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On the time differential dual-phase-lag thermoelastic model

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Abstract This paper studies the time differential dual-phase-lag model of a thermoelastic material, where the elastic deformation is accompanied by thermal effects governed by a time differential equation for the heat flux with dual phase lags. This coupling gives rise to a complex differential system requiring a special treatment. Uniqueness and continuous dependence results are established for the solutions of the mixed initial boundary value problems associated with the model of the linear theory of thermoelasticity with dual-phase-lag for an anisotropic and inhomogeneous material. Two methods are developed in this paper, both being based on an identity of Lagrange type and of a conservation law applied to appropriate initial boundary value problems associated with the model in concern. The uniqueness results are established under mild constitutive hypotheses (right like those in the classical linear thermoelasticity), without any restrictions upon the delay times (excepting the class of thermoelastic materials for which the delay time of phase lag of the conductive temperature gradient is vanishing and the delay time in the phase lag of heat flux vector is strictly positive, when an ill-posed model should be expected). The continuous dependence results are established by using a conservation law and a Gronwall inequality, under certain constitutive restrictions upon the thermoelastic coefficients and the delay times.

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1 Introduction

The dual-phase-lag model incorporates the microstructural interaction effect in the fast-transient process of heat transport. It describes the finite time required for the various microstructural interactions to take place, including the phonon-electron interaction in metals, the phonon scattering in dielectric crystals, insulators, and semiconductors, and the activation of molecules at extremely low temperature, by the resulting phase-lag (time-delay) in the process of heat transport (see for example, Tzou [1]).

In 1995, Tzou [2–4] (see also [1, 5] and the references therein) proposed, instead of Fourier's law, the following time differential constitutive law for the heat flux vector q_i

$$q_i(\mathbf{x},t) + \tau_q \frac{\partial q_i}{\partial t}(\mathbf{x},\mathbf{t}) + \frac{1}{2}\tau_q^2 \frac{\partial^2 q_i}{\partial t^2}(\mathbf{x},t) = -k_{ij}(\mathbf{x})T_{,j}(\mathbf{x},t) - \tau_T k_{ij}(\mathbf{x})\frac{\partial T_{,j}}{\partial t}(\mathbf{x},t), \tag{1}$$

where $\tau_q \geq 0$ and $\tau_T \geq 0$ are positive delay times and T is the temperature variation. When this constitutive equation is coupled with the energy equation

$$-q_{i,i} + \varrho r = a\frac{\partial T}{\partial t}, \tag{2}$$

then we obtain the following governing equation of hyperbolic type for temperature field T

$$\left(1 + \tau_q \frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2 \frac{\partial^2}{\partial t^2}\right)\left(a\frac{\partial T}{\partial t} - \varrho r\right) = \left[k_{ij}\left(1 + \tau_T \frac{\partial}{\partial t}\right)T_{,j}\right]_{,i}. \tag{3}$$

Such an equation was studied intensively in literature in many papers (see for example, [6–18]). In particular, it was established by Fabrizio and Lazzari [15] that the restrictions imposed by thermodynamics on the constitutive Eq. (1), within the framework of a linear rigid conductor, implies that the delay times have to satisfy the inequality $0 \leq \tau_q \leq 2\tau_T$. Quintanilla and Racke [11] have shown that the Eq. (3), together with suitable initial and boundary conditions, leads to an exponentially stable system when $0 < \tau_q < 2\tau_T$ and to an unstable system when $0 < 2\tau_T < \tau_q$. While in [6–8] is studied the well-posedness problem, provided some appropriate restrictions upon the parameters τ_q and τ_T are assumed.

Recently, the ability of the dual-phase-lag model as a new modified constitutive equation replacing the Fourier law to simulate the heat transport in some special cases such as micro/nanoscales [19, 20], ultra fast laser-pulsed processes [21, 22], living tissues [23], and Carbon nanotube [24] have been tested.

On the other side, when the mechanical deformation is accompanied by the thermal effects described by the constitutive Eq. (1) we are lead to a more complex mathematical problem who requires a special treatment. It is the main aim of the present paper to solve this requirement. In fact, in the present paper we formulate the initial boundary value problem of thermoelastic model based on the constitutive Eq. (1) and then we study the uniqueness and continuous data dependence results. In this connection we define two adequate initial boundary value problems associated with the problem in concern. Further, we use these auxiliary problems and the Lagrange identity method (see, e. g. [25, 26]) in order to establish uniqueness results under mild assumptions upon the thermoelastic coefficients and upon the delay times. Consequently, the uniqueness results are established under the same constitutive hypotheses like in the classical linear thermoelasticity, without any restrictions upon the delay times. However, there is an open problem for the class of materials characterized by zero delay time of phase lag of the conductive temperature gradient and for which the delay time in the phase lag of heat flux vector is strictly positive. In such a case it should be expected to have an ill-posed model.

Furthermore, we establish appropriate conservation laws and then we use the Gronwall's inequality in order to establish some estimates describing the continuous dependence of solutions with respect to the prescribed initial data and with respect to the given supply terms. Such results are established under the assumption that $0 \leq \tau_q \leq 2\tau_T$, which is in accord with the thermodynamic restriction established by Fabrizio and Lazzari [15].

2 Mathematical model

We assume that a regular region B is filled by an inhomogeneous and anisotropic thermoelastic material with dual phase lag. The basic equations of the model in concern are:

- the equations of motion

$$t_{ji,j} + \varrho f_i = \varrho \ddot{u}_i, \tag{4}$$

- the equation of energy

$$\varrho T_0 \dot{\eta} = -q_{i,i} + \varrho r, \tag{5}$$

in $B \times (0,\infty)$,

- the constitutive equations

$$t_{ij} = C_{ijkl}e_{kl} - \beta_{ij}T, \tag{6}$$

$$\varrho\eta = \beta_{ij}e_{ij} + aT, \tag{7}$$

$$q_i + \tau_q \dot{q}_i + \frac{1}{2}\tau_q^2 \ddot{q}_i = -k_{ij}T_{,j} - \tau_T k_{ij}\dot{T}_{,j}, \tag{8}$$

in $\overline{B} \times [0, \infty)$, and

- the geometrical equations

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{9}$$

in $\overline{B} \times [0, \infty)$. Here we have used the following notations: t_{ij} are the components of the stress tensor, η is the entropy per unit mass, q_i are the components of the heat flux vector, e_{ij} are the components of the strain tensor, ϱ is the mass density of the medium, u_r are the components of the displacement vector, T is the change in temperature from the constant ambient temperature $T_0 > 0$, f_i are the components of the body force per unit mass, r is the heat supply per unit mass, a superposed dot denotes time differentiation and a coma denotes the partial differentiation with respect to the corresponding Cartesian coordinate. The constitutive coefficients C_{ijkl}, β_{ij}, a and k_{ij} are continuous differentiable functions on the spatial variable $\mathbf{x}$ and have the following symmetries

$$C_{ijkl} = C_{klij} = C_{jikl}, \quad \beta_{ij} = \beta_{ji}. \tag{10}$$

Furthermore, we will assume the symmetry of the conductivity tensor, that is

$$k_{ij} = k_{ji}, \tag{11}$$

and, moreover, we assume that it is a positive definite tensor. Then we can define K_{ij} so that

$$k_{ij}K_{jk} = K_{ij}k_{jk} = \delta_{ik}. \tag{12}$$

Then, the constitutive Eq. (8) can be written as

$$T_{,i} + \tau_T \dot{T}_{,i} = -K_{ij}\left(q_j + \tau_q \dot{q}_j + \frac{1}{2}\,\tau_q^2 \ddot{q}_j\right). \tag{13}$$

Throughout this paper we consider the initial boundary value problem $\mathcal{P}$ defined by the field Eqs. (4) to (9), the initial conditions

$$\begin{aligned} u_i(\mathbf{x},0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x},0) = \dot{u}_i^0(\mathbf{x}), \quad T(\mathbf{x},0) = T^0(\mathbf{x}), \\ q_i(\mathbf{x},0) = q_i^0(\mathbf{x}), \quad \dot{q}_i(\mathbf{x},0) = \dot{q}_i^0(\mathbf{x}), \quad on \quad \overline{B}, \end{aligned} \tag{14}$$

and the boundary conditions

$$\begin{aligned} u_i(\mathbf{x},t) &= \omega_i(\mathbf{x},t) \quad on \quad \overline{\Sigma}_1 \times [0,\infty), \\ t_{ji}(\mathbf{x},t)n_j &= \psi_i(\mathbf{x},t) \quad on \quad \Sigma_2 \times [0,\infty), \\ T(\mathbf{x},t) &= \vartheta(\mathbf{x},t) \quad on \quad \overline{\Sigma}_3 \times [0,\infty), \\ q_i(\mathbf{x},t)n_i &= \xi(\mathbf{x},t) \quad on \quad \Sigma_4 \times [0,\infty). \end{aligned} \tag{15}$$

Here $u_i^0(\mathbf{x})$, $\dot{u}_i^0(\mathbf{x})$, $T^0(\mathbf{x})$, $q_i^0(\mathbf{x})$, $\dot{q}_i^0(\mathbf{x})$ and $\omega_i(\mathbf{x},t)$, $\psi_i(\mathbf{x},t)$, $\vartheta(\mathbf{x},t)$, $\xi(\mathbf{x},t)$ are prescribed smooth functions. Moreover, Σ_r $(r = 1,2,3,4)$ are subsets of the boundary ∂B such that $\overline{\Sigma}_1 \cup \Sigma_2 = \overline{\Sigma}_3 \cup \Sigma_4 = \partial B$ and $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$.

By a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; \omega_i, \psi_i, \vartheta, \xi\}$ we mean the ordered array $\mathcal{S} = \{u_i, T, e_{ij}, t_{ij}, \eta, q_i\}$ defined on $\overline{B} \times [0, \infty)$ with the properties that $u_i(\mathbf{x},t) \in C^{1,2}(B \times (0,\infty))$, $T(\mathbf{x},t) \in C^{1,1}(B \times (0,\infty))$, $e_{ij}(\mathbf{x},t) = e_{ji}(\mathbf{x},t) \in C^{0,0}(B \times (0,\infty))$, $t_{ij}(\mathbf{x},t) = t_{ji}(\mathbf{x},t) \in C^{1,0}(B \times (0,\infty))$, $\eta(\mathbf{x},t) \in C^{0,1}(B \times (0,\infty))$, $q_i(\mathbf{x},t) \in C^{1,2}(B \times (0,\infty))$ and which satisfy the field Eqs. (4) to (9), the initial conditions (14) and the boundary conditions (15). For further convenience we denote by $\mathcal{P}_0$ the initial boundary value problem $\mathcal{P}$ corresponding to the zero given data $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; \omega_i, \psi_i, \vartheta, \xi\} = 0$.

It is worth to note that if $\mathcal{S} = \{u_i, T, e_{ij}, t_{ij}, \eta, q_i\}$ is a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; \omega_i, \psi_i, \vartheta, \xi\}$, then we have

$$\begin{aligned} &\int_0^t \int_0^s q_i(z)dzds + \tau_q \int_0^t q_i(z)dz + \frac{1}{2}\,\tau_q^2 q_i(t) \\ &\quad = -k_{ij} \int_0^t \int_0^s T_{,j}(z)dzds - \tau_T k_{ij} \int_0^t T_{,j}(z)dz \\ &\qquad + \left(\tau_T k_{ij} T_{,j}^0 + \tau_q q_i^0 + \frac{1}{2}\,\tau_q^2 \dot{q}_i^0\right)t + \frac{1}{2}\,\tau_q^2 q_i^0 \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\int_0^t T_{,i}(z)dz + \tau_T T_{,i}(t) \\ &\quad = -K_{ij}\left[\int_0^t q_j(z)dz + \tau_q q_j(t) + \frac{1}{2}\,\tau_q^2\, \dot{q}_j(t)\right] \\ &\qquad + \tau_T T_{,i}^0 + K_{ij}\left(\tau_q q_j^0 + \frac{1}{2}\,\tau_q^2 \dot{q}_j^0\right). \end{aligned} \tag{17}$$

3 Some auxiliary operators

For further convenience, throughout this paper, we will use the following notations: for any continuous function f of time variable t, we denote by $f'(t)$ the integral over [0, t] of that function, that is

$$f'(t)=\int_0^t f(z)dz,\quad f''(t)=\int_0^t\int_0^s f(z)dzds,\ldots, \tag{18}$$

for any continuous function $g(t)$ we will denote by $g^*(t)$ the following function

$$g^*(t)=g''(t)+\tau_q g'(t)+\frac{1}{2}\,\tau_q^2 g(t), \tag{19}$$

for any continuous function $h(t)$ we will denote by $\tilde{h}(t)$ the following function

$$\tilde{h}(t)=h'(t)+\tau_T h(t). \tag{20}$$

Further, we note that

$$g^*(0)=\frac{1}{2}\,\tau_q^2 g(0),\quad \frac{dg^*}{dt}(0)=\tau_q g(0)+\frac{1}{2}\,\tau_q^2\dot{g}(0), \tag{21}$$

and

$$\tilde{h}(0)=\tau_T h(0),\quad \frac{d\tilde{h}}{dt}(0)=h(0)+\tau_T\dot{h}(0). \tag{22}$$

Then, the relation (16) can be written in the following way

$$\begin{aligned} q_i^*(t)=&-k_{ij}\left(T''_{,j}(t)+\tau_T T'_{,j}(t)\right)\\ &+\left(\tau_T k_{ij}T^0_{,j}+\tau_q q_i^0+\frac{1}{2}\,\tau_q^2\dot{q}_i^0\right)t+\frac{1}{2}\,\tau_q^2 q_i^0, \end{aligned} \tag{23}$$

while the relation (17) implies

$$\begin{aligned} \tilde{T}_{,i}(t)=&-K_{ij}\left(q'_j(t)+\tau_q q_j(t)+\frac{1}{2}\,\tau_q^2\dot{q}_j(t)\right)\\ &+\tau_T T^0_{,i}+K_{ij}\left(\tau_q q_j^0+\frac{1}{2}\,\tau_q^2\dot{q}_j^0\right). \end{aligned} \tag{24}$$

The following results can be readily verified.

Lemma 1 *Suppose that g is twice continuously differentiable. Then, we have*

$$\begin{aligned} \left(\frac{dg}{dt}\right)'(t)&=\frac{dg'}{dt}(t)-g(0),\\ \left(\frac{dg}{dt}\right)^*(t)&=\frac{dg^*}{dt}(t)-\left(\tau_q+t\right)g(0), \end{aligned} \tag{25}$$

$$\left(\frac{d^2g}{dt^2}\right)^*(t)=\frac{d^2g^*}{dt^2}(t)-\left[g(0)+\left(\tau_q+t\right)\dot{g}(0)\right], \tag{26}$$

$$\widetilde{\frac{dg}{dt}}(t)=\frac{d\tilde{g}}{dt}(t)-g(0),\quad \widetilde{\frac{d^2g}{dt^2}}(t)=\frac{d^2\tilde{g}}{dt^2}(t)-\dot{g}(0). \tag{27}$$

Moreover, when $g(0)=0$ and $\dot{g}(0)=0$ then, we have

$$\left(\frac{dg}{dt}\right)'(t)=\frac{dg'}{dt}(t), \tag{28}$$

$$\left(\frac{dg}{dt}\right)^*(t)=\frac{dg^*}{dt}(t),\quad \left(\frac{d^2g}{dt^2}\right)^*(t)=\frac{d^2g^*}{dt^2}(t), \tag{29}$$

$$\widetilde{\frac{dg}{dt}}(t)=\frac{d\tilde{g}}{dt}(t),\quad \widetilde{\frac{d^2g}{dt^2}}(t)=\frac{d^2\tilde{g}}{dt^2}(t). \tag{30}$$

Lemma 2 *If g is a continuous function satisfying*

$$g^*(t)=0,\quad \textit{for all}\quad t>0, \tag{31}$$

then we have

$$g(t)=0,\quad \textit{for all}\quad t\geq 0. \tag{32}$$

Proof By setting

$$g''(t)=h(t), \tag{33}$$

then the relation (31) can be written as

$$\frac{1}{2}\,\tau_q^2\ddot{h}(t)+\tau_q\dot{h}(t)+h(t)=0, \tag{34}$$

and moreover, we have

$$h(0)=0,\quad \dot{h}(0)=0. \tag{35}$$

Now it is easy to see that the Cauchy problem defined by the differential Eq. (34) and the initial conditions (35) has only the zero solution, that is

$$\int_0^t\int_0^s g(z)dzds=0,\quad \textit{for all}\quad t\geq 0, \tag{36}$$

and hence we get the conclusion expressed by relation (32) and the proof is complete.

In a similar way we can prove the following result:

Lemma 3 *Suppose that g is a continuous function satisfying*

$$\tilde{g}(t)=0, \quad \textit{for all} \quad t>0. \tag{37}$$

Then, we have

$$g(t)=0, \quad \textit{forall} \quad t\geq 0. \tag{38}$$

On the basis of the above results we can prove the following two theorems.

Lemma 4 *Let $\mathcal{S}=\{u_i,T,e_{ij},t_{ij},\eta,q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $\mathcal{D}=\{f_i,r;u_i^0,\dot{u}_i^0,T^0,q_i^0,\dot{q}_i^0;\omega_i,\psi_i,\vartheta,\xi\}$. Then $\mathcal{S}^*=\{u_i^*,T^*,e_{ij}^*,t_{ij}^*,\eta^*,q_i^*\}$ satisfies the initial boundary value problem $\mathcal{P}^*$ defined by the basic equations*

$$t^*_{ji,j}(t)+F_i(t)=\varrho\,\frac{\partial^2 u_i^*}{\partial t^2}(t), \tag{39}$$

$$\varrho\,\frac{\partial \eta^*}{\partial t}(t)=-\frac{1}{T_0}\,q^*_{i,i}(t)+R(t), \tag{40}$$

in $B\times(0,\infty)$,

$$t^*_{ij}(t)=C_{ijkl}e^*_{kl}(t)-\beta_{ij}T^*(t), \tag{41}$$

$$\varrho\eta^*(t)=\beta_{ij}e^*_{ij}(t)+aT^*(t), \tag{42}$$

$$q^*_i(t)=-k_{ij}\Big(T''_j(t)+\tau_T T'_j(t)\Big)+Q^0_i(t), \tag{43}$$

and

$$e^*_{ij}(t)=\frac{1}{2}\Big(u^*_{i,j}(t)+u^*_{j,i}(t)\Big), \tag{44}$$

in $\overline{B}\times[0,\infty)$, the initial conditions

$$\begin{aligned}
&u_i^*(\mathbf{x},0)=\frac{1}{2}\,\tau_q^2u_i^0(\mathbf{x}),\\
&\frac{\partial u_i^*}{\partial t}(\mathbf{x},0)=\tau_q u_i^0(\mathbf{x})+\frac{1}{2}\,\tau_q^2\dot{u}_i^0(\mathbf{x}),\\
&T^*(\mathbf{x},0)=\frac{1}{2}\,\tau_q^2T^0(\mathbf{x}),\quad q_i^*(\mathbf{x},0)=\frac{1}{2}\,\tau_q^2q_i^0(\mathbf{x}),\\
&\frac{\partial q_i^*}{\partial t}(\mathbf{x},0)=\tau_q q_i^0(\mathbf{x})+\frac{1}{2}\,\tau_q^2\dot{q}_i^0(\mathbf{x}),\quad \textit{on}\quad \overline{B},
\end{aligned} \tag{45}$$

and the boundary conditions

$$\begin{aligned}
&u_i^*(\mathbf{x},t)=\omega_i^*(\mathbf{x},t)\quad \textit{on}\quad \overline{\Sigma}_1\times[0,\infty),\\
&t^*_{ji}(\mathbf{x},t)n_j=\psi_i^*(\mathbf{x},t)\quad \textit{on}\quad \Sigma_2\times[0,\infty),\\
&T^*(\mathbf{x},t)=\vartheta^*(\mathbf{x},t)\quad \textit{on}\quad \overline{\Sigma}_3\times[0,\infty),\\
&q_i^*(\mathbf{x},t)n_i=\xi^*(\mathbf{x},t)\quad \textit{on}\quad \Sigma_4\times[0,\infty),
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
&F_i(\mathbf{x},t)=\varrho f_i^*(\mathbf{x},t)+\varrho\big[\big(t+\tau_q\big)\dot{u}_i^0(\mathbf{x})+u_i^0(\mathbf{x})\big],\\
&R(\mathbf{x},t)=\frac{1}{T_0}\,\varrho r^*(\mathbf{x},t)+\big(t+\tau_q\big)\Big[\beta_{ij}u^0_{i,j}(\mathbf{x})+aT^0(\mathbf{x})\Big],\\
&Q_i^0(\mathbf{x},t)=\left(\tau_T k_{ij}T_j^0(\mathbf{x})+\tau_q q_i^0(\mathbf{x})+\frac{1}{2}\,\tau_q^2\dot{q}_i^0(\mathbf{x})\right)t\\
&+\frac{1}{2}\,\tau_q^2q_i^0(\mathbf{x}).
\end{aligned} \tag{47}$$

Lemma 5 *Let $\mathcal{S}=\{u_i,T,e_{ij},t_{ij},\eta,q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $\mathcal{D}=\{f_i,r;u_i^0,\dot{u}_i^0,T^0,q_i^0,\dot{q}_i^0;\omega_i,\psi_i,\vartheta,\xi\}$. Then $\tilde{\mathcal{S}}=\{\tilde{u}_i,\tilde{T},\tilde{e}_{ij},\tilde{t}_{ij},\tilde{\eta},\tilde{q}_i\}$ satisfies the initial boundary value problem $\tilde{\mathcal{P}}$ defined by the basic equations*

$$\tilde{t}_{ji,j}(t)+G_i(t)=\varrho\,\frac{\partial^2\tilde{u}_i}{\partial t^2}(t), \tag{48}$$

$$\varrho\,\frac{\partial\tilde{\eta}}{\partial t}(t)=-\frac{1}{T_0}\,\tilde{q}_{i,i}(t)+P(t), \tag{49}$$

in $B\times(0,\infty)$,

$$\tilde{t}_{ij}(t)=C_{ijkl}\tilde{e}_{kl}(t)-\beta_{ij}\tilde{T}(t), \tag{50}$$

$$\varrho\tilde{\eta}(t)=\beta_{ij}\tilde{e}_{ij}(t)+a\tilde{T}(t), \tag{51}$$

$$\tilde{T}_{,i}(t)=-K_{ij}\left(q'_j(t)+\tau_q q_j(t)+\frac{1}{2}\,\tau_q^2\dot{q}_j(t)\right)+\Theta_i^0, \tag{52}$$

and

$$\tilde{e}_{ij}(t)=\frac{1}{2}\big(\tilde{u}_{i,j}(t)+\tilde{u}_{j,i}(t)\big), \tag{53}$$

in $\overline{B}\times[0,\infty)$, the initial conditions

$$\begin{aligned}
&\tilde{u}_i(\mathbf{x},0)=\tau_T u_i^0(\mathbf{x}),\quad \frac{\partial\tilde{u}_i}{\partial t}(\mathbf{x},0)=u_i^0(\mathbf{x})+\tau_T\dot{u}_i^0(\mathbf{x}),\\
&\tilde{T}(\mathbf{x},0)=\tau_T T^0(\mathbf{x}),\quad \tilde{q}_i(\mathbf{x},0)=\tau_T q_i^0(\mathbf{x}),\\
&\frac{\partial\tilde{q}_i}{\partial t}(\mathbf{x},0)=q_i^0(\mathbf{x})+\tau_T\dot{q}_i^0(\mathbf{x}),\quad \textit{on}\quad \overline{B},
\end{aligned} \tag{54}$$

and the boundary conditions

$$\begin{aligned} &\tilde{u}_i(\mathbf{x},t) = \tilde{\omega}_i(\mathbf{x},t) \quad on \quad \overline{\Sigma}_1 \times [0,\infty), \\ &\tilde{t}_{ji}(\mathbf{x},t)n_j = \tilde{\psi}_i(\mathbf{x},t) \quad on \quad \Sigma_2 \times [0,\infty), \\ &\tilde{T}(\mathbf{x},t) = \tilde{\vartheta}(\mathbf{x},t) \quad on \quad \overline{\Sigma}_3 \times [0,\infty), \\ &\tilde{q}_i(\mathbf{x},t)n_i = \tilde{\xi}(\mathbf{x},t) \quad on \quad \Sigma_4 \times [0,\infty), \end{aligned} \tag{55}$$

where

$$\begin{aligned} &G_i(\mathbf{x},t) = \varrho\tilde{f}_i(\mathbf{x},t) + \varrho\dot{u}_i^0(\mathbf{x}), \\ &P(\mathbf{x},t) = \frac{1}{T_0}\,\varrho\tilde{r}(\mathbf{x},t) + \beta_{ij}u_{i,j}^0(\mathbf{x}) + aT^0(\mathbf{x}), \\ &\Theta_i^0(\mathbf{x}) = \tau_T T_{,i}^0(\mathbf{x}) + K_{ij}\left(\tau_q q_j^0(\mathbf{x}) + \frac{1}{2}\,\tau_q^2\dot{q}_j^0(\mathbf{x})\right). \end{aligned} \tag{56}$$

4 Analysis of the initial boundary value problem $\mathcal{P}$

4.1 First method

We address first the uniqueness problem of solutions of the initial boundary value problem $\mathcal{P}$. Thus, we have

Theorem 1 *Suppose that meas* $\Sigma_3 \neq 0$ *and*

$$\begin{aligned} &\varrho > 0, \quad k_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i, \quad k_0 > 0, \quad for\ all \quad \xi_i, \\ &\{0 \leq \frac{1}{2}\,\tau_q \leq \tau_T\} \cup \{0 < \tau_T < \frac{1}{2}\,\tau_q\}. \end{aligned} \tag{57}$$

Then the initial boundary value problem $\mathcal{P}$ *has at most one solution.*

Proof In order to prove the uniqueness result it is sufficient to prove that the zero external given data, that is $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; \omega_i, \psi_i, \vartheta, \xi\} = 0$, implies that the corresponding solution $\mathcal{S} = \{u_i, T, e_{ij}, t_{ij}, \eta, q_i\}$ is vanishing on $\overline{B} \times [0,\infty)$. That means we have to prove that the initial boundary value problem $\mathcal{P}_0$ has only the banal solution.

Thus, we consider here that $\mathcal{S} = \{u_i, T, e_{ij}, t_{ij}, \eta, q_i\}$ is a solution of the initial boundary value problem $\mathcal{P}_0$. In view of the Lemma 4, it follows that $\mathcal{S}^* = \{u_i^*, T^*, e_{ij}^*, t_{ij}^*, \eta^*, q_i^*\}$ is a solution of the initial boundary value problem $\mathcal{P}^*$ with zero given data, denoted in what follows by $\mathcal{P}_0^*$.

We proceed first to establish an identity of Lagrange type for the solutions of the initial boundary value problem $\mathcal{P}_0^*$ associated with $\mathcal{P}_0$. To this end we start with the following identity for all $t > 0, s \in (0,t)$

$$\begin{aligned} &\frac{\partial}{\partial s}\left[\varrho u_i^*(t+s)\frac{\partial u_i^*}{\partial t}(t-s) + \varrho\frac{\partial u_i^*}{\partial t}(t+s)u_i^*(t-s)\right] \\ &\quad = \varrho\left[\frac{\partial^2 u_i^*}{\partial t^2}(t+s)u_i^*(t-s) - u_i^*(t+s)\frac{\partial^2 u_i^*}{\partial t^2}(t-s)\right], \end{aligned} \tag{58}$$

which integrated with respect to $(\mathbf{x}, s)$ over $B \times (0,t)$ and by using the zero initial conditions (45), gives

$$2\int_B \varrho u_i^*(t)\dot{u}_i^*(t)dv = \int_0^t \int_B \varrho\left[\frac{\partial^2 u_i^*}{\partial s^2}(t-s)u_i^*(t+s) - u_i^*(t-s)\frac{\partial^2 u_i^*}{\partial s^2}(t+s)\right]dvds. \tag{59}$$

We now use the relations (39) and (44), the divergence theorem and the zero boundary conditions (46), in order to transform (59) into

$$2\int_B \varrho u_i^*(t)\dot{u}_i^*(t)dv = \int_0^t \int_B \left[e_{ij}^*(t-s)t_{ij}^*(t+s) - e_{ij}^*(t+s)t_{ij}^*(t-s)\right]dvds. \tag{60}$$

In view of the constitutive Eq. (41), we obtain

$$2\int_B \varrho u_i^*(t)\dot{u}_i^*(t)dv = \int_0^t \int_B \left[T^*(t-s)\beta_{ij}e_{ij}^*(t+s) - T^*(t+s)\beta_{ij}e_{ij}^*(t-s)\right]dvds, \tag{61}$$

so that, by means of the relation (42), we get

$$2\int_B \varrho u_i^*(t)\dot{u}_i^*(t)dv = \int_0^t \int_B [T^*(t-s)\varrho\eta^*(t+s) - T^*(t+s)\varrho\eta^*(t-s)]dvds. \tag{62}$$

At this time we integrate the Eq. (40) with respect to time variable and use the zero initial conditions (45) to obtain

$$\varrho\eta^*(t) = -\frac{1}{T_0}\int_0^t q_{i,i}^*(z)dz, \tag{63}$$

so that, with the aid of the divergence theorem and the zero boundary conditions (46), from (62) we deduce

$$2\int_B \varrho u_i^*(t)\dot{u}_i^*(t)dv = \int_0^t \int_B \frac{1}{T_0}\left[T_{,i}^*(t-s)\int_0^{t+s} q_i^*(z)dz - T_{,i}^*(t+s)\int_0^{t-s} q_i^*(z)dz\right]dvds. \quad (64)$$

We use the Eqs. (43) into (64), to obtain

$$2\int_B \varrho u_i^*(t)\dot{u}_i^*(t)dv = \frac{\tau_T}{T_0}\int_0^t \int_B \left[k_{ij}T_{,i}^*(t+s) \cdot \int_0^{t-s} T'_{,j}(z)dz - k_{ij}T_{,i}^*(t-s) \int_0^{t+s} T'_{,j}(z)dz\right]dvds + \frac{1}{T_0}\int_0^t \int_B \left[k_{ij}T_{,i}^*(t+s)\int_0^{t-s} T''_{,j}(z)dz - k_{ij}T_{,i}^*(t-s)\int_0^{t+s} T''_{,j}(z)dz\right]dvds. \quad (65)$$

Further, we use the notations (18) and (19) in order to write

$$k_{ij}T_{,i}^*(t+s)\int_0^{t-s} T'_{,j}(z)dz - k_{ij}T_{,i}^*(t-s) \cdot \int_0^{t+s} T'_{,j}(z)dz = k_{ij}T''_{,j}(t-s)\left[T''_{,i}(t+s) + \tau_q T'_{,i}(t+s) + \frac{1}{2}\tau_q^2 T_{,i}(t+s)\right] - k_{ij}T''_{,j}(t+s)\left[T''_{,i}(t-s) + \tau_q T'_{,i}(t-s) + \frac{1}{2}\tau_q^2 T_{,i}(t-s)\right]$$
$$= \frac{\partial}{\partial s}\left[\tau_q k_{ij}T''_{,j}(t-s)T''_{,i}(t+s)\right] + \frac{\partial}{\partial s}\left[\frac{1}{2}\tau_q^2 k_{ij}T'_{,i}(t+s)T''_{,j}(t-s) + \frac{1}{2}\tau_q^2 k_{ij}T'_{,i}(t-s)T''_{,j}(t+s)\right], \quad (66)$$

and

$$k_{ij}T_{,i}^*(t+s)\int_0^{t-s} T''_{,j}(z)dz - k_{ij}T_{,i}^*(t-s) \cdot \int_0^{t+s} T''_{,j}(z)dz = k_{ij}T'''_{,j}(t-s)\left[T''_{,i}(t+s) + \tau_q T'_{,i}(t+s) + \frac{1}{2}\tau_q^2 T_{,i}(t+s)\right] - k_{ij}T'''_{,j}(t+s) \cdot \left[T''_{,i}(t-s) + \tau_q T'_{,i}(t-s) + \frac{1}{2}\tau_q^2 T_{,i}(t-s)\right]$$
$$= \frac{\partial}{\partial s}\left[k_{ij}T'''_{,j}(t-s)T'''_{,i}(t+s)\right] + \tau_q \frac{\partial}{\partial s}\left[k_{ij}T''_{,j}(t+s) \cdot T'''_{,i}(t-s) + k_{ij}T''_{,j}(t-s)T'''_{,i}(t+s)\right] + \frac{1}{2}\tau_q^2 \frac{\partial}{\partial s}\left[k_{ij}T'_{,i}(t+s)T'''_{,j}(t-s) + k_{ij}T'_{,i}(t-s) \cdot T'''_{,j}(t+s)\right] + \frac{1}{2}\tau_q^2 \frac{\partial}{\partial s}\left[k_{ij}T''_{,i}(t+s)T''_{,j}(t-s)\right]. \quad (67)$$

Finally, by replacing the relations (66) and (67) into identity (65), we obtain the following Lagrange identity

$$\frac{d}{dt}\left\{\int_B \varrho u_i^*(t)u_i^*(t)dv + \frac{\tau_T\tau_q^2}{2T_0}\int_B k_{ij}T''_{,j}(t)T''_{,i}(t)dv + \frac{\tau_q}{T_0}\int_B k_{ij}T'''_{,j}(t)T'''_{,i}(t)dv + \frac{\tau_q^2}{T_0}\int_B k_{ij}T''_{,j}(t)T'''_{,i}(t)dv\right\} + \frac{1}{T_0}\int_B k_{ij}T'''_{,j}(t)T'''_{,i}(t)dv + \frac{\tau_q}{2T_0}\left(2\tau_T - \tau_q\right)\int_B k_{ij}T''_{,i}(t)T''_{,j}(t)dv = 0. \quad (68)$$

Further, we integrate (68) twice with respect to time variable and use the zero initial conditions in order to obtain

$$\int_0^t \int_B \varrho u_i^*(s)u_i^*(s)dvds + \frac{\tau_q}{T_0}\int_0^t \int_B k_{ij}T'''_{,i}(s) \cdot T'''_{,j}(s)dvds + \frac{\tau_q^2}{2T_0}\int_B k_{ij}T'''_{,i}(t)T'''_{,j}(t)dv + \frac{1}{T_0}\int_0^t \int_0^s \int_B k_{ij}T'''_{,i}(z)T'''_{,j}(z)dvdzds + \frac{\tau_T\tau_q^2}{2T_0}\int_0^t \int_B k_{ij}T''_{,i}(s)T''_{,j}(s)dvds + \frac{\tau_q}{2T_0}\left(2\tau_T - \tau_q\right) \cdot \int_0^t \int_0^s \int_B k_{ij}T''_{,i}(z)T''_{,j}(z)dvdzds = 0. \quad (69)$$

Let us suppose first that

$$0 \le \frac{1}{2}\tau_q \le \tau_T. \quad (70)$$

Then it follows that all integral terms in (69) are positive and hence (69) implies that

$$u_i^*(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty), \tag{71}$$

and

$$T'''_{,i}(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty). \tag{72}$$

The last relation implies

$$T_{,i}(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty), \tag{73}$$

and since $meas\,\Sigma_3\neq 0$, we can deduce that

$$T(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty). \tag{74}$$

While the relation (71) and the Lemma 2 give

$$u_i(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty). \tag{75}$$

If we substitute (73) into relation (43) then we get

$$q_i^*(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty), \tag{76}$$

and therefore, by means of the Lemma 2, we have

$$q_i(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty). \tag{77}$$

Consequently, we have $\mathcal{S}=\{u_i,T,e_{ij},t_{ij},\eta,q_i\}=0$ and so we have the uniqueness result.

Let us now consider the case when

$$0<\tau_T<\frac{1}{2}\tau_q. \tag{78}$$

Then the identity (69) implies that

$$\frac{\tau_T\tau_q^2}{2T_0}\int_0^t\int_B k_{ij}T''_{,i}(s)T''_{,j}(s)dvds \leq \frac{\tau_q}{2T_0}\left(\tau_q-2\tau_T\right)\int_0^t\int_0^s\int_B k_{ij}T''_{,i}(z)T''_{,j}(z)dvdzds, \tag{79}$$

and hence we have

$$\Phi(t)\leq\left(\frac{1}{\tau_T}-\frac{2}{\tau_q}\right)\int_0^t\Phi(s)ds, \quad t\in(0,\infty), \tag{80}$$

with

$$\Phi(t)=\int_0^t\int_B k_{ij}T''_{,i}(s)T''_{,j}(s)dvds. \tag{81}$$

By the Gronwall's lemma, from the inequality (80) we deduce that

$$\Phi(t)=0, \quad t\in(0,\infty), \tag{82}$$

and hence

$$T_{,i}(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty), \tag{83}$$

and since $meas\,\Sigma_3\neq 0$, we can obtain that

$$T(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty). \tag{84}$$

Further, the identity (69) implies that

$$u_i^*(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty), \tag{85}$$

and by using an argument like in the above case we obtain the uniqueness result again. Thus, the proof is complete.

Let us now address the question of continuous dependence of solutions of the initial boundary value problem $\mathcal{P}$ with respect to the given data. To this aim we consider the solution $\mathcal{S}=\{u_i,T,e_{ij},t_{ij},\eta,q_i\}$ satisfying the initial boundary value problem $\mathcal{P}$ corresponding to the data $\mathcal{D}=\{f_i,r;u_i^0,\dot{u}_i^0,T^0,q_i^0,\dot{q}_i^0;0,0,0,0\}$ and we introduce the following functional

$$\begin{aligned}\mathcal{E}(t)=&\frac{1}{2}\int_0^t\int_B\Big[\varrho\frac{\partial u_i^*}{\partial s}(s)\frac{\partial u_i^*}{\partial s}(s)+C_{ijkl}e_{ij}^*(s)e_{kl}^*(s)\\&+a(T^*(s))^2\Big]dvds+\frac{\tau_T+\tau_q}{2T_0}\int_0^t\int_B k_{ij}T''_{,j}(s)T''_{,i}(s)dvds\\&+\frac{1}{T_0}\int_0^t\int_0^s\int_B k_{ij}T''_{,j}(z)T''_{,i}(z)dvdzds\\&+\frac{\tau_q^2}{4T_0}\int_B k_{ij}T''_{,j}(t)T''_{,i}(t)dv\\&+\frac{\tau_T\tau_q^2}{4T_0}\int_0^t\int_B k_{ij}T'_{,j}(s)T'_{,i}(s)dvds\\&+\frac{\tau_q}{T_0}\left(\tau_T-\frac{\tau_q}{2}\right)\int_0^t\int_0^s\int_B k_{ij}T'_{,j}(z)T'_{,i}(z)dvdzds,\end{aligned} \tag{86}$$

for all $t\geq 0$. It is a straightforward task to verify the following result.

Lemma 6 *Assume that the conductivity tensor k_{ij} is a positive definite tensor, the elasticity tensor is a positive semi-definite tensor and moreover, the following constitutive hypotheses hold*

$$\varrho>0, \quad a>0, \quad 0\leq\tau_q\leq 2\tau_T. \tag{87}$$

Then $\mathcal{E}(t)$ can be considered as a measure of $\mathcal{S} = \{u_i, T, e_{ij}, t_{ij}, \eta, q_i\}$ in the sense that $\mathcal{E}(t) \geq 0$ for all $t \geq 0$ and $\mathcal{E}(t) = 0$ for all $t \geq 0$ implies that $\mathcal{S} = \{u_i, T, e_{ij}, t_{ij}, \eta, q_i\} = 0$.

The continuous dependence of solutions of the initial boundary value problem $\mathcal{P}$, with respect to the initial data and the given body supplies, is described by the following result.

Theorem 2 *Suppose that the constitutive hypotheses of Lemma 6 hold true and moreover, meas $\Sigma_4 = 0$. Let $\mathcal{S} = \{u_i, T, e_{ij}, t_{ij}, \eta, q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; 0, 0, 0, 0\}$. Then, for any finite time $S > 0$ and for every $t \in [0, S]$, we have the following estimate describing the continuous dependence of solution with respect to the initial given data and with respect to the given supply terms*

$$\sqrt{\mathcal{E}(t)} \leq \left\{ \frac{S}{2} \int_B \left[\varrho \frac{\partial u_i^*}{\partial t}(0) \frac{\partial u_i^*}{\partial t}(0) + C_{ijkl} e_{ij}^*(0) e_{kl}^*(0) + a(T^*(0))^2 \right] dv \right\}^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \int_0^t b(s) ds, \tag{88}$$

where

$$b(t) = \left(\int_0^t \int_B \left[\frac{1}{\varrho} F_i(s) F_i(s) + \frac{1}{a} R^+(s)^2 \right] dv ds \right)^{\frac{1}{2}}, \tag{89}$$

and

$$R^+(\mathbf{x}, t) = \frac{1}{T_0} \left[R(\mathbf{x}, t) - Q_{i,i}^0(\mathbf{x}, t) \right]. \tag{90}$$

Proof In view of Lemma 4 it follows that $\mathcal{S}^* = \{u_i^*, T^*, e_{ij}^*, t_{ij}^*, \eta^*, q_i^*\}$ is a solution of the initial boundary value $\mathcal{P}^*$ with $\omega_i^* = 0$, $\psi_i^* = 0$, $\vartheta^* = 0$ and $\xi^* = 0$. Then the relations (39)–(47) lead to the following identity

$$\frac{d}{dt} \int_B \frac{1}{2} \left[\varrho \frac{\partial u_i^*}{\partial t}(t) \frac{\partial u_i^*}{\partial t}(t) + C_{ijkl} e_{ij}^*(t) e_{kl}^*(t) + a(T^*(t))^2 \right] dv = \int_B \left[F_i(t) \frac{\partial u_i^*}{\partial t}(t) + \frac{1}{T_0} R(t) T^*(t) \right] dv + \int_B \frac{T_{,i}^*(t)}{T_0} \left[-k_{ij} \left(T_{,j}''(t) + \tau_T T_{,j}'(t) \right) + Q_i^0(t) \right] dv. \tag{91}$$

By using the divergence theorem and the zero boundary condition on ∂B we get

$$\int_B \frac{T_{,i}^*(t)}{T_0} Q_i^0(t) dv = - \int_B \frac{T^*(t)}{T_0} Q_{i,i}^0(t) dv. \tag{92}$$

Moreover, we replace $T_{,i}^*(t)$ by means of (19) in order to obtain

$$\begin{aligned} &- \frac{1}{T_0} \int_B k_{ij} T_{,i}^*(t) \left[T_{,j}''(t) + \tau_T T_{,j}'(t) \right] dv \\ &= - \frac{d}{dt} \left\{ \frac{\tau_T \tau_q^2}{4T_0} \int_B k_{ij} T_{,j}'(t) T_{,i}'(t) dv + \frac{\tau_T + \tau_q}{2T_0} \int_B k_{ij} T_{,j}''(t) T_{,i}''(t) dv + \frac{d}{dt} \left(\frac{\tau_q^2}{4T_0} \int_B k_{ij} T_{,j}''(t) T_{,i}''(t) dv \right) \right\} \\ &\quad - \frac{\tau_q}{2T_0} \left(2\tau_T - \tau_q \right) \int_B k_{ij} T_{,j}'(t) T_{,i}'(t) dv - \frac{1}{T_0} \int_B k_{ij} T_{,j}''(t) T_{,i}''(t) dv. \end{aligned} \tag{93}$$

Thus, by replacing the relations (92) and (93) into (91) and by integrating twice with respect to time variable, we are led to the following conservation law

$$\mathcal{E}(t) = \frac{t}{2} \int_B \left[\varrho \frac{\partial u_i^*}{\partial t}(0) \frac{\partial u_i^*}{\partial t}(0) + C_{ijkl} e_{ij}^*(0) e_{kl}^*(0) + a(T^*(0))^2 \right] dv + \int_0^t \int_0^s \int_B \left[F_i(z) \frac{\partial u_i^*}{\partial z}(z) + R^+(z) T^*(z) \right] dv dz ds, \tag{94}$$

for all $t \in [0, S]$. By means of the Cauchy–Schwarz inequality, from (94) we obtain

$$\mathcal{E}(t) \leq \frac{S}{2} \int_B \left[\varrho \frac{\partial u_i^*}{\partial t}(0) \frac{\partial u_i^*}{\partial t}(0) + C_{ijkl} e_{ij}^*(0) e_{kl}^*(0) + a(T^*(0))^2 \right] dv + \int_0^t b(s) \left[\int_0^s \int_B \left(\varrho \frac{\partial u_i^*}{\partial z}(z) \frac{\partial u_i^*}{\partial z}(z) + a(T^*(z))^2 \right) dv dz \right]^{\frac{1}{2}} ds, \tag{95}$$

so that we have the following Gronwall inequality

$$\mathcal{E}(t) \leq \frac{S}{2} \int_B \left[\varrho \frac{\partial u_i^*}{\partial t}(0) \frac{\partial u_i^*}{\partial t}(0) + C_{ijkl} e_{ij}^*(0) e_{kl}^*(0) + a(T^*(0))^2 \right] dv + \int_0^t b(s) \sqrt{2\mathcal{E}(s)} ds, \quad t \in [0, S]. \tag{96}$$

By integrating the differential inequality (96) we obtain the estimate (88) and the proof is complete. □

4.2 Second method

In this section we use the Lemma 5 in order to prove the uniqueness and the continuous dependence of solutions of the initial boundary value problem $\mathcal{P}$. We first give a proof of the Theorem 1 based on the Lemma 5. In this aim we consider $\mathcal{S}=\{u_i,T,e_{ij},t_{ij},\eta,q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}_0$. In view of the Lemma 5, it follows that $\tilde{\mathcal{S}}=\{\tilde{u}_i,\tilde{T},\tilde{e}_{ij},\tilde{t}_{ij},\tilde{\eta},\tilde{q}_i\}$ is a solution of the initial boundary value problem $\tilde{\mathcal{P}}$ with zero given data, denoted in what follows by $\tilde{\mathcal{P}}_0$. Starting with a Lagrange identity of type (58) for the problem $\tilde{\mathcal{P}}_0$, we obtain the following identity

$$\int_B 2\varrho\tilde{u}_i(t)\dot{\tilde{u}}_i(t)dv=\int_0^t\int_B\frac{1}{T_0}\Big[\tilde{T}_{,i}(t-s)\cdot\int_0^{t+s}\tilde{q}_i(z)dz-\tilde{T}_{,i}(t+s)\int_0^{t-s}\tilde{q}_i(z)dz\Big]dvds, \tag{97}$$

which, when we replace $\tilde{T}_{,i}$ from relation (52) and we substitute $\tilde{q}_i$ by means of (20), gives

$$\begin{aligned}\int_B 2\varrho\tilde{u}_i(t)\dot{\tilde{u}}_i(t)dv=&\frac{1}{T_0}\int_0^t\int_B\Big[K_{ij}q_i''(t-s)q_j'(t+s)\\&-K_{ij}q_i''(t+s)q_j'(t-s)\Big]dvds\\&+\frac{\tau_q}{T_0}\int_0^t\int_B\big[K_{ij}q_i''(t-s)q_j(t+s)\\&-K_{ij}q_i''(t+s)q_j(t-s)\big]dvds\\&+\frac{\tau_q^2}{2T_0}\int_0^t\int_B\big[K_{ij}q_i''(t-s)\dot{q}_j(t+s)\\&-K_{ij}q_i''(t+s)\dot{q}_j(t-s)\big]dvds\\&+\frac{\tau_q\tau_T}{T_0}\int_0^t\int_B\big[K_{ij}q_i'(t-s)q_j(t+s)\\&-K_{ij}q_i'(t+s)q_j(t-s)\big]dvds\\&+\frac{\tau_q^2\tau_T}{2T_0}\int_0^t\int_B\big[K_{ij}q_i'(t-s)\dot{q}_j(t+s)\\&-K_{ij}q_i'(t+s)\dot{q}_j(t-s)\big]dvds.\end{aligned} \tag{98}$$

Further, we observe that

$$\begin{aligned}&K_{ij}q_i''(t-s)q_j'(t+s)-K_{ij}q_i''(t+s)q_j'(t-s)\\&=\frac{\partial}{\partial s}\Big[K_{ij}q_i''(t-s)q_j''(t+s)\Big],\end{aligned} \tag{99}$$

$$\begin{aligned}&K_{ij}q_i''(t-s)q_j(t+s)-K_{ij}q_i''(t+s)q_j(t-s)\\&=\frac{\partial}{\partial s}\Big[K_{ij}q_i''(t-s)q_j'(t+s)\\&\quad+K_{ij}q_i''(t+s)q_j'(t-s)\Big],\end{aligned} \tag{100}$$

$$\begin{aligned}&K_{ij}q_i''(t-s)\dot{q}_j(t+s)-K_{ij}q_i''(t+s)\dot{q}_j(t-s)\\&=\frac{\partial}{\partial s}\big[K_{ij}q_i''(t-s)q_j(t+s)\\&\quad+K_{ij}q_i''(t+s)q_j(t-s)+K_{ij}q_i'(t-s)q_j'(t+s)\big],\end{aligned} \tag{101}$$

$$\begin{aligned}&K_{ij}q_i'(t-s)q_j(t+s)-K_{ij}q_i'(t+s)q_j(t-s)\\&=\frac{\partial}{\partial s}\Big[K_{ij}q_i'(t-s)q_j'(t+s)\Big],\end{aligned} \tag{102}$$

$$\begin{aligned}&K_{ij}q_i'(t-s)\dot{q}_j(t+s)-K_{ij}q_i'(t+s)\dot{q}_j(t-s)\\&=\frac{\partial}{\partial s}\big[K_{ij}q_i'(t-s)q_j(t+s)\\&\quad+K_{ij}q_i'(t+s)q_j(t-s)\big].\end{aligned} \tag{103}$$

Finally, by substituting the relations (99)–(103) into (98) and then by integrating twice with respect to time variable the result, we get

$$\begin{aligned}&\int_0^t\int_B\varrho\tilde{u}_i(s)\tilde{u}_i(s)dvds\\&+\frac{1}{T_0}\int_0^t\int_0^s\int_B K_{ij}q_i''(z)q_j''(z)dvdzds\\&+\frac{\tau_q}{T_0}\int_0^t\int_B K_{ij}q_i''(s)q_j''(s)dvds\\&+\frac{\tau_q^2}{2T_0}\int_B K_{ij}q_i''(t)q_j''(t)dv\\&+\frac{\tau_T\tau_q^2}{2T_0}\int_0^t\int_B K_{ij}q_i'(s)q_j'(s)dvds+\frac{\tau_q}{2T_0}\left(2\tau_T-\tau_q\right)\\&\cdot\int_0^t\int_0^s\int_B K_{ij}q_i'(z)q_j'(z)dvdzds=0.\end{aligned} \tag{104}$$

Under the assumption (70) all terms of the identity (104) are positive so that we conclude that

$$\tilde{u}_i(\mathbf{x},t)=0\quad in\quad \overline{B}\times[0,\infty), \tag{105}$$

and

$$q_i''(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty). \tag{106}$$

By means of Lemma 3, from (105) we readily obtain the conclusion (75), while (106) implies that

$$q_i(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty), \tag{107}$$

and hence, by means of the constitutive Eq. (52), we obtain

$$\tilde{T}_{,i}(\mathbf{x},t)=0 \quad in \quad \overline{B}\times[0,\infty). \tag{108}$$

This last relation when coupled with the Lemma 3 implies the relation (74) and then we can obtain the uniqueness result described in Theorem 1.

When the relation (78) holds true, the identity (104) implies the inequality (80) where now Φ is defined by

$$\Phi(t)=\int_0^t\int_B K_{ij}q_i'(s)q_j'(s)dvds, \tag{109}$$

and therefore, we can follow the same way as in the proof of Theorem 1 to prove the uniqueness result.

Let us now address the continuous dependence question. Let $\mathcal{S}=\{u_i,T,e_{ij},t_{ij},\eta,q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the data $\mathcal{D}=\{f_i,r;u_i^0,\dot{u}_i^0,T^0,q_i^0,\dot{q}_i^0;0,0,0,0\}$. In view of Lemma 5 it follows that $\tilde{\mathcal{S}}=\{\tilde{u}_i,\tilde{T},\tilde{e}_{ij},\tilde{t}_{ij},\tilde{\eta},\tilde{q}_i\}$ satisfies the initial boundary value problem $\tilde{\mathcal{P}}$ with $\tilde{\omega}_i=0$, $\tilde{\psi}_i=0$, $\tilde{\vartheta}=0$ and $\tilde{\xi}=0$. Under these conditions we have the following conservation law

$$\begin{aligned}\mathcal{F}(t)=&\int_0^t\int_0^s\int_B\Big\{G_i(z)\frac{\partial\tilde{u}_i}{\partial z}(z)+P(z)\tilde{T}(z)\\&+\frac{1}{T_0}\left[q_i'(z)+\tau_Tq_i(z)\right]\Theta_i^0\Big\}dvdzds\\&+\frac{t}{2}\int_B\Big[\varrho\frac{\partial\tilde{u}_i}{\partial t}(0)\frac{\partial\tilde{u}_i}{\partial t}(0)+C_{ijkl}\tilde{e}_{ij}(0)\tilde{e}_{kl}(0)\\&+a\big(\tilde{T}(0)\big)^2+\frac{\tau_T\tau_q^2}{2T_0}K_{ij}q_i(0)q_j(0)\Big]dv,\end{aligned} \tag{110}$$

for all $t\geq 0$, where

$$\begin{aligned}\mathcal{F}(t)=&\frac{1}{2}\int_0^t\int_B\Big[\varrho\frac{\partial\tilde{u}_i}{\partial s}(s)\frac{\partial\tilde{u}_i}{\partial s}(s)+C_{ijkl}\tilde{e}_{ij}(s)\tilde{e}_{kl}(s)\\&+a\big(\tilde{T}(s)\big)^2\Big]dvds+\frac{\tau_T+\tau_q}{2T_0}\int_0^t\int_B K_{ij}q_i'(s)q_j'(s)dvds\\&+\frac{\tau_T\tau_q^2}{4T_0}\int_0^t\int_B K_{ij}q_i(s)q_j(s)dvds+\frac{\tau_q^2}{4T_0}\int_B K_{ij}q_i'(t)\\&\cdot q_j'(t)dv+\frac{1}{T_0}\int_0^t\int_0^s\int_B K_{ij}q_i'(z)q_j'(z)dvdzds\\&+\frac{\tau_q}{T_0}\left(\tau_T-\frac{\tau_q}{2}\right)\int_0^t\int_0^s\int_B K_{ij}q_i(z)q_j(z)dvdzds.\end{aligned} \tag{111}$$

Theorem 3 *Suppose that the constitutive hypotheses of Lemma 6 hold true. Let $\mathcal{S}=\{u_i,T,e_{ij},t_{ij},\eta,q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $\mathcal{D}=\{f_i,r;u_i^0,\dot{u}_i^0,T^0,q_i^0,\dot{q}_i^0;0,0,0,0\}$. Then $\mathcal{F}(t)$ can be considered as a measure for $\mathcal{S}$ and moreover, for any finite time $S>0$ and for every $t\in[0,S]$, we have the following estimate*

$$\begin{aligned}\sqrt{\mathcal{F}(t)}\leq&\Big[\frac{S}{2}\int_B\Big[\varrho\frac{\partial\tilde{u}_i}{\partial t}(0)\frac{\partial\tilde{u}_i}{\partial t}(0)+C_{ijkl}\tilde{e}_{ij}(0)\tilde{e}_{kl}(0)\\&+a\big(\tilde{T}(0)\big)^2+\frac{\tau_T\tau_q^2}{2T_0}K_{ij}q_i(0)q_j(0)\Big]dv\Big]^{\frac{1}{2}}\\&+\frac{1}{\sqrt{2}}\int_0^t\beta(s)ds,\end{aligned} \tag{112}$$

where

$$\begin{aligned}\beta(t)=&\Big(\int_0^t\int_B\Big[\frac{1}{\varrho}G_i(s)G_i(s)+\frac{1}{a}(P(s))^2\\&+\frac{1}{T_0K_0}\left(\frac{1}{\tau_T+\tau_q}+\frac{4\tau_T}{\tau_q^2}\right)\Theta_i^0\Theta_i^0\Big]dvds\Big)^{\frac{1}{2}},\end{aligned} \tag{113}$$

and K_0 is related to the lower eigenvalue of the tensor K_{ij}.

Proof By using the Cauchy–Schwarz inequality, from the identity (110) we obtain the following inequality

$$\mathcal{F}(t) \leq \frac{S}{2}\int_B \Big[\varrho \frac{\partial \tilde{u}_i}{\partial t}(0)\frac{\partial \tilde{u}_i}{\partial t}(0) + C_{ijkl}\tilde{e}_{ij}(0)\tilde{e}_{kl}(0) + a\big(\tilde{T}(0)\big)^2 + \frac{\tau_T \tau_q^2}{2T_0} K_{ij} q_i(0) q_j(0) \Big] dv + \int_0^t \beta(s)\sqrt{2\mathcal{F}(s)}ds, \tag{114}$$

which, when integrated, leads to the estimate (112) and the proof is complete. □

5 Concluding remarks

Our analysis establishes the uniqueness and continuous data dependence of solutions of the initial boundary value problems within the context of the time differential model of dual-phase-lag thermoelastic theory. The key point of our proof consists to introduce the two initial boundary value problems $\mathcal{P}^*$ and $\tilde{\mathcal{P}}$ inspired by the two operators involved into basic constitutive Eq. (1). The uniqueness results are established either when the conducting media is such that $\tau_T > \tau_q$ that is when the heat flux vector precedes the temperature gradient in the time-history, implying that the heat flux vector is the cause while the temperature gradient is the effect of the heat flow as well as for media with $\tau_q > \tau_T$, when the temperature gradient becomes the cause while the heat flux vector becomes the effect. It can be seen that our uniqueness theorems cannot cover the case when

$$\tau_T = 0, \quad \tau_q > 0. \tag{115}$$

In fact, in this case the identity (69) becomes

$$\int_0^t \int_B \varrho u_i^*(s) u_i^*(s) dv ds + \frac{\tau_q}{T_0}\int_0^t \int_B k_{ij} T'''_{,i}(s) T'''_{,j}(s) dv ds + \frac{\tau_q^2}{2T_0}\int_B k_{ij} T'''_{,i}(t) T'''_{,j}(t) dv + \frac{1}{T_0}\int_0^t \int_0^s \int_B k_{ij} T'''_{,i}(z) T'''_{,j}(z) dv dz ds - \frac{\tau_q^2}{2T_0}\int_0^t \int_0^s \int_B k_{ij} T''_{,i}(z) T''_{,j}(z) dv dz ds = 0, \tag{116}$$

and it is no clear how we can handle it to get the uniqueness of solution. Namely, the last integral term in the identity (116) is negative one and it cannot be convenably related to the any other integral terms in order to get the uniqueness. Such a case remains an open problem and it is expected that it can lead to an ill-posed model. In fact, the case was studied by Fabrizio and Franchi [16] and it was concluded that such a model is not compatible with the Second Law of Thermodynamics. However, when $\tau_T = 0$, $\tau_q > 0$ and τ_q^2 can be neglected (that is for the Cattaneo–Maxwell model or the Lord-Shulman model), the identity (116) yet furnishes the uniqueness result.

The uniqueness of solutions of the mixed initial boundary value problem in the linear theory of thermoelasticity with dual phase-lags was established by Kothari and Mukhopadhyay [27] under more restrictive assumptions upon the thermoelastic coefficients (the elasticity tensor C_{ijkl} is positive definite and the specific heat is strictly positive) and upon the delay times ($0 < \tau_q < 2\tau_T$).

The continuous dependence of solutions with respect to the initial given data and given supply terms is described by the estimates of Theorems 2 and 3. They are established under the assumption $0 \leq \tau_q \leq 2\tau_T$ which agrees with the thermodynamic restrictions found by Fabrizio and Lazzari in [15]. We have to remark that the continuous dependence results can by obtained by means of the Lagrange identity method, as described in Sect. 4, under relaxed conditions upon the delay times, provided some suitable classes of solutions are considered. Finally, we have to mention that the conditions under which the continuous dependence results are established into Theorems 2 and 3 can be relaxed by replacing, for example, the condition that $a > 0$ by $meas\ \Sigma_3 \neq 0$.

Concluding, we can see that the thermodynamic restrictions found by Fabrizio and Lazzari [15] allow to prove the well position of the related initial boundary value problems.

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