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Steady planar ideal flow of anisotropic materials

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Abstract This paper extends the ideal flow theory, which is well known for isotropic rigid perfectly plastic materials, to quite general orthotropic materials which comply with the principle of maximum plastic dissipation. The new theory is restricted to steady planar flow. The original ideal flow theory is widely used as the basis for inverse methods for the preliminary design of metal forming processes driven by minimum plastic work. The new theory extends this area of application to orthotropic materials. Moreover, another design criterion based on the Cockroft–Latham ductile fracture criterion is incorporated in the theory. To this end, the extended Bernoulli's theorem relating pressure and velocity along any streamline during the steady planar flow of rigid perfectly plastic solids when the streamline is coincident everywhere with a principal stress trajectory is used. In particular, this theorem and the concept of ideal flow combine to evaluate the integral involved in the ductile fracture criterion. The final result is a simple relation between process parameters and the constitutive parameter involved in the ductile fracture criterion. The simplicity of this relation makes it suitable for quick design of metal forming processes.

Keywords Ductile fracture · Ideal flow · Bernoulli's theorem · Orthotropic material · Rigid plastic material

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1 Introduction

Ideal plastic flows are those for which all material elements undergo minimum work paths [1]. The theory of bulk ideal flow has been developed for rigid perfectly plastic solids satisfying Tresca's yield condition and its associated flow rule. In particular, the existence of steady three-dimensional ideal flows in such solids has been demonstrated in [2]. This result has been extended to non-steady flows in [3]. On the other hand, it has been clarified in [4] that an additional condition of the existence of steady three-dimensional ideal flow is that the perimeter of the product cross-section is not larger than that of the input cross-section. A comprehensive overview on the ideal flow theory has been provided in [5]. This theory has been used as the basis for inverse methods for the preliminary design of bulk metal forming processes driven by minimum plastic work [6–8]. In particular, optimal dies for extrusion (or drawing) have been found in these papers. Necessary conditions for the optimality of extrusion dies for rigid plastic materials obeying strictly convex yield criteria have been derived in [9].

The development of simplified methods for analysis and design of metal forming processes is of importance for applications since standard finite element simulations are too slow for many purposes [10]. It is worthy of note that the ideal flow theory results in rather a general method of analyzing and designing metal forming processes, whereas many other simplified methods deal with a specific process (see, for example, [11] for plate rolling and [12] for wire drawing). A disadvantage of the existing ideal flow theory is that it is in general restricted to isotropic rigid perfectly plastic materials. It has been shown in [13] that elasticity can be incorporated in the theory for steady planar flow. However, in many cases elasticity is not so important for analyzing metal forming processes. In particular, rigid plastic models are used even in conjunction with finite element methods (see, for example, [14, 15]). On the other hand, plastic anisotropy is very common to metallic materials [16]. This material property has been incorporated in the ideal flow theory for sheet metal forming [17, 18]. The present paper concerns with the theory of bulk ideal flow for plane strain deformation of anisotropic materials assuming that the model of anisotropic plasticity proposed in [19] is valid.

In most bulk forming processes, formability is limited by ductile fracture [20]. It is therefore of importance to incorporate a ductile fracture model in the ideal flow theory. Empirical ductile fracture criteria are widely used to predict the initiation of fracture in metal forming processes. In particular, such criteria are included in modern commercial finite element packages. Reviews of empirical ductile fracture criteria are provided, for example, in [21–23]. In the present paper, the fracture criterion proposed in [24] is adopted. Note that a modified version of this criterion has been introduced in [25]. However, the original and modified criteria coincide in the case under consideration. Therefore, both criteria are referred to as the Cockroft and Latham criterion in the present paper. This criterion has been used and/or evaluated for several metals in [23, 26–33]. In particular, it has been mentioned in [26, 27, 29] that the Cockroft and Latham criterion predicts the initiation of ductile fracture more accurately than the other ductile fracture criteria considered in that papers. The approach developed in the present paper to incorporate the Cockroft and Latham criterion in the ideal flow theory is based on the extended Bernoulli theorem proven in [34]. A remarkable property of this approach is that there is no need to know the solution of a plasticity problem to apply the Cockroft and Latham criterion. The final expression is very simple and can be directly used for preliminary design driven by ductile fracture.

2 Material model and deformation process

It has been shown in [35] that the plane strain yield criterion of any incompressible anisotropic material which complies with the principle of maximum plastic dissipation is expressed solely in terms of the stress variables $s = (\sigma_{\alpha\alpha} - \sigma_{\beta\beta})/2$ and $\tau = \sigma_{\alpha\beta}$ where $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$ and $\sigma_{\alpha\beta}$ are the components of the stress tensor in an arbitrary curvilinear orthogonal coordinate system (α, β). Therefore, the yield criterion may be represented as

$$F(s, \tau) = 0. \tag{1}$$

The function $F(s, \tau)$ must satisfy the standard requirements imposed on the yield criteria in the theory of rigid plastic materials based on the associated flow rule. It is evident from the definition for s and τ that the yield function is independent of the mean stress. The subsequent investigation is restricted to the orthotropic form of initial anisotropy. This form is most common to metallic materials [16]. In order to apply the associated flow rule, τ should be represented as $\tau = 1/2(\sigma_{\alpha\beta} + \sigma_{\beta\alpha})$ where $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$. Then, Eq. (1) and this rule combine to give

$$\xi_{\alpha\alpha} = \lambda \frac{\partial F}{\partial s}, \quad \xi_{\beta\beta} = -\lambda \frac{\partial F}{\partial s}, \quad \xi_{\alpha\beta} = \lambda \frac{\partial F}{\partial \tau}. \tag{2}$$

Here $\xi_{\alpha\alpha}$, $\xi_{\beta\beta}$ and $\xi_{\alpha\beta}$ are the components of the strain rate tensor in the (α, β) coordinate system and λ is a non-negative multiplier. It is assumed that there is no Bauschinger effect. Then, the yield criterion (1) does not contain linear terms. Quadratic terms in which the shear stress occurs linearly are rejected in view of the symmetry restriction [36]. In this case $\partial F/\partial \tau = 0$ at $\tau = 0$ and it follows from (2) that

$$\xi_{\alpha\beta} = 0 \quad \text{if} \quad \sigma_{\alpha\beta} = 0. \tag{3}$$

It also follows from (2) that

$$\xi_{\alpha\alpha} + \xi_{\beta\beta} = 0. \tag{4}$$

It is evident that this is the equation of incompressibility. It is supposed that the evolution of anisotropy obeys the model proposed in [19].

Quite a general steady plane strain process of deformation of rigid plastic material is shown schematically in Fig. 1. There are two rigid zones and one plastic zone. Let U_1 be the velocity of rigid zone 1 and U_2 be the velocity of rigid zone 2. Then, it follows from the incompressibility Eq. (4) that

$$U_1 H_1 = U_2 H_2 \tag{5}$$

where H_1 is the thickness of the strip in rigid zone 1 and H_2 is the thickness of the strip in rigid zone 2. The surface of tool is frictionless and the velocity vector is continuous across rigid plastic boundaries in steady ideal flow [2].

The complete system of equations to solve comprises Eqs. (1), (2) and the equilibrium equations.

3 Steady planar ideal flow of orthotropic material

The ideal flow theory deals with non-standard boundary value problems of plasticity. In particular, the shape of tool is unknown and should be found such that all material elements undergo minimum work paths. The latter is advantageous for a number of metal forming processes. Therefore, the ideal flow theory deals with the design of such processes. This kind of boundary value problems is difficult to solve by standard numerical methods. In addition, the system of equations is hyperbolic [35]. This greatly adds to the difficulties of numerical solutions, unless the method of characteristics is used.

Consider three curvilinear orthogonal right-handed coordinate systems; namely, (x_1, y_1), (x_2, y_2) and (x_3, y_3). The (x_1, y_1) coordinate system is defined by the condition that the x_1-coordinate curves coincide with streamlines, the (x_2, y_2) coordinate system by the condition that the x_2-coordinate curves coincide with trajectories of the greatest principal stress, and the (x_3, y_3) coordinate system by the condition that the x_3-coordinate curves coincide with one of the curvilinear principal axes of anisotropy. In general, these three coordinate systems are different (Fig. 2a). However, by definition (see, for example, [6]), the (x_1, y_1) and (x_2, y_2) coordinate systems coincide in steady planar ideal flow (Fig. 2b). Let us make an additional assumption that the (x_1, y_1) and (x_3, y_3) coordinate systems coincide at point M of a generic streamline

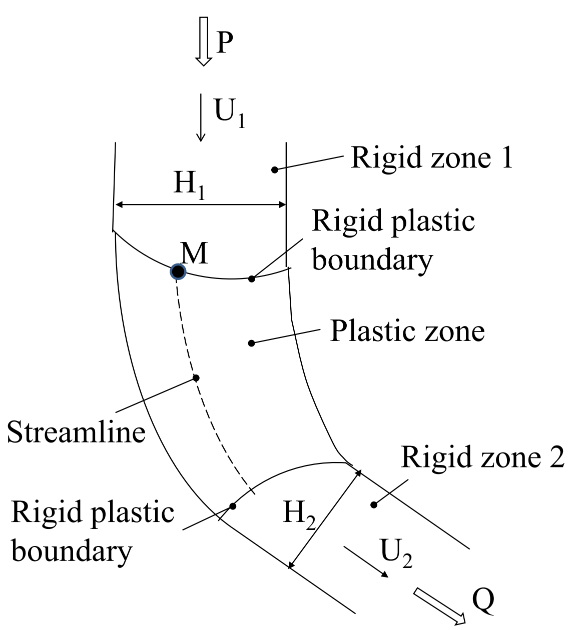

Fig. 1 Schematic diagram of a typical steady plane strain process

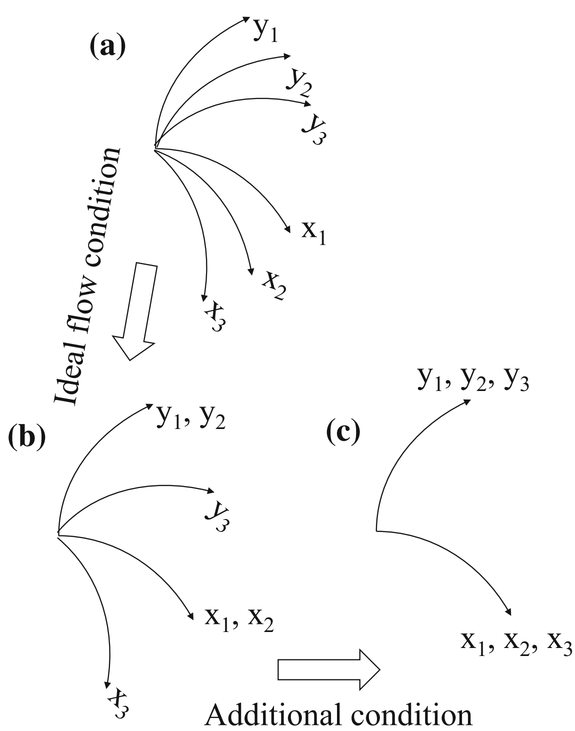

Fig. 2 Influence of the ideal flow and additional assumptions on the orientation of $(x_1\ y_1)$, $(x_2\ y_2)$ and $(x_3\ y_3)$ coordinate systems

(Fig. 1). Then, the constitutive equation proposed in [19] shows that the (x_1, y_1) and (x_3, y_3) coordinate systems coincide along this streamline and, therefore, everywhere in the plastic zone (Fig. 2c). This means that the yield locus is invariant along the motion. The importance of this property of constitutive equations has been emphasized in [37, 38]. The velocity vector in rigid zone 1 is parallel to the walls of the container. A requirement of ideal flow is that there is no velocity discontinuity across rigid plastic boundaries. Therefore, the tangent to the streamline at M is parallel to the walls of the container (Fig. 1). Thus the additional assumption made requires that the principal axes of anisotropy in rigid zone 1 be parallel and orthogonal to the walls of the container. Using the same arguments it is possible to demonstrate that the principal axes of anisotropy in rigid zone 2 are parallel and orthogonal to the direction of the velocity vector of this zone. The coincidence of the (x_1, y_1), (x_2, y_2) and (x_3, y_3) coordinate systems is an additional condition imposed on the standard system of equations of the anisotropic rigid plastic material under consideration. The existence of non-trivial steady ideal flows depends on the possibility to satisfy this additional condition without violating the standard system of equations for a sufficiently large class of problems. The present section provides a proof of the existence of steady planar ideal flow for the material model chosen. In what follows, it is assumed that $x_1 \equiv x_2 \equiv x_3 \equiv \alpha$ and $y_1 \equiv y_2 \equiv y_3 \equiv \beta$. In this case $\tau = \sigma_{\alpha\beta} = 0$ everywhere. Therefore, Eq. (3) results in

$$\xi_{\alpha\beta} = 0 \tag{6}$$

and the yield criterion (1) reduces to

$$\sigma_{\alpha\alpha} - \sigma_{\beta\beta} = mK \tag{7}$$

where K is a material constant, $m = +1$ if $\sigma_{\alpha\alpha} > \sigma_{\beta\beta}$ and $m = -1$ if $\sigma_{\alpha\alpha} < \sigma_{\beta\beta}$. Equations (4) and (6) replace two equations of the associated flow rule (2). The third equation determines λ. Since $\sigma_{\alpha\beta} = 0$, the equilibrium equations are [39]

$$h_\beta \frac{\partial \sigma_{\alpha\alpha}}{\partial \alpha} + \left(\sigma_{\alpha\alpha} - \sigma_{\beta\beta}\right) \frac{\partial h_\beta}{\partial \alpha} = 0,$$
$$h_\alpha \frac{\partial \sigma_{\beta\beta}}{\partial \beta} + \left(\sigma_{\beta\beta} - \sigma_{\alpha\alpha}\right) \frac{\partial h_\alpha}{\partial \beta} = 0. \tag{8}$$

Here h_α and h_β are the scale factors for the α- and β-lines, respectively. Eliminating $\sigma_{\alpha\alpha} - \sigma_{\beta\beta}$ in (8) by means of (7) and integrating yield

$$\frac{h_\beta}{H_\beta(\beta)} = \exp\left(-\frac{\sigma_{\alpha\alpha}}{mK}\right), \quad \frac{h_\alpha}{H_\alpha(\alpha)} = \exp\left(\frac{\sigma_{\beta\beta}}{mK}\right). \tag{9}$$

Here $H_\alpha(\alpha)$ is an arbitrary function of α and $H_\beta(\beta)$ is an arbitrary function of β. However, different choices of these functions merely change the scale of the coordinate curves. Therefore, without loss of generality it is possible to choose $H_\alpha(\alpha) = H_\beta(\beta) = \sqrt{e}$. Then, Eq. (9) reduces to

$$\frac{\sigma_{\alpha\alpha}}{mK} = \frac{1}{2} - \ln h_\beta, \quad \frac{\sigma_{\beta\beta}}{mK} = \ln h_\alpha - \frac{1}{2}. \tag{10}$$

Substituting (10) into (7) gives

$$h_\alpha h_\beta = 1. \tag{11}$$

Let u_α and u_β be the velocity components referred to the (α, β) coordinate system. The component u_β vanishes everywhere since the α-lines coincide with streamlines. Therefore, the strain rate components are given by [39]

$$\xi_{\alpha\alpha} = \frac{\partial u_\alpha}{h_\alpha \partial \alpha}, \quad \xi_{\beta\beta} = \frac{u_\alpha}{h_\alpha h_\beta} \frac{\partial h_\beta}{\partial \alpha},$$
$$2\xi_{\alpha\beta} = \frac{\partial u_\alpha}{h_\beta \partial \beta} - \frac{u_\alpha}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta}. \tag{12}$$

Equations (4) and (12) combine to yield

$$\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_\alpha}{h_\beta} \frac{\partial h_\beta}{\partial \alpha} = 0.$$

This equation can be immediately integrated to give

$$u_\alpha = \frac{V_1(\beta)}{h_\beta}. \tag{13}$$

Here $V_1(\beta)$ is an arbitrary function of β. Equations (6) and (12) combine to yield

$$\frac{\partial u_\alpha}{\partial \beta} - \frac{u_\alpha}{h_\alpha} \frac{\partial h_\alpha}{\partial \beta} = 0.$$

This equation can be immediately integrated to give

$$u_\alpha = V_2(\alpha) h_\alpha. \tag{14}$$

Here $V_2(\alpha)$ is an arbitrary function of α. It follows from Eqs. (13) and (14) that

$$h_\alpha h_\beta = \frac{V_1(\beta)}{V_2(\alpha)}. \tag{15}$$

It is evident from Eqs. (11) and (15) that the stress and velocity solutions are compatible if $V_1(\beta) = V_2(\alpha) = V_0 = \text{constant}$. Then, Eqs. (13) and (14) become

$$u_\alpha = V_0 h_\alpha. \tag{16}$$

Equations (10), (11) and (16) connect the velocity and stress components. In particular,

$$\frac{\sigma_{\alpha\alpha}}{mK} = \frac{1}{2} + \ln\left(\frac{u_\alpha}{V_0}\right), \quad \frac{\sigma_{\beta\beta}}{mK} = -\frac{1}{2} + \ln\left(\frac{u_\alpha}{V_0}\right).$$

This is a restriction imposed by the ideal flow conditions on all possible solutions for the material model chosen.

4 Design driven by ductile fracture

In the case under consideration the ductile fracture criterion proposed in [24] has the form

$$\begin{aligned} &\int_{\alpha_M}^{\alpha_f} \frac{\sigma_{\alpha\alpha}\xi_{eq}h_\alpha}{Ku_\alpha} d\alpha = C \quad \text{if } m = 1 \\ &\int_{\alpha_M}^{\alpha_f} \frac{\sigma_{\beta\beta}\xi_{eq}h_\alpha}{Ku_\alpha} d\alpha = C \quad \text{if } m = -1. \end{aligned} \tag{17}$$

Here C is a constitutive parameter, ξ_{eq} is the equivalent strain rate, α_M is the value of α at M (Fig. 1), and α_f is the value of α at the site of the initiation of ductile fracture. The definition for the equivalent strain rate is usually associated with the plastic work rate [36]. Using (4), (6) and (7) the plastic work rate is represented as

$$\frac{dw}{dt} = \sigma_{\alpha\alpha}\xi_{\alpha\alpha} + \sigma_{\beta\beta}\xi_{\beta\beta} + 2\sigma_{\alpha\beta}\xi_{\alpha\beta} = mK\xi_{\alpha\alpha} \tag{18}$$

where d/dt denotes the convected derivative. Therefore, it is natural to put $\xi_{eq} = m\xi_{\alpha\alpha}$. Then, Eq. (17) becomes

$$\begin{aligned} &\int_{\alpha_M}^{\alpha_f} \frac{\sigma_{\alpha\alpha}\xi_{\alpha\alpha}h_\alpha}{Ku_\alpha} d\alpha = C \quad \text{if } m = 1 \\ &\int_{\alpha_M}^{\alpha_f} \frac{\sigma_{\alpha\alpha}\xi_{\alpha\alpha}h_\alpha}{Ku_\alpha} d\alpha + \int_{\alpha_M}^{\alpha_f} \frac{\xi_{\alpha\alpha}h_\alpha}{u_\alpha} d\alpha = -C \quad \text{if } m = -1. \end{aligned} \tag{19}$$

Here Eq. (7) has been taken into account. In the case under consideration Eq. (18) becomes

$$u_\alpha \frac{\partial w}{\partial \alpha} = mK \frac{\partial u_\alpha}{\partial \alpha}. \tag{20}$$

Here Eq. (12) for $\xi_{\alpha\alpha}$ has been used. Equation (20) can be immediately integrated to give

$$w = mK \ln\left(\frac{u_\alpha}{U_1}\right). \tag{21}$$

It has been taken into account here that $u_\alpha = U_1$ and $w = 0$ at the rigid plastic boundary between the plastic zone and rigid zone 1 (Fig. 1). In the case of steady ideal flow the following relation is immediate from the extended Bernoulli's theorem [34],

$$\frac{\partial \sigma_{\alpha\alpha}}{\partial \alpha} - \frac{\partial w}{\partial \alpha} = 0.$$

This equation can be immediately integrated to give

$$\sigma_{\alpha\alpha} - w = Kp. \tag{22}$$

Here p is a constant of integration. Eliminating w in (21) by means of (22) yields

$$\frac{\sigma_{\alpha\alpha}}{K} = p + m \ln\left(\frac{u_\alpha}{U_1}\right). \tag{23}$$

Substituting Eq. (12) for $\xi_{\alpha\alpha}$ and (23) into (19) results in the following condition for flow without the initiation of ductile fracture

$$\begin{aligned} &\int_{U_1}^{U_2} \left[p + \ln\left(\frac{u_\alpha}{U_1}\right)\right] \frac{du_\alpha}{u_\alpha} < C \quad \text{if } m = 1 \\ &\left| \int_{U_1}^{U_2} \left[p - \ln\left(\frac{u_\alpha}{U_1}\right)\right] \frac{du_\alpha}{u_\alpha} + \int_{U_1}^{U_2} \frac{du_\alpha}{u_\alpha} \right| < C \quad \text{if } m = -1. \end{aligned} \tag{24}$$

It has been taken into account here that $u_\alpha = U_1$ at the rigid plastic boundary between the plastic zone and rigid zone 1 and $u_\alpha = U_2$ at the rigid plastic boundary between the plastic zone and rigid zone 2 (Fig. 1). Integrating Eq. (24) gives

$$\begin{aligned} &p\ln\left(\frac{U_2}{U_1}\right)+\frac{1}{2}\ln^2\left(\frac{U_2}{U_1}\right)<C \qquad \text{if } m=1 \\ &\left|(p+1)\ln\left(\frac{U_2}{U_1}\right)-\frac{1}{2}\ln^2\left(\frac{U_2}{U_1}\right)\right|<C \quad \text{if } m=-1 \end{aligned}. \tag{25}$$

Using (5) Eq. (25) may be transformed to

$$\begin{aligned} &p\ln\left(\frac{H_1}{H_2}\right)+\frac{1}{2}\ln^2\left(\frac{H_1}{H_2}\right)<C \qquad \text{if } m=1 \\ &\left|(p+1)\ln\left(\frac{H_1}{H_2}\right)-\frac{1}{2}\ln^2\left(\frac{H_1}{H_2}\right)\right|<C \quad \text{if } m=-1. \end{aligned} \tag{26}$$

This result is valid for any steady planar ideal flow of the anisotropic material under consideration.

5 Fracture in drawing

A schematic diagram of the drawing/extrusion process is shown in Fig. 1. The shape of the die is determined by an ideal flow solution. This shape can be calculated using the general theory developed in Sect. 3. However, the initiation of ductile fracture can be predicted without knowing the exact shape of the die. In the case of drawing $m = 1$, $P = 0$ and $Q \neq 0$. Therefore, $\sigma_{\alpha\alpha} = 0$ along the rigid plastic boundary between the plastic zone and rigid zone 1. By definition, $w = 0$ along this boundary. Then, it follows from (22) that $p = 0$. Substituting this value of p into (26) gives

$$\ln^2\left(\frac{H_1}{H_2}\right)<2C. \tag{27}$$

Ductile fracture does not initiate if this inequality is satisfied.

6 Conclusions

It has been shown that non-trivial steady planar ideal flow solutions exist in anisotropic plasticity assuming that the model proposed in [19] is valid. In this case the yield locus is invariant along the motion. The importance of this property of constitutive equations has been emphasized in [37, 38]. An additional requirement, as compared to ideal flow in isotropic plasticity, is that the principal axes of anisotropy in rigid zone 1 (Fig. 1) are parallel and perpendicular to the walls of the container. This requirement is not so restrictive. For example, the principal axes of anisotropy induced by flat rolling are parallel and perpendicular to the sides of products. The theory developed in Sect. 3 provides an efficient method of metal forming design driven by minimum plastic work. Using this method an optimal shape of tool can be found in the same manner as in isotropic plasticity (see, for example, [6]). In addition, it has been shown that the design based on the ideal flow theory can be supplemented with the Cockroft–Latham ductile fracture criterion by means of Eq. (26). This simple equation relates the constitutive parameter C, which is supposed to be known for a given material, geometric parameters of the deformation process and p. The latter is determined by stress boundary conditions. In particular, in the case of drawing Eq. (26) reduces to Eq. (27). The simplicity of Eq. (26) makes it suitable for quick design of the process. This preliminary design can also be used as an initial guess for sophisticated design solutions based on numerical methods.

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