma^*_{\varphi\varphi}$ tends to zero for $\rho \to \infty$.

The dependences of $\sigma^*_{\max} = 100\sigma^*_{\varphi\varphi}(1,0)/T_{\max}$ with respect of $T_{\max} = T(0,0)$ are shown in Fig. 5a, b, c. Figure 5a presents the dimensionless stresses $\sigma^*_{\max}$ for a $\gamma = -0.0005\ \mathrm{K}^{-1}$, $\beta = -0.001;\ -0.0005;\ 0;\ 0.0005;\ 0.001\,\mathrm{K}^{-1}$ as a function of $T_{\max}$. The dependences are almost linear and the highest values are obtained for $\beta = 0.001\,\mathrm{K}^{-1}$. Figure 5b shows the dimensionless stresses $\sigma^*_{\max}$ for the same values of parameter β as Fig. 5a, but different value of parameter γ, namely $\gamma = 0K^{-1}$, as well as Fig. 5c where it assumes that $\gamma = 0.0005\,\mathrm{K}^{-1}$. From these figures it can be observed small differences of values $\sigma^*_{\max}$ for the same values of β.

5 Final remarks

The axisymmetric problems of the thermoelastic half-space heated by a normal heat flux acting on a circle on the boundary plane are considered. Two cases of the boundary conditions on the outside of heated region are assumed: the thermal insulation or zero temperature. The second case leads to the mixed boundary values problem.

The half-plane is the body with thermal conductivity and coefficient of linear expansion in the form of given functions of temperature as well as constants of Young modulus and Poisson ratio. The problems are solved for arbitrary forms of dependency of heat conductivity on temperature and arbitrary form of the boundary heat flux. The obtained stress components in the half-space are presented in the exact forms by multiple integrals. The detailed analysis of stresses on the boundary is presented for linear forms of dependencies of α and K on temperature and the boundary heat flux given by (4.8). For this case the multiple integrals are calculated partially analytically and by using numerical methods and the results are presented in the form of graphics. It can be underlined that in the case of thermal conductivity K proportional to the coefficient of linear expansion the temperature and stresses distributions are analogous to the corresponding problems of homogenous half-space within the framework of the linear theory of thermal stresses.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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