

# A dynamic viscoelastic analogy for fluid-filled elastic tubes

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**Abstract** In this paper we evaluate the dynamic effects of the fluid viscosity for fluid filled elastic tubes in the framework of a linear uni-axial theory. Because of the linear approximation, the effects on the fluid inside the elastic tube are taken into account according to the Womersley theory for a pulsatile flow in a rigid tube. The evolution equations for the response variables are derived by means of the Laplace transform technique and they all turn out to be the very same integro-differential equation of the convolution type. This equation has the same structure as the one describing uni-axial waves in linear viscoelastic solids characterized by a relaxation modulus or by a creep compliance. In our case, the analogy is connected with a peculiar viscoelastic solid which exhibits creep properties similar to those of a fractional Maxwell model (of order  $1/2$ ) for short times, and of a standard Maxwell model for long times. The present analysis could find applications in biophysics concerning the propagation of pressure waves within large arteries.

**Keywords** Elastic tubes · Viscoelasticity · Transient waves · Bessel functions · Dirichlet series

**Mathematics Subject Classification** 35Q35 · 76A10 · 76D05

## 1 Introduction

It is well known that the dynamic theory for fluid filled elastic tubes might find applications in the analysis of the propagation of pressure waves within large arteries. In the framework of a uni-axial theory, the effect of the viscosity of the fluid inside the tube (e.g. blood) is usually taken into account including a friction term in the one-dimensional momentum equation, due to the Poiseuille's formula for steady flows in rigid tubes, see e.g. [2, 12].

Restricting ourselves to a linear approach, here we find it more appropriate to evaluate the friction term for a non-stationary flow, which is easily derived starting from a sinusoidal flow, as in the renowned Womersley model for pulsatile flow of viscous (Newtonian) fluids in rigid tubes [21], see also [5, 11].

In Sect. 2 we follow the approach displayed in [3] considering the Navier–Stokes equations for an incompressible Newtonian fluid in cylindrical coordinates. For such system of PDEs we neglect the motion in the circumferential direction and we assume the flow to be quasi-one-dimensional. In particular, the radial velocity is assumed to be very small with respect to the axial one. The resulting equations are then integrated over the radial coordinate, reducing the number of independent variables to two (i.e. time  $t$  and axial distance  $x$ ) and the dependent variables to three (pressure, averaged

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velocity, and cross-sectional area). A pressure-area relation for a uniform elastic tube, longitudinally tethered, is then used to eliminate one of the remaining dependent variables. However, the final equations contain two quantities which depend on the velocity profile.

In Sect. 3, working within the framework of a linear uni-axial theory, we reduce the system of PDEs discussed in Sect. 2 to a system of two linear equations that contains only an additional unknown function, i.e. the friction term, that we denote by  $f_0(x, t)$  and which still depends on the velocity profile. Then, because of our linearity assumption, we find it reasonable to evaluate  $f_0$  in terms of the averaged velocity. This can be done by assuming the velocity profile to be described as in the case of an unsteady flow in a rigid tube with an unperturbed radius based on the Womersley theory for pulsatile flow. As a consequence, we are able to obtain the friction term corresponding to any time history of the pressure-gradient by means of a time convolution integral between a certain memory function  $\Phi(t)$  and the pressure-gradient itself. Finally, we obtain the evolution equation for the pressure waves that appears to have the same structure as the one known for stress waves in a viscoelastic solid (in the relaxation representation), see e.g. [14].

In Sect. 4 we discuss in detail this dynamic analogy in the framework of a linear theory of viscoelasticity. For further details about mathematical and historical aspects of this theory we recommend the interested reader to refer to specialized articles and treatises, e.g. [8, 13, 14, 18, 19]. The memory function  $\Phi(t)$  is then shown to be a complete monotonic function expressed in terms of a Dirichlet series. Therefore, the latter has to be intended as the rate of the relaxation modulus of a peculiar viscoelastic model. We then calculate the corresponding rate of creep  $\Psi(t)$ , which enables us to obtain the wave equation in the creep representation. Finally, by a suitable time integration of  $\Phi(t)$  and  $\Psi(t)$ , we derive the corresponding relaxation modulus  $G(t)$  and the creep compliance  $J(t)$ , providing a full characterization of the viscoelastic analogy.

In Sect. 5 we derive the asymptotic representations of the memory functions  $\Phi(t)$  and  $\Psi(t)$  for short and long times. The resulting functions show that the corresponding viscoelastic model is governed by a stress–strain relation of Maxwell type with a fractional order changing from 1/2 to 1 as time evolves from zero

to infinity. For sake of clarity, we show the plots versus time of the functions  $\Phi(t)$ ,  $\Psi(t)$  and of  $G(t)$  and  $J(t)$ .

Finally, in Sect. 6 we complete the paper with concluding remarks and hints for future research. Furthermore, a detailed mathematical description of the technical results can be found in the “Appendix”.

## 2 The basic non-linear equations

Let us consider in a cylindrical coordinate system the Navier–Stokes equations of motion for an incompressible Newtonian fluid. If the motion in the circumferential direction is neglected, these equations read (denoting the partial derivatives with subscripts)

$$\begin{cases} u_t + wu_r + uu_x + \frac{p_x}{\rho} = v \left( u_{rr} + \frac{u_r}{r} + u_{xx} \right) \\ w_t + ww_r + uw_x + \frac{p_r}{\rho} = v \left( w_{rr} + \frac{w_r}{r} - \frac{w}{r^2} + w_{xx} \right) \end{cases} \tag{1}$$

where  $t$  is the time,  $x, r$  denote the axial and radial directions,  $u, w$  the corresponding components of the fluid velocity and  $p$  the pressure. The constants  $\rho$  and  $v$  are the density and kinematic viscosity of the fluid.

Then, following the approach shown in [3], the flow is assumed to be quasi one dimensional, i.e. the radial velocity  $w$  is very small with respect to the axial velocity  $u$  and the resulting equations are integrated over the radial coordinate  $r$ . As a consequence we arrive at Eqs. (14) and (15) in [3], which in our notation read:

$$\begin{cases} U_t + \frac{U}{A} (1 - \chi) A_t + \chi U U_x + \frac{p_x}{\rho} = \frac{2v}{R} u_r \Big|_R, \\ A_t + \frac{\partial}{\partial x} (UA) = 0. \end{cases} \tag{2}$$

In these equations

$$U = \frac{1}{R^2} \int_0^R 2ru \, dr \tag{3}$$

is the averaged (uni-axial) velocity (the quantity measured in most experiments),

$$A = \pi R^2 \tag{4}$$

is the cross-sectional area of the tube, and

$$\chi = \frac{1}{R^2 U^2} \int_0^R 2ru^2 \, dr. \tag{5}$$

As a consequence,  $\{U(x, t), A(x, t)\}$ , the basic dependent variables for a quasi-one dimensional flow, turn out to be related also to the pressure gradient  $A = p_x$ , to the parameter  $\chi$  and to the friction term defined by

$$f(x, t) = \frac{2\nu}{R} u_r \Big|_R. \tag{6}$$

As a matter of fact, both quantities depend on the variation of the axial velocity  $u$  with  $r$  (the velocity profile) so they could not be properly determined without carrying out a full two-dimensional non-linear treatment of the problem. It is worth stressing that when the fluid is inviscid the velocity profile is flat ( $\chi = 1$ ) and the friction term is absent ( $f(x, t) \equiv 0$ ).

According to Barnard et al. [3], in the presence of viscosity, a particularly simple assumption is the parabolic form of the velocity profile:

$$u = 2U(1 - r^2/R^2), \tag{7}$$

so that  $\alpha = 4/3$  and

$$f(x, t) = -8\nu U/R^2. \tag{8}$$

This assumption, however, is too simple because it is satisfied only for stationary flows related to the Poiseuille approximation.

### 3 The basic linear equations

Linearizing Eq. (2) we get the following systems of PDEs:

$$\begin{cases} U_t + p_x/\rho = f_0(x, t), \\ A_t + A_0 U_x = 0, \end{cases} \tag{9}$$

from which we get

$$\begin{cases} A_0 U_t + c_0^2 A_x = A_0 f_0(x, t), \\ A_t + A_0 U_x = 0, \end{cases} \tag{10}$$

where  $A_0 = \pi R_0^2$  is the unperturbed cross-sectional area of radius  $R_0$ ,

$$c_0^2 = \frac{A_0}{\rho} \left( \frac{dp}{dA} \right) \Big|_0 \tag{11}$$

denotes the corresponding Moens–Korteweg velocity (for pressure waves in elastic tubes filled with an inviscid fluid) and

$$f_0 = \frac{2\nu}{R_0} u_r \Big|_{R_0} \tag{12}$$

is the corresponding friction term due to the viscosity, evaluated at the same unperturbed radius.

In the case of a parabolic velocity profile (Poiseuille approximation of steady flow), linearizing Eqs. (7) and (8), we get

$$\begin{aligned} u &= 2U(x, t) (1 - r^2/R_0^2) \\ \Rightarrow f_0(x, t) &= -\frac{8U(x, t)}{\tau}, \quad \tau = \frac{R_0^2}{\nu}. \end{aligned} \tag{13}$$

We note that the time constant  $\tau$ , introduced here for steady flow, would provide a proper time scale also in all profiles occurring for any unsteady flow. As a consequence, in every function of time appearing in our approach the constant  $\tau$  is to be interpreted as a proper time scale introduced by the viscosity of the fluid; in future, even when we will omit  $\tau$  in the time dependence, this scale-parameter is understood to be equal to one.

In order to eliminate the velocity component in the radial direction, in the framework of a linearized theory for unsteady flow, and evaluate the corresponding friction term  $f_0$  in terms of the uni-axial velocity  $U(x, t)$ , we find it reasonable to take profit of the well established Womersley linear model for pulsatile flow of a viscous (Newtonian) fluid in rigid tubes [21].

Because of the linearity, we can easily compute the friction term corresponding to a pressure-gradient which is sinusoidal in time and oscillates with a frequency  $\omega$ ,

$$\hat{\lambda}(x, t; \omega) = A_0(x) \exp(-i\omega t)$$

as derived by the Womersley model. Introducing the Womersley parameter<sup>1</sup>

$$\alpha := R_0 \sqrt{\frac{\omega}{\nu}} = \sqrt{\omega\tau}, \tag{14}$$

we get the friction expressed in terms of  $(\sqrt{\omega\tau})$ :

$$\hat{f}_0(x, t; \omega) = \hat{\Phi}(\omega) A_0(x) \exp(-i\omega t)/\rho, \tag{15}$$

with

$$\hat{\Phi}(\omega) = \frac{2}{i^{3/2} \sqrt{\omega\tau}} \frac{J_1(i^{3/2} \sqrt{\omega\tau})}{J_0(i^{3/2} \sqrt{\omega\tau})}, \tag{16}$$

<sup>1</sup> In our analysis we have preferred to express the Womersley parameter  $\alpha$  in terms of the time scale  $\tau$  introduced in Eq. (13).

where  $J_0, J_1$  denote the Bessel functions of order 0, 1, respectively, see e.g. [1].

Then, being interested to transient motion, it is worth to pass from Fourier transform to Laplace transform with parameter  $s = i\omega$  so that we get

$$\tilde{\Phi}(s) = \frac{2}{\sqrt{s\tau}} \frac{I_1(\sqrt{s\tau})}{I_0(\sqrt{s\tau})}, \tag{17}$$

where  $I_0, I_1$  denote the modified Bessel functions of order 0, 1, respectively. We have used the known relation between Bessel and modified Bessel functions of any order  $\nu, J_\nu(iz) = i^\nu I_\nu(z)$ , see e.g. [1].

As a consequence we can formally obtain the friction term corresponding to any time history of the pressure-gradient  $A(x, t)$  by means of a Laplace convolution integral. Indeed we get:

$$f_0(x, t) = \Phi(t) * A(x, t)/\rho \tag{18}$$

from which we derive

$$U_t(x, t) = -[1 - \Phi(t) *] A(x, t)/\rho \tag{19}$$

where  $\Phi(t)$  denotes the inverse Laplace Transform of  $\tilde{\Phi}(s)$  and  $*$  denotes the time Laplace convolution<sup>2</sup>.

So doing we have properly modified the relation between acceleration  $U_t$  and pressure gradient  $A$  for inviscid flow (the Euler equation) in terms of a Laplace convolution with a characteristic function  $\Phi(t)$  in order to take into account the *memory effects of the viscosity*. We agree to call  $\Phi(t)$  the *relaxation memory function*, whose meaning will be clarified in the following with respect to our wave process.

Indeed, denoting with  $Y = Y(x, t)$  a generic response variable (such as  $\{U, A, p, \Lambda\}$ ), after simple manipulations we get the evolution equation

$$Y_{tt}(x, t) = c_0^2 [1 - \Phi(t) *] Y_{xx}(x, t) \tag{20}$$

which, in the absence of viscosity ( $f_0(x, t) \equiv 0$ , i.e.  $\Phi(t) \equiv 0$ ), reduces to the classical D'Alembert wave equation

<sup>2</sup> We recall the definition of the time convolution between two locally integrable functions  $f(t), g(t)$ :

$$\begin{aligned} f(t) * g(t) &:= \int_0^t f(\tau)g(t - \tau) d\tau \\ &= \int_0^t f(t - \tau)g(\tau) d\tau, \end{aligned}$$

so that its Laplace transform reads  $\tilde{f}(s)\tilde{g}(s)$ .

$$Y_{tt}(x, t) = c_0^2 Y_{xx}(x, t). \tag{21}$$

As a matter of fact, Eq. (20) is an integro-partial differential equation, which could be taken into account in order to investigate wave processes in fluid filled compliant tubes of biophysical interest. In particular, this evolution equation could be solved for  $x \geq 0, t \geq 0$  together with a known input condition  $Y(0, t) = Y_0(t)$  at  $x = 0$ , assuming that the tube is initially quiescent, and that there are no waves coming from  $x = +\infty$ . This initial boundary-value problem is usually referred to as a *signalling problem*. The above simplifying assumptions are expected to explain the qualitative features of the distortion (due to the viscosity of the fluid) of the transient waves generated at the accessible end of the tube as they propagate along the tube.

#### 4 The dynamic viscoelastic analogy

From Eq. (20) we easily recognize an analogy with the wave equation of linear viscoelasticity in the *Relaxation Representation*, see [14]. For this purpose we will prove that the memory function  $\Phi(t)$  is completely monotonic (i.e. it is a non-negative, non-increasing function with infinitely many derivatives alternating in sign for  $t \geq 0$ ), as required to represent (with opposite sign) the non-dimensional rate of the relaxation modulus  $G(t)$  scaled with its initial value  $0 < G(0^+) < \infty$  according to

$$\Phi(t) = -\frac{1}{G(0^+)} \frac{dG}{dt}, \tag{22}$$

and consequently in Laplace domain

$$s\tilde{G}(s) = G(0^+) [1 - \tilde{\Phi}(s)]. \tag{23}$$

We recall that  $G(t)$  represents the stress response to a unit step of strain, see [14] and it is usually assumed to be completely monotone as well. Being  $G(0^+)$  finite and positive, the corresponding viscoelastic solid exhibits stress waves with a finite wave front velocity  $c_0 = \sqrt{G(0^+)/\rho}$ .

In fact, carrying out the inversion of the Laplace transform (17), by means of the Bromwich formula and applying the residues theorem, we get the memory function in terms of a convergent Dirichlet series as

$$\Phi(t) = \frac{4}{\tau} \sum_{n=1}^{\infty} \exp(-\lambda_n^2 t / \tau) \tag{24}$$

where  $\lambda_n$  are the zeros of the oscillating Bessel function  $J_0$  on the positive real axis. Further details about the proof can be found in the “Appendix”. So, we recognize that the relaxation memory function, resulting from the sum of a convergent Dirichlet series with positive coefficients, is indeed a complete monotone function, ensuring the complete monotonicity of the corresponding relaxation modulus  $G(t)$ , with  $G(+\infty) = 0$ .

In Linear Viscoelasticity, the relaxation modulus  $G(t)$  is coupled with another material function, the *creep compliance*  $J(t)$ , which represents the strain response to a unit step of stress. Hence, to complete the dynamic viscoelastic analogy we now consider, in addition to the *relaxation representation* of the wave equation (20), also the corresponding *creep representation*.

We first observe that the two material functions  $G(t)$  and  $J(t)$  are connected through their Laplace transforms by the following fundamental equation

$$s \tilde{J}(s) = \frac{1}{s \tilde{G}(s)} \tag{25}$$

that implies in the time domain

$$J(0^+) = \frac{1}{G(0^+)} > 0, \quad J(+\infty) = \frac{1}{G(+\infty)} = +\infty. \tag{26}$$

This means, according with the classification of viscoelastic media (see [14]), the model of our concern can be understood as of Type II.

Introducing the non-dimensional rate of creep  $\Psi(t)$ , defined as

$$\Psi(t) := \frac{1}{J(0^+)} \frac{dJ}{dt}, \tag{27}$$

we have

$$s \tilde{J}(s) = J(0^+) \left[ 1 + \tilde{\Psi}(s) \right]. \tag{28}$$

Then, in the Laplace domain, the functions  $\Phi(t)$  and  $\Psi(t)$  turn out to be related as

$$1 + \tilde{\Psi}(s) = \left[ 1 - \tilde{\Phi}(s) \right]^{-1} \tag{29}$$

as a consequence of Eqs. (23), (25) and (28).

Now, recalling the *recurrence relations* for the modified Bessel functions of the first kind,  $I_\nu(z)$ , i.e.

$$I_{\nu-1}(z) - \frac{2\nu}{z} I_\nu(z) = I_{\nu+1}(z),$$

with  $\nu = 1$ , after a simple manipulation we get:

$$\tilde{\Psi}(s) = \frac{2}{\sqrt{s\tau}} \frac{I_1(\sqrt{s\tau})}{I_2(\sqrt{s\tau})}. \tag{30}$$

The inversion of the Laplace transform can be carried out using again the Bromwich formula. Then, applying the residues theorem we get:

$$\Psi(t) = \frac{8}{\tau} + \frac{4}{\tau} \sum_{n=1}^{\infty} \exp(-\mu_n^2 t / \tau) \tag{31}$$

where  $\mu_n$  are the zeros of the oscillating Bessel function  $J_2$  on the positive real axis.

Therefore, we recognize that the  $\Psi$  function, resulting from the sum of a convergent Dirichlet series with positive coefficients, is also a complete monotone function like  $\Phi(t)$ . As a consequence, the creep compliance  $J(t)$  turns out to be a Bernstein function (that is a non negative function with a completely monotone first derivative). We note, however, the presence of a positive constant term added to the Dirichlet series. This implies that the asymptotic expression for the creep compliance  $J(t)$ , for long times, contains an extra term which is linear in time. The latter is a typical feature of the classical Maxwell model of viscoelasticity.

Using the analogy with the linear dynamical theory of viscoelasticity, the wave equation corresponding to the creep representation reads, see [14],

$$\left[ 1 + \Psi(t) * \right] Y_{tt} = c_0^2 Y_{xx}, \tag{32}$$

where  $\Psi(t)$  can be referred to as the *creep memory function*. Given the previous results, we can now trivially compute the relaxation modulus  $G(t)$  and the creep compliance  $J(t)$ .

In view of Eqs. (22) and (27), we get  $G(t)$  and  $J(t)$

$$G(t) = G(0^+) \left[ 1 - \int_0^t \Phi(t') dt' \right], \tag{33}$$

$$J(t) = J(0^+) \left[ 1 + \int_0^t \Psi(t') dt' \right]. \tag{34}$$

Integrating term by term we get

$$G(t) = G(0^+) \left\{ 1 - \sum_{n=1}^{\infty} \left[ \frac{4}{\lambda_n^2} - \frac{4}{\lambda_n^2} e^{-\lambda_n^2 t/\tau} \right] \right\}, \tag{35}$$

$$J(t) = J(0^+) \left\{ 1 + \frac{8t}{\tau} + \sum_{n=1}^{\infty} \left[ \frac{4}{\mu_n^2} - \frac{4}{\mu_n^2} e^{-\mu_n^2 t/\tau} \right] \right\}. \tag{36}$$

Due to the results for the the zeros of Bessel functions shown in [20], i.e.

$$\sum_{n=1}^{\infty} \frac{4}{\lambda_n^2} = 1, \quad J_0(\lambda_n) = 0, \quad \forall n \in \mathbb{N}, \tag{37}$$

$$\sum_{n=1}^{\infty} \frac{4}{\mu_n^2} = \frac{1}{3}, \quad J_2(\mu_n) = 0, \quad \forall n \in \mathbb{N}, \tag{38}$$

we get

$$G(t) = G(0^+) \sum_{n=1}^{\infty} \frac{4}{\lambda_n^2} \exp(-\lambda_n^2 t/\tau), \tag{39}$$

$$J(t) = J(0^+) \left\{ \frac{4}{3} + \frac{8t}{\tau} - \sum_{n=1}^{\infty} \frac{4}{\mu_n^2} \exp(-\mu_n^2 t/\tau) \right\}. \tag{40}$$

As we see from Eq. (35), together with the condition  $G(+\infty) = 0$ , we recover the identity in Eq. (37). This result gave us a relevant hint to derive Sneddon’s result about series involving zeros of Bessel functions of the first kind, by means of a technique based on the Laplace transform (see [7]).

### 5 Asymptotic and numerical results

In this section we exhibit the plots (versus time) of the relaxation and creep memory functions as computed taking a suitable number of terms in the corresponding Dirichlet series (24) and (31), respectively. From our numerical experiments, we found that 100 terms are surely suitable to get stable and confident results.

We also compare these plots with the asymptotic representations of the two functions for short and long time in order to check their matching with the numerical solutions. According to the Tauberian theorems, the asymptotic representations of  $\Phi(t)$  and

$\Psi(t)$ , for short times, are *formally* derived by inverting their Laplace transforms (15) and (30), approximated as  $s\tau \rightarrow \infty$  respectively, see “Appendix”. For large times, the asymptotic representations are promptly obtained by taking the first term of the corresponding Dirichlet series. We get:

$$\Phi(t) \sim \begin{cases} \frac{2}{\sqrt{\pi\tau}} t^{-1/2}, & t \rightarrow 0, \\ \frac{4}{\tau} \exp(-\lambda_1^2 t/\tau), & t \rightarrow \infty, \end{cases} \tag{41}$$

with  $\lambda_1^2 \approx 5.78$  and

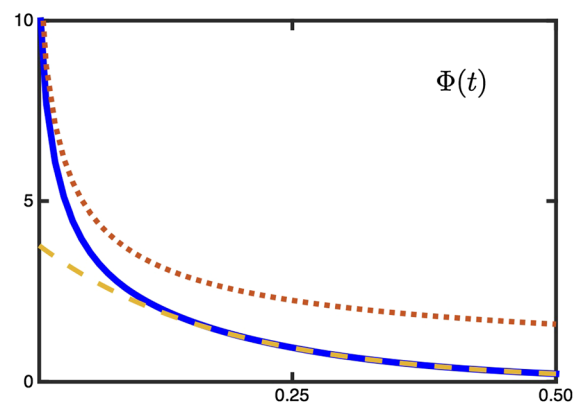
$$\Psi(t) \sim \begin{cases} \frac{2}{\sqrt{\pi\tau}} t^{-1/2}, & t \rightarrow 0, \\ \frac{8}{\tau} + \frac{4}{\tau} \exp(-\mu_1^2 t/\tau), & t \rightarrow \infty, \end{cases} \tag{42}$$

with  $\mu_1^2 \approx 26.37$ .

In Figs. 1 and 2 we show the plots of the relaxation and creep memory functions versus time, by assuming  $\tau = 1$  and we compare them with their asymptotic representations for short and long times, printed with dotted and dashed lines, respectively.

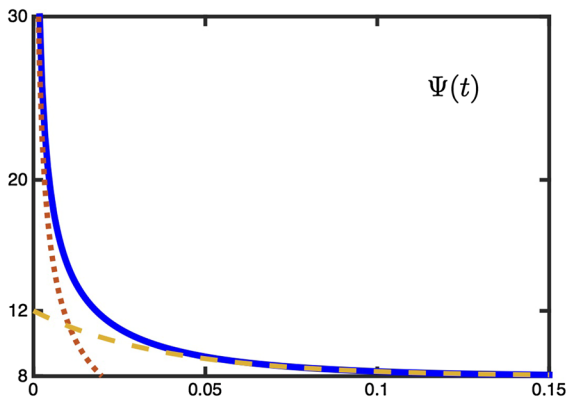
Finally, in Figs. 3 and 4 we show the plots of the  $G(t)$  and  $J(t)$  respectively, versus time, by assuming  $\tau = 1$ ,  $G(0^+) = J(0^+) = 1$  in Eqs. (39) and (40) and taking 100 terms in the corresponding Dirichlet series.

Considering the asymptotic and numerical results, as well as the results in [14, 16], we observe that the

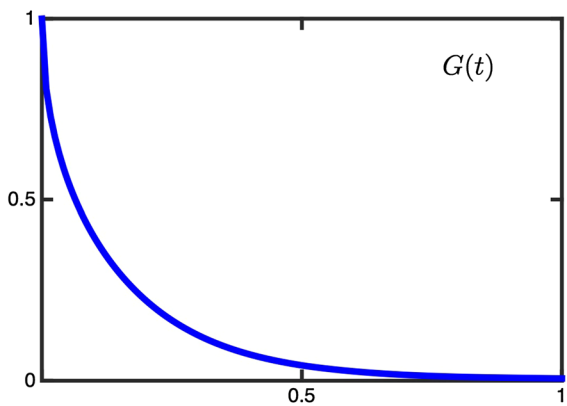


**Fig. 1** The relaxation-memory function  $\Phi(t)$  (continuous line) with its asymptotic representations (dotted and dashed lines) versus time scaled with  $\tau$

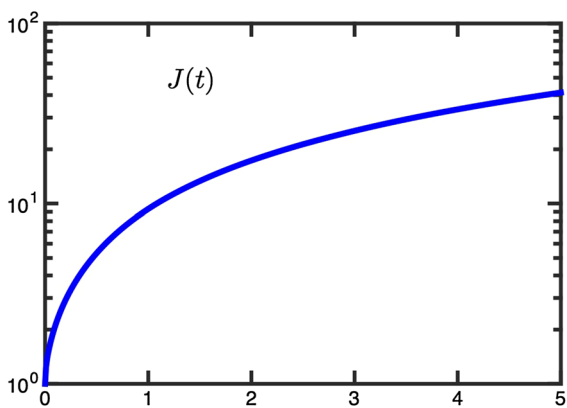




**Fig. 2** The creep-memory function  $\Psi(t)$  (continuous line) with its asymptotic representations (dotted and dashed lines) versus time scaled with  $\tau$



**Fig. 3** The normalized relaxation modulus  $G(t)$  versus time scaled with  $\tau$



**Fig. 4** The normalized creep compliance  $J(t)$  versus time scaled with  $\tau$

relaxation and creep properties of this model are consistent with those of a peculiar viscoelastic medium that behaves since short times as a fractional Maxwell solid of order 1/2 up to as a standard Maxwell solid for long times.

### 6 Conclusions and final remarks

In this paper we have shown that, from the dynamic point of view of uni-axial wave propagation, an elastic tube filled with a viscous fluid is equivalent to a viscoelastic solid with particular rheological properties that evolve as shown in Sect. 5. We note that this analogy suffers of subtleties due to the presence of Dirichlet series. In any case this model is worth to be investigated in future with respect to the evolution of the transient waves propagating through it. In fact, as formerly pointed out by Buchen and Mainardi [4], see also [14], the singular behavior of the creep function at  $t = 0$  induces a wave-front smoothing of any initial discontinuity, like a diffusion effect. For more details on this smoothing effect found for transient waves propagating in media with a singular memory we refer to [9].

For that concerning the profile of the fluid inside the tube, we understand that the Poiseuille parabolic profile, corresponding to the Maxwell model, found for long times is preceded by a boundary layer profile, corresponding to the fractional Maxwell model of order 1/2 for short times. This fact is compatible with what expected in a pulsatile blood flow in large arteries for slow and high frequencies and it is essentially due to the effect of the blood viscosity. For further research we propose to investigate the evolution of the shock formation due to this boundary layer following the preliminary non-linear analysis by Mainardi and Buggisch [15].

Finally, we note that more realistic models for 1D blood flow take into account the viscoelastic nature of the arterial wall. On this respect the literature is wide but more recent articles require fractional viscoelastic models as well, see e.g. [17] and references therein. So we point out the necessity to include in the available 1D models for the arterial wall viscoelasticity the effects of the blood viscosity discussed in the present paper.

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**Appendix: Mathematical discussions**

We provide the details for the proof of some statements found in the text.

Proof of the expressions (24) and (31)

Consider the Laplace Transform of the *Relaxation-memory function*  $\tilde{\Phi}(s)$

$$\tilde{\Phi}(s) = \frac{2 I_1(\sqrt{s\tau})}{\sqrt{s\tau} I_0(\sqrt{s\tau})} \tag{43}$$

where we will consider  $\tau = 1$  for sake of simplicity (it can be restored by make the substitution  $s \leftrightarrow s\tau$ ). Then, the Relaxation-memory function is given by

$$\Phi(t) = 4 \sum_{n=1}^{\infty} \exp(-\lambda_n^2 t) \tag{44}$$

where  $\lambda_n \in \mathbb{R}$  are such that  $J_0(\lambda_n) = 0, \forall n = 1, 2, 3, \dots$  and  $t > 0$ .

*Proof* Firstly, consider the power series representation for the Modified Bessel functions of the First Kind:

$$\begin{aligned} I_0(\sqrt{s}) &= 1 + \frac{s}{4} + \frac{s^2}{64} + O(s^3), \\ I_1(\sqrt{s}) &= \sqrt{s} \left( \frac{1}{2} + \frac{s}{16} + \frac{s^2}{384} \right) + O(s^{7/2}), \end{aligned} \tag{45}$$

one can eventually deduce that

$$\tilde{\Phi}(s) = \frac{2 I_1(\sqrt{s})}{\sqrt{s} I_0(\sqrt{s})} = 2 \frac{\frac{1}{2} + \frac{s}{16} + \frac{s^2}{384} + \dots}{1 + \frac{s}{4} + \frac{s^2}{64} + \dots} \tag{46}$$

which means that the function of our concern is regular in  $s = 0$  and it does not have any branch cuts. Then we can obtain the required function by means of the Bromwich Integral:

$$\Phi(t) = \frac{1}{2\pi i} \int_{Br} \tilde{\Phi}(s) e^{st} ds. \tag{47}$$

In particular,  $\tilde{\Phi}(s)$  has simple poles such that:  $I_0(\sqrt{s}) = 0$ . Now, if we rename  $\sqrt{s}$  as  $\sqrt{s} = -i\lambda$ , then

$$I_0(\sqrt{s}) = 0 \iff J_0(\lambda) = 0.$$

Moreover,

$$\sqrt{s_n} = -i\lambda_n \iff s_n = -\lambda_n^2, \quad n \in \mathbb{N}. \tag{48}$$

From the previous statements, we can then conclude that:

$$\begin{aligned} \Phi(t) &= \sum_{s_n} \mathcal{R}es \left\{ \tilde{\Phi}(s) e^{st} \right\}_{s_n} \\ &= \sum_{n=1}^{\infty} \mathcal{R}es \left\{ \frac{2 I_1(\sqrt{s})}{\sqrt{s} I_0(\sqrt{s})} e^{st} \right\}_{s=-\lambda_n^2}. \end{aligned} \tag{49}$$

It is quite straightforward that

$$\begin{aligned} \mathcal{R}es \left\{ \frac{2 I_1(\sqrt{s}) e^{st}}{\sqrt{s} I_0(\sqrt{s})} \right\}_{s=s_n} \\ = \lim_{s \rightarrow s_n} (s - s_n) \frac{2 I_1(\sqrt{s}) e^{st}}{\sqrt{s} I_0(\sqrt{s})} \\ = 4 \exp(s_n t). \end{aligned} \tag{50}$$

Thus,

$$\Phi(t) = \sum_{n=1}^{\infty} \mathcal{R}es \left\{ \tilde{\Phi}(s) e^{st} \right\}_{s=-\lambda_n^2} = 4 \sum_{n=1}^{\infty} e^{-\lambda_n^2 t}. \tag{51}$$

□

Let us now consider the Laplace Transform of the **Creep-memory function**  $\tilde{\Psi}(s)$

$$\tilde{\Psi}(s) = \frac{2 I_1(\sqrt{s})}{\sqrt{s} I_2(\sqrt{s})}. \tag{52}$$

Then  $\Psi(t)$  is given by

$$\Psi(t) = 8 + 4 \sum_{n=1}^{\infty} \exp(-\mu_n^2 t) \tag{53}$$

where  $\mu_n$  are such that  $J_2(\mu_n) = 0$  and  $\mu_n \neq 0$ , for  $n \in \mathbb{N}$ .

*Proof* By means of the same procedure shown before we are able to point out that



$$\begin{aligned} \Psi(t) &= \sum_{s_n} \text{Res} \left\{ \tilde{\Psi}(s) e^{st} \right\}_{s_n} \\ &= \sum_{n=0}^{\infty} \text{Res} \left\{ \frac{2}{\sqrt{s}} \frac{I_1(\sqrt{s})}{I_2(\sqrt{s})} e^{st} \right\}_{s=-\mu_n^2} \end{aligned} \tag{54}$$

with  $\mu_n$  such that  $J_2(\mu_n) = 0$ ,  $\mu_n \neq 0$ , for  $n \in \mathbb{N}$  and  $\mu_0 \equiv 0$ .

Now, we have to distinguish two cases:

If  $s_n \neq 0$ , then

$$\text{Res} \left\{ \tilde{\Psi}(s) e^{st} \right\}_{s_n} = 4 \exp(s_n t). \tag{55}$$

Otherwise, if  $s_n = \mu_0 = 0$ ,

$$\text{Res} \left\{ \tilde{\Psi}(s) e^{st} \right\}_{s_n=0} = \lim_{s \rightarrow 0} s \tilde{\Psi}(s) e^{st} = 8. \tag{56}$$

Thus,

$$\begin{aligned} \Psi(t) &= \sum_{n=0}^{\infty} \text{Res} \left\{ \tilde{\Psi}(s) e^{st} \right\}_{s=-\mu_n^2} \\ &= 8 + 4 \sum_{n=1}^{\infty} \exp(-\mu_n^2 t). \end{aligned} \tag{57}$$

□

From the above results we are able to conclude that representation of both memory function  $\Phi(t)$  and  $\Psi(t)$  are given by Dirichlet series (44) and (53), whose convergence is proved in the following.

On the convergence of the Dirichlet series (44) and (53)

The series (44) is convergent for  $t > 0$ .

*Proof* Consider a Generalized Dirichlet Series:

$$f(z) = \sum_{n=1}^{\infty} a_n \exp(-\alpha_n z), \quad z \in \mathbb{C}. \tag{58}$$

In general, we have that the abscissa of convergence and the abscissa of absolute convergence would be different, i.e.  $\sigma_c \neq \sigma_a$ , but they will satisfy the following condition:

$$0 \leq \sigma_a - \sigma_c \leq d = \limsup_{n \rightarrow \infty} \frac{\ln n}{\alpha_n}. \tag{59}$$

If  $d = 0$ , then

$$\sigma \equiv \sigma_c = \sigma_a = \limsup_{n \rightarrow \infty} \frac{\ln |a_n|}{\alpha_n}. \tag{60}$$

In our case  $a_n = 1$  and  $\alpha_n = \lambda_n^2 \neq 0$ . Then, we have to understand the behavior of the coefficients  $\lambda_n$  for  $n \gg 1$ , where  $J_0(\lambda_n) = 0$ ,  $\forall n \in \mathbb{N}$ .

Considering the asymptotic representation

$$J_0(x) \underset{x \gg 1}{\sim} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \tag{61}$$

we get to the following conclusion:

$$J_0(\lambda_n) = 0, \quad \text{for } n \gg 1 \implies \lambda_n \propto n, \quad \text{for } n \gg 1. \tag{62}$$

Thus,

$$\frac{\ln n}{\alpha_n} = \frac{\ln n}{\lambda_n^2} \underset{n \gg 1}{\sim} \frac{\ln n}{n^2} \xrightarrow{n \rightarrow \infty} 0 \tag{63}$$

which tells us that  $d = 0$ . Finally,

$$\sigma \equiv \sigma_c = \sigma_a = \limsup_{n \rightarrow \infty} \frac{\ln |a_n|}{\alpha_n} = 0 \tag{64}$$

being  $a_n = 1$ .

This result implies that the series (44), with  $a_n = 1$  and  $\alpha_n = \lambda_n^2 \neq 0$ , converges for  $\text{Re}\{z\} = t > 0$ . □

In a similar way a we can prove that the series (53) is convergent for  $t > 0$ .

On the asymptotic representations

Finally, we derive the asymptotic representations for  $\Phi(t)$  and  $\Psi(t)$  as  $t \rightarrow 0^+$  applying the Tauberian theorems to the corresponding Laplace transforms:

$$\tilde{\Phi}(s) = \frac{2}{\sqrt{s\tau}} \frac{I_1(\sqrt{s\tau})}{I_0(\sqrt{s\tau})}, \quad \tilde{\Psi}(s) = \frac{2}{\sqrt{s\tau}} \frac{I_1(\sqrt{s\tau})}{I_2(\sqrt{s\tau})}. \tag{65}$$

Then, in view of the asymptotic representation of the modified Bessel functions as  $z \rightarrow \infty$  with  $|\arg(z)| < \pi/2$  and for any  $\nu$

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi}} z^{-1/2} \exp(z),$$

see e.g. [1], we conclude that for  $z = \sqrt{s\tau} \rightarrow \infty$  ( $s > 0$ ) we get

$$\tilde{\Phi}(s) \sim \frac{2}{\sqrt{st}}, \quad \tilde{\Psi}(s) \sim \frac{2}{\sqrt{st}},$$

so that

$$\begin{aligned} \Phi(t) &\sim \frac{2}{\sqrt{\pi\tau}} t^{-1/2}, \quad t \rightarrow 0^+, \\ \Psi(t) &\sim \frac{2}{\sqrt{\pi\tau}} t^{-1/2}, \quad t \rightarrow 0^+. \end{aligned} \quad (66)$$

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